Discrete Mathematics and Graph Theory

Module 3 - Counting Techniques

Aarthy B

Division of Mathematics,
School of Advanced Sciences,
Vellore Institute of Technology,
Chennai campus.

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Contents

Introduction

- Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics.
- Enumeration, the counting of objects with certain properties, is an important part of combinatorics.
- We must count objects to solve many different types of problems.
- For instance, counting is required to determine the complexity of algorithms, to determine whether there are enough telephone numbers or Internet protocol addresses to meet the demand, in mathematical biology in sequencing DNA and also when probability of events are computed.

Counting Principles

- Product rule
- Sum rule
- Subtraction rule

Product Rule

Product rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Problem 1

A new company with just two employees, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution:

The procedure of assigning offices to these two employees consists of assigning an office to the first employee, which can be done in 12 ways, then assigning an office to the second employee different from the office assigned to the first employee, which can be done in 11 ways.

By the product rule, there are $12 \times 11 = 132$ ways to assign offices to these two employees.

Problem 2

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution:

The procedure of labeling a chair consists of two tasks:

1. assigning to the seat one of the 26 uppercase English letters and
2. assigning to it one of the 100 possible integers.

The product rule shows that there are $26 \times 100 = 2600$ different ways that a chair can be labeled.

$\therefore$ The largest number of chairs that can be labeled differently is 2600.

Problem 3

There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Solution:

The procedure of choosing a port consists of two tasks:

1. picking a computer and
2. picking a port on this computer.

Because there are 32 ways to choose the computer and 24 ways to choose the port, the product rule shows that there are $32 \times 24 = 768$ ports.

Problem 4

License plates in the canadian province of Ontario consist of four letters followed by three of the digits $0-9$ (not necessarily distinct). How many different license plates can be made in Ontario?

Solution:

There are 26 ways in which the first letter can be chosen, 26 ways in which the second can be chosen. Similarly, for the third and fourth. By the product rule, the number of ways in which the three letters can be chosen is

$$26 \times 26 \times 26 \times 26 = 26^4$$

Problem 4 Contd.

By the same reasoning, there are 10^3 ways in which the final three digits of an Ontario license plate can be selected.
Thus, $26^4 \times 10^3 = 456,976,000$ different license plates which can be manufactured by the government of Ontario.

Problem 5

If 2 distinguishable dice are rolled,in how many ways can they fall? If 5 distinguishable dice are rolled, how many possible outcomes are there? How many if 100 distinguishable dice are tossed?

Solution:

The first die can fall in 6 ways and the second can fall in 6 ways. Thus, there are $6 \times 6 = 36$ outcomes when two dice are rolled.

Problem 5 contd.

Also, the third, fourth and fifth die each have 6 possible outcomes so there are $6 \times 6 \times 6 \times 6 \times 6 = 6^5$ possible outcomes when all 5 dice are tossed.
Likewise, there are 6^{100} possible outcomes when 100 dice are tossed.

Sum Rule

Sum rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Problem 1

Suppose that either a member of the mathematics faculty or a student is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics students and no one is both a faculty and a student?

Solution:

There are 37 ways to choose a member of the mathematics faculty and 83 ways to choose a member of the mathematics student.

Choosing a mathematics faculty is never the same as choosing a mathematics student because no one is both a faculty and a student.

By the sum rule, it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

Problem 2

A student can choose a computer project from one of three lists. The three lists contain $23, 15$ and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

The student can choose a project by selecting a project from the first list, second list or the third list. Because no project is on more than one list, by the sum rule, there are $23 + 15 + 19 = 57$ ways to choose a project.

Problem 3

In how many ways can we draw a heart or a spade from an ordinary deck of playing cards? A heart or an ace? An ace or a king? A card numbered 2 through 10? A numbered card or a king?

Solution:

- Since there are 13 hearts and 13 spades, we may draw a heart or a spade in $13 + 13 = 26$ ways.
- We may draw a heart or an ace in $13 + 3 = 16$ ways since there are only 3 aces that are not hearts.
- We may draw an ace or a king in $4 + 4 = 8$ ways.

Problem 3 Contd.

- There are 9 cards numbered 2 through 10 in each of 4 suits - clubs, diamonds, hearts or spades. So, we may choose a numbered card in $9 + 9 + 9 + 9 = 36$ ways.
- We may choose a numbered card or a king in $36 + 4 = 40$ ways.

Problem 4

How many ways can we get a sum of 4 or 8 when two distinguishable dice (say one die is red and the other is white) are rolled? How many ways can we get an even sum?

Solution:

Let us label the outcome of a 1 on the red die and a 3 on the white die as the ordered pair $(1, 3)$.
Then, we see that the outcomes $(1, 3), (2, 2)$ and $(3, 1)$ are the only ones whose sum is 4.
Thus, there are 3 ways to obtain the sum 4.
Likewise, we obtain the sum 8 from the outcomes $(2, 6), (3, 5), (4, 4)$, $(5, 3)$ and $(6, 2)$.
Thus, there are $3 + 5 = 8$ outcomes whose sum is 4 or 8.

Problem 4 Contd.

The number of ways to obtain an even sum is the same as the number of ways to obtain either the sum $2, 4, 6, 8, 10$ or 12.
There is 1 way to obtain the sum 2, 3 ways to obtain the sum 4, 5 ways to obtain 6, 5 ways to obtain 8, 3 ways to obtain a 10 and 1 way to obtain a 12.
Therefore, there are $1 + 3 + 5 + 5 + 3 + 1 = 18$ ways to obtain an even sum.

Problem 5

How many ways can we get a sum of 8 when two indistinguishable dice are rolled? An even sum?

Solution:

If the dice are distinguishable, we should obtain a sum of 8 by the outcomes $(2, 6), (3, 5), (4, 4)$, $(5, 3)$ and $(6, 2)$.
But since the dice are similar, the outcomes $(2, 6)$ and $(6, 2)$ and as well $(3, 5)$ and $(5, 3)$ cannot be differentiated and thus we obtain the sum of 8 with the roll of two similar dice in only 3 ways.
Likewise, we can get an even sum in $1 + 2 + 3 + 3 + 2 + 1 = 12$ ways.

Principle of Inclusion-Exclusion

Principle of Inclusion-Exclusion

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Note:

- To correctly count the number of ways to do the two tasks, we must subtract the number of ways that are counted twice.
- This is also known as *Subtraction Rule*.

Note

1. $|A \cup B| = |A| + |B| - |A \cap B|$
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
3. $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$
4. $(A \cap B) \cap (A \cap C) = A \cap B \cap C$

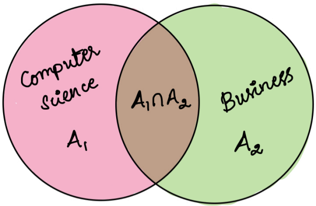
Problem 1

A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in Computer Science, 147 majored in Business, and 51 majored in both Computer Science and Business. How many of these applicants majored neither in Computer Science nor Business?

Solution:

To find the number of applicants who majored neither in Computer Science nor Business, we can subtract the number of students who majored in both Computer Science and Business from the total number of applicants.
Let A be the set of students who majored in Computer Science and B be the set of students who majored in Business. Then, $|A_1| = 220, |A_2| = 147$ and $|A_1 \cap A_2| = 51$.

Problem 1 Contd.



We have,

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= 220 + 147 - 52 \\ &= 316. \end{aligned}$$

Thus, we conclude that $350 - 316 = 34$ of the applicants majored neither in Computer Science nor Business.

Problem 2

There are 250 students in an engineering college. Of these 188 have taken a course in Fortran, 100 have taken a course in C and 35 have taken a course in Java. Further 88 have taken courses in both Fortran and C. 23 have taken courses in both C and Java and 29 have taken courses in both Fortran and Java. If 19 of these students have taken all the three courses, how many of these 250 students have not taken a course in any of these three programming languages?

Solution:

Let F, C and J denote the students who have taken the languages Fortran, C and Java respectively. Then,
$|F| = 188, |C| = 100, |J| = 35, |F \cap C| = 88, |F \cap J| = 29, |C \cap J| = 23, |F \cap C \cap J| = 19.$

Problem 2 Contd.

Then the number of students who have taken at least one of the three languages is given by,

$$\begin{aligned} |F \cup C \cup J| &= |F| + |C| + |J| - |F \cap C| \\ &- |F \cap J| - |C \cap J| - |F \cap C \cap J| \\ &= 188 + 100 + 35 - 88 - 23 - 29 + 19 \\ &= 323 - 140 + 19 \\ &= 202 \end{aligned}$$

No. of students who have not taken a course in any of these languages $= 250 - 202 = 48.$

Problem 3

At a university 60% of the teachers play tennis, 50% of them play bridge, 70% jog, 20% play tennis and bridge, 30% play tennis and jog and 40% play bridge and jog. What is the percentage of teachers who jog, play tennis and play bridge?

Solution:

Let us assume that there are 100 teachers in the university. Let T, B and J be the set of teachers who play tennis, who play bridge and who jog respectively. We have

$$|T| = 60, |B| = 50, |J| = 70$$
$$|T \cap B| = 20, |T \cap J| = 30, |B \cap J| = 40$$
$$|T \cup B \cup J| = 100, |T \cap B \cap J| = ?$$

Problem 3 Contd.

From the formula,

$$\begin{aligned} |T \cup B \cup J| = |T| + |B| + |J| - |T \cap B| \\ - |T \cap J| - |B \cap J| + |T \cap B \cap J| \\ 100 = 60 + 50 + 70 - 20 - 30 - 40 + |T \cap B \cap J| \end{aligned}$$

We have,

$$|T \cap B \cap J| = 100 - 90 = 10$$

Therefore, there are 10% of the teachers who jog, play tennis and play bridge.

Problem 4

Find the number of integers between 1 and 250 both inclusive that are not divisible by any of the integers $2, 3, 5$ and 7.

Solution:

Let A, B, C, D be the sets of integers that lie between 1 and 250 and that are divisible by $2, 3, 5,$ and 7 respectively.

The elements of A are $2, 4, 6, ..., 250$.

$\therefore |A| = 125$ which is the same as $\lfloor \frac{250}{2} \rfloor$.

