Accelerated evolution of convective simulations

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We present a method for coupling boundary value problems with initial value problems in order to achieve Accelerated Evolution (AE) of convective solutions on dynamical timescales, rather than the long thermal timescale. We study this method in the context of Rayleigh-Bénard convection. We demonstrate that the solution reached by AE and Standard Evolution (SE) are similar, and that this method works at a large range of supercriticalities. The AE method is used to achieve converged solutions at high supercriticality (10^7), and its extensions to more complex systems are briefly discussed.

I. INTRODUCTION

Natural convection occurs in the presence of disparate timescales which prohibit numericists from studying realistic models of natural systems. For example, flows in the convection zone of stars like the Sun are characteristically low Mach number (Ma) in the deep interior. Explicit timestepping methods which are bound by the Courant-Friedrich-Lewy (CFL) timestep limit must resolve the fastest motions (sound waves), resulting in timesteps which are prohibitively small for studies of the deep, low-Ma motions. These systems are numerically stiff, and the difference between the sound crossing time and the convective overturn time have made studies of low-Ma stellar convection difficult. Traditionally, approximations such as the anelastic approximation, in which sound waves are explicitly filtered out, have been used to study low-Ma flows [1, 2]. More recently, advanced numerical techniques which use implicit or mixed implicit-explicit timestepping mechanisms have made it feasible to study convection at low Mach numbers [3–8], and careful studies of deep convection which would have been impossible a decade ago are now widely accessible.

While solutions to the problem of divergent *dynamical* timescales have proven useful, the difference between convective timescales and thermal timescales, which characterize system relaxation, remain a significant problem facing studies of convection. Solar convection is a prime example of this phenomenon; dynamical timescales in the solar convective zone are relatively short (10 min overturn at solar surface, one month solar rotation rate) compared to the Kelvin-Helmholtz timescale of $3 \cdot 10^7$ years [9]. In such a system, it is impossible to resolve the convective dynamics while also evolving the thermal structure of the system in a meaningful fashion. As modern simulations aim to model natural convection by increasing into the high-Rayleigh-number (Ra) regime, the ratio of the thermal diffusion timescale to dynamical timescales grows [7]. Furthermore, as dynamical and thermal timescales separate, simulations become more turbulent. Increasingly turbulent motions require finer grid meshes and smaller timesteps to capture advective dynamics. Thus, the progression of simulations into the high-Ra regime of natural convection is slowed by two simultaneous effects: timestepping through a single convective overturn time becomes more computationally expensive while the number of overturn times required for systems to reach thermal equilibration grows.

The vast difference between convective and thermal timescales has long plagued numericists studying convection, and an abundance of approaches has been employed to study thermally converged solutions. One popular method for accelerating the convergence of high-Ra solutions is by "bootstrapping" – the process of using the flow fields in a converged solution at lower Ra as initial conditions for a simulation at a higher Ra. This method has been used with great success [10, 11], but it is not without its faults. Bootstrapped solutions are succeptible to hysteresis effects, in which large-scale convective structures present in the low Ra solution imprint onto the dynamics of the new, high Ra solution. Another commonly-used tactic in moderate-Ra simulations is to use a simple model of the full convective state as initial conditions. For example, past studies have used a linear eigenvalue solve to set the initial convective state [12] or used an axisymmetric solution as initial conditions for convection in a 3D cylinder [11]. In certain systems, the approximate state of the evolved solution can be estimated. There, an appropriate set of initial conditions can either be solved for analytically [13] or by using knowledge of Mixing Length Theory or other convective theories to adjust the initial profile towards the proper adiabatic state [14].

Despite the numerous methods that have been used, the most straightforward way to achieve a thermally converged solution is to evolve a convective simulation through a thermal timescale. Some modern studies do just that [2], but it is becoming increasingly difficult. Such evolution is *expensive*, and state-of-the-art simulations at the highest values of Ra can only be reasonably run for tens to hundreds of buoyancy times [15], much less the thousands of buoyancy

timescales required for thermal convergence.

In this work, we study a method of achieving accelerated evolution of a convective simulation through adjusting the thermodynamic state using information from resolved, convective dynamics. We couple measurements of the (non-converged) convective solution with knowledge about energy balances in the eventual solution to self-consistently adjust the mean thermodynamic profile towards its final state. While such a method has been used previously [16], we find no explanation in the current literature of the steps involved in employing this method, nor any study into the accuracy of such a method. In section II, we describe our convective simulations, our numerical methods, and our method for achieving accelerated evolution. In section III, we compare solutions reached through the accelerated evolution method to those that have been evolved through a full diffusive timescale, and we examine select simulations at very high Ra which have achieved accelerated evolution. Finally, in section IV, we discuss extensions of the methods presented here, and we offer concluding remarks.

II. EXPERIMENT

We study incompressible Rayleigh-Bénard convection under the Oberbeck-Boussinesq approximation, such that our fluid has a constant kinematic viscosity (ν), thermal diffusivity (κ), and coefficient of thermal expansion (α). The density of the fluid is a constant, ρ_0, except on the term where the constant gravitational acceleration, $\boldsymbol{g} = -g\hat{z}$, acts in the vertical momentum equation, where $\rho = \rho_0(1 - \alpha T_1)$. Under these constraints, the equations of motion are [17]

