Additions to ch3 postscript

I. CONCERNS

These concerns are copied and pasted from Brad's email on march 23:

1. How do you calculate your Rossby number? Is it a volumetric average? Temporal average? Are you averaging the enstrophy and taking the square root? Are you averaging the modulus of the vorticity?
2. So, your epsilon is rather tiny. You informed me previously that the models barely restratify for small epsilon. Is that true when the model is rotationally constrained?
3. How big of an unstable band of wavenumber do you expect to see? Your boxes seem a little narrow to me. Your horizontal spectral resolution is pretty poor in the sense that you should only get three discrete nonzero wavenumbers that are smaller than the critical onset wavenumber. I can't decide if this is a problem or not.
4. The equation at the end of page 56 has got something wrong. Are you saying that Keith's expression (Nu-1) $\sim$ $\mathrm{Ra}^{3/2}$ Ta^{-1} is equivalent to Ra / $\mathrm{Ra}_{\mathrm{crit}}^{1/6}$? That is clearly not right since Ta $\sim$ $\mathrm{Ra}_{\mathrm{crit}}^{3/2}$. I have lost your argument somewhere in the middle of this equation.
5. So, you mention repeated the scaling achieved by King et al. (2012), which has a theoretical argument for why it exists. Are your models consistent with the assumptions that King et al. use? i.e., Do your boundary layers scale like $\delta_s \sim \mathrm{Ra}^{-1/3}$?
6. You also discuss the Ra$\sim$$\mathrm{Ta}^{3/4}$ scaling of Julien et al. 2012. I am completely confused. Both of the Rayleigh and Taylor numbers are inputs. They can't scale.
7. You might want to derive the scaling law that figure 3.5 suggests. My quick estimate gives Ro $\;$ $\mathrm{Ek}^{1/3}$. That doesn't seem "weak" to me since lots of dynamics scale with similar power laws. For example, the most unstable wavenumber scales like $\mathrm{Ek}^{-1/3}$.

II. QUICK RESPONSES

1. We calculate:

$$\mathrm{Ro} = \frac{\sqrt{\boldsymbol{\omega}\cdot\boldsymbol{\omega}}}{2\Omega}, \qquad \text{with} \qquad \boldsymbol{\omega} = \nabla\times\boldsymbol{u}.$$

   everywhere in the domain. We then take a volume average

$$\langle\mathrm{Ro}\rangle = \frac{1}{L_x L_y L_z}\iiint \mathrm{Ro}\,dx\,dy\,dz,$$

   and output that quantity to file. To get the values reported in the paper, we take a time-average of the volume average,

$$\mathrm{Ro}_{\mathrm{reported}} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\langle\mathrm{Ro}\rangle\,dt.$$

2. Even in rotationally constrained systems, the initial stratification is the same as evolved stratification to within $\sim\epsilon$ (see Fig. 1). So the flows do feel $\sim$3 density scale heights in this work.
3. See Fig. 2. Intuitively, I agree – our boxes seem a little narrow, however, the most unstable and fastest-growing horizontal modes are certainly contained within our simulation domain. Visually, flows in our simulations do not appear to be dominated by low-aspect-ratio effects (e.g., development of mean flows driven by convective elements not having enough room to horizontally expand). From a private conversation with Jon Aurnou (2019 APS Division of Fluid Dynamics meeting), I expressed a similar concern: perhaps our simulation boxes were too narrow, and that led to confusing results. He has quite a bit more experience and intuition for rotating convection than I do, and he didn't seem concerned by our choice. His intuition is essentially that rotationally

