CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 3

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1. (a) $x(t) = \frac{1}{2} + \frac{e^{jw_0t} + e^{-jw_0t}}{2}$. Thus; $a_0 = \frac{1}{2}, a_1 = \frac{1}{2}, a_{-1} = \frac{1}{2}$ and all other $a_k = 0$ when $k \neq 0, 1, -1$.

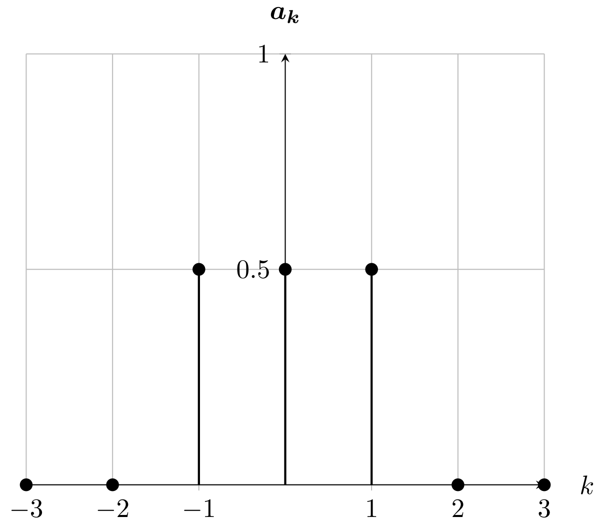

Figure 1: a_k vs k.

(b) $y(t) = \frac{3}{2} + 2\frac{(e^{jw_0t} - e^{-jw_0t})}{2j} = \frac{3}{2} + j(e^{-jw_0t} - e^{jw_0t})$. Thus; $b_0 = \frac{3}{2}, b_1 = -j, b_{-1} = j$ and all other $b_k = 0$ when $k \neq 0, 1, -1$.

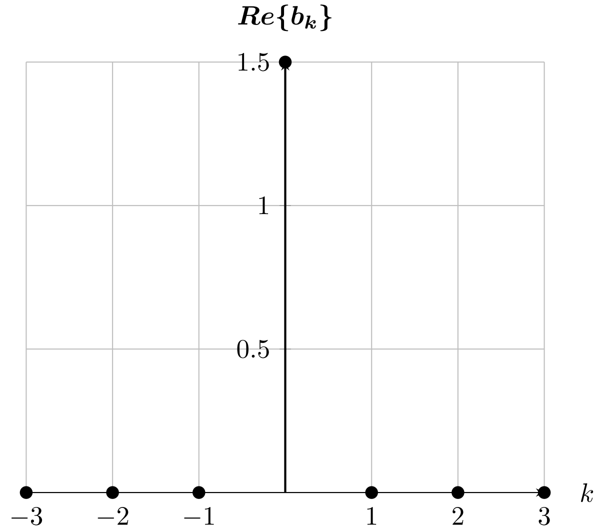

Figure 2: $Re\{b_k\}$ vs k.

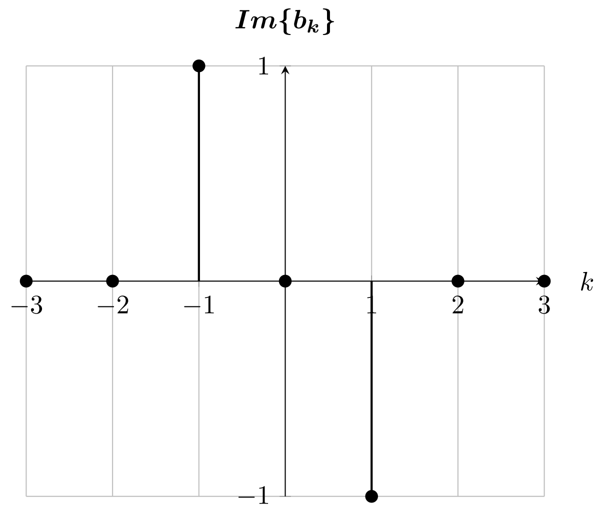

Figure 3: $Im\{b_k\}$ vs k.