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_0}\nabla P - g(1 - \alpha T_1)\hat{z} + \nu \nabla^2 \boldsymbol{u}, \tag{2}$$

$$\frac{\partial T_1}{\partial t} + \boldsymbol{u} \cdot \nabla (T_0 + T_1) = \kappa \nabla^2 T_1, \tag{3}$$

where $\boldsymbol{u} = u\hat{x} + v\hat{y} + w\hat{z}$ is the velocity, $T = T_0 + T_1$ are the initial and fluctuating components of temperature, and P is the pressure. We non-dimensionalize these equations such that the dimensionless unit of length is the layer height (L_z), temperature is in units of the initial temperature jump across the layer ($\Delta T_0 = L_z \nabla T_0$), and velocity is in units of the freefall velocity ($v_{\rm ff} = \sqrt{\alpha g L_z^2 \nabla T_0}$). by these choices, one time unit is a freefall time (L_z/v_{ff}). We introduce a reduced kinematic pressure, $\varpi \equiv P/\rho_0 + \phi + |\boldsymbol{u}|^2/2$, where the gravitational potential, ϕ, is defined such that $\boldsymbol{g} = -\nabla\phi$. As P is a Langrange multiplier under the Oberbeck-Boussinesq approximation, ϖ can be treated straightforwardly as a linear variable. In non-dimensional form, Eqns. 2 & 3 become

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \varpi - T_1 \hat{z} + \mathcal{R} \nabla \times \boldsymbol{\omega} = \boldsymbol{u} \times \boldsymbol{\omega}, \tag{4}$$

$$\frac{\partial T_1}{\partial t} - \mathcal{P} \nabla^2 T_1 + w \frac{\partial T_0}{\partial z} = -\boldsymbol{u} \cdot \nabla T_1. \tag{5}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity. The dimensionless control parameters $\mathcal{P}$ and $\mathcal{R}$ are set by the Rayleigh and Prandtl numbers,

$$\mathcal{R} \equiv \sqrt{\frac{\rm Pr}{\rm Ra}}, \qquad \mathcal{P} \equiv \frac{1}{\sqrt{\rm Pr\,Ra}}, \qquad {\rm Ra} = \frac{g\alpha L_z^4 \nabla T_0}{\nu\kappa} = \frac{(L_z\, v_{\rm ff})^2}{\nu\kappa}, \qquad {\rm Pr} = \frac{\nu}{\kappa}. \tag{6}$$

We hold Pr= 1 constant throughout this work, such that $\mathcal{P} = \mathcal{R}$.

We study 2D and 3D convection in which the domain is a cartesian box, whose dimensionless vertical extent is $z \in [-1/2, 1/2]$, and which is horizontally periodic with an extent of $x, y \in [0, \Gamma]$ where $\Gamma = 2$ is the aspect ratio. In the 2D cases, we set $v = \partial_y = 0$. We specify no-slip, impenetrable boundary conditions at both the top and bottom boundary and we use mixed thermal boundary conditions, such that

$$u = v = w = 0 \text{ at } z = \pm 1/2, \qquad T_1 = 0 \text{ at } z = +1/2, \qquad \frac{\partial T_1}{\partial z} = 0 \text{ at } z = -1/2. \tag{7}$$

For this choice of boundary conditions, the critical value of Ra at which the onset of convection occurs is ${\rm Ra_{crit}} = 1295.78$, and the supercriticality of a run is defined as $S \equiv {\rm Ra/Ra_{crit}}$. Studies of convection which aim to model astrophysical systems such as stars often employ mixed thermal boundary conditions [12, 18, 19], as we do here; however, our choice of thermal boundary conditions here reflects the fact that the conditions in Eqn. 7 are the simplest to implement in the process of accelerated evolution (see section II A) we study here.

We utilize the Dedalus[1] pseudospectral framework [20] to evolve Eqns. (1), (4), & (5) forward in time using an implicit-explicit (IMEX), third-order, four-step Runge-Kutta timestepping scheme RK443 [21]. The linear terms (on the LHS of the equations) are solved implicitly, while the nonlinear terms (RHS) are explicitly solved. Variables are time-evolved on a dealiased Chebyshev (vertical) and Fourier (horizontal, periodic) domain in which the physical grid dimensions are 3/2 the size of the coefficient grid.

As initial conditions, we fill T_1 with random white noise whose magnitude is $10^{-6}\mathcal{P}$. This ensures that the initial perturbations are much smaller than the evolved convective temperature perturbations, even at large Ra. We filter this noise spectrum in coefficient space, such that only the lower 25% of the coefficients have power.

A. The method of accelerated evolution

Here we describe a method of Accelerated Evolution (AE). We use this method to quickly evolve the thermodynamic state of our solutions. We compare this method to Standard Evolution (SE), in which we naively evolve the atmosphere for one thermal diffusion time, $t_\kappa = \mathcal{P}^{-1}$. As Ra increases, SE solutions become intractable, while the timeframe of convergence for an AE solution remains nearly constant (in units of freefall times). For an example of time saving achieved by using AE, we compare energy traces at $S = 10^5$ from a SE run in Fig 1a to an AE run in Fig. 1b.

The horizontally averaged profiles of the vertical conductive flux, $\mathrm{F}_{\mathrm{cond}} = \langle -\kappa\nabla(T_0 + T_1)\rangle_{x,y}$, and the vertical convective flux, $\mathrm{F}_{\mathrm{conv}} = \langle w(T_0 + T_1)\rangle_{x,y}$, where $\langle\rangle_{x,y}$ represent a horizontal average, are the basis of the AE method. We measure both of these quantities early in a simulation, as in Fig. 1c. At these early stages in the simulation, these flux profiles are highly asymmetric, with much more flux leaving through the upper boundary than entering through the lower boundary as the atmosphere approaches the proper isotherm selected by the fixed temperature upper boundary condition. However, by calculating the total flux, $\mathrm{F}_{\mathrm{tot}}$ =$\mathrm{F}_{\mathrm{conv}}$+ $\mathrm{F}_{\mathrm{cond}}$, and then calculating the profiles

$$f_{\mathrm{conv}}(z) = \frac{F_{\mathrm{conv}}}{F_{\mathrm{tot}}}, \qquad f_{\mathrm{cond}}(z) = \frac{F_{\mathrm{cond}}}{F_{\mathrm{tot}}}, \tag{8}$$

