# EM384: Analytical Methods for Engineering Management

## Lesson 16: Sensitivity Analysis I

27 February 2023

# Table of contents

1. Lesson Objectives

2. Practical Exercise

3. Conclusion

# Lesson Objectives

# Lesson Objectives

Create and interpret Sensitivity Reports, to include:

- Shadow Price
- Reduced Cost
- Constraint Outcomes: Binding and Nonbinding

# Review

1. Resource Allocation Problem:
   - Constraints: $\leq$
   - Objective: Maximize
2. Cost-benefit Trade-Off Problem:
   - Constraints: $\geq$
   - Objective: Minimize
3. Mixed Problem:
   - Constraints: $\leq$, $\geq$, or $=$
   - Objective: Either Maximize or Minimize

# Review of Sensitivity Analysis / What if Analysis

- In Block 1 we covered sensitivity analysis for spreadsheet models. The concept in Block 2 is the same!
  - What would happen to the optimal solution if different assumptions were made?
  - Looks at how sensitive the optimal solution is to the value of each parameter.
  - Provides valuable guidance to management regarding the impact of changing a decision
- We can apply these same concepts from spreadsheet models to Linear Programming models!

# Sensitivity Analysis on Decision Variable Coefficients

- Basic Variable: A decision variable that is included in the optimal solution (and therefore has a non-zero value).
- Non-basic Variable: A decision variable that is *not* included in the optimal solution (and therefore has a value of zero)
- Reduced Cost: For any nonbasic variable, **the reduced cost for the variable is the amount by which its objective function coefficient must be improved before that variable will be a basic variable in some optimal solution to the LP.**[1] If a decision variable is basic (already in the solution), its reduced cost is zero.
- Allowable Increase and Allowable Decrease: This provides the range the objective function coefficient can vary while the optimal solution (decision variables) remains the same. For changes outside of the given range, the linear program would have to be re-solved.

[1] "Introduction to Mathematical Programming" by Wayne Winston

# Sensitivity Analysis on Constraint ”RHS”

- Shadow Price: **The shadow price of a linear program constraint is the amount by which the optimal objective function value will change if the right-hand-side is increased by one.**[2]
- Binding Constraint: A constraint which has a non-zero shadow price.
- Non-binding Constraint: A constraint which has a shadow price equal to zero.
- Allowable Increase and Allowable Decrease: This provides the range the right-hand side constraint can change before the shadow price becomes unreliable (or changes). The shadow price is valid within the given range between Allowable Increase and Allowable Decrease for the right-hand side of the constraint.

[2] ”Introduction to Mathematical Programming” by Wayne Winston

# Example Problem From Lesson 14

Plan a Spaghetti Dinner for the Corps of Cadets that minimizes cost while meeting nutritional requirements.

| | Pasta | Meatballs | Sauce | Garlic Bread | Requirement |
|---|---|---|---|---|---|
| Calories | 600 | 500 | 100 | 300 | 2000 |
| Protein | 0 | 10 | 0 | 3 | 30 |
| Carbs | 10 | 2 | 2 | 20 | 60 |
| Cost/Serving | $0.75 | $1.75 | $1.25 | $1.00 | |

**REQUIREMENT:** Formulate the LP (Objective Function, Decision Variables, and Constraints) and solve using Excel Solver.

# Algebraic Formulation

Decision variables:

$x_1$: Servings of Pasta

$x_2$: Servings of Meatballs

$x_3$: Servings of Sauce

$x_4$: Servings of Garlic Bread

Objective function:

Maximize $Z = 0.75x_1 + 1.75x_2 + 1.25x_3 + x_4$ (Cost, in \$)

Constraints:

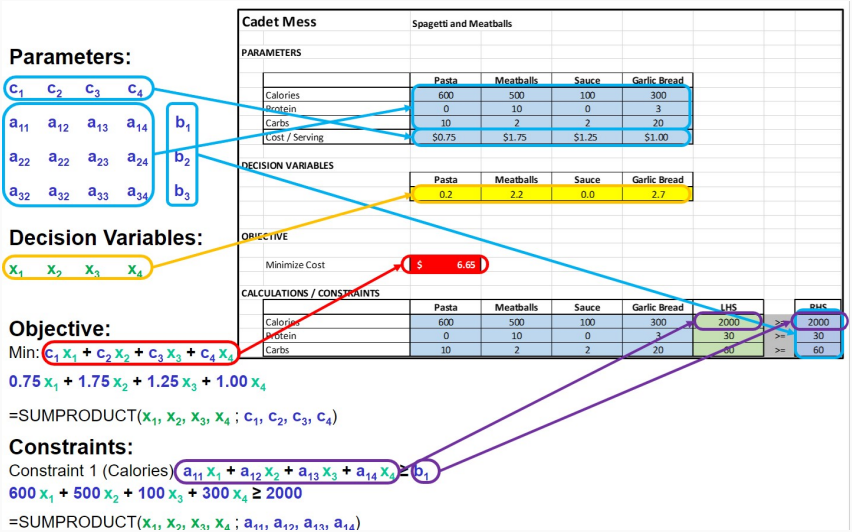
$600x_1 + 500x_2 + 100x_3 + 300x_4 \geq 2000$ (Calories)

$10x_2 + 3x_4 \geq 30$ (Protein)

$10x_1 + 2x_2 + 2x_3 + 20x_4 \geq 60$ (Carbs)

$x_1, x_2 \geq 0$ (non-negativity)

# Excel Formulation

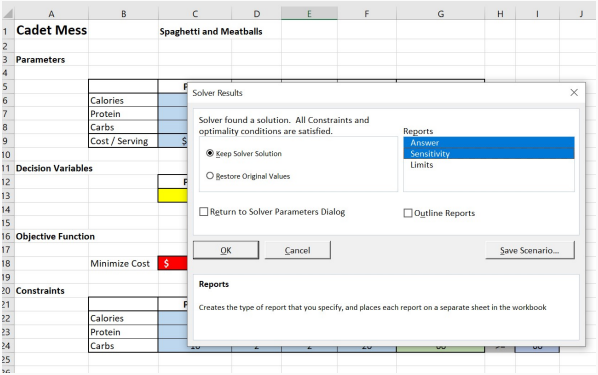


# Let’s do some Sensitivity Analysis

Open the Lesson 14 - All PE (Solution) file on Teams (In the PE folder).

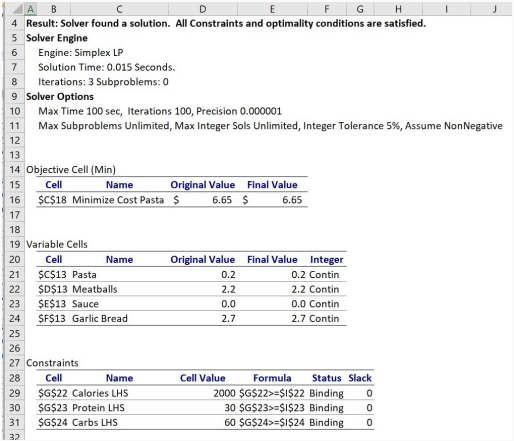
Run the Excel Solver.

Click on ”Answer” and ”Sensitivity” on the right side before you click OK.



# Answer Report

The Answer Report gives you the optimal solution, objective function value, and whether constraints are binding or not.



**Result: Solver found a solution. All Constraints and optimality conditions are satisfied.**

**Solver Engine**

Engine: Simplex LP

Solution Time: 0.015 Seconds.