(c) $cos(2w_0t+\frac{\pi}{4}) = cos(2w_0t)cos(\frac{\pi}{4}) - sin(2w_0t)sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}(cos2w_0t - sin2w_0t)$. Thus;
$z(t) = \frac{1}{2} + \frac{e^{jw_0t}+e^{-jw_0t}}{2} + \frac{3}{2} + j(e^{-jw_0t} - e^{jw_0t}) + \frac{\sqrt{2}}{2}\left[\frac{e^{j2w_0t}+e^{-j2w_0t}}{2} - \frac{e^{j2w_0t}-e^{-j2w_0t}}{2j}\right]$. Thus;
$c_0 = 2, c_1 = \frac{1}{2} - j, c_{-1} = \frac{1}{2} + j, c_2 = \frac{\sqrt{2}}{4}(1+j)$, $c_{-2} = \frac{\sqrt{2}}{4}(1-j)$ and all other $c_k = 0$ when $k \neq 0, 1, -1, 2, -2$

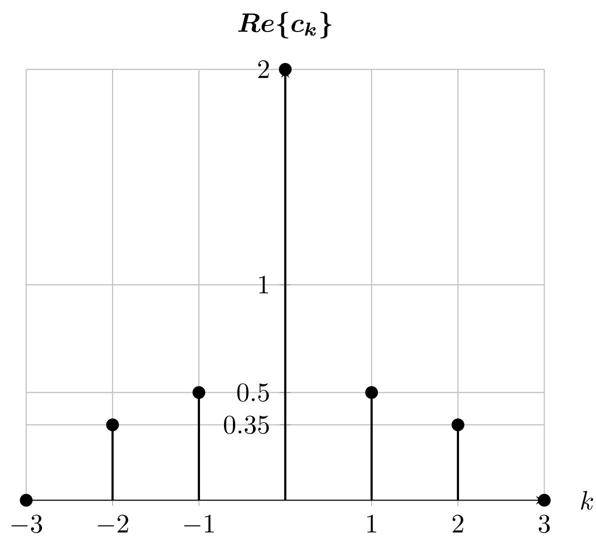

Figure 4: $Re\{c_k\}$ vs k.

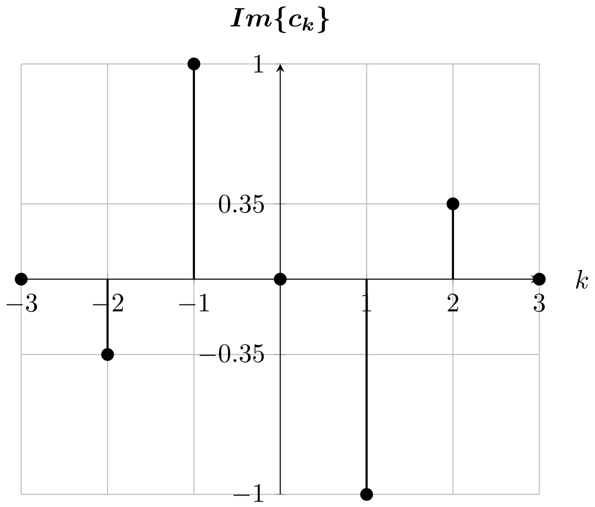

Figure 5: $Im\{c_k\}$ vs k.

2. $A_0 = \dfrac{2}{T}\int_0^{T_1} x(t)dt = \dfrac{2}{T}\int_0^{T_1} Mdt = \dfrac{2MT_1}{T}$

   $A_k = \dfrac{2}{T}\int_0^{T_1} x(t)coskw_0tdt = \dfrac{2}{T}\int_0^{T_1} Mcoskw_0tdt = \dfrac{2M}{Tkw_0}sinkw_0T_1$

   $B_k = \dfrac{2}{T}\int_0^{T_1} x(t)sinkw_0tdt = \dfrac{2}{T}\int_0^{T_1} Msinkw_0tdt = -\dfrac{2M}{Tkw_0}(coskw_0T_1 - 1) = \dfrac{2M}{Tkw_0}(1 - coskw_0T_1)$

