Additions to ch2 postscript

I. CONCERNS

This document includes text that will be cleaned up and added to the postscript to Ch. 2 of my thesis. It aims to address Brad's concerns regarding this work, which I'll attempt to summarize here.

A. Overarching Concerns

The biggest concern is that Ra and ϵ are not independent control parameters. Put differently,

1. Is ϵ in the initial state similar to ϵ in the final state, and
2. Is Ra in the initial state similar to Ra in the final state?

B. Smaller concerns and questions

1. Is the lack of scaling of Mach vs. Ra in 3D a result of entropy restratification?

I will not show in deep analytical detail what causes the differences in Mach number scaling in 2D and 3D. Bluntly, this is because I don't know what causes this, and figuring this out and showing it in careful detail is a new paper's worth of work. However, I will comment on the evidence that we have on hand that suggests that this is a 2D vs. 3D effect and not something else.

II. ADDRESSING THE OVERARCHING CONCERNS

A. A clearer nondimensionalization of the equations

The (dimensional) form of the fully compressible equations solved in this chapter is

$$\frac{\partial \ln \rho}{\partial t} + \nabla \cdot \boldsymbol{u} = -\boldsymbol{u} \cdot \nabla \ln \rho, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + R\nabla T - \nu \nabla \cdot \bar{\bar{\boldsymbol{\sigma}}} - \bar{\bar{\boldsymbol{\sigma}}} \cdot \nabla \nu = -\boldsymbol{u} \cdot \nabla \boldsymbol{u} - RT\nabla \ln \rho + \boldsymbol{g} + \nu \bar{\bar{\boldsymbol{\sigma}}} \cdot \nabla \ln \rho, \tag{2}$$

$$\frac{\partial T}{\partial t} - \frac{1}{c_V}\left(\chi \nabla^2 T + \nabla T \cdot \nabla \chi\right) = -\boldsymbol{u} \cdot \nabla T - (\gamma - 1)T\nabla \cdot \boldsymbol{u} + \frac{1}{c_V}\left(\chi \nabla T \cdot \nabla \ln \rho + \nu\, [\bar{\bar{\boldsymbol{\sigma}}} \cdot \nabla] \cdot \boldsymbol{u}\right), \tag{3}$$

Here, $\boldsymbol{u}$ is the velocity, ρ is the density, T is the temperature, R is the ideal gas constant (where the pressure $P = R\rho T$), $\boldsymbol{g}$ is the gravitational acceleration, $\bar{\bar{\boldsymbol{\sigma}}}$ is the viscous stress tensor (in units of inverse time), c_V is the specific heat at constant volume, and ν and χ are respectively the viscous and thermal diffusivities (in units of length^2/time). In constructing these equations, we have assumed that the dynamic diffusivities are defined as $\mu = \rho\nu$ (the dynamic viscosity) and $\kappa = \rho\chi$ (the thermal conductivity).

Before analyzing these equations further, I will assume that the background state is in hydrostatic equilibrium and thermal equilibrium. This means that

$$\nabla T_0 + T_0 \nabla \ln \rho_0 = \frac{\boldsymbol{g}}{R}, \tag{4}$$

and

$$\chi(\nabla^2 T_0 + \nabla T_0 \cdot \nabla \ln \rho_0) + \nabla T_0 \cdot \nabla \chi = 0 \tag{5}$$

Throughout this work, we assume that the diffusivities are functions of depth but not time, and can be expressed in terms of a constant value and the initial density stratification,

$$\chi(z) = \chi_t \frac{\rho_t}{\rho_0(z)}, \qquad \nu(z) = \nu_t \frac{\rho_t}{\rho_0(z)},$$

where χ_t, ν_t, and ρ_t are respectively the values of the thermal diffusivity, viscous diffusivity, and density at the top of the atmosphere, and are constant values. Throughout this work, we set $\rho_t = 1$, and I will drop it going forward. These diffusivity profiles can be plugged into Eqn. 5,

$$\frac{\chi_t}{\rho_0(z)}(\nabla^2 T_0 + \nabla T_0 \cdot \nabla \ln \rho_0) - \frac{\chi_t}{\rho_0(z)}\nabla T_0 \cdot \nabla \ln \rho_0 = 0 \qquad \rightarrow \qquad \nabla^2 T_0 = 0,$$

which is to say that for this choice of diffusivities, the only requirement for thermal equilibrium in the initial, static, conductive state is that the temperature profile have no second derivative.