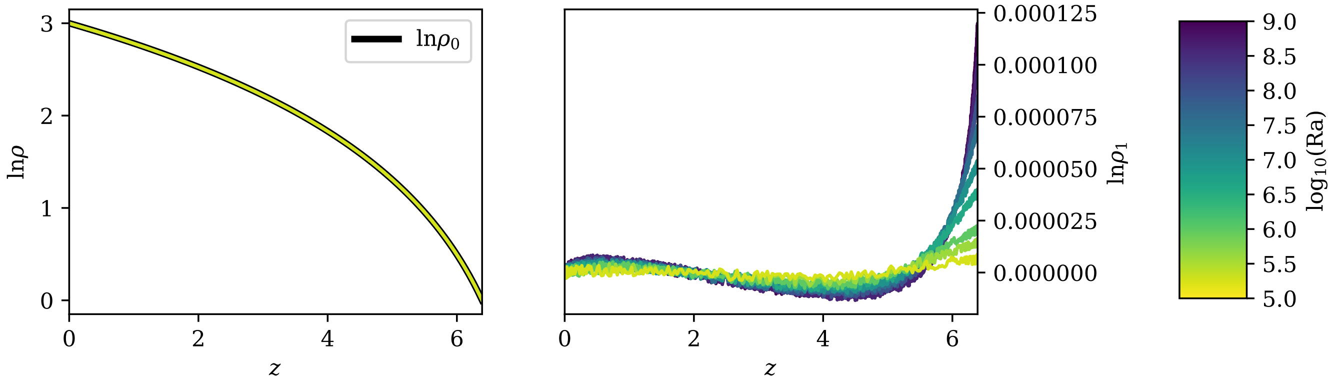

FIG. 1. Density profiles for all simulations with $\mathrm{Ro_p} = 0.6$ are shown. (Left) The log of the full density profile compared to the initial profile. All profiles are, to first order, indistinguishable from the initial profile. (Right) Deviations in $\ln\rho$ away from the initial state. As Ra increases (from yellow to purple), density differences, particularly in the upper boundary layer, become more extreme. The noise in these profiles is a result of the accuracy of the output data that I have on hand (which was the full rho profile, not the fluctuations), and does not reflect noise or accuracy in the simulations themselves.