   For $M = 1, T = 4T_1 \Rightarrow A_0 = \dfrac{2T_1}{4T_1} = \dfrac{1}{2}, A_k = \dfrac{2}{Tk\frac{2\pi}{T}}sink\frac{2\pi}{4T_1}T_1 = \dfrac{sink\frac{\pi}{2}}{k\pi}$ for $k \neq 0, B_k = \dfrac{2}{Tk\frac{2\pi}{T}}(1 - cosk\frac{2\pi}{4T_1}T_1) = \dfrac{1 - cosk\frac{\pi}{2}}{k\pi}$ for $k \neq 0$. Graphs are plotted based on the values $M = 1, T = 4T_1$:

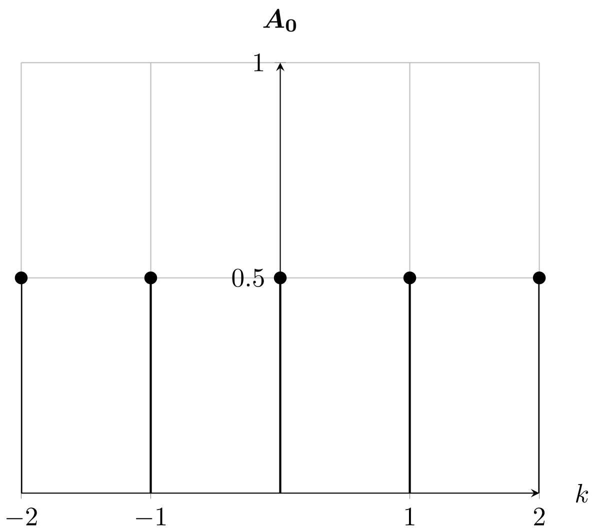

Figure 6: A_0 vs k.

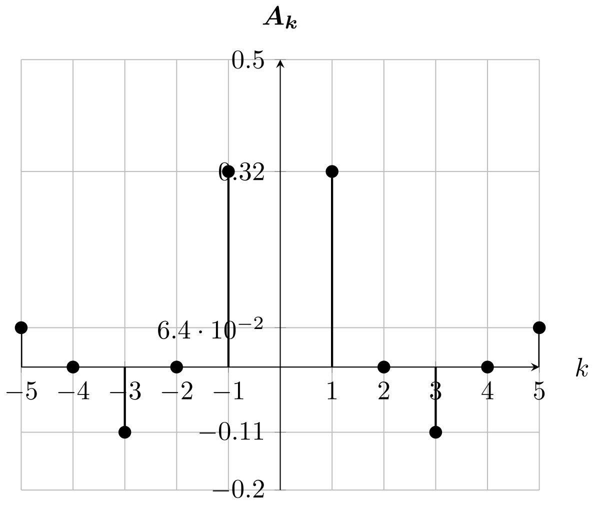

Figure 7: A_k vs k.

A_k is symmetric about the y-axis.

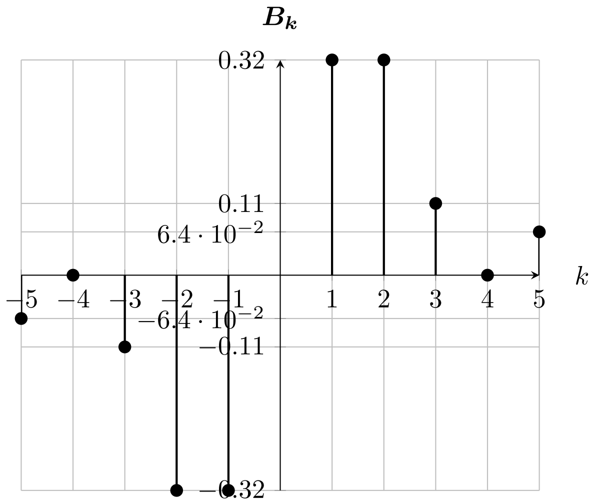

Figure 8: B_k vs k.

B_k is symmetric about the origin.