Since hydrostatic and thermal equilibrium are always satisfied by the equations, we can remove them from the momentum and temperature equation and we can plug in our definition of the diffusivities. I'll also rearrange for easier reading (our former setup of the equations was set up to show LHS / RHS splitting of linear and nonlinear terms),

$$\frac{\partial \ln \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \ln \rho + \nabla \cdot \boldsymbol{u} = 0 \tag{6}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -R(\nabla T_1 + T_1 \nabla \ln \rho_0 + T_0 \nabla \ln \rho_1 + T_1 \nabla \ln \rho_1) + \frac{\nu_t}{\rho_0}\left(\bar{\bar{\boldsymbol{\sigma}}} \cdot \nabla \ln \rho_1 + \nabla \cdot \bar{\bar{\boldsymbol{\sigma}}}\right), \tag{7}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T + (\gamma - 1) T \nabla \cdot \boldsymbol{u} = \frac{\chi_t}{c_V \rho_0}(\nabla^2 T_1 + \nabla T_0 \cdot \nabla \ln \rho_1 + \nabla T_1 \cdot \nabla \ln \rho_1) + \frac{\nu_t}{c_V \rho_0}[\bar{\bar{\boldsymbol{\sigma}}} \cdot \nabla] \cdot \boldsymbol{u}, \tag{8}$$

B. The scale of thermodynamic fluctuations and velocities

In our convective system, to find the magnitude of thermodynamic fluctuations, we turn to the entropy equation,

$$\frac{1}{c_P}\nabla s = \frac{1}{\gamma}\nabla \ln T - \frac{\gamma - 1}{\gamma}\nabla \ln \rho, \tag{9}$$

where s is the specific entropy, c_P is the specific heat at constant pressure, and $\gamma = c_P/c_V = 5/3$ is the adiabatic index. For consistency with our work in this chapter, we will decompose our thermodynamic variables as follows:

$$T = T_0 + T_1, \qquad s = s_0 + s_1, \qquad \ln \rho = \ln \rho_0 + \ln \rho_1.$$

Note that due to our somewhat unintuitive decomposition on $\ln \rho$, the density fluctuations are of the form

$$\rho = \rho_0 + \rho' = \rho_0 e^{\ln \rho_1} \rightarrow \ln \rho_1 = \ln\left(1 + \frac{\rho'}{\rho_0}\right).$$

We assume that the background entropy gradient is slightly negative,

$$\frac{1}{c_P}\nabla s_0 = \frac{1}{\gamma}\nabla \ln T_0 - \frac{\gamma - 1}{\gamma}\nabla \ln \rho_0 = -O(\epsilon).$$

Furthermore, we will assume that convective motions will aim to drive the atmosphere towards an adiabat, wiping out the superadiabaticity of the initial state. Put differently, we assume that $\nabla s = 0$ most places in the domain such that $\nabla s_1 = O(\epsilon)$ (except in boundary layers). This means that

$$\frac{1}{c_P}\nabla s_1 = \frac{1}{\gamma}\frac{\nabla T_1}{T_0 + T_1} - \frac{\gamma - 1}{\gamma}\nabla \ln \rho_1 \approx \epsilon.$$

If ϵ is small, we expect thermodynamic fluctuations from the background to be small. Under this assumption, we can assume that $T_0 + T_1 \approx T_)$, and we can integrate the former equation to find the magnitude of fluctuations,

$$\frac{s_1}{c_P} \approx \frac{1}{\gamma}\frac{T_1}{T_0} - \frac{\gamma - 1}{\gamma}\ln \rho_1 \approx \epsilon, \tag{10}$$

which states that fluctuations in thermodynamic quantities are O(ϵ) compared to the background atmosphere.

