EM384: Analytical Methods for Engineering Management

Lesson 10: Introduction to Linear Programs

6 February 2023

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Lesson Objectives

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- Solve a linear program with two decision variables graphically by enumerating extreme points or using level curves.
- Understand and apply the following terminology for Linear Programs: Feasible region, Infeasible region, Feasible solution, Infeasible solution, Optimal solution, Boundary, Extreme Points.

Review: Steps to formulating an Linear Program (LP)

- Define the decision variables: these represent quantities that you are trying to determine.
- Define the objective function: This is a linear function of the decision variables. Is it a minimization or maximization problem?
- Define the constraints: What values can the decision variables take? What equalities or inequalities define these constraints?

Example Problem Revisited

Example Problem (Review)

Kalo Fertilizer Company makes a fertilizer using two ingredients that provide nitrogen, phosphate, and potassium.

A pound of ingredient A contributes 10 pounds of nitrogen and 6 pounds of phosphate, while a pound of ingredient B contributes 2 pounds of nitrogen, 6 pounds of phosphate, and 1 pound of potassium.

Ingredient A costs \$3 per pound, and ingredient B costs \$5 per pound.

The company want to know how many pounds of each chemical ingredient to put into each bag of fertilizer to meet the minimum requirement of 20 pounds of nitrogen, 36 pounds of phosphate, and 0.5 pounds of potassium while minimizing cost.

Mathematical Formulation (Review)

Decision Variables:

x_1: Amount of ingredient A to put in each bag of fertilizer (in pounds)

x_2: Amount of ingredient B to put in each bag of fertilizer (in pounds)

Objective Function

Minimize $Z = 3x_1 + 5x_2$

Constraints

$10x_1 + 2x_2 \geq 20$ (Nitrogen constraint)

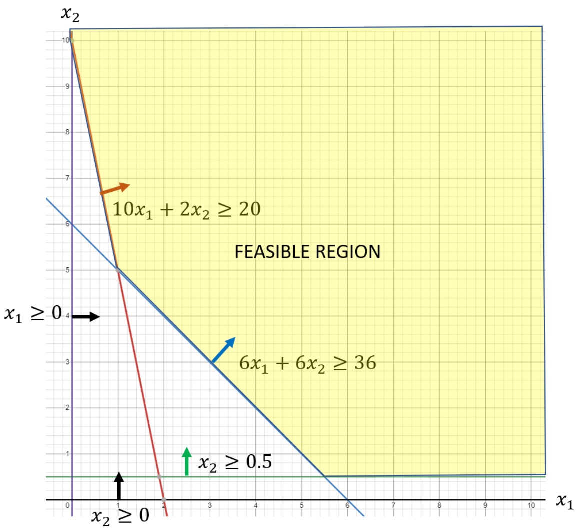
$6x_1 + 6x_2 \geq 36$ (Phosphate constraint)

$x_2 \geq 0.5$ (Potassium constraint)

$x_1, x_2 \geq 0$ (Non-negativity constraint)

Graphical Solution

Begin by plotting the feasible region:



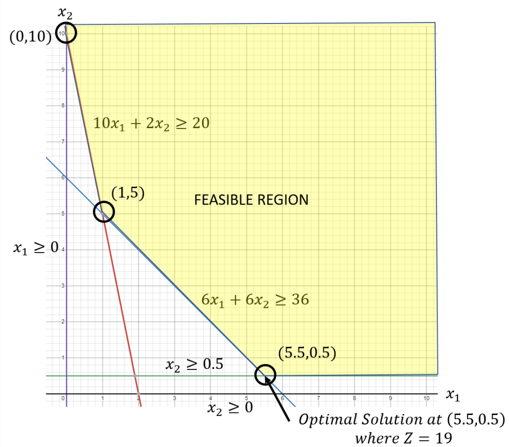
Enumeration of Extreme Points Method

- Draw the constraint boundary line for each constraint.
- Find the feasible region by determining where all constraints are satisfied simultaneously.
- Determine the extreme points of the feasible region.
- Solve for the coordinates of the extreme points using either substitution or elimination.
- Evaluate the objective function using the coordinates of each extreme point.
- Whichever point maximizes/minimizes the objective function is the optimal solution.
- If the feasible region is unbounded (as in this example), then for a minimization problem with non-negative objective function coefficients, the LP will have one (or possibly) infinite optimal solutions. In this case, *the objective function minimizing direction is towards the boundary.*

Graphical Solution

Solution method 1: Enumerate the extreme points and calculate the objective function value Z. The value of Z that is minimum or maximum (depending on problem) is an optimal solution.