3. (a)

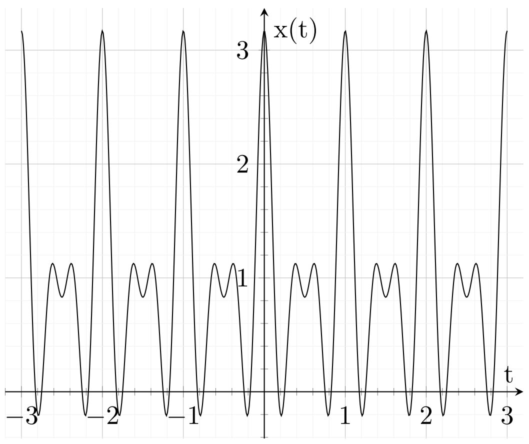

Figure 9: x(t) vs t

(b) $w_0 = 2\pi$, $x(t) = 1 + \dfrac{e^{j2\pi t} + e^{-j2\pi t}}{4} + \dfrac{e^{j4\pi t} + e^{-j4\pi t}}{2} + \dfrac{e^{j6\pi t} + e^{-j6\pi t}}{3}$. Thus; $a_0 = 1, a_1 = a_{-1} = \dfrac{1}{4}, a_2 = a_{-2} = \dfrac{1}{2}, a_3 = a_{-3} = \dfrac{1}{3}$ and all other $a_k = 0$ when $k \neq 0, 1, -1, 2, -2, 3, -3$

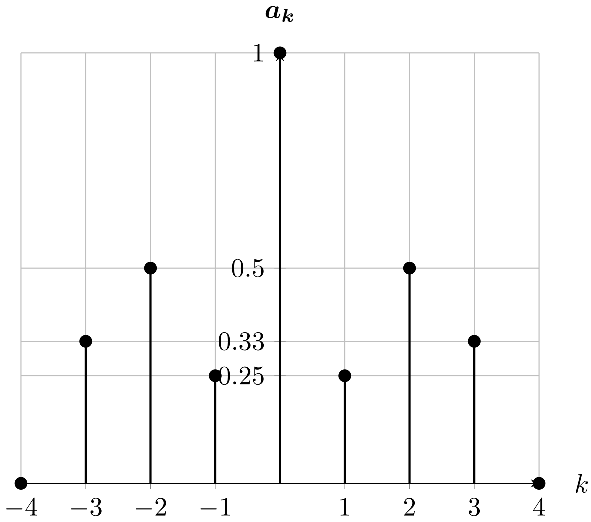

Figure 10: a_k vs k.

(c) The plot of x(t) is in time domain and it is a continuous graph. The plot of a_k is in frequency domain and it is a discrete graph. Plots are similar since both of them are symmetric about y-axis. The term k=0 is 1, constant. The terms for k=1 and k=-1 both have fundamental frequency $w_0 = 2\pi$ and they are referred to as fundamental components. The terms for k=2 and k=-2 are periodic with half the period and have twice the frequency (4π) of the fundamental components. The terms for k=3 and k=-3 are periodic with one third the period and have triple the frequency (6π) of the fundamental components.

(d) We can write $x(t) = \sum_{k=-3}^{3} a_k e^{j2\pi t}$ from part (b). In order to find spectral coefficients of y(t), first compute frequency response $H(jw) = \int_0^{\infty} e^{-2\tau} e^{-jw\tau} d\tau = \dfrac{1}{2+jw}$. We can write $y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0) e^{jkw_0 t}$. Thus; $y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$. $b_k = a_k H(jk2\pi)$. Therefore; $b_0 = \dfrac{1}{2}, b_1 = \dfrac{1}{4}(\dfrac{1}{2+j2\pi}) = \dfrac{1}{8}(\dfrac{1-j\pi}{1+\pi^2}), b_{-1} = \dfrac{1}{4}(\dfrac{1}{2-j2\pi}) = \dfrac{1}{8}(\dfrac{1+j\pi}{1+\pi^2}), b_2 = \dfrac{1}{2}(\dfrac{1}{2+j4\pi}) = \dfrac{1}{4}(\dfrac{1-j2\pi}{1+4\pi^2}), b_{-2} = \dfrac{1}{2}(\dfrac{1}{2-j4\pi}) = \dfrac{1}{4}(\dfrac{1+j2\pi}{1+4\pi^2}), b_3 = \dfrac{1}{3}(\dfrac{1}{2+j6\pi}) = \dfrac{1}{6}(\dfrac{1-j3\pi}{1+9\pi^2}), b_{-3} = \dfrac{1}{3}(\dfrac{1}{2-j6\pi}) = \dfrac{1}{6}(\dfrac{1+j3\pi}{1+9\pi^2})$ and all other $b_k = 0 \ \forall k \in Z$