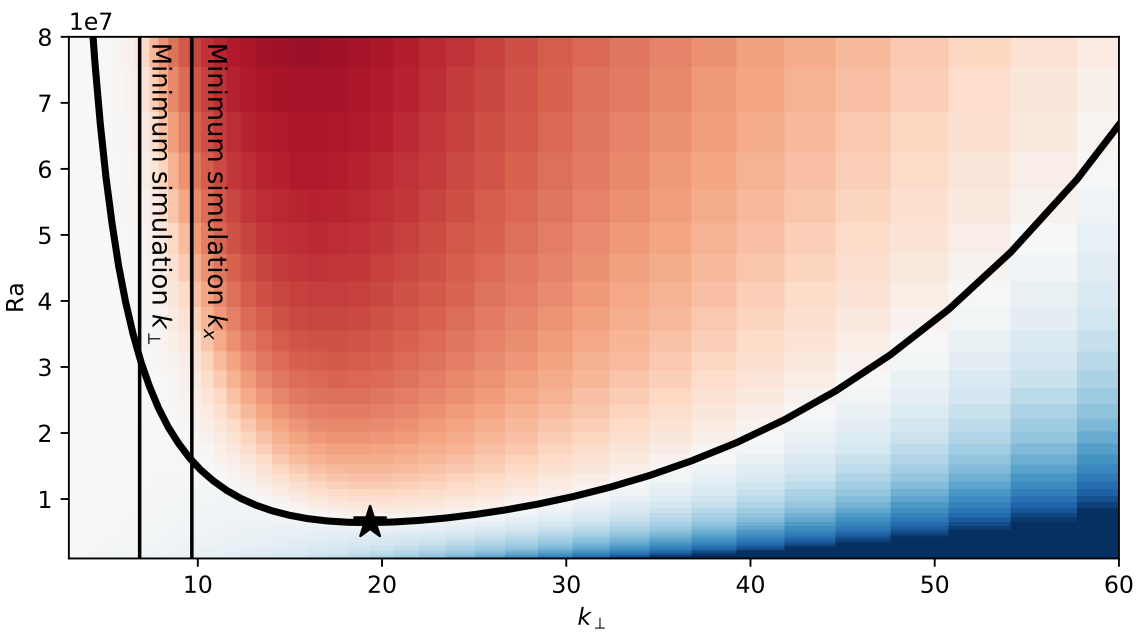

FIG. 2. Critical curve of convective onset at $\mathrm{Ta} = 10^{10}$, $\epsilon = 10^{-4}$, $n_\rho = 3$. Red colors indicate a value of $(k_\perp, \mathrm{Ra})$ that is convectively unstable – that is, where the linear solution is exponential growth. Blue colors indicate a linear solution of exponential decay. The thick black line shows the critical curve that separates convectively unstable modes from convectively stable modes. The star indicates the true value of the critical Rayleigh number and indicates the point $(k_{\perp,\mathrm{crit}}, \mathrm{Ra_{crit}})$. In our simulation domains, we set $L_x = L_y = 2\lambda_{\mathrm{crit}} = 4\pi/k_{\perp,\mathrm{crit}}$. For this choice, we have annotated the maximum wavenumber which is contained along a given horizontal direction (x or y) with a line ("Minimum simulation k_x"), as well as the maximum wavenumber contained along a diagonal of our simulation ("Minimum simulation $k_\perp$").

constrained convective domains become "infinitely large" compared to convective elements very quickly. Due to rotationally constrained structures becoming essentially vertically invariant, the difference between a large box and a small box is that you basically fit more or less of those vertically invariant structures into your box. But – the solution and heat transport per horizontal area should be the same. It's unclear if that intuition fits in stratified convection, but our highly-rotationally constrained results (right panels of Fig. 3.2) suggest that it might be true.