Similarly, $|B| = \lfloor \frac{250}{3} \rfloor = 83$

$|C| = \lfloor \frac{250}{5} \rfloor = 50$

$|D| = \lfloor \frac{250}{7} \rfloor = 35$

Problem 4 Contd.

The set of integers between 1 and 250 which are divisible by 2 and 3, that is, $A \cap B$ is the same as that which is divisible by 6, since 2 and 3 are relatively prime numbers.

$\therefore |A \cap B| = \lfloor \frac{250}{6} \rfloor = 41$

$|A \cap C| = \lfloor \frac{250}{10} \rfloor = 25$

$|A \cap D| = \lfloor \frac{250}{14} \rfloor = 17$

$|B \cap C| = \lfloor \frac{250}{15} \rfloor = 16$

$|B \cap D| = \lfloor \frac{250}{21} \rfloor = 11$

$|C \cap D| = \lfloor \frac{250}{35} \rfloor = 7$

Problem 4 Contd.

$|A \cap B \cap C| = \lfloor \frac{250}{30} \rfloor = 8$

$|A \cap B \cap D| = \lfloor \frac{250}{42} \rfloor = 5$

$|A \cap C \cap D| = \lfloor \frac{250}{70} \rfloor = 3$

$|B \cap C \cap D| = \lfloor \frac{250}{105} \rfloor = 2$

$|A \cap B \cap C \cap D| = \lfloor \frac{250}{210} \rfloor = 1$

By the Principle of Inclusion-Exclusion, the number of integers between 1 and 250 that are divisible by at least one of $2, 3, 5$ and 7 is given by,

$$\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| \\
& - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| \\
& + |B \cap C \cap D| - |A \cap B \cap C \cap D|
\end{aligned}$$

Problem 4 Contd.

$$= 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7$$
$$+ 8 + 5 + 3 + 2 - 1$$
$$= 193$$

Number of integers between 1 and 250 that are not divisible by any of the integers $2, 3, 5$ and 7 is

$$= \text{Total no. of integers } - |A \cup B \cup C \cup D|$$
$$= 250 - 193$$
$$= 57.$$

Problem 5

Of 30 PCs owned by faculty members in a certain university department, 20 run windows, 8 have 21 inch monitors, 25 have CD-ROM drives, 20 have at least two of these features, and 6 have all three.

a) How many PCs have at least one of these features?
b) How many PCs have none of these?
c) How many have exactly one feature?
d) How many have exactly two of three features described?

Solution:

Let W be the set of PCs running windows, M the set of PCs with 21 inch monitors and C the set with CD-ROM drives. Then,

$|W| = 20, |M| = 8, |C| = 25,$

$|(W \cap M) \cup (W \cap C) \cup (M \cap C)| = 20$ and $|W \cap M \cap C| = 6.$

Problem 5 Contd.

By the principle of inclusion-exclusion,

$$\begin{aligned}20 &= |(W \cap M) \cup (W \cap C) \cup (M \cap C)| \\ &= |W \cap M| + |W \cap C| + |M \cap C| - |(W \cap M) \cap (W \cap C)| \\ &- |(W \cap C) \cap (M \cap C)| - |(W \cap M) \cap (M \cap C)| \\ &+ |(W \cap M) \cap (W \cap C) \cap (M \cap C)|\end{aligned}$$

Since each of the last four terms here is $|W \cap M \cap C|$,
We obtain,

$$20 = |W \cap M| + |W \cap C| + |M \cap C| - 2|W \cap M \cap C|$$

Therefore,

$$|W \cap M| + |W \cap C| + |M \cap C| = 20 + 2(6) = 32.$$

Problem 5 Contd.

a) The number of PCs with at least one of these features is

$$\begin{aligned}|W \cup M \cup C| &= |W| + |M| + |C| - |W \cap M| \\ &\quad - |W \cap C| - |M \cap C| + |W \cap M \cap C| \\ &= 20 + 8 + 25 - (|W \cap M| + |W \cap C| + |M \cap C|) + 6 \\ &= 59 - 32 \\ &= 27.\end{aligned}$$

b) It follows that $30 - 27 = 3$ PCs have none of the specified features.

Problem 5 Contd.

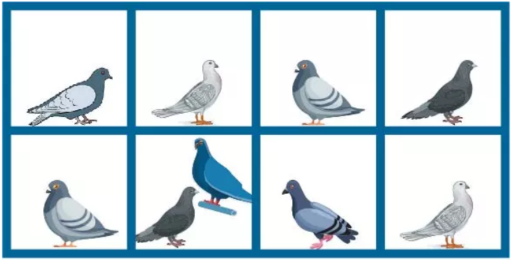
c) Since the number of computers with exactly one feature is the number with at least one feature less the number with at least two, the number with exactly one is $27 - 20 = 7$.

d) The number of computers with exactly two features is the number with at least two less the number with exactly three. That is, $20 - 6 = 14$.

Pigeonhole Principle

The Pigeonhole principle

If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



Examples

1. Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 possible birthdays.
2. In any group of 27 English words, there must be atleast two words that begin with the same letter, because there are 26 letters in the English alphabets.

Generalisation of the Pigeonhole principle

If n pigeons are accomodated in m pigeonholes and $n > m$, then one of the pigeonholes must contain at least $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$ pigeons where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, which is a real number.

Problem 1

Show that among 100 people, at least 9 of them were born in the same month.

Solution:

Here, No. of Pigeons = No. of People = n = 100.

No. of Pigeonholes = No. of Month = m = 12.

Then by generalized pigeon hole principles,

$\left\lfloor \frac{(100-1)}{12} \right\rfloor + 1 = 9$, were born in the same month.

Problem 2

Show that if seven colors are used to paint 50 cycles, at least 8 cycles will be the same color.

Solution:

Here, No. of Pigeons = No. of Cycle = n = 50.

No. of Pigeonholes = No. of Colors = m = 7.

Then by generalized pigeon hole principles,

$\left\lfloor \frac{(50-1)}{7} \right\rfloor + 1 = 8.$ Thus, at least 8 cycles will have the same color.

Problem 3

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade of there are five possible grade A, B, C, D and F.

Solution:

Here, No. of Pigeons = No. of students in DM class = n.

No. of Pigeonholes = No. of grades = m = 5.

Then by generalized pigeon hole principles,

$$= \left\lfloor \frac{(n-1)}{5} \right\rfloor + 1 = 6$$

$$= \left\lfloor \frac{(n-1)}{5} \right\rfloor = 5$$

. $n - 1 = 25 \implies n = 26$. Thus, the total number of students is 26.

Problem 4

Prove that in any group of six people, at least three people must be mutual friends or at least three must be mutual strangers.

Solution:

Let A be one of the six people. Let the remaining 5 people be accomodated in 2 rooms labeled "A's friends" and "strangers to A".

Treating 5 people (n) as 5 pigeons and 2 rooms (m) as pigeonholes, by the generalised pigeonhole principle, one of the rooms must contain $\left\lfloor \frac{(5-1)}{2} \right\rfloor + 1 = 3$ people.

Problem 4 Contd.

Let the room labeled "A's friends" contain 3 people. If any two of these 3 people are friends, then together with A, we have a set of 3 mutual friends.

If no of these 3 people are friends, then these 3 people are mutual strangers.

In either case, we get the required conclusion.

Permutations and Combinations

Permutations

An ordered arrangement of r elements of a set containing n distinct elements is called an $r-$permutation of n elements and is denoted by $P(n, r)$ or nP_r where $r \leq n$.

Example:

Let $S = \{a, b, c\}$. The $2-$ permutations of S are the ordered arrangements $a, b; a, c; b, a; b, c; c, a;$ and c, b. There are six $2-$ permutations of this set with three elements. There are three ways to choose the first element and two ways to choose the second element because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \times 2 = 6$.

Theorem

If n and r are integers with $0 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Problem 1

How many ways are there to select a first prize winner, a second prize winner, and a third prize winner from 100 different people who have entered a contest?

Solution:

The number of ways to pick the three prize winners is the number of ordered selections of three elements from a set 100 elements, that is, the number of 3- permutations of a set of 100 elements. Consequently, the answer is,

$$P(100, 3) = 100 \times 99 \times 98 = 9, 70, 200.$$

Problem 2

Suppose there are 8 runners in a race. The winner receives a gold medal, the second place finisher receives a silver medal, and the third place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution:

The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are

$$P(8,3) = 8 \times 7 \times 6 = 336$$

possible ways to award the medals.

Problem 3

Suppose that a salesman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting the cities?

Solution:

The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. So, there are

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths.

Problem 4

How many permutations of the letters $ABCDEFGH$ contain the string ABC?

Solution:

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects namely, the block ABC and the individual letters D, E, F, G and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters $ABCDEFGH$ in which ABC occurs as a block.

Problem 5

Find the number of positive integers greater than a million that can be formed with the digits $2, 3, 0, 3, 4, 2$ and 3.

Solution:

The numbers greater than a million must be of 7 digits.
In the given set of digits, 2 appear twice, 3 appear thrice and all others are distinct.
The total number of 7 digit numbers that can be formed with given digits is

$$\frac{7!}{2!3!} = 420.$$

Problem 5 Contd.

The set of 420 positive integers, include some numbers which begin with 0.
Clearly, these numbers are less than a million and they must not be counted for the answer.
The number of such numbers is given by the permutations of 6 non zero digits and is equal to

$$\frac{6!}{2!3!} = 60.$$

Therefore, the number of positive integers greater than a million that can be formed using the digits is $420 - 60 = 360$.

Problem 6

In how many ways can the letters of the word $MONDAY$ be arranged? How many of them begin with M and end with Y? How many of them do not begin with M but end with?

Solution:

The word $MONDAY$ consists of 6 letters which can be arranged in $P(6,6) = 6! = 720$ ways.
If M occupies the first place and Y the last place, then there are 4 letters (O, N, D, A) left to be arranged in 4 places in between M and Y. This can be done in $4! = 24$ ways.
If M does not occupy the first place but Y occupies the last place, the first place can be occupied in 4 ways by any one of the letters O, N, D, A.

Problem 6 Contd.

For the second place, again 4 letters are available including M.
The third, fourth and fifth places can be filled by $3, 2, 1$ ways.
Hence, by product rule, the required number of arrangements are $4 \times 4 \times 3 \times 2 \times 1 = 96$.