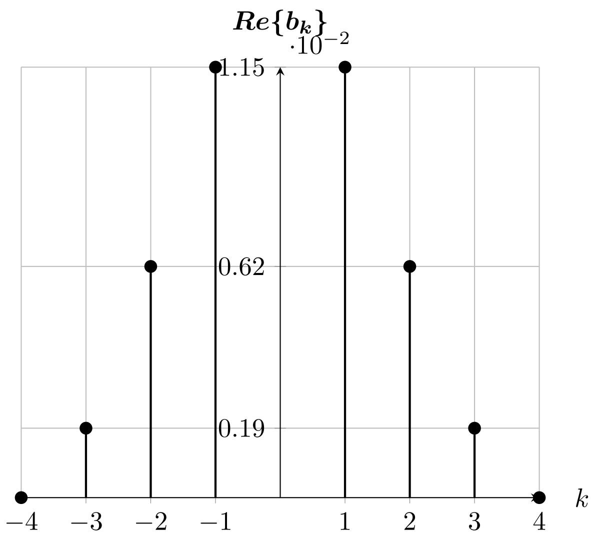

Figure 11: $Re\{b_k\}$ vs k.

Note that $b_0 = \dfrac{1}{2}$ but in order not to break the graph, it is not on the graph.

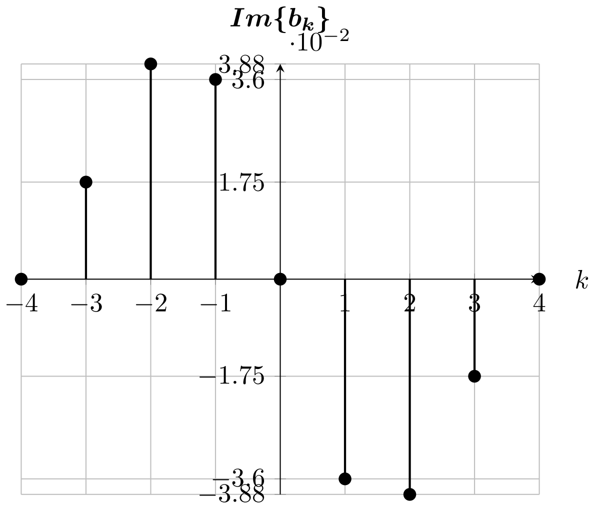

Figure 12: $Im\{b_k\}$ vs k.

4. (a) $b_k = \frac{1}{3}e^{-3jkw_0}a_k - \frac{2}{7}a_{-k}$

   (b) $\frac{d^3x(t)}{dt^3} = \sum_{-\infty}^{\infty} -jk^3w_0^3a_ke^{jkw_0t}$. Thus; $c_k = -jk^3w_0^3a_k$

5. (a) $w_0 = \frac{2\pi}{N} = \frac{\pi}{2}, N = 4$

   $x[n] = \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j}$. Thus; $a_{1+4k} = \frac{1}{2j}, a_{-1+4k} = \frac{-1}{2j}\ \forall k \in Z$

   (b) $w_0 = \frac{2\pi}{N} = \frac{\pi}{2}, N = 4$

   $y[n] = 1 + \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}$. Thus; $b_{4k} = 1, b_{1+4k} = \frac{1}{2}, b_{-1+4k} = \frac{1}{2}\ \forall k \in Z$

   (c) $N = 4$. Circular convolution: x[n]y[n] $\leftarrow FS \rightarrow c_k = \sum_{l=0}^{3} a_lb_{k-l}$

   $c_{1+4k} = a_0b_1 + a_1b_0 + a_2b_{-1} + a_3b_{-2} = \frac{1}{2j}$

   $c_{-1+4k} = a_0b_{-1} + a_1b_{-2} + a_2b_{-3} + a_3b_{-4} = \frac{-1}{2j}$

   $c_{3+4k} = a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 = \frac{-1}{2j}$