Iterations: 3 Subproblems: 0

**Solver Options**

Max Time 100 sec, Iterations 100, Precision 0.000001

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 5%, Assume NonNegative

Objective Cell (Min)

| Cell | Name | Original Value | Final Value |
|---|---|---|---|
| $C$18 | Minimize Cost Pasta | $ 6.65 | $ 6.65 |

Variable Cells

| Cell | Name | Original Value | Final Value | Integer |
|---|---|---|---|---|
| $C$13 | Pasta | 0.2 | 0.2 | Contin |
| $D$13 | Meatballs | 2.2 | 2.2 | Contin |
| $E$13 | Sauce | 0.0 | 0.0 | Contin |
| $F$13 | Garlic Bread | 2.7 | 2.7 | Contin |

Constraints

| Cell | Name | Cell Value | Formula | Status | Slack |
|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | $G$22>=$I$22 | Binding | 0 |
| $G$23 | Protein LHS | 30 | $G$23>=$I$23 | Binding | 0 |
| $G$24 | Carbs LHS | 60 | $G$24>=$I$24 | Binding | 0 |

# Sensitivity Report

The Sensitivity Report gives you the reduced costs, shadow prices, and allowable increases/ decreases.

It also includes all of the information from the Answer report (how do you know if a constraint is binding?).

**Microsoft Excel 16.0 Sensitivity Report**
**Worksheet: [Lesson 13 PE - All PEs (Solution).xlsx]1. Cadet Mess**
**Report Created: 9/23/2022 7:05:13 AM**

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

Optimal solution (optimal value for each decision variable)

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

The Reduced Cost for each basic and non-basic variable

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
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Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

Decision variable coefficients in the objective function

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

Allowable increase for Objective Function coefficients without a change to which variables are included in the optimal solution

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

Allowable decrease for Objective Function coefficients without a change to which variables are included in the optimal solution

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

Final value for LHS of constraints

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

The shadow price of each constraint

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

Right hand side of each constraint

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

Amount by which we can increase the right-hand side of each constraint and still use the shadow price to find the new objective function value.

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$13 | Pasta | 0.157790927 | 0 | 0.75 | 1.15 | 0.505154639 |
| $D$13 | Meatballs | 2.189349112 | 0 | 1.75 | 0.958333333 | 1.388888889 |
| $E$13 | Sauce | 0 | 1.119329389 | 1.25 | 1E+30 | 1.119329389 |
| $F$13 | Garlic Bread | 2.702169625 | 0 | 1 | 0.98 | 0.2875 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $G$22 | Calories LHS | 2000 | 0.000966469 | 2000 | 2740 | 82.4742268 |
| $G$23 | Protein LHS | 30 | 0.123274162 | 30 | 1.70212766 | 24.66666667 |
| $G$24 | Carbs LHS | 60 | 0.017011834 | 60 | 10.66666667 | 45.66666667 |

Amount by which we can decrease the right-hand side of each constraint and still use the shadow price to find the new objective function value.

# Practical Exercise

# Practical Exercise

The Wyndor Glass Co. is preparing to introduce two new products:

- An 8-foot glass door with aluminum framing
- A 4-foot x 6-foot double hung wood-framed window

The company has three plants:

- Plant 1 produces aluminum frames and hardware.
- Plant 2 produces wood frames.
- Plant 3 produces glass and assembles the windows and doors.

**Production Time Used for Each Unit Produced (in hrs)**

| | Doors | Windows | Available Hours per Week |
|---|---|---|---|
| Plant 1 | 1 | 0 | 4 |
| Plant 2 | 0 | 2 | 12 |
| Plant 3 | 3 | 2 | 18 |
| Unit Profit | $300 | $500 | |

Which combination of products would maximize company profit?

1) Formulate by Hand below.
2) Create a model and solve in Excel.
3) Produce an Answer Report and Sensitivity Report. Be prepared to answer verbal questions from your instructor.

# Wyndor Formulation Solution

## Decision Variables

$x_1$: Number of doors produced.

$x_2$: Number of windows produced.