Problem 7

A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find

a) the total number of passwords that can be formed.

b) the number of passwords in which no digit repeats.

Solution:

a) Since there are 26 alphabets and 10 digits and the digits can be repeated, by product rule, the number of 4- character password is $26 \times 10 \times 10 \times 10 = 26000$.
Similarly, the number of 5- character password is

$$26 \times 10 \times 10 \times 10 \times 10 = 260000.$$

Hence, the total number of passwords is

$$26000 + 260000 = 286000.$$

Problem 7 Contd.

b) Since the digits are not repeated, he first digit after alphabet can be taken from any one out of 10, the second digit from remaining 9 digits and so on. Thus, the number of 4- character password is

$$26 \times 10 \times 9 \times 8 = 18720$$

and the number of 5- character password is

$$26 \times 10 \times 9 \times 8 \times 7 = 131040$$

by the product rule.
Hence, the total number of passwords is 149760.

Problem 8

How many bit strings of length 10 contains

a) exactly four 1's

b) at most four 1's

c) an equal number of 0's and 1's

Solution:

a) A bit string of length 10 can be considered to have 10 positions. These 10 positions should be filled with four 1's and six 0's.

$$\text{No. of required bit strings} = \frac{10!}{4!6!} = 210$$

Problem 8 Contd.

b) The 10 positions should be filled with

- no 1's and ten 0's
- one 1's and nine 0's
- two 1's and eight 0's
- three 1's and seven 0's
- four 1's and six 0's

$$\text{No. of required bit strings} = \frac{10!}{0!10!} + \frac{10!}{1!9!} + \frac{10!}{2!8!} + \frac{10!}{3!7!} + \frac{10!}{4!6!} = 386.$$

Problem 8 Contd.

c) The 10 positions should be filled with 5 men and 5 women

$$\text{No. of required bit strings} = \frac{10!}{5!5!} = 252.$$

Circular Permutation - Problem 9

If 6 people A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from other by rotation? If A, B, C are females and the others are males, in how many arrangements do the genders alternate?

Solution:

The no. of different circular arrangements of n objects in $(n-1)!$.
$\therefore$ The required number of circular arrangements is $= 5! = 120$.

Circular Permutation - Problem 9 Contd.

Since rotation does not alter the circular arrangement, we can assume that A occupies the top position as shown in the figure.
Of the remaining places, positions $1, 3, 5$ must be occupied by the 3 males. This can be achieved in $P(3,3) = 3! = 6$ ways.
The remaining two places 2 and 4 should be occupied by the remaining two females. This can be achieved in $P(2,2) = 2! = 2$ ways.
$\therefore$ Total no.of required circular arrangements $= 6 \times 2 = 12$.

Problem 10

Five boys and five girls are to be arranged around in a circular table for a discussion so that the boys and girls alternate. In how many ways can they be seated?

Solution:

First arrange the 5 boys around the table. This can be done in $(n-1)! = (5-1)! = 4! = 24$ ways. In each of these arrangements, there are 5 gaps in which the girls can be arranged in $5!$ ways. Hence the total number of ways are $5! \times 4! = 2880$ ways.

Problem 11

A round table conference is to be held between 10 delegates of 10 countries. In how many ways can they be seated if

1. two particular delegates are always together
2. two particular delegates are on either side of the chairperson.

Problem 11 Contd.

Solution:

1. Since two particular delegates are always together, treat them as one unit and with the other 8 people, we have 9 units. They can be arranged in $(n-1)! = (9-1)! = 8!$. In each of these arrangements the two can be arranged among themselves in $2!$ ways. Thus the total number of arrangements is $8! \times 2! = 80,640$ ways.
2. Since two particular delegates are on either side of the chairperson, these three are always together. So treat them as a single unit and the other 7, we have in total 8 units. They can be arranged in $(n-1)! = (8-1)! = 7!$. In each of these arrangements the two can be interchanged among themselves in 2 ways. Thus the total number of arrangements is $7! \times 2 = 10,080$ ways.

Combinations

An unordered selection of r elements of a set containing n distinct elements is called an $r-$combination of n elements and is denoted by $C(n,r)$ or nC_r or $\binom{n}{x}$ where $r \leq n$.

Theorem

The number of r- combinations of a set with n elements, where n is a non negative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Problem 1

How many ways are there to select five players from a 10- member tennis team to make a trip to a match at another school?

Solution:

The answer is given by the number of 5- combinations of a set with 10 elements. The number of such combinations is,

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Problem 2

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of 6 people to go on this mission?

Solution:

The number of ways to select a crew of 6 from 30 people is the number of 6- combinations of a set with 30 elements, because the order in which these people are chosen does not matter. The number of such combinations is,

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \times 29 \times 28 \times 27 \times 26 \times 25}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 5,93,775.$$

Problem 3

Suppose there are 9 faculty members in the Mathematics department and 11 in the Computer Science department. How many ways are there to select a committee to develop a Discrete Mathematics course at a school if the committee is to consist of 3 faculty members from the Mathematics department and 4 faculty from the Computer Science department?

Solution:

By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4- combinations of a set with 11 elements. The number of ways to select a committee is,

$$C(9,3) \times C(11,4) = \frac{9!}{3!6!} \times \frac{11!}{4!7!} = 84 \times 330 = 27,720.$$

Problem 4

Laura is going to toss a coin eight times. In how many ways can she get five heads and three tails?

Solution:

Laura might get a string of five heads followed by three tails $HHHHHTTT$ or a string of three tails followed by five heads $TTTHHHHH$, or the sequence $HTHHTHTH$, and so on.
The number of such sequences is the number of ways of selecting five occasions (from the eight) in which the heads should come up or equivalently, the number of ways of selecting the three occasions on which tails should come up. The answer is,

$$C(8,5) = C(8,3) = 56.$$

Problem 5

Reynold has 10 children but his car holds only 5 people (including driver). When he goes to the circus, in how many ways can he select four children to accompany him?

Solution:

The question involves choosing, not order. There are,

$$C(10,4) = \frac{10!}{4!6!} = 210$$

different ways.

Problem 6

How many committees of 5 people can be chosen from 20 men and 12 women.

a) if exactly 3 men must be on each committee?

b) if at least 4 women must be on each committee?

Solution:

a) We must choose 3 men from 20 and then 2 women from 12. The answer is,

$$C(20,3) \times C(12,2) = 1140 \times 66 = 75,240.$$

b) We calculate the case of 4 women and 5 women separately and add the results (using Sum rule). The answer is,

$$C(12,4) \times C(20,1) + C(12,5) \times C(20,0) = 495 \times 20 + 792 = 10,692.$$

Problem 7

A man has 7 relatives, 4 ladies and 3 gentlemen. His wife has 7 relatives, 3 ladies and 4 gentlemen. In how many ways can they invite them to a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?

Solution:

They can invite in 4 possible ways:

1. 3 ladies from husband's side and 3 men from wife's side.
2. 3 men from husband's side and 3 ladies from wife's side.
3. 2 ladies and 1 man from husband's side; 1 lady and 2 men from wife's side.
4. 1 lady and 2 men from husband's side; 2 ladies and 1 man from wife's side.

Problem 7 Contd.

1. No.of ways $= C(4,3) \times C(4,3) = 16.$
2. No.of ways $= C(3,3) \times C(3,3) = 1.$
3. No.of ways $= [C(4,2) \times C(3,1)] \times [C(3,1) \times C(4,2)] = 324.$
4. No.of ways $= [C(4,1) \times C(3,2)] \times [C(3,2) \times C(4,1)] = 144.$

The total no.of ways $= 16 + 1 + 324 + 144 = 485.$

Problem 8

There are six men and five women in a room. Find the number of ways four persons can be drawn from the room.

1. They can be male or female
2. Two men and two women
3. They must all be of same gender

Solution:

1. Four persons can be drawn from 11 persons is $C(11,4) = 330$ ways.
2. Two men can be selected in $C(6,2)$ ways and two women can be selected in $C(5,2)$ ways. Hence the number of ways of selecting 2 men and 2 women are $C(6,2) \times C(5,2) = 75$.
3. Number of ways of selecting 4 of same gender is $C(6,4) + C(5,4) = 20$.

Problem 9

From a group consisting of 6 men and 7 women, in how many ways can we select a committee of

a) 3 men and 4 women
b) 4 members which has at least one women
c) 4 persons that has at most one men
d) 4 persons of both genders

Solution:

a) Three men can be selected from 6 men in $C(6,3)$ ways, 4 women can be selected in $C(7,4)$ ways. By product rule, we have $C(6,3) \times C(7,4) = 700$.

Problem 9 Contd.

b) For a committee of at least one women we have the following possibilities

- 1 women and 3 men
- 2 women and 2 men
- 3 women and 1 men
- 4 women and 0 men

Thus, the selection can be done in

$$= (7C_1 \times 6C_3) + (7C_2 \times 6C_2) + (7C_3 \times 6C_1) + (7C_4 \times 6C_0)$$
$$= 140 + 315 + 210 + 35 = 700.$$

Problem 9 Contd.

c) For a committee of almost one men we have the following possibilities

- 1 men and 3 women
- 0 men and 4 women

Thus, the selection can be done in

$$= (6C_1 \times 7C_3) + (6C_0 \times 7C_4) = 245.$$

Problem 9 Contd.

d) For a committee of both genders we have the following possibilities

- 1 men and 3 women
- 2 men and 2 women
- 3 men and 1 women

Thus, the selection can be done in

$$= (6C_1 \times 7C_3) + (6C_2 \times 7C_2 + (6C_3 \times 7C_1) = 665.$$

Combination with repetition - Problem 10

In how many ways can 2 letters be selected from the set $\{a, b, c, d\}$ when repetition of the letters is allowed, if

a) the order of the letters matters

b) the order does not matter?

Solution:

a) When the order of the selected letters matters, the number of possible selections $= 4^2 = 16$, which are listed below:

$$\begin{aligned} aa, ab, ac, ad \\ ba, bb, bc, bd \\ ca, cb, cc, cd \\ da, db, dc, dd \end{aligned}$$

Problem 10 Contd.

In general, the number of $r-$permutations of n objects, if repetition of the objects is allowed, is equal to n^r, since there are n ways to select an object from the set for each of the $r-$ positions.