   $c_{-3+4k} = a_0b_{-3} + a_1b_{-4} + a_2b_{-5} + a_3b_{-6} = \frac{1}{2j}$ and all other $c_k = 0\ \forall k \in Z$

   (d) x[n]y[n]= $sin\frac{\pi}{2}n + (sin\frac{\pi}{2}n)(cos\frac{\pi}{2}n) = sin\frac{\pi}{2}n + \frac{sin\pi n}{2}$

   $c_k = \frac{1}{4}\sum_{n=0}^{3}(sin\frac{\pi}{2}n + \frac{sin\pi n}{2})e^{-jkw_0n} = \frac{1}{4}[e^{-jkw_0} - e^{-3jkw_0}] = \frac{1}{4}[e^{-jk\frac{\pi}{2}} - e^{-3jk\frac{\pi}{2}}] = \frac{1}{4}[(cosk\frac{\pi}{2} - jsink\frac{\pi}{2}) - (cos3k\frac{\pi}{2} - jsin3k\frac{\pi}{2})]$

   $c_{1+4k} = \frac{-j}{2} = \frac{1}{2j}, c_{-1+4k} = \frac{j}{2} = \frac{-1}{2j}, c_{3+4k} = \frac{j}{2} = \frac{-1}{2j}, c_{-3+4k} = \frac{-j}{2} = \frac{1}{2j}$ and all other $c_k = 0\ \forall k \in Z$. The spectral coefficients for x[n]y[n] are exactly same for part(c) and part(d).

6. $a_k = a_k^{(1)} + a_k^{(2)}$

   w_0 for $a_k^{(1)} = \frac{\pi}{6} = \frac{2\pi}{N_1} \Rightarrow N_1 = 12$

   w_0 for $a_k^{(2)} = \frac{5\pi}{6} = \frac{2\pi m}{N_2} \Rightarrow N_2 = 12$ for $m = 5$. Thus; $N = 12$ for a_k and x[n].

   $a_k = cos\frac{k\pi}{6} + sin\frac{5k\pi}{6} = \frac{1}{2}e^{jk\frac{\pi}{6}} + \frac{1}{2}e^{-jk\frac{\pi}{6}} + \frac{1}{2j}e^{jk\frac{5\pi}{6}} - \frac{1}{2j}e^{-jk\frac{5\pi}{6}}$ and $a_k = \frac{1}{12}\sum_{n=0}^{11} x[n]e^{-jk\frac{\pi}{6}n} = \frac{1}{12}(x[0] + x[1]e^{-jk\frac{\pi}{6}} + x[2]e^{-jk\frac{\pi}{3}} + x[3]e^{-jk\frac{\pi}{2}} + x[4]e^{-jk\frac{2\pi}{3}} + x[5]e^{-jk\frac{5\pi}{6}} + x[6]e^{-jk\pi} + x[7]e^{-jk\frac{7\pi}{6}} + x[8]e^{-jk\frac{4\pi}{3}} + x[9]e^{-jk\frac{3\pi}{2}} + x[10]e^{-jk\frac{5\pi}{3}}x[11]e^{-jk\frac{11\pi}{6}})$. Thus; $x[n] = 6\delta[n-1] + 6\delta[n-11] + 6j\delta[n-5] - 6j\delta[n-7]$ for $0 \le n \le 11$

7. (a) $N = 4, w_0 = \frac{\pi}{2}$. $a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jkw_0n} = \frac{1}{4}[e^{-jkw_0} + 2e^{-2jkw_0} + e^{-3jkw_0}] = \frac{1}{4}[cos\frac{k\pi}{2} - jsin\frac{k\pi}{2} + 2(cosk\pi - jsink\pi) + cos\frac{3k\pi}{2} - jsin\frac{3k\pi}{2}]$

   $a_{4k} = 1, a_{1+4k} = \frac{-1}{2}, a_{3+4k} = \frac{-1}{2}$ and all other $a_k = 0\ \forall k \in Z$.

