EM384: Analytical Methods for Engineering Management

Lesson 11: Solving Linear Programs Using a Graphical Method

9 February 2023

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Lesson Objectives

Lesson 11 Objectives

- Understand and be able to define the four assumptions of linear programming: Proportionality, Additivity, Divisibility, and Certainty

Assumptions of Linear Programs

Linear Programming requires linearity in the objective function as shown in the generic linear program (LP) below:

Maximize

$$Z = a_1x_1 + a_2x_2 + ... + a_nx_n$$

Subject to Constraints:

$$b_{11}x_1 + b_{12}x_2 + ... + b_{1n}x_n \leq c_1$$

$$b_{21}x_1 + b_{22}x_2 + ... + b_{2n}x_n \leq c_2$$

$$...$$

$$b_{m1}x_1 + b_{m2}x_2 + ... + b_{mn}x_n \leq c_m$$

$$x_1, x_2, ..., x_n \geq 0$$

Assumptions of Linear Programming

Assumptions of Linear Programs

Linearity in the objective function and constraints requires the following four assumptions:

- **Proportionality**: A change in a variable results in a proportionate change in that variable's contribution to the function.
- **Additivity**: The function value is the sum of the contributions of each term.
- **Divisibility:** The decision variables can be divided into non-integer, or fractional, values (Integer Programming can be used if the divisibility assumption does not hold).
- **Certainty**: The coefficients of the decision variables are known and constant.

Constraint Outcomes

Constraint Outcomes

Every constraint of a linear program is either binding or non-binding on the optimal solution.

1. A **binding constraint** is one where some optimal solution is on the line for the constraint. Thus if this constraint were to be changed slightly (in a certain direction), this optimal solution would no longer be feasible.
2. A **non-binding constraint** is one where no optimal solution is on the line for the constraint.

Practical Exercises

Practical Exercise

1) Plot the feasible region of a linear program described by the following constraints:

$$x_1 + x_2 \geq 6$$

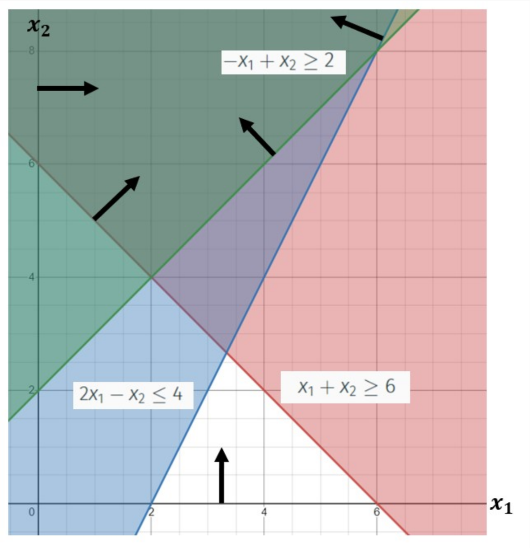
$$2x_1 - x_2 \leq 4$$

$$-x_1 + x_2 \geq 2$$

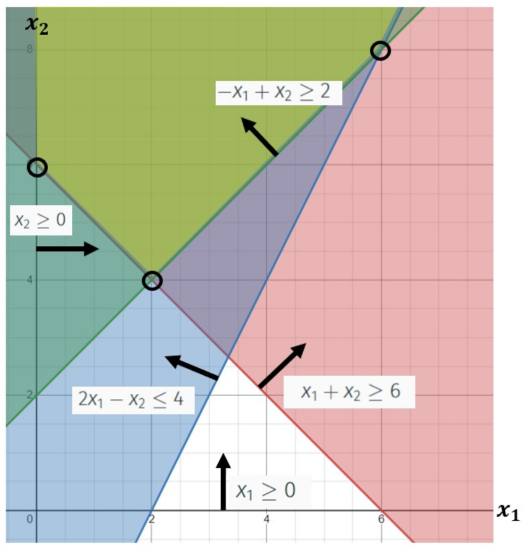
$$x_1, x_2 \geq 0$$

2) Circle the extreme points

Practical Exercise



Practical Exercise



Extreme Points:

- $(0, 6)$
- $(2, 4)$
- $(6, 8)$

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why:

$$min \qquad Z = -x_1 + 2x_2$$

Subject to constraints:

$$x_1 + x_2 \geq 6$$

$$2x_1 - x_2 \leq 4$$

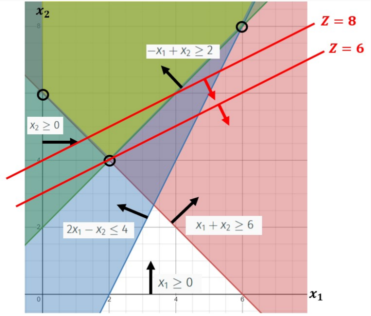
$$-x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Practical Exercise

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why. $min \quad Z = -x_1 + 2x_2$

Method of Level Curves:



$Z^*(2,4) = 2 + 4 = 10$ so $x_1^* = 2$ and $x_2^* = 4$

Method of Extreme Points:

- $Z(0,6) = 0 + 2(6) = 12$
- $Z(2,4) = -2 + 2(4) = 6$
- $Z(6,8) = -6 + 2(8) = 10$
- $Z(0,8) = 0 + 2(8) = 16$
- $Z(6.5,9) = -6.5 + 2(9) = 11.5$

$Z^*(2,4) = 6$ The optimal solution is $x_1^* = 2$ and $x_2^* = 4$ Note: If either of these two solutions were the smallest, then the problem would have no solution (could make obj function infinitely large)

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why:

$$min \qquad Z = x_1 + x_2$$

Subject to constraints:

$$x_1 + x_2 \geq 6$$

$$2x_1 - x_2 \leq 4$$

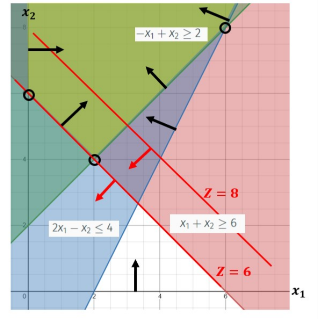
$$-x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Practical Exercise

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why. $\min \quad Z = x_1 + x_2$

Method of Level Curves:



Infinite optimal solutions

$Z^*(2,4) = 2 + 4 = 10$ so $x_1^* = 2$ and $x_2^* = 4$

Method of Extreme Points:

- $Z(0,6) = 0 + 6 = 6$
- $Z(2,4) = 2 + 4 = 6$
- $Z(6,8) = -6 + 8 = 14$
- $Z(0,8) = 0 + 8 = 8$
- $Z(6.5,9) = -6.5 + 9 = 15.5$

$Z^*(2,4) = 6$ The optimal solution is $x_1^* = 2$ and $x_2^* = 4$ (There are infinite optimal solutions)

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why:

$$max \qquad Z = -x_1 + 2x_2$$

Subject to constraints:

$$x_1 + x_2 \geq 6$$

$$2x_1 - x_2 \leq 4$$

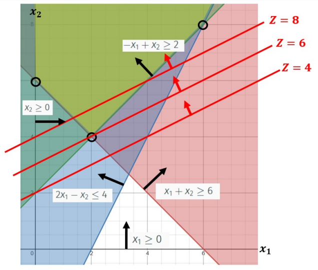
$$-x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Practical Exercise

Using the feasible region, find the optimal solution for the following problem, if it exists. If it does not, state why. $\quad min \quad Z = -x_1 + 2x_2$

Method of Level Curves:



No optimal solution

Method of Extreme Points:

- $Z(0,6) = 0 + 2(6) = 12$
- $Z(2,4) = -2 + 2(4) = 6$
- $Z(6,8) = -6 + 2(8) = 10$
- $Z(0,8) = 0 + 2(8) = 16$
- $Z(6.5,9) = -6.5 + 2(9) = 11.5$

No optimal solution (In this max problem, we found a point on a ray where Z is larger than Z at any extreme point)

Practical Exercise 2

You are the S3 Air of a battalion deploying to the Joint Readiness Training Center. Your battalion commander wants to deploy as many soldiers as possible.

- There are twelve C130 and ten C17 aircraft available from the Air Force at a cost of $4K and $5K per aircraft, respectively.
- $80K is budgeted for airlift. • For this type mission, the maximum pax load is 3 dozen for a C130 and 4 dozen for a C17.
- Only 36 hours of ground support are available to support your missions at the arrival airfield. A C130 requires 2 hours for service and a C17 requires 3 hours.

The S3 wants your recommendation for an airlift plan to support the deployment.

REQUIREMENT: Formulate the LP (Objective Function, Decision Variables, and Constraints) and solve graphically:

Conclusion

Next Class

Homework:

Homework Set 3

Next Lesson:

- WPR 1 (See course director email)