Ok, with that in mind, let's return to the momentum equation and assume that the dominant force balance is between advection and buoyancy,

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -R(\nabla T_1 + T_1 \nabla \ln \rho_0 + T_0 \nabla \ln \rho_1 + T_1 \nabla \ln \rho_1).$$

If we assume $\nabla = L^{-1}$, a length scale, and we multiply the RHS by T_0/T_0, we retrieve

$$\frac{u^2}{L} \approx \frac{RT_0}{L}\left(\frac{\nabla T_1}{T_0} + \frac{T_1}{T_0}\nabla \ln\rho_0 + \nabla\ln\rho_1 + \frac{T_1}{T_0}\nabla\ln\rho_1\right).$$

Using our above scaling arguments, the first three terms in the RHS parenthesis are O(ϵ), and the last term is O(ϵ^2). Plugging in the definition of the isothermal sound speed for the background atmosphere, $c_s^2 = RT_0$, we get

$$u^2 \sim c_s^2[O(\epsilon) + O(\epsilon^2)],$$

Assuming that $\epsilon \leq 1$, or that thermodynamic fluctuations aren't larger than the background state, we can drop the O(ϵ^2) term, and we retrieve

$$\mathrm{Ma}^2 = \frac{u^2}{c_s^2} = O(\epsilon).$$

C. Nondimensionalization on the freefall velocity

While this is *not* the nondimensionalization we used in this work, I think it is perhaps more clear than the one that we used in the work. Let's nondimensionalize the velocity on the freefall velocity scale at the top of the domain (in the published work we nondimensionalized on the sound speed scale). We'll use the same thermodynamic nondimensionalization as in the published work (so that all the initial atmosphere has all thermodynamic quantities equal to unity at the top of the atmosphere). This nondimensionalization is

$$\nabla^* \to \frac{1}{L}\nabla, \qquad \partial_{t^*} \to \frac{1}{\tau}\partial_t, \qquad \boldsymbol{u}^* \to u_{\mathrm{ff}}\boldsymbol{u}\ (\text{with}\ u_{\mathrm{ff}} = \sqrt{\epsilon R T_t}), \qquad T^* \to T_t T, \qquad \rho^* \to \rho_t\rho,$$

where here, quantities with (*) are "dimensionful," as in the previous equations, and going forward quantities without stars will be nondimensional. In this nondimensionalization, convective velocities and times will be O(1), and thermodynamic fluctuations ($T_1, \ln\rho_1$) will be O(ϵ), because background thermodynamic quantities are O(1).

1. *Continuity equation*

The contintuity equation as we write it is already nondimensional,

$$\frac{\partial \ln\rho_1}{\partial t} + \nabla\cdot\boldsymbol{u} + w\partial_z\ln\rho_0 = -\boldsymbol{u}\cdot\nabla\ln\rho_1. \tag{11}$$

In this nondimensionalization, it is immediately clear that this equation has two O(1) terms:

$$\nabla\cdot\boldsymbol{u} + w\partial_z\ln\rho_0,$$

and the remaining terms are O(ϵ). The anelastic approximation is the approximation in which $\epsilon \to 0$ and the O(ϵ) terms drop out of the equation. Note that it is crucial that we solve the linear, O(1) pieces of this equation implicitly in order to avoid hard CFL constraints from sound waves on our low-Mach flows.

2. *Momentum equation*

Nondimensionalizing the momentum equation, we find that

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{RT_t}{u_{\mathrm{ff}}^2}\left(\nabla T_1 + T_1\nabla\ln\rho_0 + T_0\nabla\ln\rho_1 + T_1\nabla\ln\rho_1\right) + \frac{1}{\rho_0}\frac{\nu_t}{u_{\mathrm{ff}}L}\left(\bar{\bar{\boldsymbol{\sigma}}}\cdot\nabla\ln\rho_1 + \nabla\cdot\bar{\bar{\boldsymbol{\sigma}}}\right) \tag{12}$$

Defining the freefall Reynolds number at the top of the domain as $\mathrm{Re}_{\mathrm{ff}} = u_{\mathrm{ff}}L/\nu_t$, and remembering that $u_{\mathrm{ff}}^2 = \epsilon R T_t$, the momentum equation is

$$\frac{D\boldsymbol{u}}{Dt} = -\epsilon^{-1}\left(\nabla T_1 + T_1\nabla\ln\rho_0 + T_0\nabla\ln\rho_1 + T_1\nabla\ln\rho_1\right) + \frac{1}{\rho_0}\frac{1}{\mathrm{Re}_{\mathrm{ff}}}\left(\bar{\bar{\boldsymbol{\sigma}}}\cdot\nabla\ln\rho_1 + \nabla\cdot\bar{\bar{\boldsymbol{\sigma}}}\right). \tag{13}$$

Under this nondimensionalization, the LHS terms are O(1). The first three buoyancy terms are O(1) and the fully nonlinear buoyancy term is O(ϵ). The viscous term's magnitude depends primarily on the Reynolds number.