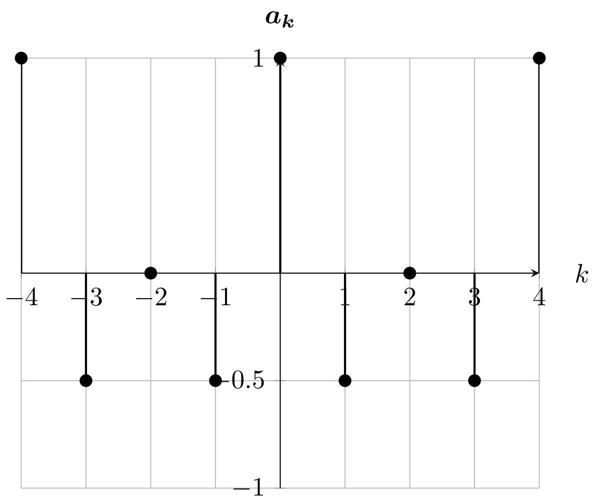

Figure 13: a_k vs k.

(b) i. $y[n] = x[n] - \sum_{-\infty}^{\infty} \delta[n+1-4k]$ since $y[-1+4k] = 0 \ \forall k \in Z$

ii. $N = 4, w_0 = \dfrac{\pi}{2}$. $b_k = \dfrac{1}{4}\sum_{n=0}^{3} y[n]e^{-jkw_0 n} = \dfrac{1}{4}[e^{-jkw_0} + 2e^{-2jkw_0}] = \dfrac{1}{4}[cos\frac{k\pi}{2} - jsin\frac{k\pi}{2} + 2(cosk\pi - jsink\pi)]$

Thus; $b_{4k} = \dfrac{3}{4}, b_{1+4k} = \dfrac{-2-j}{4}, b_{2+4k} = \dfrac{1}{4}, b_{3+4k} = \dfrac{-2+j}{4}$ and all other $b_k = 0 \ \forall k \in Z$

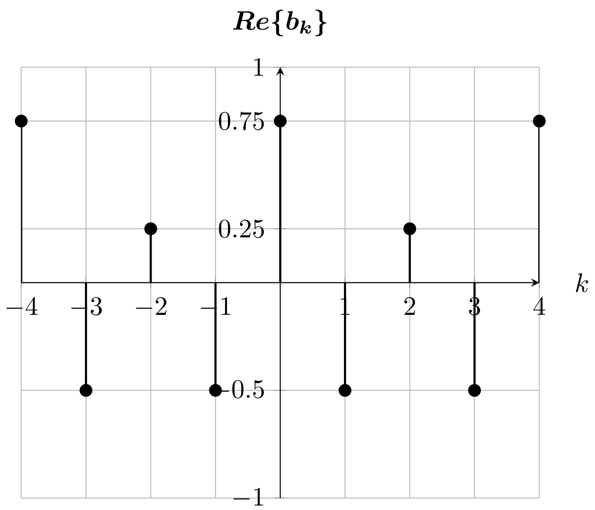

Figure 14: $Re\{b_k\}$ vs k.

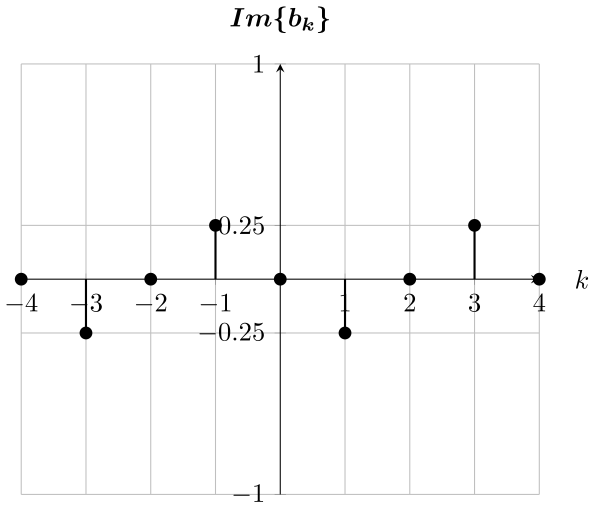

Figure 15: $Im\{b_k\}$ vs k.