we remove the asymmetry and examine which parts of the atmosphere are dominated by conduction, and which are not. We presume that the early convection occupies roughly the same volume as the evolved convection, and thus that the early thermal boundary layers are roughly the proper width. Where $f_{\mathrm{conv}} = 1$, convection dominates all transport, and where $f_{\mathrm{cond}} = 1$, conduction dominates all transport (in the boundary layers). Under this assumption, the proper evolved atmospheric flux profiles are $\mathrm{F}_{\mathrm{conv,\,ev}} = \mathrm{F}_{\mathrm{bot}} \cdot f_{\mathrm{conv}}$ and $\mathrm{F}_{\mathrm{cond,\,ev}} = \mathrm{F}_{\mathrm{bot}} \cdot f_{\mathrm{cond}}$, where $\mathrm{F}_{\mathrm{bot}} = \mathcal{P}$ is the amount of flux entering the bottom of the atmosphere.

In the evolved, time-stationary state, the horizontal- and time-average of Eqns. (4) and (5), neglecting terms that vanish due to symmetry, are

$$\frac{\partial}{\partial z}\langle\varpi\rangle_{x,y} - \langle T_1\rangle_{x,y}\hat{z} = \langle \boldsymbol{u}\times\boldsymbol{\omega}\rangle_{x,y}, \tag{9}$$

$$\frac{\partial}{\partial z}\mathrm{F}_{\mathrm{conv,\,ev}} - \mathcal{P}\frac{\partial^2}{\partial z^2}\langle T_1\rangle_{x,y} = 0. \tag{10}$$

Convective flows are perturbations around a thermal profile defined by these equations in the proper evolved, statistically stationary state. Furthermore, under the specification of $\mathrm{F}_{\mathrm{conv,\,ev}}$ and $\langle \boldsymbol{u}\times\boldsymbol{\omega}\rangle_{x,y}$, *the mean thermodynamic structure of the system* ($\langle\varpi\rangle_{x,y}$, $\langle T_1\rangle_{x,y}$) *is fully specified.*

The AE method is thus simple: we construct $\mathrm{F}_{\mathrm{conv,\,ev}}$ as described above. Then we calculate a profile, $\xi(z) = \mathrm{F}_{\mathrm{conv,\,ev}}/\mathrm{F}_{\mathrm{conv}}$, which is the amount that the flux in the system needs to be reduced by, as a function of height. We multiply the velocities and the thermal fluctuations, $T - \langle T\rangle$, by $\sqrt{\xi}$, such that the product of all fluctuations (which carry the convective flux) are diminished by a factor of ξ. We solve Eqns. (9-10) with $\mathrm{F}_{\mathrm{conv,\,ev}}$ and $\xi\cdot\langle \boldsymbol{u}\times\boldsymbol{\omega}\rangle_{x,y}$ for the mean thermodynamic state, and then continue evolving in time. This adjustment of the mean profile and the diminishing of velocities and fluctuations is the AE method, and it can generally be applied tens of buoyancy times after the peak of convective transient. For specifics on the precise implementation of the AE method, we refer the reader to appendix A.

The AE method specified here quickly converges the fluxes within the system to within a few percent of the fluxes in a system converged through SE (Fig. 1d & e).

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[1] http://dedalus-project.org/