3. Energy Equation

A similar nondimensionalization of the energy equation gives us

$$\frac{DT}{Dt} + (\gamma - 1)T\nabla\cdot\boldsymbol{u} = \frac{\chi_t}{u_{\text{ff}}L}\frac{1}{\rho_0 c_V}(\nabla^2 T_1 + \nabla T_0\cdot\nabla\ln\rho_1 + \nabla T_1\cdot\nabla\ln\rho_1) + \frac{\nu_t}{\tau T_t}\frac{1}{\rho_0 c_V}(\bar{\bar{\boldsymbol{\sigma}}}\cdot\nabla)\cdot\boldsymbol{u}. \tag{14}$$

The thermal diffusion term straightforwardly has a freefall Péclet number in it ($\text{Pe}_{\text{ff}} = u_{\text{ff}}L/\chi_t$). The viscous heating term is a bit more confusing, but

$$\frac{\nu_t}{\tau T_t} = \frac{1}{T_t}\frac{\nu_t\tau}{L^2}\frac{L^2}{\tau^2} = \frac{1}{T_t}\frac{u_{\text{ff}}^2}{\text{Re}_{\text{ff}}} = \frac{\epsilon R}{\text{Re}_{\text{ff}}}.$$

Unsurprisingly, the viscous heating term (which is composed of O(1) velocities) has an ϵ appear in front of it to reflect that its magnitude is of the order of the temperature fluctuations. Plugging these back in, the full energy equation is

$$\begin{aligned}\frac{\partial T_1}{\partial t} &+ (w\partial_z T_0 + [\gamma-1]T_0\nabla\cdot\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla T_1 + [\gamma-1]T_1\nabla\cdot\boldsymbol{u}) \\ &= \frac{1}{\rho_0 c_V}\left[\frac{1}{\text{Pe}_{\text{ff}}}(\nabla^2 T_1 + \nabla T_0\cdot\nabla\ln\rho_1 + \nabla T_1\cdot\nabla\ln\rho_1) + \frac{\epsilon R}{\text{Re}_{\text{ff}}}(\bar{\bar{\boldsymbol{\sigma}}}\cdot\nabla)\cdot\boldsymbol{u}\right].\end{aligned} \tag{15}$$

I've written the LHS of this equation to call attention to the fact that there are two groups of terms on the LHS: O(1) terms which include velocities and T_0 (sound wave terms which must be implicitly timestepped), and O(ϵ) terms which include T_1 and are on the scale of convective dynamics. Note also that in our nondimensionalization where $P = \rho T$, $R = 1$, so the viscous heating term just has $\epsilon/\text{Re}_{\text{ff}}$ in front of it.

D. Control Parameters

From the above nondimensional equations, we can see fairly straightforwardly that there are three primary control parameters in these equations:

- Re_{ff}, the freefall Reynolds number,
- Pe_{ff}, the freefall Péclet number, and
- ϵ, the superadiabaticity.

The first two of these terms are set in our convective domains by an input Rayleigh and Prandtl number,

$$\text{Re}_{\text{ff}} = \sqrt{\frac{\text{Ra}}{\text{Pr}}}, \qquad \text{Pe}_{\text{ff}} = \sqrt{\text{Ra}\,\text{Pr}}.$$

The third of these parameters, ϵ, is initially set by the superadiabatic excess, but because we are not fixing the *entropy* at the top and bottom of the domain (we're fixing the temperature), it's possible that this value is evolving over time. Let's explore this possibility to see whether or not the input value of ϵ is truly the value of ϵ in the evolved flows.