## Objective Function

Maximize $Z = 300x_1 + 500x_2$ (Profit in \$)

## Constraints

$x_1 \leq 4$ (Plant 1 hours)

$2x_2 \leq 12$ (Plant 2 hours)

$3x_1 + 2x_2 \leq 18$ (Plant 3 hours)

$x_1, x_2 \geq 0$ (non-negativity)

# Wyndor Excel Solution

| Parameters | | | | | | |
|---|---|---|---|---|---|---|
| | | Doors | Windows | | | |
| | Plant 1 | 1 | 0 | | | |
| | Plant 2 | 0 | 2 | | | |
| | Plant 3 | 3 | 2 | | | |
| | Profit | $300 | $500 | | | |
| | | | | | | |
| **Decision Variables** | | | | | | |
| | | Doors | Windows | | | |
| | # produced | 2 | 6 | | | |
| | | | | | | |
| **Objective** | | | | | | |
| | Max. Profit | $3,600 | | | | |
| | | | | | | |
| **Constraints** | | | | | | |
| | | Doors | Windows | LHS | | RHS |
| | Plant 1 | 1 | 0 | 2 | <= | 4 |
| | Plant 2 | 0 | 2 | 12 | <= | 12 |
| | Plant 3 | 3 | 2 | 18 | <= | 18 |

# Sensitivity Report

Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $C$10 | # produced Doors | 2 | 0 | 300 | 450 | 300 |
| $D$10 | # produced Windows | 6 | 0 | 500 | 1E+30 | 300 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $E$17 | Plant 1 LHS | 2 | 0 | 4 | 1E+30 | 2 |
| $E$18 | Plant 2 LHS | 12 | 150 | 12 | 6 | 6 |
| $E$19 | Plant 3 LHS | 18 | 100 | 18 | 6 | 6 |

# Interpreting the Sensitivity Report

- What does a reduced cost of 0 tell us about the decision variables?
- What does a positive shadow price tell us about a constraint?
- What does a zero shadow price tell us about a constraint?
- What does the allowable increase/decrease tell us for decision variables and constraints?
- What is the new value of the obj function if we increase The Plant 2 hours from 12 to 16? from 12 to 24?

# Interpreting the Sensitivity Report

- What does a reduced cost of 0 tell us about the decision variables? It tells us that the decision variable is a basic variable and is therefore non-zero in the optimal solution.
- What does a positive shadow price tell us about a constraint? It tells us that the constraint is binding.
- What does a zero shadow price tell us about a constraint? It tells us that the constraint is non-binding.
- What does the allowable increase/decrease tell us for decision variables and constraints? For the decision variables, the allowable increase/decrease tells us the range in which we can change the objective function coefficients and not change the basic and non-basic variables. For the constraints, the allowable increase/decrease tells us the range within which we can change the RHS value and still use the shadow price to calculate the change in the objective function value.

# Interpreting the Sensitivity Report

- What is the new value of the obj function if we increase The Plant 2 hours from 12 to 16?
- What is the new value of the obj function if we increase The Plant 2 hours from 12 to 24?
- What happens to the basic variables if the profit per door decreases to to $250?

# Interpreting the Sensitivity Report

- What is the new value of the obj function if we increase The Plant 2 hours from 12 to 16? The objective function will improve by $4x150 = \$600$. **Verify by changing the parameter and re-solving the linear program.**
- What is the new value of the obj function if we increase The Plant 2 hours from 12 to 24? The objective function will improve but we have to re-solve because an increase of 8 is outside the allowable increase. **Calculate the new objective function value by changing the parameter and re-solving the linear program.**
- What happens to the basic variables if the profit per door decreases to to $250? The basic variables will not change because the change is within the allowable decrease.

# Conclusion

# Next Class

**Homework**:

- Review Chapter 7.3
- Work on Homework Set 5

**Next Lesson: Same Lesson Objectives**

- Create and interpret Sensitivity Reports, to include:
  - Shadow Price
  - Reduced Cost
  - Constraint outcomes: binding and nonbinding