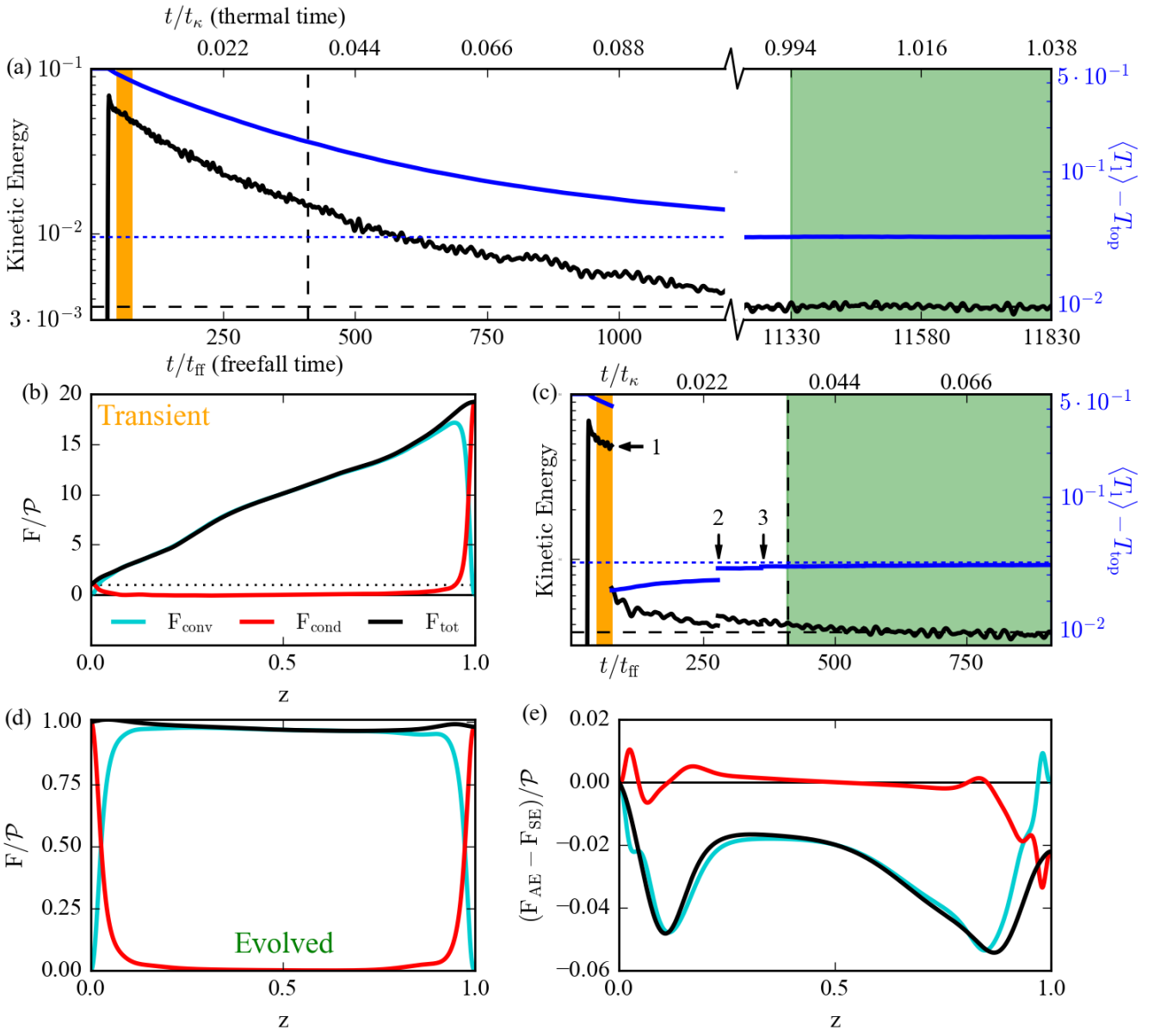

FIG. 1. A time trace is shown of kinetic energy (black) and mean temperature (blue) vs. time for an SE run at $S = 10^5$ (a). The mean evolved values of kinetic energy and mean temperature are denoted by the horizontal dashed black and blue lines. By taking an average of the fluxes early in the simulation (b), and appropriately using the AE method described here, the evolution of these atmospheres can be greatly accelerated. The time trace of such an accelerated atmosphere, which uses these methods, is shown in (c). The y-axes and x-axes are scaled identically as in (a). Here, the AE method is used three times, marked by the numbered arrows. The fluxes averaged over the green region which begins at the vertical dashed line shortly after AE 3 are shown in (c) are shown in (d), and they are nearly in the fully evolved state, sampled over a similarly long region that occurs much later in the simulation in (a). The difference between the fluxes in the AE and SE ssolutions is shown in (e), and it is small. The dashed vertical line in (a) denotes the simulation time at which evolved flux measurements begin to be taken in (c).

III. RESULTS

We study evolved solutions achieved through the SE method from convective onset up to $S = 10^5$ in 2D and $S = 10^4$ in 3D. These SE runs are compared to AE runs spanning from onset up to $S = 10^7$ in 2D and $S = 10^4$ in 3D. For a full list of simulations, we refer the reader to appendix B.

We report the time- and volume-averaged values of select measurements of the evolved solutions in Fig. 2. We report the scaling of heat transport in the evolved solution, as quantified by the Nusselt number, in Fig. 2a. The volume averaged Nusselt number is defined as

$$\mathrm{Nu} = \frac{\langle F_{\mathrm{conv}} + F_{\mathrm{cond}} \rangle}{\langle F_{\mathrm{cond,\ ref}} \rangle} = \frac{\langle wT - \mathcal{P}\partial_z T \rangle}{\langle -\mathcal{P}\partial_z T \rangle}. \tag{11}$$

In 3D and in 2D when $S < 10^{3+2/3}$, the evolved system is defined by a clear value of Nu and the convection reaches a statistically stationary state. In 2D and at larger values of S, the value of Nu oscillates as a function of time as the plumes in the solution oscillate horizontally. Our choice of no-slip boundary conditions prevent the fluid from entering a full-fledged shearing state [22], but the oscillatory motions which do arise cause the system to vary between periods of low heat transport and high heat transport. The SE simulations which span up to $S = 10^5$ exhibit the same horizontally oscillatory motion as the AE solutions for the same initial conditions. The scaling of the mean value of Nu is roughly $\mathrm{Nu} \propto \mathrm{Ra}^{1/5}$, weaker than that reported in similar systems with fixed-T and fixed-flux boundary conditions [10]. We attribute this weaker Nu scaling to the oscillatory nature of the plumes, which may have been avoided by previous studies using bootstrapping techniques as initial conditions.

In Fig. 2b, we report the rms Reynolds number in the AE solutions, where $\mathrm{Re} = \langle |\boldsymbol{u}| \rangle / \mathcal{R}$. This measure scales roughly as $\mathrm{Re} \propto \mathrm{Ra}^{0.45}$, and shows little variance with time.

In Fig. 2c, we report the volume averaged temperature of the AE solutions, with the value at the upper (fixed T) boundary removed. This measurement probes the thickness of the boundary layers of the solutions, and should show scaling which is inversely proportional to Nu in converged solutions where fixed-flux boundary conditions are used [23]. We find here that $(\langle T \rangle - T_{\mathrm{top}}) \propto \mathrm{Ra}^{-1/5}$, precisely the inverse scaling of Nu, giving confidence that these solutions are in a converged state.

In Fig. 2d-f, we report the fractional difference between measurements from the AE solutions and measurements from the SE solutions. We find that the mean value of Nu from AE solutions is accurate to the values from SE solutions to within $\sim 1\%$, and the same is true for $\langle T \rangle$ measurements. Re measurements show slightly greater error, with AE measurements being on average $\leq 2\%$ away from the SE measurements.

The measurements presented in Fig. 2 demonstrate that the AE method can be powerfully employed in parameter space studies in which large numbers of simulations are compared in a volume-averaged sense. We now turn our examination to a more direct comparison of AE and SE for convection at $S = 10^5$, as has been introduced in Fig. 1.