When the order of the selected letter does not matter, the number of possible selections $C(4+2–1,2) = C(5,2) = 10$, which are listed below:

$$\begin{aligned} aa, ab, ac, ad \\ bb, bc, bd \\ cc, cd \\ dd \end{aligned}$$

Combination with repetition

In general, the number of $r-$combinations of n kinds of objects, if repetitions of the objects is allowed $= C(n + r–1, r)$.

Combination with repetition - Problem 11

There are 3 piles of identical red, blue and green balls, where each pile contains at least 10 balls. In how many ways can 10 balls be selected:

a) if there is no restriction?

b) if at least one red ball must be selected?

c) if at least one red ball, at least 2 blue balls and at least 3 green balls must be selected?

d) if exactly one red ball must be selected?

e) if exactly one red ball and at least one blue ball must be selected?

f) if at most one red ball is selected?

g) if twice as many red balls as green balls must be selected?

Problem 11 Contd.

a) There are $n = 3$ kinds of balls and we have to select $r = 10$ balls, when repetitions are allowed.
$\therefore$ No. of ways of selecting $= C(n + r - 1, r) = C(12, 10) = 66$.

b) We take one red ball and keep it aside. Then we have to select 9 balls from the 3 kinds of balls and include the first red ball in the selections.
$\therefore$ No. of ways of selecting $= C(11, 9) = 55$.

Problem 11 Contd.

c) We take away 1 red, 2 blue and 3 green balls and keep them aside. Then we select 4 balls from the 3 kinds of balls and include the 6 already chosen bolls in each selection.

$\therefore$ No. of ways of selecting $= C(3+4–1,4) = 15.$

d) We select 9 balls from the piles containing blue and green balls and include 1 red ball in each selection.

$\therefore$ No. of ways of selecting $= C(2+9–1,9) = 10.$

Problem 11 Contd.

e) We take away one red ball and one blue ball and keep them aside. Then we select 8 balls from the blue and green piles and include the already reserved red and blue balls to each selection.
$\therefore$ No. of ways of selecting $= C(2+8–1, 8) = 9$.

f) The selections must contain no red ball or 1 red ball.
$\therefore$ No. of ways of selecting

$$\begin{aligned} &= C(2+10–1, 10) + C(2+9–1, 9) \\ &= 11 + 10 = 21 \end{aligned}$$

Problem 11 Contd.

g) The selections must contain 0 red and 0 green balls or 2 red and 1 green balls or 4 red and 2 green balls or 6 red and 3 green balls.
$\therefore$ No. of ways of selecting

$$= C(1+10–1,10) + C(1+7–1,7) + C(1+4–1,4) + C(1+1–1,1)$$
$$= C(10,10) + C(7,7) + C(4,4) + C(1,1)$$
$$= 1+1+1+1 = 4.$$

Recurrence Relation

Recurrence Relation

An equation that expresses a_n namely the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \cdots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer is called a **recurrence relation for $\{a_n\}$** or a **difference equation.**

Examples

1. Consider a geometric progression $4, 12, 36, 108, \cdots$. The common ratio is 3.
2. If $\{a_n\}$ represents this infinite sequence, we see that $\frac{a_{n+1}}{a_n} = 3 \implies a_{n+1} = 3a_n, n \geq 0$ is the recurrence relation corresponding to the geometric sequence.
3. However, the above recurrence relation does not represent a unique geometric sequence. Since the sequence $5, 15, 45, \cdots$ also satisfies the above relation.

Examples

1. In order that the recurrence relation $a_{n+1} = 3a_n, n \geq 0$ may represent a unique sequence, we should know one of the terms of the sequence, say, $a_0 = 4$.
2. The value $a_0 = 4$ is called the **initial condition.**.
3. Consider a Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$ which can be represented by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0, F_1 = 1$.

Recurrence Relation - General form

1. A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

   is called a **recurrence relation of degree k** with constant coefficients, where $c_0, c_1, c_2, \cdots, c_k$ are real numbers and $c_k \neq 0$.
2. The recurrence relation is called **linear** because each a_r is raised to the power one and there are no products such as $a_r.a_s$.
3. The **degree** is the difference between the greatest and least subscripts of the members of the sequence occurring in the recurrence relation.

Solution of Recurrence relation

- Characteristics Root method
  - Linear Homogeneous recurrence relation with constant coefficients
  - Linear Non-homogeneous recurrence relation with constant coefficients
- Generating function method

Homogeneous Linear recurrence relation

Homogeneous Linear recurrence relation

Characteristics Root method - Homogeneous

Definition:

A recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \qquad (1)$$

where $c_1, c_2, \cdots, c_k$ are real numbers and $c_k \neq 0$ is called a linear homogeneous recurrence relation of degree k with constant coefficients.

The characteristic equation of (1) is given by

$$c_0 r^2 + c_1 r + c_2 = 0 \qquad (2)$$

The roots r_1 and r_2 of (2) are called the characteristic roots.

Characteristics Root method - Homogeneous - General Solution

Roots	**General solution**
r_1 and r_2 are real and distinct	$a_n = Ar_1^n + Br_2^n$
r_1 and r_2 are real and equal	$a_n = (A + Bn)r^n$
$r_1 = \alpha + i\beta, r_2 = \alpha - i\beta,$ $r = \sqrt{\alpha^2 + \beta^2}, \theta = tan^{-1}(\frac{\beta}{\alpha})$	$a_n = r^n[Acosn\theta + Bsinn\theta]$

To determine the unique solution, the constants A and B are to be determined using the initial conditions.

Problem 1

What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution:

The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$.
Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

for some constants α_1 and α_2.

Problem 1 Contd.

From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$
$$a_1 = 7 = \alpha_1.2 + \alpha_2.(-1)$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$.
Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3.2^n - (-1)^n.$$

Problem 2

What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} \text{ with initial conditions } a_0 = 1 \text{ and } a_1 = 6?$$

Solution:

The only root of $r^2 - 6r + 9 = 0$ is $r = 3$.
Hence, the solution to this recurrence relation is $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2.

Problem 2 Contd.

From the initial conditions, it follows that

$$a_0 = 1 = \alpha_1$$
$$a_1 = 6 = \alpha_1.3 + \alpha_2.3$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

Problem 3

Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

Solution:

The characteristic polynomial of this recurrence relation is $r^3 - 6r^2 + 11r - 6 = 0$.
The characteristic roots are $r = 1$, $r = 2$ and $r = 3$.
Since $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$.
Hence, the solutions to this recurrence relation are of the form $a_n = \alpha_1.1^n + \alpha_2.2^n + \alpha_3.3^n$.

Problem 3 Contd.

To find constants α_1, α_2 and α_3, use the initial conditions.
This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$
$$a_1 = 5 = \alpha_1 + \alpha_2.2 + \alpha_3.3$$
$$a_2 = 15 = \alpha_1 + \alpha_2.4 + \alpha_3.9$$

When these three simultaneous equations are solved for α_1, α_2 and α_3, we find that $\alpha_1 = 1, \alpha_2 = -1$ and $\alpha_3 = 2$.
Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with $a_n = 1 - 2^n + 2.3^n$.

Problem 4

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$.

Solution:

The characteristic equation of this recurrence relation is $r^3 + 3r^2 + 3r + 1 = 0$.
Since $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation.
The solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}.n.(-1)^n + \alpha_{1,2}.n^2.(-1)^n$$

To find the constants $\alpha_{1,0}, \alpha_{1,1}$ and $\alpha_{1,2}$, use the initial conditions.

Problem 4 Contd.

This gives

$$a_0 = 1 = \alpha_{1,0}$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$$
$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$$

The simultaneous solutions of these three equations is,

$$\alpha_{1,0} = 1, \alpha_{1,1} = 3 \text{ and } \alpha_{1,2} = -2.$$

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with $a_n = (1 + 3n - 2n^2)(-1)^n$.

Problem 5

Find the explicit formula for the Fibonacci numbers.

Solution:

The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{(1+\sqrt{5})}{2} \text{ and } r_2 = \frac{(1-\sqrt{5})}{2}$$

Therefore, it follows that the Fibonacci numbers are given by,

$$f_n = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right]^n$$

for some constants α_1 and α_2.

Problem 5 Contd.

The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants.
We have, $f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1 \left[\frac{1+\sqrt{5}}{2}\right] + \alpha_2 \left[\frac{1-\sqrt{5}}{2}\right] = 1$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = \frac{1}{\sqrt{5}} \text{ and } \alpha_2 = -\frac{1}{\sqrt{5}}.$$

Consequently, the Fibonacci numbers are given by,

$$f_n = \frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}} \left[\frac{1-\sqrt{5}}{2}\right]^n$$

Problem 6

Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, n \geq 2$$

given $a_0 = 1, a_1 = 4$.

Solution:

The characteristic polynomial, $x^2 - 5x + 6$, has distinct roots $x_1 = 2, x_2 = 3$.
The solution is given by $a_n = C_1(2^n) + C_2(3^n)$.
Since $a_0 = 1$, we must have $C_1(2^0) + C_2(3^0) = 1$ and since $a_1 = 4$, we have $C_1(2^1) + C_2(3^1) = 4$.
Therefore,

$$C_1 + C_2 = 1$$
$$2C_1 + 3C_2 = 4.$$

Problem 6 Contd.

Solving, we have

$$C_1 = -1, C_2 = 2.$$

The solution is

$$a_n = -2^n + 2(3^n)$$

Problem 7

Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}, n \geq 2$$

with initial conditions $a_0 = 1, a_1 = 4$.

Solution:

The characteristic polynomial, $x^2 - 4x + 4$, has repeated root $x = 2$.
The solution is given by $a_n = C_1(2^n) + C_2 n(2^n)$.
The initial conditions yield

$$C_1 = 1, 2C_1 + 2C_2 = 4. \text{ So, } C_2 = 1$$

Thus, $a_n = 2^n + n(2^n) = (n+1)2^n$.