1. The magnitude of Δs

In order to find the imposed entropy jump across our convective domain, we integrate the horizontally-averaged entropy equation,

$$\frac{1}{c_P}\nabla s = \frac{1}{\gamma}\nabla\ln T - \frac{\gamma-1}{\gamma}\nabla\ln\rho, \tag{16}$$

from the bottom to the top of our domain, to find

$$\frac{\Delta s}{c_P} = \int_0^{L_z}\frac{\partial s}{\partial z}dz = \frac{1}{\gamma}\left(\ln T\Big|_{z=0}^{z=L_z} - (\gamma-1)\ln\rho\Big|_{z=0}^{z=L_z}\right).$$

Under our choice of fixed-temperature boundary conditions, $T(z=0)=T_b$ and $T(z=L_z)=T_t$. Furthermore, we note that $\ln(\rho_t/\rho_b)=-n_\rho$

$$\frac{\Delta s}{c_P}=\frac{1}{\gamma}\ln\left(\frac{T_t}{T_b}\right)+\frac{\gamma-1}{\gamma}n_\rho(t). \tag{17}$$

For our nondimensional polytropes, $T_t=1$ and $T_b=1+L_z$, where we define $L_z=e^{n_\rho/m}-1$. This means $\ln(T_t/T_b)=\ln(e^{-n_{\rho,0}/m})=-n_{\rho,0}/m$, where $n_{\rho,0}$ is the value of n_ρ in the initial atmosphere. Decomposing $n_\rho(t)=n_{\rho,0}+\Delta n_\rho(t)$, we find

$$\frac{\Delta s}{c_P}=\frac{1}{\gamma}\left((\gamma-1)n_{\rho,0}-\frac{n_{\rho,0}}{m}\right)+\frac{\gamma-1}{\gamma}\Delta n_\rho(t) \tag{18}$$

Also recall that $m=m_{\text{ad}}-\epsilon$ and $m_{\text{ad}}=(\gamma-1)^{-1}$, such that we can simplify this expression to

$$\frac{\Delta s}{c_P}=\frac{\gamma-1}{\gamma}\left(-\epsilon\frac{n_{\rho,0}}{m}+\Delta n_\rho(t)\right). \tag{19}$$

In other words, this is some $\mathrm{O}(\epsilon)$ quantity that evolves with the number of density scale heights of the evolved atmosphere. So – let's try to put some boundaries on the value of $\Delta n_\rho(t)$.

a. Estimates for changes in n_ρ We can't know this a priori – if we did, we wouldn't need to run the convection simulation because we'd know the boundary layer structure (and thus stuff like the Nusselt number) in advance. But – we can build a simple model to get a feel for how n_ρ will evolve. We know that the atmospheric density stratification will evolve under two constraints:

1. Mass is conserved, and
2. Adiabaticity is achieved throughout most of the domain (the convective interior, not the boundary layers).

Our choice of fixing the temperature at the top and bottom boundaries means that the adiabat that our simulation "chooses" in the convective interior will be bounded by two adiabats that we know about: the adiabat the crosses the top boundary temperature ($T=1$ at $z=L_z$) and the adiabat that crosses the bottom boundary temperature ($T=1+L_z$ at $z=0$). We further know the adiabatic temperature gradient for our nondimensionalization, $\nabla_{\text{ad}}=-g/c_P=-1+\epsilon/c_P$. Let's therefore set up models where

$$\partial_z T_{\text{top-BL}}=\begin{cases}\nabla_{\text{ad}} & z<(1-\delta)L_z,\\ \nabla_{\text{s}} & z\geq(1-\delta)L_z,\ |\nabla_{\text{s}}|>|\nabla_{\text{ad}}|\end{cases},\qquad \partial_z T_{\text{bot-BL}}=\begin{cases}\nabla_{\text{ad}} & z>\delta L_z,\\ \nabla_{\text{s}} & z\leq\delta L_z,\ |\nabla_{\text{s}}|>|\nabla_{\text{ad}}|\end{cases}, \tag{20}$$

or where the temperature gradient is perfectly adiabatic everywhere except for, respectively, a superadiabatic top or bottom boundary layer. The boundary layer will extend through some fraction of the domain, $\delta<1$, and its superadiabatic temperature gradient, ∇_{s}, is constrained by the fact that the temperature is fixed at both boundaries by the initial superadiabatic polytrope. For a given value of ϵ and δ, a temperature profile with a top boundary layer ($T_{\text{top-BL}}$) or a bottom boundary layer ($T_{\text{bot-BL}}$) can be constructed. To understand how these boundary layers respectively affect the density stratification, and therefore the entropy jump across the domain, we can solve a simple mass-conserving boundary value problem for hydrostatic equilibrium,