As the AE method primarily serves to adjust the thermodynamic structure of the solution, we compare the temperature profile attained by AE and SE in Fig. 3. We see that the boundary layer length scale is nearly identical in the AE and SE solution (Fig. 3a), but that the mean temperature in the interior differs by about 0.5% on average (Fig. 3b). The probability distribution function of point-by-point temperature values are compared in Fig. 3c. We construct this PDF by interpolating our temperature field onto a regular grid, determing the frequency distribution of all T values, and then properly normalizing such that the integral of the PDF is unity. In addition to the difference in mode which can be seen in the mean vertical profile, the SE case has a broader spread of temperature values. This can likely be attributed to the slight overstabilization of the temperature profile obtained by the AE method (as can be seen in Fig. 1a&b). One means of comparing two probability distributions to determine if they are drawn from the same underlying sample is through the use of a Kolmogorov-Smirnov (KS) [24]. In general, a KS test must be conducted on independent, uncorrelated data, which poorly describes the point-by-point values of flow in a fluid simulation. Thus, we will merely use the KS statistic, the maximum difference between the cumulative distribution functions (CDFs) of the two sample distributions, as a numerical method of directly comparing the two PDFs. For the distributions shown in Fig. 3b, we find a KS statistic of 0.495 near the modes, which is very large, and implies that about half of the data points examined in the AE run are at lower temperature than at the SE run. However, as shown in Fig. 3d, aside from the large difference near the modes, the two CDFs differ by less than one part in a hundred.

In addition to comparing the thermodynamic state achieved by the SE and AE methods, we examine the velocities found in the evolved states. We compute PDFs in the same manner as in Fig. 3c for the vertical velocity (Fig. 4a), horizontal velocity (Fig. 4b), and the nonlinear convective flux (Fig. 4c). We report KS statistics of 0.00615, 0.0349, and 0.0204, respectively. Perhaps unsurpisingly, the nonlinear convective transport between the SE and AE methods are very similar, as is captured in a volume-averaged sense in the Nu measurements of Fig. 2a&d). In general, the velocity boundary conditions (Eqn. 7) more strongly dominate the flow fields for the SE cases than the AE cases in Fig. 4a&c. This suggests that the velocity boundary layers are not yet fully formed in the AE solutions. However, aside from this difference, the two solutions show a similar distribution of realized velocity values.

Despite differences between the SE and AE solutions for the case studied in Figs. 1, 3, & 4, the AE method is still extremely powerful. The first application of the AE method ($t \approx 70$ in Fig. 1b) immediately increases the average time step of our solver by a factor of 2-3. At higher supercriticality ($S = 10^7$), the AE solve immediately boosts the

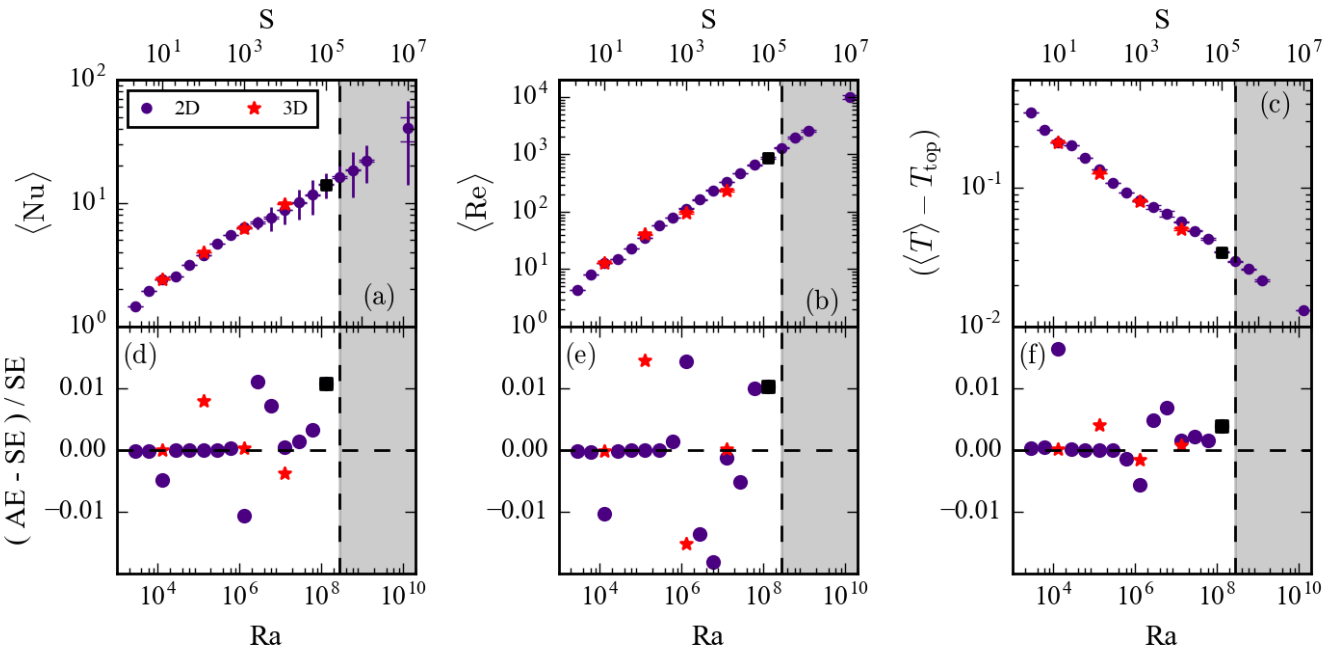

FIG. 2. Volume- and time-averaged measurements of the Nusselt number, the RMS Reynolds number, and the mean temperature for AE runs are shown in (a)-(c). Symbols are located at the mean value of each measurement, vertical lines represent the standard deviation of the measurement, which shows natural variation over the averaging window, and error bars represent the change in the mean value over the averaging window, such that very large error bars represent solutions which are not fully converged when averaging begins. (a) The Nusselt number scales as $\mathrm{Ra}^{1/5}$, and above $\mathrm{S} \geq 10^{3+2/3}$, simulations exhibit oscillating plume structures whose heat transport varies over time. (b) The RMS Reynolds number, which measures the level of turbulence in the evolved solution, scales as $\mathrm{Ra}^{0.45}$. (c) Temperature, with its value at the upper boundary removed, is shown, and scales as $\mathrm{Ra}^{-1/5}$. Relative error for (d) Nu, (e) Re, and (f) T are shown between measurements taken in AE solutions and SE solutions. The greyed area of the plot indicates the region in which only AE runs were carried out due to computational expense. The run at $S = 10^5$ marked as a black square is examined in more details in Figs. 1, 3, & 4.

timestep by nearly a factor of 4. Thus, not only does this method evolve the solution into nearly the correct state, but further time evolution (either to achieve precisely the correct thermodynamic state or to take measurements of fluid quantities) happens more efficiently.