Problem 8

Solve the recurrence relation

$$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$$

Solution:

Let $a_n = r^n$ be a solution of the given equation.
The characteristic equation is,

$$\begin{aligned} r^3 - 8r^2 + 21r - 18 &= 0 \\ r^3 - 2r^2 - 6r^2 + 12r + 9r - 18 &= 0 \\ r^2(r-2) - 6r(r-2) + 9(r-2) &= 0 \\ (r-2)(r^2 - 6r + 9) &= 0 \\ (r-2)(r-3)^2 &= 0 \end{aligned}$$

which gives $r = 2, 3, 3$.
So, the general solution is $a_n = (b_1 + b_2 n)3^n + b_3 2^n$.

Problem 9

Solve $a_n = a_{n-1} + 2a_{n-2}, n \geq 2$ with the initial conditions $a_0 = 0, a_1 = 1$.

Solution:

The given recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$
$$a_n - a_{n-1} - 2a_{n-2} = 0 \qquad (1)$$

is a 2nd order linear homogeneous recurrence relation with constant coefficients.
Let $a_n = r^n$ be a solution of (1).
The characteristic equation is

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$\text{or} \quad r = 2, -1 \quad \text{distinct real roots.}$$

Problem 9 Contd.

So the general solution is $a_n = b_1 2^n + b_2(-1)^n$.
Again, $a_0 = 0$ implies $b_1 + b_2 = 0$.
Add $a_1 = 1$ implies $2b_1 - b_2 = 1$.
The solutions of these two equations are $b_1 = \frac{1}{3}$ and $b_2 = \frac{-1}{3}$.
Hence, the explicit solution is given by,

$$a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n.$$

Problem 10

Solve the recurrence relation $a_n = 2(a_{n-1} - a_{n-2});\ n \geq 2$ and $a_0 = 1, a_1 = 2$.

Solution:

The given recurrence relation is

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

The characteristic equation is $r^2 - 2r + 2 = 0$.
Solving, we have $r = 1 \pm i$.

The general solution is given by, $a_n = \sqrt{2}\left(Acos\frac{n\pi}{4} + Bsin\frac{n\pi}{4}\right) \quad (1).$

Problem 10 Contd.

Using the condition $a_0 = 1$ in (1), we get $A = 1$.
Using $a_1 = 2$ in (1), we get

$$\sqrt{2}\left(\frac{1}{\sqrt{2}} + B\frac{1}{\sqrt{2}}\right) = 2$$
$$i.e. B = 1$$

$\therefore$ The required solution is

$$a_n = (\sqrt{2})^n\left(cos\frac{n\pi}{4} + sin\frac{n\pi}{4}\right)$$

Non-Homogeneous Linear recurrence relation

Non-Homogeneous Linear recurrence relation

Characteristics Root method - Non Homogeneous

Definition:

A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(x) \qquad (1)$$

where $c_0, c_1, c_2, \cdots, c_k$ are real numbers with $c_0 \neq 0$, $c_k \neq 0$ is called a linear non homogeneous recurrence relation of degree k with constant coefficients.

The recurrence relation $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \quad (2)$ is called the associated homogeneous recurrence relation.

The solution of (1) depends on the solution of (2). Let $a_n^{(h)}$ be the general solution of (2).

Suppose $a_n^{(p)}$ is a particular solution of (1), then the general solution of (1) is $a_n = a_n^{(h)} + a_n^{(p)}$.

Particular Solution

Although there is no general method for finding particular solution for every function $f(n)$, there are methods which will work for polynomial in n and powers of constants. One such method is the method of undetermined coefficients or trial sequence method.

Particular Solution

No.	$f(n)$	Trial function
1.	b^n (if b is not a root of the characteristic equation)	Ab^n
2.	Polynomial $P(n)$ of degree m	$A_0 + A_1 n + A_2 n^2 + \ldots + A_n n^m$
3.	$c^n P(n)$ (if c is not a root of characteristic equation)	$c^n (A_0 + A_1 n + A_2 n^2 + \ldots + A_m n^m)$
4.	b^n (if b is a root of the characteristic equation with multiplicity s)	$A n^s b^n$
5.	$c^n P(n)$ (if c is a root of the characteristic equation with multiplicity t)	$n^t (A_0 + A_1 n + A_2 n^2 + \ldots + A_m n^m) c^n$

Problem 1

Solve the recurrence relation $a_n - 2a_{n-1} = 2^n, a_0 = 2$.

Solution:

Given recurrence relation is $a_n - 2a_{n-1} = 2^n$.
The homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$.
Since $n - (n-1) = 1$, it is first order equation.
The characteristic equation is $r - 2 = 0 \implies r = 2$.
Solution of homogeneous recurrence equation is $a_n^{(h)} = c.2^n$.
Given $f(n) = 2^n$, where 2 is the root of the characteristic equation.
$\therefore\ a_n = An2^n$ is the particular solution and the constant A is to be determined using $a_n - 2a_{n-1} = 2^n$.

Problem 1 Contd.

$$\begin{aligned} A.n2^n - 2.A(n-1)2^{n-1} &= 2^n \\ \implies 2^n[A(n-(n-1))] &= 2^n \\ \implies A(n-n+1) &= 1 \\ \implies A &= 1 \\ \therefore a_n^{(p)} &= n.2^n \end{aligned}$$

Hence, the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\implies a_n = c.2^n + n.2^n \qquad (1)$$

we have $a_0 = 2$.

Problem 1 Contd.

$\therefore$ Putting $n = 0$ in (1), we get

$$a_0 = c.2^0 + 0$$
$$2 = c \implies c = 2$$

$\therefore$ The general solution is $a_n = 2.2^n + n.2^n, n \geq 0$.
$\implies a_n = (n+2)2^n, n \geq 0$.

Problem 2

Solve $a_n - 3a_{n-1} = 2n, a_1 = 3$.

Solution:

Given $a_n - 3a_{n-1} = 2n, a_1 = 3$.
The homogeneous recurrence relation is $a_n - 3a_{n-1} = 0$.
Since $n - (n-1) = 1$, the order is 1.
The characteristic equation is $r - 3 = 0 \implies r = 3$.
Solution of homogeneous recurrence equation is $a_n^{(h)} = c.3^n$.
Given $f(n) = 2n$, which is a polynomial of degree 1.
$\therefore a_n = A_0 + A_1 n$ is the particular solution.

Problem 2 Contd.

We determine A_0 and A_1 satisfying the given equation.

$$\therefore A_0 + A_1 n - 3(A_0 + A_1(n-1)) = 2n$$
$$\implies -2nA_1 + 3A_1 - 2A_0 = 2n$$

This is true for all n, so equating like coefficients on both sides, we get
$-2A_1 = 2 \implies A_1 = -1$
and $\quad 3A_1 - 2A_0 = 0.$

$$\implies 2A_0 = 3A_1 = -3 \implies A_0 = -\frac{3}{2}$$
$$\therefore a_n^{(p)} = -\frac{3}{2} - n = -\frac{1}{2}(3+2n)$$

$\therefore$ the general solution of the given recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$\implies a_n = c.3^n - \frac{1}{2}(3+2n), n \geq 0 \qquad (1)$$

Problem 2 Contd.

Putting $n = 1$ in (1) we get

$$a_1 = c.3 - \frac{1}{2}(3+2)$$
$$\implies 3 = 3c - \frac{5}{2} \implies 3c = 3 + \frac{5}{2} = \frac{11}{2}$$
$$\implies c = \frac{11}{6}$$

$\therefore$ The general solution is

$$a_n = \frac{11}{6}.3^n - \frac{1}{2}(3+2n), n \geq 0.$$

Problem 3

Solve the recurrence relation

$$a_n - 5a_{n-1} + 6a_{n-2} = 8n^2, a_0 = 4, a_1 = 7.$$

Solution:

Given $a_n - 5a_{n-1} + 6a_{n-2} = 8n^2 \quad (1)$.
The homogeneous recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$.
Since $n - (n-2) = 2$, the order is 2.
The characteristic equation is $r^2 - 5r + 6 = 0 \implies r = 2, 3$.
Solution of homogeneous recurrence equation is $a_n^{(h)} = A.2^n + B.3^n$.
Given $f(n) = 8n^2$, which is a polynomial of degree 2 and 2 is a root of the characteristic equation.
$\therefore a_n = A_0 + A_1 n + A_2 n^2$ is the particular solution where A is to be determined using the given equation (1).

Problem 3 Contd.

Substituting in (1), we get

$$A_0 + A_1 n + A_2 n^2 - 5(A_0 + A_1(n-1) + A_2(n-1)^2)$$
$$+6(A_0 + A_1(n-2) + A_2(n-2)^2) = 8n^2$$
$$\implies A_0 + A_1 n + A_2 n^2 - 5[A_0 + A_1 n - A_1 + A_2(n^2 - 2n + 1)]$$
$$+6[A_0 + A_1 n - 2A_1 + A_2(n^2 - 4n + 4)] = 8n^2$$
$$\implies n^2(A_2 - 5A_2 + 6A_2) + n(A_1 - 5A_1 + 10A_2 + 6A_1 - 24A_2)$$
$$+2A_0 - 7A_1 + 19A_2 = 8n^2$$
$$\implies 2A_2 n^2 + (2A_1 - 14A_2)n$$
$$+2A_0 - 7A_1 + 19A_2 = 8n^2$$

Problem 3 Contd.

This is true for all n, so equating like coefficients, we get

$$\begin{aligned} 2A_2 = 8 &\implies A_2 = 4 \\ 2A_1 - 14A_2 = 0 \implies A_1 - 7A_2 = 0 &\implies A_1 = 7A_2 = 28 \\ \text{and } 2A_0 - 7A_1 + 19A_2 &= 0 \\ \implies 2A_0 = 7A_1 - 19A_2 = 7 \times 28 - 19 \times 4 &= 120 \\ \implies A_0 &= 60 \\ \therefore a_n^{(p)} &= 60 + 28n + 4n^2 \end{aligned}$$

$\therefore$ the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$\implies a_n = A.2^n + B.3^n + 60 + 28n + 4n^2 \qquad (2)$$

Problem 3 Contd.