$$\begin{aligned}\frac{\partial M}{\partial z}&=\rho,\\ T\frac{\partial\ln\rho}{\partial z}+\frac{\partial T}{\partial z}&=-g,\end{aligned} \tag{21}$$

under the constraints that $M=0$ at $z=0$ and $M=\int_0^{L_z}\rho_0 dz$ at $z=L_z$. In the left two panels of Fig. 1, some examples of these temperature and density profiles can be seen. All cases shown have $\epsilon=0.5$, to ensure that the changes away from the initial state are appreciable and observable. The orange lines show profiles of $T_{\text{top-BL}}$ and the purples lines show profiles of $T_{\text{bot-BL}}$; for both cases, we solve out for $\delta=\{0.2,0.02\}$. The changes in $\ln\rho$ are qualitatively similar to the changes observed in Fig. 4 of our published work, and we generally find that: bottom boundary layers aim to *increase* n_ρ while upper boundary layers aim to *decrease* n_ρ. We performed this analysis for a broad range of epsilon in the range $\epsilon=[10^{-7},0.5]$, and we show the magnitude of the change of n_ρ as a function of ϵ in the upper right panel. In all cases, Δn_ρ for the orange (upper boundary layer) points are negative while the purple (lower boundary layer) points are positive. In the lower right panel, we solve Eqn. 19 for ΔS, and normalize it by