IV. EXTENSIONS & CONCLUSION

In this work we have used Rayleigh-Bénard convection as a test case for the AE method, but we argue that the true power of this method is in its extensions to more complicated studies. Such extensions are beyond the scope of this paper, but we will briefly describe areas in which extensions to the AE method should be explored. We note that a strength of the AE method is its generality. One need only derive the steady-state, horizontally-averaged equations for a given equation set (e.g., Eqns. (9) & (10)) and couple that information with knowledge of the boundary conditions and current dynamics (as described in section II A and appendix A).

One such area in which this method can be applied is to simple studies of internally heated convection [25]. The simplest of these studies examine the incompressible Boussinesq equations, as presented in this work, but include a constant source term in the energy equation. The evolved convective state in such internally heated systems includes a flux profile whose value linearly increases with height, as the volumetric heating requires the convection to carry more flux towards the top of the domain. These systems can be straightforwardly studied using the methods that we examined here, but scaling the profiles derived in Eqn. (8) by the proper, height-dependent flux.

Studies of stratified convection are likely to gain the most through developing and employing the AE method. In many studies of stratified convection, the thermal diffusivity is inversely proportional to the density [7]. Thus, the thermal timescale is much larger deep in the atmosphere, where density is high and diffusivity is very low, than it is near the surface. As such, runs at similar supercriticalities but with different degrees of stratification can have thermal timescales which differ by orders of magnitude. Fortunately, the AE method is easily adapted to the simplest

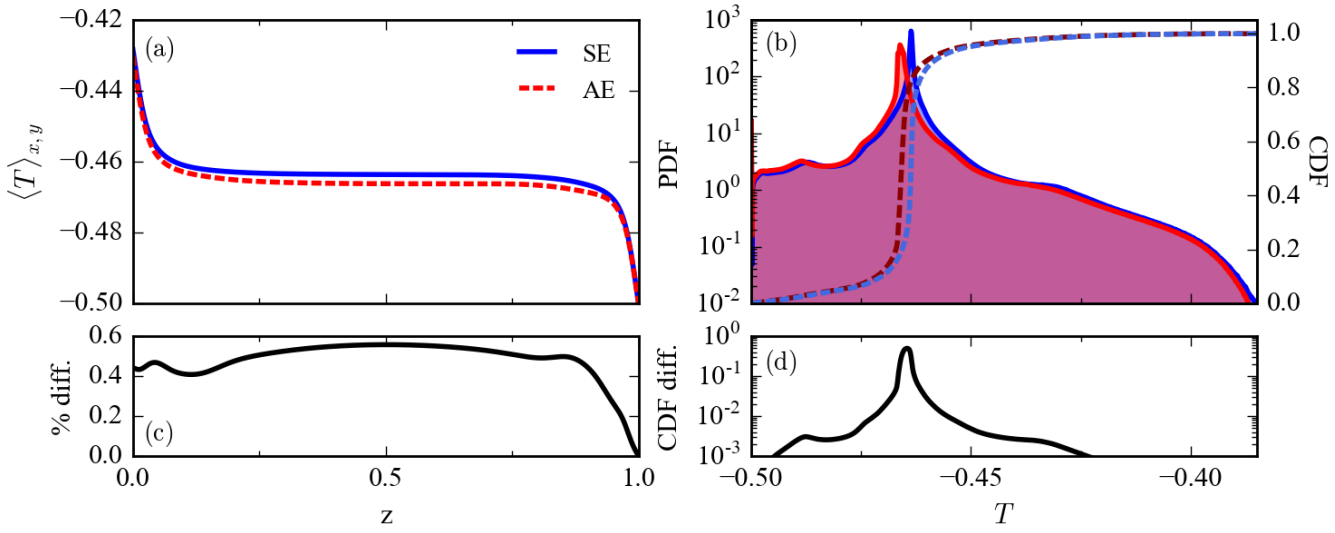

FIG. 3. Comparisons of the evolved thermodynamic states of an AE and SE run at $S = 10^5$ are shown. (a) Evolved horizontally- and time-averaged temperature profiles, as a function of height. (b) Probability Distribution Functions (PDFs) and their integrated Cumulative Distribution Functions (CDFs) of point-by-point measurements of the temperature field. (c) The percentage difference between the mean temperature profiles as a function of height. (d) The value of the Kolmogorov-Smirnov (KS) statistic, or the difference between the AE and SE CDFs, as a function of temperature. T is sampled every 0.1 time units for 500 total time units, and was interpolated onto an evenly spaced grid before sampling. The difference between the mean profiles is very small O(0.5%), but this small difference results in a large difference between the two temperature CDFs near the values of the temperature mode. The spread of temperature around the mode, or the fluctuations that drive convection, are nearly identical between the two runs.

stratified, compressible systems. Rather than studying the total flux through the system, one must only study the flux in excess of the adiabatic while constructing the fractions of Eqn. (8). Furthermore, in addition to solving for hydrostatic balance and thermal equilibrium, as in Eqns. (9) & (10), it is essential to simultaneously ensure that mass is conserved. Thus, additional equations and boundary conditions must be used to ensure that mass is not created or destroyed, in a manner similar to how mass appears in equations in stellar structure codes [26].