To find A and B, we use $a_0 = 4, a_1 = 7$
Putting $n = 0, 1$ in (2), we get

$$a_0 = A + B + 60$$

$$\implies 4 = A + B + 60 \implies A + B = -56$$

$$\text{and } a_1 = A.2 + B.3 + 60 + 28 + 4$$

$$\implies 7 = 2A + 3B + 92 \implies 2A + 3B = -85 \qquad (4)$$

$$\implies (3) \times 2 \implies 2A + 2B = -112$$

$$- - - - - - - - - - - - - - -$$

$$\text{subtracting } B = 27$$

$$\therefore A = -56 - B = -56 - 27 = -83$$

$\therefore$ the general solution is

$$a_n = (-83)^n + 27.3^n + 4n^2 + 28n + 60 \forall n \geq 0.$$

Problem 4

Solve the recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 3.2^n + 7.3^n$$

where $n \geq 0$ and $a_0 = 1, a_1 = 4$.

Solution:

Given $a_{n+2} - 6a_{n+1} + 9a_n = 3.2^n + 7.3^n \quad (1)$
We shall workout treating forward from a_n.
The homogeneous recurrence relation is $a_{n+2} - 6a_{n+1} + 9a_n = 0$.
Since $n + 2 - n = 0$, it is a second order difference equation.
The characteristic equation is $r^2 - 6r + 9 = 0 \implies r = 3, 3$.
Solution of homogeneous recurrence equation is $a_n^{(h)} = (A + Bn)3^n$.
Given $f(n) = 3.2^n + 7.3^n$, where 3 is a double root of the characteristic equation.

Problem 4 Contd.

So, we assume the particular solution as $a_n = A_0.2^n + A_1 n^2.3^n$ [corresponding to each term of $f(n)$] where A_0, A_1 are constants to be determined using the given equation (1).

$$\therefore A_0 2^{n+2} + A_1 (n+2)^2 . 3^{n+2} - 6\left(A_0 2^{n+1} + A_1 (n+1)^2 . 3^{n+1}\right) + 9\left(A_0 2^n + A_1 n^2 . 3^n\right) = 3.2^n + 7.3^n$$

$$\Rightarrow A_0\left[2^{n+2} - 6.2^{n+1} + 9 . 2^n\right] + A_1\left[(n+2)^2 3^{n+2} - 6(n+1)^2 3^{n+1} + 9n^2 . 3^n\right] = 3.\,2^n + 7.3^n$$

$$\Rightarrow 2^n A_0 [4 - 12 + 9] + 3^n . A_1 \left[9(n+2)^2 - 18(n+1)^2 + 9n^2\right] = 3.2^n + 7.3^n$$

$$\Rightarrow A_0 . 2^n + A_1 . 3^n \left[9(n^2 + 4n + 4) - 18(n^2 + 2n + 1) + 9n^2\right] = 3.2^n + 7.3^n$$

$$\Rightarrow A_0 . 2^n + 18 A_1 . 3^n = 3.2^n + 7.3^n$$

Problem 4 Contd.

This is true for all $n \geq 0$, so equating like coefficients, we get

$$A_0 = 3 \text{ and } 18A_1 = 7 \implies A_1 = \frac{7}{18}$$

$\therefore$ particular solution is $a_n^{(p)} = 3.2^n + \frac{7}{18}.n^2.3^n$.

Hence the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$\implies a_n = (A + 3B)3^n + 3.2^n + \frac{7}{18}.n^2.3^n \qquad (2)$$

To find A and B, we use the initial conditions $a_0 = 1, a_1 = 4$.

Problem 4 Contd.

$\therefore$ putting $n = 0, 1$ in (2), we get

$$a_0 = A + 3 \Rightarrow 1 = A + 3 \Rightarrow A = -2$$

and $$a_1 = (A + B) . 3 + 3.2 + \frac{7}{18} .3$$

$$\Rightarrow \quad 4 = (-2 + B) . 3 + 6 + \frac{7}{6}$$

$$\Rightarrow \quad 4 = 3B + \frac{7}{6}$$

$$\Rightarrow \quad 3B = 4 - \frac{7}{6} = \frac{24 - 7}{6} = \frac{17}{6}$$

$$\Rightarrow \quad B = \frac{17}{18}$$

Problem 4 Contd.

The general solution is

$$a_n = \left(-2 + \frac{17}{18}n \right) 3^n + 3.2^n + \frac{7}{18}n^2.3^n \ \forall n \geq 0.$$

$$\implies a_n = \frac{1}{18}(7n^2 + 17n - 36)3^n + 3.2^n \ \forall n \geq 0.$$

Problem 5

Solve the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} + 4(n+1).3^n, \text{ where } a_0 = 2, a_1 = 3.$$

Solution:

Given $a_n - 6a_{n-1} + 9a_{n-2} = 4(n+1).3^n \qquad (1)$
The homogeneous recurrence relation is $a_n - 6a_{n-1} + 9a_{n-2} = 0$.
Since $n - (n-2) = 2$, the order is 2.
The characteristic equation is $r^2 - 6r + 9 = 0 \implies r = 3, 3$.
Solution of homogeneous recurrence equation is $a_n^{(h)} = (A + 3B)3^n$.
Given $f(n) = 4(n+1)3^n$ and 3 is a root of the characteristic equation of multiplicity 2.

Problem 5 Contd.

The trial particular solution corresponding to 4.3^n is $cn^2 3^n$ and corresponding to $(n+1)$ is $A_0 + A_1 n$.

$$\therefore \quad a_n = n^2 3^n (A_0 + A_1 n) \qquad \text{[Merging } c \text{ with } A_0 \text{ and } A_1]$$

where A_0 and A_1 are to be determined, using the given equation (1)

$$\therefore n^2 3^n (A_0 + A_1 n) - 6\left[(n-1)^2 3^{n-1}\left(A_0 + A_1 (n-1)\right)\right] + 9\left[(n-2)^2 3^{n-2}\left(A_0 + A_1(n-2)\right)\right] = 4(n+1)3^n$$

$$\Rightarrow \quad n^2 . 3^n (A_0 + A_1 n) - 2.3^n\left[(n-1)^2\left(A_0 + A_1(n-1)\right)\right] + 3^n\left[(n-2)^2\left(A_0 + A_1(n-2)\right)\right] = 4(n+1)3^n$$

Problem 5 Contd.

$$\Rightarrow n^2(A_0+A_1 n)-2\left[A_0(n-1)^2+A_1(n-1)^3\right]+\left[A_0(n-2)^2+A_1(n-2)^3\right] = 4(n+1)$$

$$\Rightarrow A_0 n^2 + A_1 n^3 - 2\left[A_0(n^2-2n+1)+A_1(n^3-3n^2+3n-1)\right] + A_0(n^2-4n+4)+A_1(n^3-6n^2+12n-8) = 4(n+1)$$

$$\Rightarrow n^3[A_1-2A_1+A_1]+n^2[A_0-2A_0+6A_1+A_0-6A_1] + n[4A_0-6A_1-4A_0+12A_1]-2A_0+2A_1+4A_0-8A_1 = 4(n+1)$$

$$\Rightarrow \quad 6A_1 n + 2A_0 - 6A_1 = 4(n+1)$$

Problem 5 Contd.

This is true for all n, so equating like terms, we get,

$$aA_1 = 4 \implies A_1 = \frac{2}{3}$$
$$\text{and } 2A_0 - 6A_1 = 4$$
$$\implies A_0 - 3A_1 = 2 \implies A_0 = 2 + 3.\frac{2}{3} = 4$$
$$\therefore a_n^{(p)} = n^2.3^n\left(4 + \frac{2}{3}n\right)$$

$\therefore$ the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$.

$$\implies a_n = (A + Bn)3^n + \left(4 + \frac{2n}{3}\right)n^2.3^n \qquad (3)$$

To find A and B, we use the initial conditions $a_0 = 2, a_1 = 3$.

Problem 5 Contd.

Putting $n = 0, 1$ in (2) we get

$$a_0 = A \Rightarrow A = 2$$

and $$a_1 = (A+B).3 + \left(4 + \frac{2}{3}.1\right)3$$

$$\Rightarrow \quad 3 = 3A + 3B + 14$$

$$\Rightarrow \quad 3B = -3.2 - 11 = -17$$

$$\therefore B = -\frac{17}{3}$$

$\therefore$ the general solution is $a_n = \left(2 - \frac{17}{3}n\right)3 + \left(4 + \frac{2n}{3}\right)n^2.3^n,\ n \geq 0$

$$= (6-17n)\,3^{n-1} + (12+2n)\,n^2\,3^{n-1},\ n \geq 0$$

Problem 6

Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution:

To solve this linear non-homogeneous recurrence relation with constant coefficients we need to solve its associated linear homogeneous equation and to find a particular solution for the given non-homogeneous equation.
The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.
We now find a particular solution. Since $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n, say $P_n = cn + d$ where c and d are constants.

Problem 6 Contd.

To determine whether there are any solutions of this form, suppose that $P_n = cn + d$ is such a solution.
Then, the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.
Simplifying and combining like terms gives $(2 + 2c)n + (2d–3c) = 0$.
It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d–3c = 0$.
This shows that $cn + d$ is a solution if and only if $c = −1$ and $d = -\frac{3}{2}$.
Consequently, $a_n^{(p)} = -n - \frac{3}{2}$ is a particular solution.

Problem 6 Contd.

All solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha.3^n$$

where α_n is a constant.
To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general solution. We find that $3 = -1 - \frac{3}{2} + 3\alpha$, which implies that $\alpha = \frac{11}{6}$.
The solution we seek is,

$$a_n = -n - \frac{3}{2} + \frac{11}{6}3^n.$$

Problem 7

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution:

This is a linear homogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ are

$$a_n^{(h)} = \alpha_1.3^n + \alpha_2.2^n$$

where α_1 and α_2 are constants.
Since $F(n) = 7^n$, a reasonable trial solution is $a_n^{(P)} = C.7^n$, where C is a constant.
Substituting the terms of this sequence into the recurrence relation implies that

$$C.7^n = 5C.7^{n-1} - 6C.7^{n-2} + 7^n$$

Problem 7 Contd.

Factoring out 7^{n-2}, this equation becomes

$49C = 35C - 6C + 49,$ which implies that $20C = 49$ or that $C = \dfrac{49}{20}$

Hence, $a_n^{(P)} = (\frac{49}{20})7^n$ is a particular solution.
All solutions are of the form

$$a_n = \alpha_1.3^n + \alpha_2.2^n + \left(\frac{49}{20}\right)7^n.$$

Problem 8

Solve the recurrence equation

$$a_r - 7a_{r-1} + 10a_{r-2} = 2^r$$

with initial condition $a_0 = 0$ and $a_1 = 6$.

