EM384: Analytical Methods for Engineering Management

Lesson 17: Sensitivity Analysis II

1 March 2023

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Lesson Objectives

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Create and interpret Sensitivity Reports, to include:

- Shadow Price
- Reduced Cost
- Constraint Outcomes: Binding and Nonbinding

Review

Sensitivity Analysis on Decision Variable Coefficients

- Basic Variable: A decision variable that is included in the optimal solution (and therefore has a non-zero value).
- Non-basic Variable: A decision variable that is *not* included in the optimal solution (and therefore has a value of zero)
- Reduced Cost: For any nonbasic variable, **the reduced cost for the variable is the amount by which its objective function coefficient must be improved before that variable will be a basic variable in some optimal solution to the LP.**[1] If a decision variable is basic (already in the solution), its reduced cost is zero.
- Allowable Increase and Allowable Decrease: This provides the range the objective function coefficient can vary while the optimal solution (decision variables) remains the same. For changes outside of the given range, the linear program would have to be re-solved.

[1] "Introduction to Mathematical Programming" by Wayne Winston

Sensitivity Analysis on Constraint ”RHS”

- Shadow Price: **The shadow price of a linear program constraint is the amount by which the optimal objective function value will change if the right-hand-side is increased by one.**[2]
- Binding Constraint: A constraint which has a non-zero shadow price.
- Non-binding Constraint: A constraint which has a shadow price equal to zero.
- Allowable Increase and Allowable Decrease: This provides the range the right-hand side constraint can change before the shadow price becomes unreliable (or changes). The shadow price is valid within the given range between Allowable Increase and Allowable Decrease for the right-hand side of the constraint.

[2] ”Introduction to Mathematical Programming” by Wayne Winston

Example Walkthrough

JRTC Depployment Algebraic Formulation

Decision variables:

x_1: Number of C130s used

x_2: Number of C17s used

Objective function:

Maximize $Z = 3x_1 + 4x_2$ (Soldiers deployed, in dozens)

Constraints:

$x_1 \leq 12$ (C130s available)

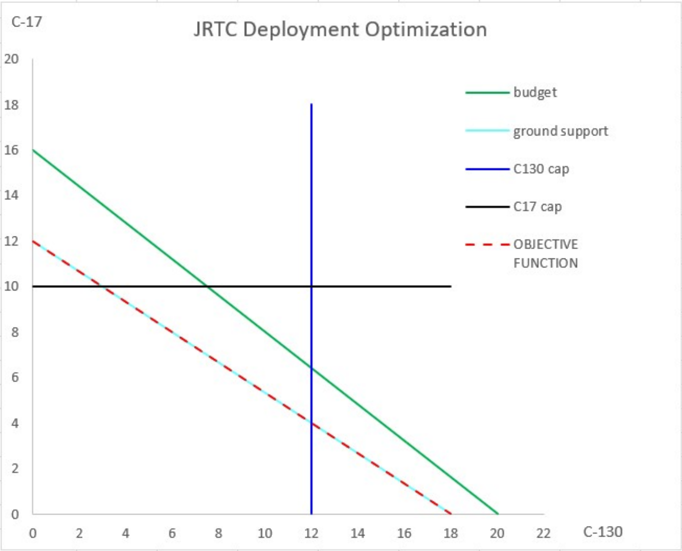
$x_2 \leq 10$ (C17s available)

$2x_1 + 3x_2 \leq 36$ (Ground support)

$4x_1 + 5x_2 \leq 80$ (Budget, in \$1000's)

$x_1, x_2 \geq 0$ (non-negativity)

Graphical Solution



Excel Solution

	A	B	C	D	E	F	G	H
1	**Parameters**							
2								
3			**C-130**	**C-17**				
4		Personnel	3	4	dozen			
5		Ground Support	2	3	hours			
6		Cost	4	5	$k			
7								
8	**Decision Variables**							
9								
10			**C-130**	**C-17**				
11		Number of Aircraft	12	4	# AC			
12								
13	**Objective Function**							
14								
15		# of Personnel	**52**	dozen				
16								
17	**Constraints**							
18								
19					**LHS**		**RHS**	
20		C-130s Available	1	0	12	<=	12	# AC
21		C-17s Available	0	1	4	<=	10	# AC
22		Ground Support	2	3	36	<=	36	hours
23		Cost Constraint	4	5	68	<=	80	$k

Excel Answer Report

Microsoft Excel 16.0 Answer Report
Worksheet: [JRTC Problem with Sentivity Analysis.xlsx]JRTC
Report Created: 9/12/2020 10:53:02 AM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.016 Seconds.
Iterations: 3 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 0%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	# of Soldiers (dozens)	52	52

Objective Value: 52 dozen Soldiers

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$3	Number of Aircraft C-130	12	12	Contin
$C$3	Number of Aircraft C-17	4	4	Contin

Optimal Solution : 12 x C130s and 4 x C17s

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$9	Budget ($ thousands)	68	$D$9<=$F$9	Not Binding	12
$D$10	Ground Support (hours)	36	$D$10<=$F$10	Binding	0
$D$11	C-130s Available	12	$D$11<=$F$11	Binding	0
$D$12	C-17s Available	4	$D$12<=$F$12	Not Binding	6

Slack (for <= constraints): For these constraints, if a slack variable is positive it means that not all of that particular resource was used. If the slack is zero for a particular constraint, then all of the available resource was used and that constraint is termed **binding.**

If a constraint has a Slack of 0, you know it is a **Binding** Constraint.