Studies of convection frequently examine paired stable and convecting regions [13, 14, 16]. When the interface between the stable region and the convecting region is stiff and motions do not cross that interface, convective motions cannot accelerate the restratification of the stable region. In fully-convective domains, such as those studied in this work, the thermodynamics evolve at a more rapid rate than the thermal diffusion time across the domain due

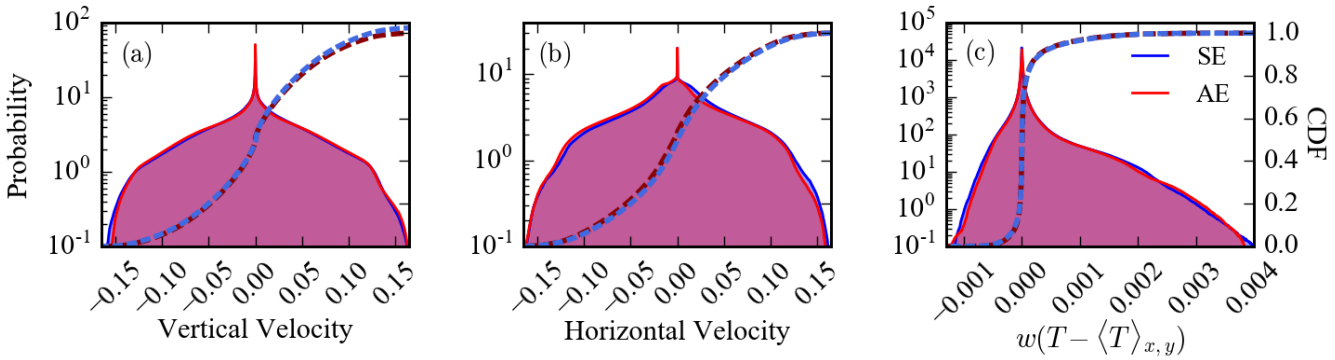

FIG. 4. Probability distribution functions of (a) the vertical velocity, (b) the horizontal velocity, and (c) nonlinear convective transport are shown for a 2D runs achieved through the SE and AE methods at $S = 10^5$. Flows are sampled every 0.1 time units for 500 total time units, and flows are interpolated onto an evenly spaced grid before sampling. The cumulative distribution function is overplotted on each plot.

to the help of convective motions. For example, in Fig. 1a, the SE solution is fully converged after $4 \cdot 10^3$ frefall time units, despite the thermal timescale being roughly 10^4 frefall units.

A final area in which we encourage extensions of this method is in the study of systems operating in the presence of different thermal boundary conditionss. We have studied solutions in which we employ mixed thermal boundary conditions, as in Eqn. (7). However, the AE method can be easily applied to fixed-temperature boundary conditions, as well. The total evolved flux ($\mathrm{F_{bot}}$) through the system is not known a priori, but it can be measured using the ratios in Eqn. (8). The integrated conductive flux is a constant which does not change with time due to the fixed temperature jump across the domain. Thus, the total flux through the system can be calculated as $F_{\mathrm{bot}} = \mathcal{P} / \int f_{\mathrm{cond}} dz$. While easily extensible to fixed-temperature boundary conditions, we hesitate to use the AE method for fixed-flux boundary conditions. In those systems, there are infinitely many degenerate solutions to the final temperature profile, and the AE method could advance the solution into an unrealistic state as a result of an arbitrary user choice on the temperature profile's offset.

In conclusion, here we have described a method for converging convective simulations using a method of Accelerated Evolution (AE), and compared it to Standard Evolution (SE) achieved by running simulations for a thermal timescale. We have shown that the AE method produces evolved solutions on short timescales whose dynamics are similar to SE solutions, and we have discussed differences that arise between the two methods.

We have shown that the AE method is valid at low values of S, in which solutions converge quickly and the thermal timescale is short, but that it is also applicable at high values of S, where SE solutions can take an intractable amount of time. This shows that the AE method not only progresses solutions towards the correct evolved state (at high S), but that it also does not remove the simulation from the correct evolved state once such a state is achieved (at low S). We have studied the AE method in both 2D and 3D, but have restricted most of our study to 2D flows. As a result of the AE method depending only on 1D information integrated across the horizontal dimensions of the atmosphere, we only examine a select small group of 3D simulations to demonstrate the method's efficacy there in addition to in 2D.

This work serves as a first step towards creating robust methods for achieving evolved convective solutions at high Ra on short timescales. In order to study how astrophysical systems change over meaningful evolutionary timescales, there are two main approaches. One-dimensional systems which parameterize dynamics can be quickly evolved over thermal timescales. However, if the experimenter wants to simultaneously study system evolution in the presence of realistic convective dynamics, we now know that it is impossible to avoid the constraints that advective dynamics place on timestep size [3–5] in both implicit and explicit methods. We must explore other avenues of evolving the system forward on thermal timescales, and this work aims as a first step in these explorations.

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Appendix A: Accelerated Evolution Recipe

In order to achieve Accelerated Evolution (AE), we pause the Direct Numerical Simulation (DNS) which is evolving the dynamics of the convection and solve a 1D Boundary Value Problem (BVP) consisting of Eqns. (9) & (10). After solving this BVP, we adjust the fields being evolved in the DNS appropriately towards their evolved state, and then we continue running the now-evolved DNS. There specific steps taken in completing the AE method are as follows:

1. After the start of the DNS, we wait some time, $t_{\mathrm{transient}}$ after the convective transient in which the dynamics vigorously break away from the hydrostatic state.