Solution:

The general solution, also called homogeneous solution, to the problem is given by homogeneous part of the given recurrence equation.
The homogeneous part of the equation

$$a_r - 7a_{r-1} + 10a_{r-2} = 2^r \quad (1)$$
$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad (2)$$

The characteristic equation of (1) is given by

$$x^2 - 7x + 10 = 0$$
$$\implies (x-2)(x-5) = 0$$
$$\implies x = 2 \text{ and } x = 5$$

Problem 8 Contd.

Since the roots 2 and 5 of characteristic equation are distinct, the homogeneous solution is given by

$$a_r = A2^r + B5^r \tag{3}$$

Now, particular solution is given by

$$a_r = rC2^r$$

Substituting the value of a_r in equation (1), we get

$$\begin{aligned} C[r2^r - 7(r-1)2^{r-1} + 10(r-2)2^{r-2}] &= 2^r \\ \text{or } C[4r - 7(r-1)2 + 10(r-2)]2^{r-2} &= 2^r \\ \text{or } C[4r - 14(r-1) + 10r - 20] &= 4 \end{aligned}$$

Problem 8 Contd.

$$\text{or } C[4r - 14r + 14 + 10r - 20] = 4$$
$$\text{or } -6C = 4 \implies C = -\frac{2}{3}$$

Particular solution is, $a_r = -\frac{2}{3}r2^r$.
The complete, also called total, solution is obtained by combining the homogeneous and particular solutions. This is given as

$$a_r = A2^r + B5^r - \frac{2}{3}r2^r \tag{4}$$

The equation (4) contains two undetermined coefficients A and B, which are to be determined. To find this, we use the given initial conditions for $r = 0$ and $r = 1$.
Since values of a_0 and a_1 are given.

Problem 8 Contd.

Putting r = 0 and r = 1 in equation (4), we get the following equations (5) and (6) respectively.

$$a_0 = A2^0 + B5^0 - \frac{2}{3} * 0 * 2^0$$

or $$0 = A + B \qquad \text{...(5)} \quad (\because \quad a_0 = 0 \text{ is given})$$

and $$a_1 = A2^1 + B5^1 - \frac{2}{3} * 1 * 2^1$$

or $$6 = 2A + 5B - \frac{4}{3} \qquad (\because \quad a_1 = 6 \text{ is given})$$

or $$2A + 5B = \frac{22}{3} \qquad \text{...(6)}$$

Solving equation (5) and (6), we get

$$A = -\frac{2}{9} \text{ and } B = \frac{22}{9}$$

Problem 8 Contd.

Replacing A and B in equation (4) by its respective values, we get the closed form formula for the given recurrence equation. Thus,

$$a_r = -\frac{22}{9}2^r + \frac{22}{9}5^r - \frac{2}{3}r2^r$$

$$a_r = \frac{22}{9}[5^r - 2^r] - \frac{2}{3}r2^r$$

Problem 9

Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with initial conditions $a_0 = 1, a_1 = -1$.

Solution:

The associated homogeneous recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \tag{5}$$

Let $a_n = r^n$ be a solution of (1).
The characteristic equation is $r^2 - 5r + 6 = 0$
$\implies r = 3, 2.$
So, the solution of (1) is $a_n^{(h)} = C_1 3^n + C_2 2^n$.
To find the particular solution of the given equation, let $a_n^{(P)} = A$.
Substituting in the given equation

$$A - 5A + 6 = 2 \implies A = 1.$$

Problem 9 Contd.

$a_n^{(P)} = 1$ which is a particular solution.
Hence, the general solution is,

$$a_n = a_n^{(h)} + a_n^{(P)} = C_1 3^n + C_2 2^n + 1 \tag{6}$$

To find C_1 and C_2, put $n = 0$ and $n = 1$ in (2)

$$a_0 = C_1 + C_2 + 1$$
$$1 = C_1 + C_2 + 1$$

$$C_1 + C_2 = 0 \tag{7}$$

Again

$$a_1 = 3C_1 + 2C_2 + 1$$
$$-1 = 3C_1 + 2C_2 + 1$$

$$3C_1 + 2C_2 = -2 \tag{8}$$

Problem 9 Contd.

Putting the values of C_1 and C_2 in (3), the required solution is

$$a_n = (-2)3^n + (2)2^n + 1.$$

Generating Functions

Generating Functions

Definition:

The generating function of the sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ of real numbers is the infinite series

$$\begin{aligned} G(x) &= a_0 + a_1 x + a - 2x^2 + \cdots + a_n x^n + \cdots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

$G(x) = a_0 + a_1 x + a - 2x^2 + \cdots + a_n x^n$ is called the generating function for the finite sequence $a_0, a_1, a_2, \cdots, a_n$.

- $G(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$
- $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \cdots = \frac{1}{1-ax}$

Problem 1

Solve the recurrence relation

$$a_{n+1} - a_n = 3n^2 - n, n \geq 0, a_0 = 3$$

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$.
Given $a_{n+1} - a_n = 3n^2 - n, n \geq 0, a_0 = 3$.
Multiply by x^n,

$$a_{n+1}x^n - a_n x^n = 3n^2 x^n - nx^n$$

$$\therefore \sum_{n=0}^{\infty} a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 3\sum_{n=0}^{\infty} n^2 x^n - \sum_{n=0}^{\infty} nx^n$$

Problem 1 Contd.

Multiply by x^n, $a_{n+1}x^n - a_n x^n = 3n^2x^n - nx^n$

$$\therefore \sum_{n=0}^{\infty} a_{n+1}\, x^n - \sum_{n=0}^{\infty} a_n\, x^n = 3 \sum_{n=0}^{\infty} n^2 x^n - \sum_{n=0}^{\infty} n\, x^n$$

$$\Rightarrow \left[a_1 x^0 + a_2 x + a_3 x^2 + \dots\right] - G(x) = 3\left[1^2 x + 2^2 x^2 + 3^2 x^3 + \dots\right] - \left[x + 2x^2 + 3x^3 + \dots\right]$$

$$\Rightarrow \frac{1}{x}\left[a_1 x + a_2 x^2 + a_3 x^3 + \dots\right] - G(x) = 3S - x\left[1 + 2x + 3x^2 + \dots\right]$$

$$= 3S - x(1-x)^{-2}$$

$$\Rightarrow \frac{1}{x}\left[a_0 + a_1 x + a_2 x^2 + \dots - a_0\right] - G(x) = 3S - x(1-x)^{-2}$$

$$\Rightarrow \frac{1}{x}[G(x) - 3] - G(x) = 3S - x(1-x)^{-2},$$

where $S = 1^2 x + 2^2 x^2 + 3^2 x^3 + \dots]$

Problem 1 Contd.

$$\Rightarrow G(x)\left[\frac{1}{x}-1\right]-\frac{3}{x} = 3S - x(1-x)^{-2}$$

$$\Rightarrow \quad G(x)\frac{[1-x]}{x} = 3S - x(1-x)^{-2} + \frac{3}{x}$$

$$\Rightarrow \quad G(x) = \frac{3x}{1-x}S - \frac{x^2}{(1-x)^3} + \frac{3}{(1-x)}$$

$$\text{Now } S = 1^2 x + 2^2 x^2 + 3^2 x^3 + \ldots = \sum_{n=0}^{\infty} n^2 x^n$$

$$= \sum_{n=0}^{\infty}\left[n(n+1)-n\right]x^n$$

$$= \sum_{n=0}^{\infty} n(n+1)x^n - \sum_{n=0}^{\infty} n x^n$$

$$= 1.2x + 2.3x^2 + 3.4x^3 + \ldots - \left[x + 2x^2 + 3x^3 + \ldots\right]$$

$$= x\left[1.2 + 2.3x + 3.4x^2 + \ldots\right] - x\left(1 + 2x + 3x^2 + \ldots\right)$$

Problem 1 Contd.

$$\text{We know } \frac{1}{1.2}\left[1.2 + 2.3x + 3.4x^2 + \dots\right] = (1-x)^{-3}$$

$$\Rightarrow \quad 1.2 + 2.3x + 3.4x^2 + \dots = 2(1-x)^{-3}$$

$$\text{and } 1 + 2x + 3x^2 + \dots = (1-x)^{-2}$$

$$\therefore S = 2x(1-x)^{-3} - x(1-x)^{-2} = \frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2}$$

$$\therefore G(x) = \frac{3x}{1-x}\left[\frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2}\right] - \frac{x^2}{(1-x)^3} + \frac{3}{1-x}$$

$$= \frac{6x^2}{(1-x)^4} - \frac{3x^2}{(1-x)^3} - \frac{x^2}{(1-x)^3} + \frac{3}{1-x}$$

$$= \frac{6x^2}{(1-x)^4} + \frac{4x^2}{(1-x)^3} + \frac{3}{1-x}$$

$$= 6x^2(1-x)^{-4} - 4x^2(1-x)^{-3} + 3(1-x)^{-1}$$

Problem 1 Contd.

$$\sum_{n=0}^{\infty} a_0 x^n = 6x^2 \cdot \frac{1}{1.2.3}\left[1.2.3 + 2.3.4x + 3.4.5x^2 + \ldots + (n-1)\,n\,(n+1)\,x^{n-2} + \ldots\right]$$
$$-4x^2 \cdot \frac{1}{1.2}\left[1.2 + 2.3x + 3.4x^2 + \ldots + (n-1)\,n\,x^{n-2} + \ldots\right]$$
$$+\ 3\left(1 + x + x^2 + \ldots + x^n + \ldots\right)$$

Equating the coefficients of x^n, we get

$$\begin{aligned} a_n &= (n-1)\,n\,(n+1) - 2.(n-1).n + 3 \\ &= n\,(n^2-1) - 2\,(n^2-n) + 3 \\ &= n^3 - 2n^2 + n + 3 = n\,(n-1)^2 + 3 \end{aligned}$$

Problem 2

Solve $a_n = 4a_{n-1}, n \geq 0, a_0 = 2$ by generating function method.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ be the generating function of the sequence $\{a_n\}$.
Given $a_n = 4a_{n-1}$.
Multiply by x^n,

$$\begin{aligned} a_n x^n &= 4a_{n-1}x^n \\ &= 4x a_{n-1} x^{n-1} \\ \therefore \sum_{n=0}^{\infty} a_n x^n &= 4x \sum_{n=0}^{\infty} a_{n-1} x^{n-1} \end{aligned}$$

Problem 2 Contd.

$$\Rightarrow \quad a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n = 2 + 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$\Rightarrow \quad G(x) = 2 + 4x\, G(x)$$

$$\Rightarrow \quad G(x)\,[1 - 4x] = 2$$

Problem 2 Contd.

$$\Rightarrow \quad G(x) = \frac{2}{1-4x}$$

$$\Rightarrow \quad G(x) = 2(1-4x)^{-1}$$

$$= 2(1 + 4x + 4^2x^2 + \ldots + 4^n x^n + \ldots]$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$= 2 + 2.4x + 2.4^2 x^2 + \ldots + 2.4^n x^n + \ldots$$

$$\therefore a_n = 2.4^n, \quad n \geq 0$$

Problem 3

Solve $a_n - 3a_{n-1} = 2 \ \forall n \geq 1, a_0 = 2$.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$.
Given $a_n - 3a_{n-1} = 2$.
Multiply by x^n,

$$a_n x^n - 3a_{n-1}x^n = 2x^n$$
$$\implies a_n x^n - 3x a_{n-1} x^{n-1} = 2x^n$$

Problem 3 Contd.