You can only improve the Objective Value by increasing resources to **Binding** constraints. Adding more resources to a constraint with slack will not affect the Objective Value

BN CDR: we need to take more Soldiers…what if I increase the budget to 100?

Excel Sensitivity Report

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [JRTC Problem with Sentivity Analysis.xlsx]
Report Created: 9/12/2020 10:53:03 AM

Reduced Cost: Indicates how much the objective function coefficient of each decision variable would have to IMPROVE before it would be included in the optimal solution. If a decision variable is already in the optimal solution, its reduced cost is 0.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficien	Allowable Increase	Allowable Decrease
$B$3	Number of Aircraft C-1	12	0	3	1E+30	0.33333333
$C$3	Number of Aircraft C-1	4	0	4	0.5	4

Allowable Increase and Allowable Decrease (Decision Variables): This provides the range the objective function coefficient can vary while the optimal solution remains the same.

What if a C17 could hold 2 dozen people instead of 4 dozen - would this change our optimal solution?

If there are any changes outside of the allowable increase of decrease, you'll have to resolve the LP

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	Budget ($ thousands)	68	0	80	1E+30	12
$D$10	Ground Support (hour	36	1.33333333	36	7.2	12
$D$11	C-130s Available	12	0.33333333	12	6	9
$D$12	C-17s Available	4	0	10	1E+30	6

Shadow Price: The shadow price for a constraint is the change in the value of the optimal solution per unit increase in the right-hand side of the constraint. i.e. Given one more unit of a constrained resource, the shadow price is how much the objective value would change

What if we had one more hour of ground support available - how would this affect our optimal solution?
What if we had one LESS hour of ground support available?

The shadow price is valid for the range given by the **Allowable Increase and Allowable Decrease** for the right-hand side (Constraints).

What if we had 8 more hours available?

Practical Exercise 1

PE1

Problem 1

University Ceramics manufactures plates, mugs, and steins that include the campus name and logo for sale in campus bookstores. The time required for each item to go through the two stages of production (molding and finishing), the material required (clay), and the corresponding unit profits are given in the following table, along with the amount of each resource available for tomorrow's eight-hour shift.

	Plates	Mugs	Steins	Available
Molding (minutes)	4	6	3	2,400
Finishing (minutes)	8	14	12	7,200
Clay (ounces)	5	4	3	3,000
Unit Profit	$3.10	$4.75	$4.00	

See Handout

PE1 Formulation

Decision Variables

x_1: Number of plates produced.

x_2: Number of mugs produced.

x_3: Number of steins produced.

Objective Function

Maximize $Z = 3.10x_1 + 4.75x_2 + 4x_3$ (Profit in \$)

Constraints

$4x_1 + 6x_2 + 3x_3 \leq 2400$ (Moulding, in minutes)

$8x_2 + 14x_2 + 12x_3 \leq 7200$ (Finishing, in minutes)

$5x_1 + 4x_2 + 3x_3 \leq 3000$ (Clay, in ounces)

$x_1, x_2, x_3 \geq 0$ (non-negativity)

PE1

Parameters							
		Plates	**Mugs**	**Steins**			
	Molding	4	6	3	minutes		
	Finishing	8	14	12	minutes		
	Clay	5	4	3	ounces		
	Profit	$3.10	$4.75	$4.00			
Decision Variables							
		Plates	**Mugs**	**Steins**			
	# produced	300	0	400			
Objective							
	Max. Profit	**$2,530.00**					
Constraints							
		Plates	**Mugs**	**Steins**	**LHS**		**RHS**
	Molding	4	6	3	2400	<=	2400
	Finishing	8	14	12	7200	<=	7200
	Clay	5	4	3	2700	<=	3000

PE1

Answer Report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$13	Max. Profit Plates	$11.85	$2,530.00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$10	# produced Plates	1	300	Contin
$D$10	# produced Mugs	1	0	Contin
$E$10	# produced Steins	1	400	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$17	Molding LHS	2400	$F$17<=$H$17	Binding	0
$F$18	Finishing LHS	7200	$F$18<=$H$18	Binding	0
$F$19	Clay LHS	2700	$F$19<=$H$19	Not Binding	300

Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$10	# produced Plates	300	0	3.1	2.233333333	0.366666667
$D$10	# produced Mugs	0	-0.458333333	4.75	0.458333333	1E+30
$E$10	# produced Steins	400	0	4	0.65	1.375