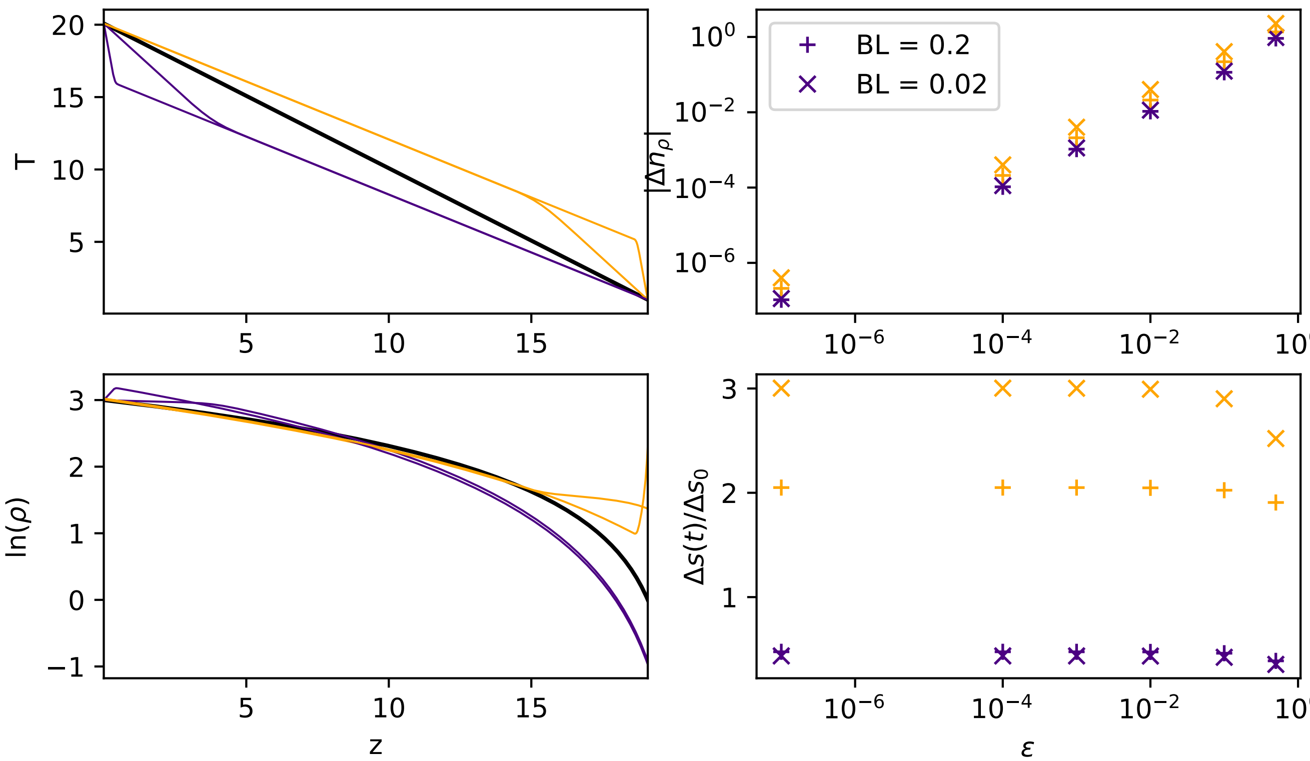

FIG. 1. (left two panels) Stratification of simple atmospheric solutions with upper and lower boundary layers are shown. Temperature (upper left) and log density (lower left) are shown, and the initial state (an $\epsilon = 0.5$ polytrope) is shown in black. Orange profiles show atmospheres which are perfectly adiabatic everywhere except for an upper boundary layer, and purple profiles show profiles which are adiabatic other than a lower boundary layer. All shown atmospheres have the same ΔT across the domain and the same mass. (upper right) Changes in n_ρ from profiles similar to those in the left panels at many different values of ϵ. Purple points correspond to lower boundary layers and are positive; orange points correspond to upper boundary layers and are negative. The shape of the marker shows the difference between a thicker boundary layer (0.2 or 20% of the domain) vs. a thinner boundary layer (0.02 or 2%). Δn_ρ scales with ϵ, as expected. In the lower right panel, we calculate the evolved Δs according to Eqn. 19 and normalize it by the initial Δs. We find that upper boundary layers make the atmosphere more superadiabatic and lower boundary layers make the atmosphere more adiabatic.

its input value. We find that upper boundary layers aim to increase the magnitude of ΔS, effectively increasing the input value of ϵ (by a factor of $\sim 2-3$). Lower boundary layers, on the other hand, aim to decrease the magnitude of ΔS, effectively decreasing the input value of ϵ (by a factor of, again, $\sim 2-3$). In our published work (Fig. 4), we find that the upper boundary layer effects tend to dominate, so it's reasonable to assume that ΔS will change by a factor of a few away from the initial conditions.

b. Experimentally measured changes in n_ρ Fortunately, when I did this work in 2017, I included the evolved value of n_ρ in a supplementary `.csv` file. Unfortunately, when I did this work, I was fairly new to computing, and I made a bit of a rookie mistake. I had my simulations output the volume-integrated value of the average value of $\ln \rho$ at the bottom minus the average value of $\ln \rho$ at the top. I did *not* output the *petrubation* value away from the intial state, but rather included the initial state plus the fluctuations. This means that I have robust measurements for high ϵ, where the fluctuations are on the same order of magnitude as the initial state. I trust that the values for $\epsilon = 10^{-4}$ are approximately correct, but the values for $\epsilon = 10^{-7}$ appear to just be spurious numerical noise, so I'm not including them in this analysis.

Using Eqn. 19, I calculated the evolved Δs in my simulations and normalized it by the input value. I've plotted this in the left panel of Fig. 2. We can see that ΔS changes by a factor of a few over many decades of Ra in both 2D and 3D. If we calculate the value of Ra in the evolved state through the formula,

$$\mathrm{Ra}_{\mathrm{out}} = \frac{\Delta s_{\mathrm{evolved}}}{\Delta s_{\mathrm{initial}}} \mathrm{Ra}_{\mathrm{in}},$$

and plot $\mathrm{Ra}_{\mathrm{out}}$ vs. $\mathrm{Ra}_{\mathrm{in}}$ in the right panel of Fig. 2, we see that while we don't perfectly achieve a 1-to-1 input-to-output