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2 \sum_{n=1}^{\infty} x^n$$

$$\Rightarrow a_0 + \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$= a_0 + 2(x + x^2 + x^3 + \ldots)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - 3x\, G(x) = 2 + \frac{2x}{1-x}$$

$$\Rightarrow \quad G(x) - 3x\, G(x) = \frac{2}{1-x}$$

$$\Rightarrow \quad G(x)\,[1-3x] = \frac{2}{1-x}$$

$$\Rightarrow \quad G(x) = \frac{2}{(1-3x)(1-x)}$$

Problem 3 Contd.

Split $\dfrac{2}{(1-3x)(1-x)}$ into partial fractions

Let $$\frac{2}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$\Rightarrow \quad 2 = A(1-3x) + B(1-x)$$

when $x = 1$, $\quad 2 = A(1-3) \quad \Rightarrow A = -1$

when $x = \dfrac{1}{3}$, $\quad 2 = B(1 - 1/3) \quad \Rightarrow B = 3$

Problem 3 Contd.

$$\therefore G(x) = \frac{-1}{1-x} + \frac{3}{1-3x}$$

$$= -(1-x)^{-1} + 3(1-3x)^{-1}$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$= -(1 + x + x^2 + x^3 + \ldots + x^n + \ldots)$$
$$+ 3(1 + 3x + 3^2 x^2 + \ldots + 3^n x^n + \ldots)$$

$$\therefore a_n = -1 + 3 . 3^n, \; n \geq 0$$

$$\Rightarrow a_n = -1 + 3^{n+1}, \; n \geq 0$$

Problem 4

Solve $a_{n+2} - 2a_{n+1} + a_n = 2^n, a_0 = 2, a_1 = 1$ by generating function method.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$.
Given $a_{n+2} - 2a_{n+1} + a_n = 2^n$.

Multiply by x^n, then $a_{n+2}x^n - 2a_{n+1}x^n + a_n x^n = 2^n x^n$

$$\therefore \sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\Rightarrow \frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} - \frac{2}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{0}^{\infty} 2^n x^n$$

Problem 4 Contd.

$$\text{But} \sum_{n=0}^{\infty} a_{n+2}\, x^{n+2} = a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots - a_0 - a_1 x$$

$$= G(x) - a_0 - a_1 x$$

$$= G(x) - 2 - x$$

$$\sum_{n=0}^{\infty} a_{n+1}\, x^{n+1} = a_1 x + a_2 x^2 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + \ldots - a_0$$

$$= G(x) - 2$$

$$\therefore \quad \frac{1}{x^2}\big(G(x) - 2 - x\big) - \frac{2}{x}\big(G(x) - 2\big) + G(x) = \sum_{0}^{\infty} 2^n x^n$$

Problem 4 Contd.

$$\Rightarrow \quad G(x)\left(\frac{1}{x^2}-\frac{2}{x}+1\right)-\frac{(2+x)}{x^2}+\frac{4}{x} = (1-2x)^{-1} = \frac{1}{1-2x}$$

$$\Rightarrow \quad G(x)\left(\frac{1-2x+x^2}{x^2}\right) = \frac{1}{1-2x}+\frac{2+x}{x^2}.\frac{4}{x}.$$

$$\Rightarrow \quad G(x)\frac{(1-x)^2}{x^2} = \frac{1}{1-2x}+\frac{2+x-4x}{x^2} = \frac{1}{1-2x}+\frac{2-3x}{x^2}$$

$$\Rightarrow \quad G(x) = \frac{x^2}{(1-x)^2}\left[\frac{1}{1-2x}+\frac{2-3x}{x^2}\right]$$

$$= \frac{x^2}{(1-2x)(1-x)^2}+\frac{2-3x}{(1-x)^2}$$

$$= \frac{x^2}{(1-2x)(1-x)^2}+\frac{3(1-x)-1}{(1-x)^2}$$

Problem 4 Contd.

$$= \frac{x^2}{(1-2x)(1-x)^2} + \frac{3}{1-x} - \frac{1}{(1-x)^2}$$

$$\text{Let } \frac{x^2}{(1-2x)(1-x)^2} = \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

$$\Rightarrow x^2 = A(1-x)^2 + B(1-2x)(1-x) + C(1-2x)$$

Put $x = \frac{1}{2}$, $\quad \frac{1}{4} = A\left(1-\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow A = 1$

Put $x = 1$, $\quad 1 = C(1-2) \qquad \Rightarrow C = -1$

Equating coefficients of x^2, $1 = A + 2B \Rightarrow B = 0 \qquad [\because A = 1]$

Problem 4 Contd.

$$\therefore \quad G(x) = \frac{1}{1-2x} - \frac{1}{(1-x)^2} + \frac{3}{1-x} - \frac{1}{(1-x)^2}$$

$$\Rightarrow a_0 + a_1 x + \dots + a_n x^n + \dots = \frac{1}{1-2x} + \frac{3}{1-x} - \frac{2}{(1-x)^2}$$

$$= (1-2x)^{-1} + 3(1-x)^{-1} - 2(1-x)^{-2}$$

$$= 1 + 2x + 2^2 x^2 + \dots + 2^n x^n + \dots$$
$$+ \ 3[1 + x + x^2 + \dots + x^n + \dots]$$
$$-2[1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots]$$

$$\therefore a_n = 2^n + 3 - 2(n+1)$$

$$= 2^n - 2n + 1, \ n \geq 0$$

Problem 5

Using the method of generating function to solve the recurrence relation $a_{n+1} - 8a_n + 16a_{n-1} = 4^n, n \geq 1, a_0 = 1, a_1 = 8$.

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $\{a_n\}$.
Given $a_{n+2} - 2a_{n+1} + a_n = 2^n$.

Multiplying by x^n, we get

$$a_{n+1}x^n - 8a_n x^n + 16a_{n-1}x^n = 4^n x^n$$

$$\therefore \quad \sum_{n=1}^{\infty} a_{n+1}x^n - 8\sum_{n=1}^{\infty} a_n x^n + 16\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} 4^n x^n$$

$$\Rightarrow \frac{1}{x}\sum_{n=1}^{\infty} a_{n+1}x^{n+1} - 8\sum_{n=1}^{\infty} a_n x^n + 16x\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=1}^{\infty} (4x)^n$$

Problem 5 Contd.

$$\text{But} \sum_{n=1}^{\infty} a_{n+1}\, x^{n+1} = a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots - a_0 - a_1 x$$

$$= G(x) - 1 - 8x \qquad [\because\ a_0 = 1,\ a_1 = 8]$$

$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + \ldots - a_0$$

$$= G(x) - 1$$

$$\therefore\ \frac{1}{x}\left[G(x) - 1 - 8x\right] - 8\left[G(x) - 1\right] + 16\, x\, G(x)$$

Problem 5 Contd.

$$\Rightarrow \quad G(x)\left[\frac{1}{x} - 8 + 16x\right] - \frac{1}{x} - 8 + 8 = \frac{1}{1-4x} - 1$$

$$\Rightarrow \quad G(x)\left[\frac{1 - 8x + 16x^2}{x}\right] - \frac{1}{x} = \frac{1-(1-4x)}{1-4x}$$

$$\Rightarrow \quad G(x)\left[\frac{(1-4x)^2}{x}\right] = \frac{4x}{1-4x} + \frac{1}{x}$$

$$\Rightarrow \quad G(x)\,\frac{(1-4x)^2}{x} = \frac{4x^2 + 1 - 4x}{x(1-4x)}$$

Problem 5 Contd.

$$\therefore \quad G(x) = \frac{1-4x+4x^2}{(1-4x)^3}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = (1-4x+4x^2)(1-4x)^{-3}$$

$$= \left(1-4x+4x^2\right)\frac{1}{1.2}\Big(1.2 + 2.3\,(4x) + 3.4\,(4x)^2$$

$$+ (n-1)\,n\,(4x)^{n-2} + n\,(n+1)\,(4x)^{n-1}$$

$$+ (n+1)\,(n+2)\,(4x)^n + \ldots\Big)$$

$$\therefore \quad a_n = \text{coefficient of } x^n$$

Problem 5 Contd.

$$\Rightarrow \quad a_n = \frac{1}{2}\left[(n+1)(n+2)\,4^n - 4n(n+1)\,4^{n-1} + 4(n-1)\,n\,4^{n-2}\right]$$

$$= \frac{1}{2}\left[4^n (n+1)(n+2-n) + n(n-1)\,4^{n-1}\right]$$

$$= \frac{1}{2}\left[2(n+1)\,4^n + (n^2-n)\,4^{n-1}\right]$$

$$= \frac{4^{n-1}}{2}\left[8(n+1) + n^2 - n\right]$$

$$\Rightarrow \quad a_n = \frac{4^{n-1}}{2}[n^2+7n+8], \; n \geq 0$$