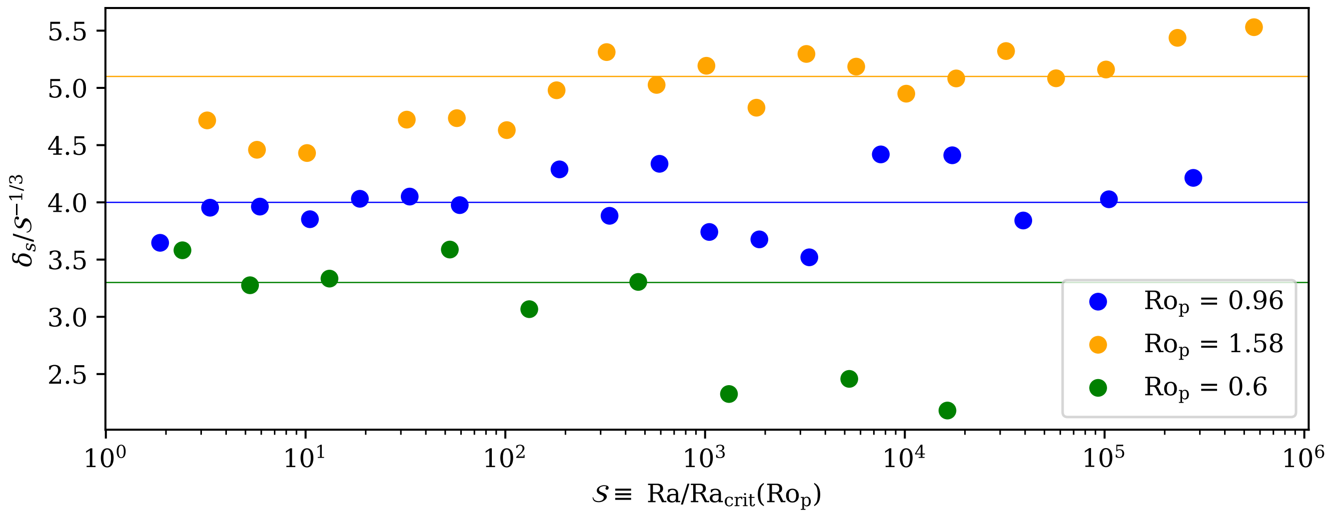

FIG. 3. The thickness of the entropy boundary layer at the top of the simulation is plotted vs. Ra at constant $\mathrm{Ro_p}$. For each case, we plot the supercriticality, $\mathcal{S}$ on the x-axis, which is Ra normalized by the value of $\mathrm{Ra_{crit}}$ along that $\mathrm{Ro_p}$ path, reported in table 3.1. The boundary layer values are compensated by $\mathcal{S}^{-1/3}$, the prediction for the scaling of boundary layers per the arguments of King et. al. To first order, across many decades of Ra, this quantity is flat, indicating that this predition describes our simulation behavior well.

4. Ah, the confusion here is due to bad notation on our part. The $\mathrm{Ra_{crit}}$ we're referring to here is a $\mathrm{Ra_{crit}}(\mathrm{Ro_p}$, not a $\mathrm{Ra_{crit}}(\mathrm{Ta})$. If you look at Fig. 3.1a, we're referring to the value of $\mathrm{Ra_{crit}}$ for a given value of $\mathrm{Ro_p}$, which has a singular value, indicated by an orange circle. We're not referring to the full black line in that figure which is $\mathrm{Ra_{crit}}(\mathrm{Ta})$. Perhaps we should have stopped that scaling before the final proportionality and left it at (Nu 1) $\propto \mathrm{Ra}^{3/2}/\mathrm{Ta} = \mathrm{Ro_p}\mathrm{Ra}^{1/6}$, meaning that we would expect a $\mathrm{Ra}^{1/6}$ scaling along a fixed $\mathrm{Ro_p}$ path.

5. To first order, $\delta_s \sim \mathrm{Ra}^{-1/3}$ is a good description of what we're seeing on our $\mathrm{Ro_p}$ paths. See Fig. 3.

6. At fixed Prandtl number (and ϵ, n_ρ), the solution is completely characterized by Ra and Ta – correct. We've posited in this work that there's a "new" input parameter,

$$\mathrm{Ro_p} = \mathrm{Ra}^{\alpha}\mathrm{Ta}^{\beta}\mathrm{Pr}^{\gamma}. \tag{1}$$

Rearranging this expression, it becomes clear that

$$\mathrm{Ta} = (\mathrm{Ra}^{\alpha}\mathrm{Pr}^{\gamma})^{1/\beta}\,, \tag{2}$$

which is to say that, at fixed Pr, even if we now have "three" input parameters (Ra, Ta, $\mathrm{Ro_p}$), we only have two degrees of freedom given those inputs. Choosing two (e.g., Ra and $\mathrm{Ro_p}$ in this work) specifies the third (Ta).

We've suggested that $\alpha = 1/2$, $\beta = -3/8$, and $\gamma = -1/4$ may be the "proper" definition of $\mathrm{Ro_p}$, where by "proper" we mean: these exponents may trace out surfaces in (Ra, Ta, Pr) space along which the Rossby number is constant. When we refer to the $\mathrm{Ra}{\sim}\mathrm{Ta}^{3/4}$ scaling, what we are referring to is really the ratio of β and α,

$$\frac{\beta}{\alpha} = \frac{-3/8}{1/2} = -\frac{3}{4}.$$

Recast differently, if you fix the values of Pr and $\mathrm{Ro_p}$, you can see that Ra and Ta are constrained by

$$\mathrm{Ra} \propto \mathrm{Ta}^{-\beta/\alpha} \qquad \rightarrow \qquad \mathrm{Ra} \propto \mathrm{Ta}^{3/4},$$

and this is the scaling we're referring to.

7. Chi-by-eye, I see Ro decreasing by less than a factor of 2 over an order of magnitude of Ek in this picture, but it'd be good to have exact values.