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$17	Molding LHS	2400	0.216666667	2400	200	600
$F$18	Finishing LHS	7200	0.279166667	7200	2400	2400
$F$19	Clay LHS	2700	0	3000	1E+30	300

Practical Exercise 2

PE2

Problem 2

Dahlby Outfitters wishes to introduce packaged trail mix as a new product. The ingredients for the trail mix are seeds, raisins, flakes, and two kinds of nuts. Each ingredient contains certain amounts of vitamins, minerals, protein, and calories. The marketing department has specified that the product be designed so that a certain minimum nutritional profile is met. The decision problem is to determine the optimal product composition – that is, to minimize the product cost by choosing the amount for each of the ingredients in the mix. The following data summarizes the parameters of the problem:

	Grams per Pound					
Component	Seeds	Raisins	Flakes	Pecans	Walnuts	Nutritional Requirement
Vitamins	10	20	10	30	20	20
Minerals	5	7	4	9	2	10
Protein	1	4	10	2	1	15
Calories	500	450	160	300	500	600
Cost/pound	$4	$5	$3	$7	$6	

See Handout

PE2 Formulation

Decision Variables

x_1: weight of seeds in mix, in pounds.

x_2: weight of raisins in mix, in pounds.

x_3: weight of flakes in mix, in pounds.

x_4: weight of pecans in mix, in pounds.

x_5: weight of walnuts in mix, in pounds.

Objective Function

Minimize $Z = 4x_1 + 5x_2 + 3x_3 + 7x_4 + 6x_5$ (cost in \$)

Constraints

$10x_1 + 20x_2 + 10x_3 + 30x_4 + 20x_5 \geq 20$ (Vitamins, in grams)

$5x_1 + 7x_2 + 4x_3 + 9x_4 + 2x_5 \geq 10$ (Minerals, in Grams)

$x_1 + 4x_2 + 10x_3 + 2x_4 + x_5 \geq 15$ (Protein, in grams)

$500x_1 + 450x_2 + 160x_3 + 300x_4 + 500x_5 \geq 600$ (Protein, in grams)

$x_1, x_2, x_3 \geq 0$ (non-negativity)

PE2

Parameters									
		Seeds	**Raisins**	**Flakes**	**Pecans**	**Walnuts**			
	Vitamins	10	20	10	30	20			
	Minerals	5	7	4	9	2			
	Protein	1	4	10	2	1			
	Calories	500	450	160	300	500			
	Cost	$4	$5	$3	$7	$6			
Decision Variables									
		Seeds	**Raisins**	**Flakes**	**Pecans**	**Walnuts**			
	lbs. of	0.48	0.33	1.32	0.00	0.00			
Objective									
	Min. Cost	**$7.54**							
Constraints									
		Seeds	**Raisins**	**Flakes**	**Pecans**	**Walnuts**	**LHS**		**RHS**
	Vitamins	10	20	10	30	20	24.642	>=	20
	Minerals	5	7	4	9	2	10	>=	10
	Protein	1	4	10	2	1	15	>=	15
	Calories	500	450	160	300	500	600	>=	600

PE2

Answer Report

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$C$14	Min. Cost Seeds	$7.54	$7.54

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$11	lbs. of Seeds	0.48	0.48	Contin
$D$11	lbs. of Raisins	0.33	0.33	Contin
$E$11	lbs. of Flakes	1.32	1.32	Contin
$F$11	lbs. of Pecans	0.00	0.00	Contin
$G$11	lbs. of Walnuts	0.00	0.00	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$18	Vitamins LHS	24.64200477	$H$18>=$J$18	Not Binding	4.642004773
$H$19	Minerals LHS	10	$H$19>=$J$19	Binding	0
$H$20	Protein LHS	15	$H$20>=$J$20	Binding	0
$H$21	Calories LHS	600	$H$21>=$J$21	Binding	0

Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$11	lbs. of Seeds	0.477326969	0	4	1.064766839	0.462962963
$D$11	lbs. of Raisins	0.334128878	0	5	0.391666667	0.849173554
$E$11	lbs. of Flakes	1.318615752	0	3	2.651612903	0.376
$F$11	lbs. of Pecans	0	1.57875895	7	1E+30	1.57875895
$G$11	lbs. of Walnuts	0	3.471360382	6	1E+30	3.471360382

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$H$18	Vitamins LHS	24.64200477	0	20	4.642004773	1E+30
$H$19	Minerals LHS	10	0.490453461	10	1.03626943	0.578512397
$H$20	Protein LHS	15	0.056085919	15	2.333333333	5.882352941
$H$21	Calories LHS	600	0.002983294	600	60.86956522	74.07407407

Conclusion

Next Class

Homework:

- Watch video on Python PuLP (will be uploaded to TEAMS)
- Finish Homework Set 5

Next Lesson: Same Lesson Objectives

- Model and solve a linear program in Python using the PuLP Library, and interpret the results to include the sensitivity analysis.