2. Then, during the DNS, we calculate time averages of the 1D profiles of F_{conv}, F_{tot}, and $\langle \boldsymbol{u} \times \boldsymbol{\omega} \rangle_{x,y}$, updating them every timestep. To calculate these averages, we use a trapezoidal-rule integration in time, and then divide by the total time elapsed over which the average is taken.

3. We pause the DNS once the averages are sufficiently converged. To ensure that an average is converged, at least some time t_{min} must have passed since the average was started to ensure that the full range of convective dynamics are probed, and the profiles must change by no more than $P\%$ on a given timestep.

4. Construct $\mathrm{F_{conv,\ ev}}$ and ξ as specified in section II A from the averaged profiles.

5. Solve the BVP for $\langle T_1 \rangle_{x,y}$ and $\langle \varpi \rangle_{x,y}$ of the evolved state. Set the horizontal average of the current DNS thermodynamic fields equal to the results of the BVP.

6. Multiply the velocity field and the temperature fluctuations, $T - \langle T \rangle_{x,y}$, by $\sqrt{\xi}$ in the DNS to properly reduce the convective flux.

7. Continue running the DNS

We refer to this process as an "AE BVP solve."

While the use of a single AE BVP solve rapidly advances the convecting state to one that is closer to the evolved state, we find that repeating this method multiple times is the best way to ensure that the AE solution is truly converged. For all runs in 2D at $S < 10^5$, we set $t_{\rm transient} = 50$, completed an AE BVP solve with $t_{\rm min} = 30$ and $P = 0.1$, and then repeated the procedure, including another wait of $t_{\rm transient} = 50$ before beginning averages. For all 3D runs and 2D runs with $S \in [10^5, 10^6]$, we did a first AE BVP solve with $t_{\rm transient} = 20$, $t_{\rm min} = 20$, and $P = 1$ in order to quickly reach a near- converged state and vastly increase our timestep size. After this first solve, we completed two AE BVP solves, with $t_{\rm transient} = 30$, $t_{\rm min} = 30$, and $P = 0.1$ to get very close to the solution (as in Fig. 1c). At very high $S = 10^7$, we ran two AE BVP solves with $t_{\rm min} = 20$ and $P = 1$. For the first solve, we set $t_{\rm transient} = 20$, and for the second we set $t_{\rm transient} = 30$. We used fewer solves at this high value of S in part to reduce the computational expense of the run, and in part because of how a third BVP generally did not modify the solution hugely (as in Fig. 1c).

Appendix B: Table of Runs

In Table I we list key properties of all simulations conducted in this work. The supercriticality, Rayleigh number, and resolution are reported. We report the simulation run time of the SE solutions and AE solutions, as well as the amount of time over which average measurements were taken, in freefall time units. The volume-averaged Nusselt number of the AE and SE solutions are shown. In the upper part of the table, information pertaining to 2D runs is reported, and below the double horizontal bars we report properties of all 3D runs.

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TABLE I. Simulation parameters

S	Ra	nz	nx, ny	$t_{\rm therm}$	$t_{\rm avg}$	$\mathrm{Nu_{SE}}$	$\mathrm{Nu_{AE}}$
$10^{1/3}$	$2.79 \cdot 10^3$	32	64	52.8	100	1.46	1.46
$10^{2/3}$	$6.01 \cdot 10^3$	32	64	77.6	100	1.95	1.95
10^{1}	$1.30 \cdot 10^4$	32	64	114	100	2.43	2.42
$10^{1+1/3}$	$2.79 \cdot 10^4$	32	64	167	100	2.54	2.54
$10^{1+2/3}$	$6.01 \cdot 10^4$	32	64	245	100	3.14	3.14
10^{2}	$1.30 \cdot 10^5$	64	128	360	100	3.8	3.8
$10^{2+1/3}$	$2.79 \cdot 10^5$	64	128	528	100	4.71	4.71
$10^{2+2/3}$	$6.01 \cdot 10^5$	64	128	776	100	5.5	5.5
10^{3}	$1.30 \cdot 10^6$	128	256	$1.14 \cdot 10^3$	200	6.4	6.33
$10^{3+1/3}$	$2.79 \cdot 10^6$	128	256	$1.67 \cdot 10^3$	500	6.87	6.95
$10^{3+2/3}$	$6.01 \cdot 10^6$	256	512	$2.45 \cdot 10^3$	500	7.54	7.59
10^{4}	$1.30 \cdot 10^7$	256	512	$3.60 \cdot 10^3$	500	8.83	8.83
$10^{4+1/3}$	$2.79 \cdot 10^7$	256	512	$5.28 \cdot 10^3$	500	10.13	10.14
$10^{4+2/3}$	$6.01 \cdot 10^7$	256	512	$7.76 \cdot 10^3$	500	11.65	11.69
10^{5}	$1.30 \cdot 10^8$	512	1024	$1.14 \cdot 10^4$	500	14.02	14.18
$10^{5+1/3}$	$2.79 \cdot 10^8$	512	1024	$1.67 \cdot 10^4$	500	–	16.21
$10^{5+2/3}$	$6.01 \cdot 10^8$	512	1024	$2.45 \cdot 10^4$	500	–	18.58
10^{6}	$1.30 \cdot 10^9$	1024	2048	$3.60 \cdot 10^4$	500	–	22.13
10^{7}	$1.30 \cdot 10^{10}$	2048	4096	$1.14 \cdot 10^5$	200	–	40.78
10^{1}	$1.30 \cdot 10^4$	32	64×64	114	100	2.42	2.42
10^{2}	$1.30 \cdot 10^5$	64	128×128	360	100	3.97	4
10^{3}	$1.30 \cdot 10^6$	128	256×256	$1.14 \cdot 10^3$	500	6.27	6.27
10^{4}	$1.30 \cdot 10^7$	256	512×512	$3.60 \cdot 10^3$	500	9.92	9.88

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