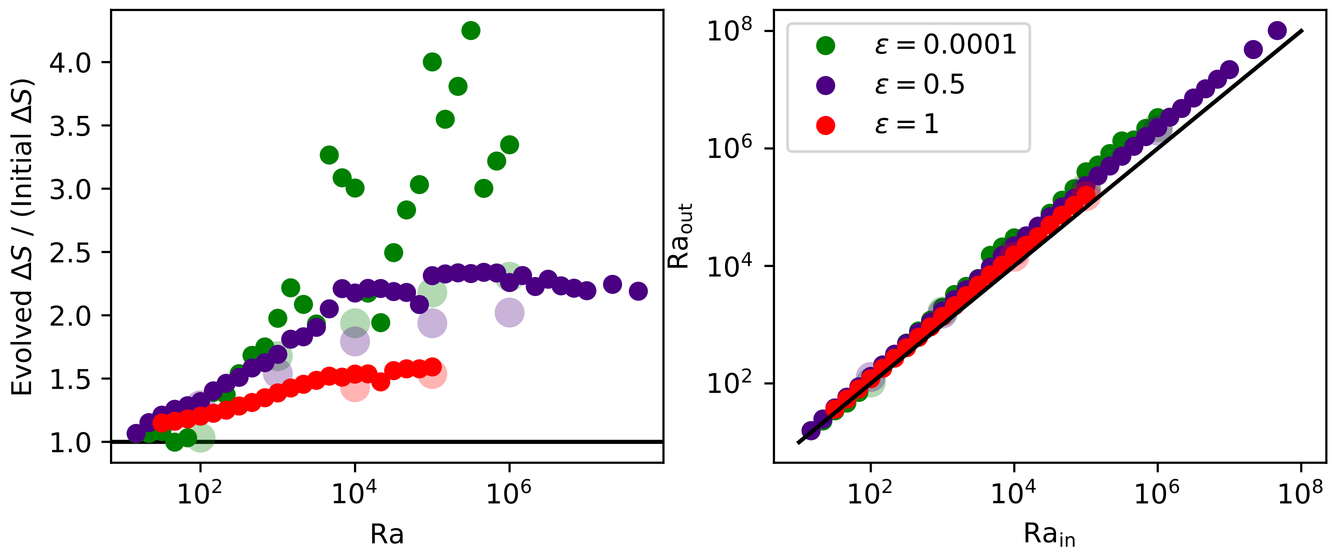

FIG. 2. (left) Measured values of Δs, from Eqn. 19, using values of Δn_ρ obtained from our simulations. 2D points are small, dark circles while 3D points are large, light-colored circles. (right) The corresponding changes in Ra resulting from the small observed changes in Δs. The ideal 1-to-1 correspondence between input and output Ra is plotted as a black line, and it is close to the achieved scaling.

ratio, we achieve something close to that. Furthermore, for the larger values of ϵ where I trust this measurement more, it seems like the evolved ΔS plateaus around Ra $\sim 10^4$ (similar to where those cases become supersonic), so ϵ and Ra are truly independent input parameters for our most turbulent simulations.

E. Mach number evolution in 2D vs. 3D

If the scaling of Mach number with Ra were a result of Δs changing with increasing Ra, our presented results would suggest that Mach number should grow with increasing Ra in both 2D and 3D. We find (Fig. 2) that the magnitude of ΔS grows with Ra in both 2D and 3D, and the Mach number should also therefore grow. However, ΔS changes only by a factor of 2-4, at most, from onset through the most turbulent simulations that we studied. We would therefore expect Mach to change by $\sqrt{2-4}$, at most, as a result of ΔS changes. This is not what we observe; in 3D, the Mach number at fixed ϵ is constant as a function of increasing Ra, while the Mach number in 2D increases by more than an order of magnitude. This disagreement suggests that changing entropy profiles are not responsible for the changes in Mach number.

Here's the data that we have on hand:

1. Velocities grow with increasing Ra in 2D but not 3D.
2. Morphologies are very different in 2D and 3D (2D exhibits a large, coherent "flywheel," 3D does not.)
3. At low Mach number, eventually 2D cases break apart these flywheels and exhibit "shearing" states, much like in Boussinesq convection.

So this tells us that something about 2D compressible convection is very similar to 2D Boussinesq convection (except at Mach 1). However, to my knowledge, no one has really done the work to specifically tease out why these high-velocity flywheel modes are so favored by 2D convection. It's possible that these modes are a result of inverse turbulent cascades in 2D, but that's pure speculation. This would be an interesting problem to pursue, but it's beyond the scope of this thesis.