$Z(0, 10) = 50 \qquad Z(1, 5) = 28 \qquad Z(5.5, 0.5) = 19$ (Optimal Solution)



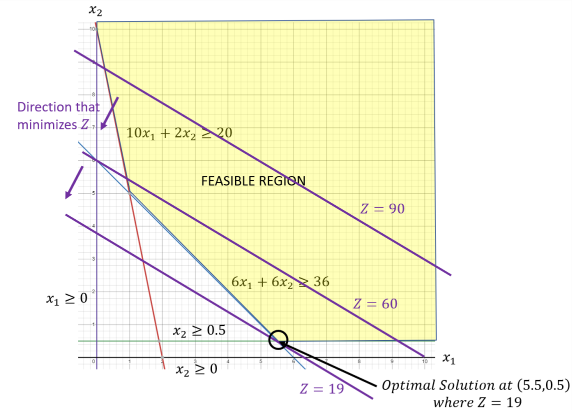
Level Curves Method

- Draw the constraint boundary line for each constraint.
- Find the feasible region by determining where all constraints are satisfied simultaneously.
- Determine one objective function line. All others will be parallel.
- Find the optimal objective function line by moving your objective function line in the direction that minimizes or maximizes Z, and stop when you reach the 'edge' of the feasible region (your line should touch a vertex/corner)
- A feasible point on the optimal objective function line is an optimal solution.

Graphical Solution

Solution Method 2: We plot an objective function line an move in the direction that minimizes Z within the feasible region. We stop when the boundary of the feasible region is reached and calculate the optimal solution.

Optimal solution is $Z(5.5, 0.5) = 3(5.5) + 5(0.5) = 19$.



Some Definitions

- The feasible region is the set of points where all constraints are *satisfied*. The infeasible region is the set of points where *at least one* constraint is not satisfied.
- The **boundary** of a feasible region are defined by constraints.
- An **extreme point** is the intersection of two constraints on the boundary of a feasible region.
    - A **bounded** feasible region is entirely enclosed by constraints.
    - An **unbounded** feasible region is a region that is not entirely enclosed by constraints.
- Both bounded and unbounded feasible regions contain an infinite number of feasible solutions.

Bounded Feasible Region

- If the feasible set is bounded, then there will be **exactly one solution** (at an extreme point), **or infinite solutions** between two extreme points (including the extreme points).
- Therefore, **for a bounded feasible region, the method of extreme points will always give you an optimal solution.**

Unbounded Feasible Region

- If the feasible region is not bounded (there is a direction in which you can travel indefinitely while staying in the feasible region) then a particular objective may or may not have an optimal solution.
  - If it is a maximization problem, there might be a maximum, or it might be possible to make the objective function arbitrarily large inside the feasible region.
  - If it is a minimization problem, there might be a minimum, or it might be possible to make the objective function arbitrarily small (big and negative) inside the feasible region.
- If the feasible region is unbounded, then the method of extreme points also requires testing points on the the infinite ray(s) leaving some extreme points.

Practical Exercise

Algebra Review

1) Plot the feasible region of a linear program described by the following constraints:

$$x_1 + x_2 \geq 6$$

$$2x_1 - x_2 \leq 4$$

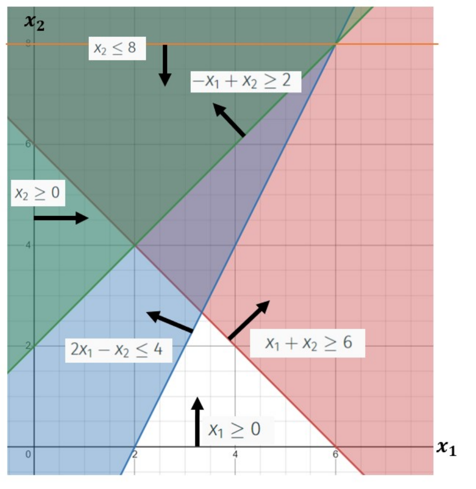
$$-x_1 + x_2 \geq 2$$

$$x_2 \leq 8$$

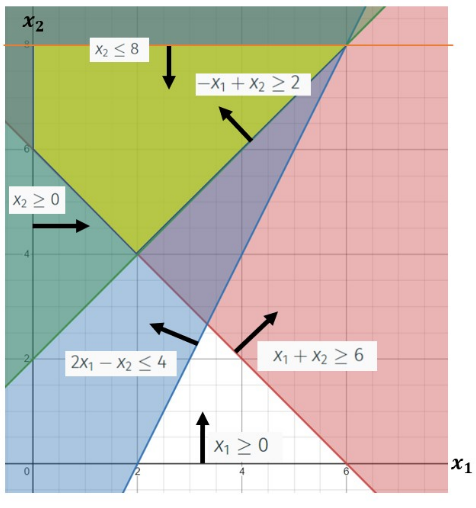
$$x_1, x_2 \geq 0$$

2) Solve using a graphical solution method.

Plot the constraints:

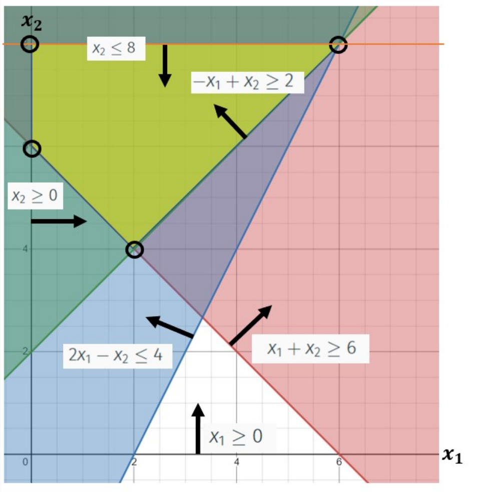


Shade the feasible region.



Practical Exercise

Identify and circle the extreme points.



Extreme Points:

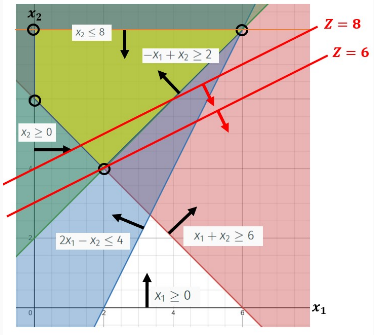
- $(0, 8)$
- $(0, 6)$
- $(2, 4)$
- $(6, 8)$

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why:

$$min \qquad Z = -x_1 + 2x_2$$

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why. $min \quad Z = -x_1 + 2x_2$

Method of Level Curves:



Method of Extreme Points:

- $Z(0, 8) = 0 + 2(8) = 16$
- $Z(0, 6) = 0 + 2(6) = 12$
- $Z(2, 4) = -2 + 2(4) = 6$
- $Z(6, 8) = -6 + 2(8) = 10$

$Z^*(2, 4) = 6$ The optimal solution is $x_1^* = 2$ and $x_2^* = 4$

$Z^*(2, 4) = 2 + 4 = 10$ so $x_1^* = 2$ and $x_2^* = 4$

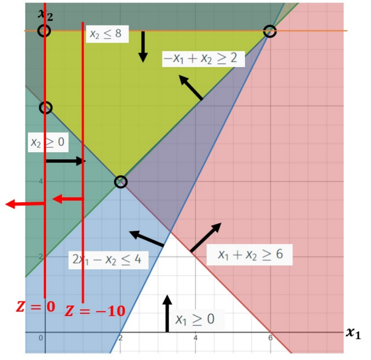
Using the feasible find the optimal solution for the following problem, if it exists. If it does not, state why:

$$max \qquad Z = -100x_1$$

Practical Exercise

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why. $max \quad Z = -100x_1$

Method of Level Curves:



Method of Extreme Points:

- $Z(0, 8) = -100(0) = 0$
- $Z(0, 6) = -100(0) = 0$
- $Z(2, 4) = -100(4) = -400$
- $Z(6, 8) = -100(8) = -800$

There are infinite optimal solutions. An optimal solution is $x_1^* = 0$ and $x_2^* = 8$

Inf. sol. where $Z^* = 0$. $x_1^* = 0$ and $x_2^* = 8$

Conclusion

Next Class

Homework:

- Read Chapter 3 (Pages 35-end of 3.2)

Next Lesson:

- Graphical solutions II - more practice with unbounded feasible regions.

WPR1 is rapidly approaching! (Lesson 12). It will cover all content from lesson 1-11, and be conducted in the lab (The room will be put out by your instructor beforehand). You will be expected to do problems on paper, as well as in Excel on the lab computer. You may only use your issued scientific calculator (no graphing calculators). No outside references are allowed.