BVP Methods

I. BRIEF DESCRIPTION OF METHOD

In a really simple sense, this is the procedure that is being done when I do the "BVP solve" for getting a more converged atmosphere.

1. Start up IVP, and run it.
2. Once the flows hit Re $= 1$, wait some time, $t_{\text{transient}}$.
3. After $t_{\text{transient}}$, start taking horizontal and time averages of specified profiles in the atmosphere.
4. Once the profiles are converged (the change in the profiles at the next timestep change, on average, less than f, where f is a fraction. If $f = 0.01$, the profiles change no more than 1% from the previous timestep.), feed them into a 1D problem.
5. In the 1D problem, adjust the profiles from the atmosphere intelligently. Feed those profiles into a BVP that obeys the same thermal boundary conditions as the IVP, then adjust the mean thermal profile of the IVP atmosphere with the result of the BVP.
6. Keep running the IVP. If desired, wait some time, t_{equil}, and then restart from step 3.

II. MORE THOROUGH WRITE-UP, BOUSSINESQ CONVECTION

In boussinesq convection, the time-steady energy equation can be written as

$$\nabla \cdot (\boldsymbol{u}T - \kappa \nabla T) = 0$$

When we take horizontal averages and time averages, and assume constant κ, this equation reduces to

$$\partial_z \langle (wT) \rangle - \kappa \partial_z^2 \langle (T_0 + T_1) \rangle = 0,$$

where angles represent a time- and horizontal average. Further, due to symmetry, when we take a time- and horizontal average, we find that most of the terms in the Rayleigh-Benard momentum equation go away. Even if they hadn't, the main thing we want from the momentum equation is an update to the hydrostatic balance of the atmosphere, and that means that we need to update

$$\partial_z \langle p \rangle = \langle T_1 \rangle .$$

So, put simply, the BVP that we need to solve in RBC is

$$\begin{aligned} \frac{\partial T_1}{\partial z} - T_{1z} &= 0 \\ \kappa \frac{\partial T_{1z}}{\partial z} &= \frac{\partial}{\partial z} \langle wT \rangle - \kappa \frac{\partial^2 T_0}{\partial z^2} \\ \frac{\partial p}{\partial z} - T_1 &= 0. \end{aligned} \tag{1}$$

These three equations, coupled with two proper thermal boundary conditions, retrieve the full thermodynamic state of the atmosphere. I've written the equations in their dedalus-like form, for clarity. I'll solve using fixed temp (top) and fixed flux (temp gradient, bottom) boundary conditions. Dual fixed-flux boundary conditions will work as well, but then the integrated temperature profile's constant offset is actually not unique, so that system has degenerate answers that work for the BVP.

In a real way, the profile of the enthalpy flux thus sets the profile of the thermal structure of the atmosphere. In order for this method to work, the profile of the enthalpy flux fed into the BVP must be the enthalpy flux profile that we expect in the fully evolved atmosphere. Getting this profile properly is really the subtle part of this method, so let's talk about that.

A. Adjusting the enthalpy flux profile

One subtlety of the mixed flux (bot) / temperature (top) boundary conditions is that they are, by definition, asymmetrical (at least during the transient). The non-adiabatic (large) temperature gradient in the system means that the system starts with too much energy, and it wants to relax to a constant temperature gradient. To do that, flux needs to leave through the top, and this takes a long time when the diffusivities are small.

To illustrate what's happening in our system, I've plotted some fluxes in Fig. 1. In the top panel of Fig. 1, the fluxes of a rayleigh-benard system at $\mathrm{Ra} = 10^8$ are shown. This plot shows the total flux in the IVP (blue), the convective flux (green), and the conductive flux (orange/yellow). The dashed-dot black line is the flux being sent through the system at the bottom boundary. The short red line is the profile that we get when we divide the enthalpy flux by the total flux, and then multiply that normalized profile by the flux entering the lower boundary.

The bottom panel of Fig. 1 shows a zoom-in on the bottom part of the top panel. Here, we show the same "properly normalized" enthalpy flux that we just described in the top panel, but it is now shown in green. This properly normalized profile is used as the enthalpy flux profile for the BVP, and this sets the temperature gradient (and thus the conductive flux) that we get out of the BVP, which is shown in yellow. We then feed the temperature profile defined by the yellow line back into the IVP, and let the velocities in the IVP naturally react to the abrupt change in the background atmosphere.

B. The actual procedure, and open questions

So basically what we do is:

(a) Run the IVP to get average fluxes, as in the top panel of Fig. 1.

(b) Normalize the enthalpy flux profile properly to get the enthalpy flux profile as shown in the bottom panel of Fig. 1

(c) Use that normalized enthalpy flux profile as a RHS constant forcing term in the BVP equations (Eqns. 1), and solve for the consistent temperature profile that satisfies those equations under the proper boundary conditions

(d) Update the full fields in the IVP such that the mean pressure and temperature profiles are the pressure / temperature profiles that come out of the BVP.

(e) Run the IVP so that the velocity field relaxes in the presence of the new atmosphere.

This seems to work well. Open questions:

(a) How long do we average over? What's the right criterion for having a converged profile?

(b) How many times do we need to do the BVP to get to the right answer? If we get the right converged profile, I think we're only going to need one BVP.

(c) How long does it take the velocity field to fully equilibrate post-BVP?

(d) How similar is the BVP final state to the IVP final state?

Note: this is not hard to implement in fully compressible, too!

III. EXTENSION TO STRATIFIED CONVECTION AND FC EQUATION

This method is, fortunately, not hard to extend to the fully compressible equations. For a system whose full energy equation is of the form

$$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{F}) = \kappa(\mathrm{IH}),$$

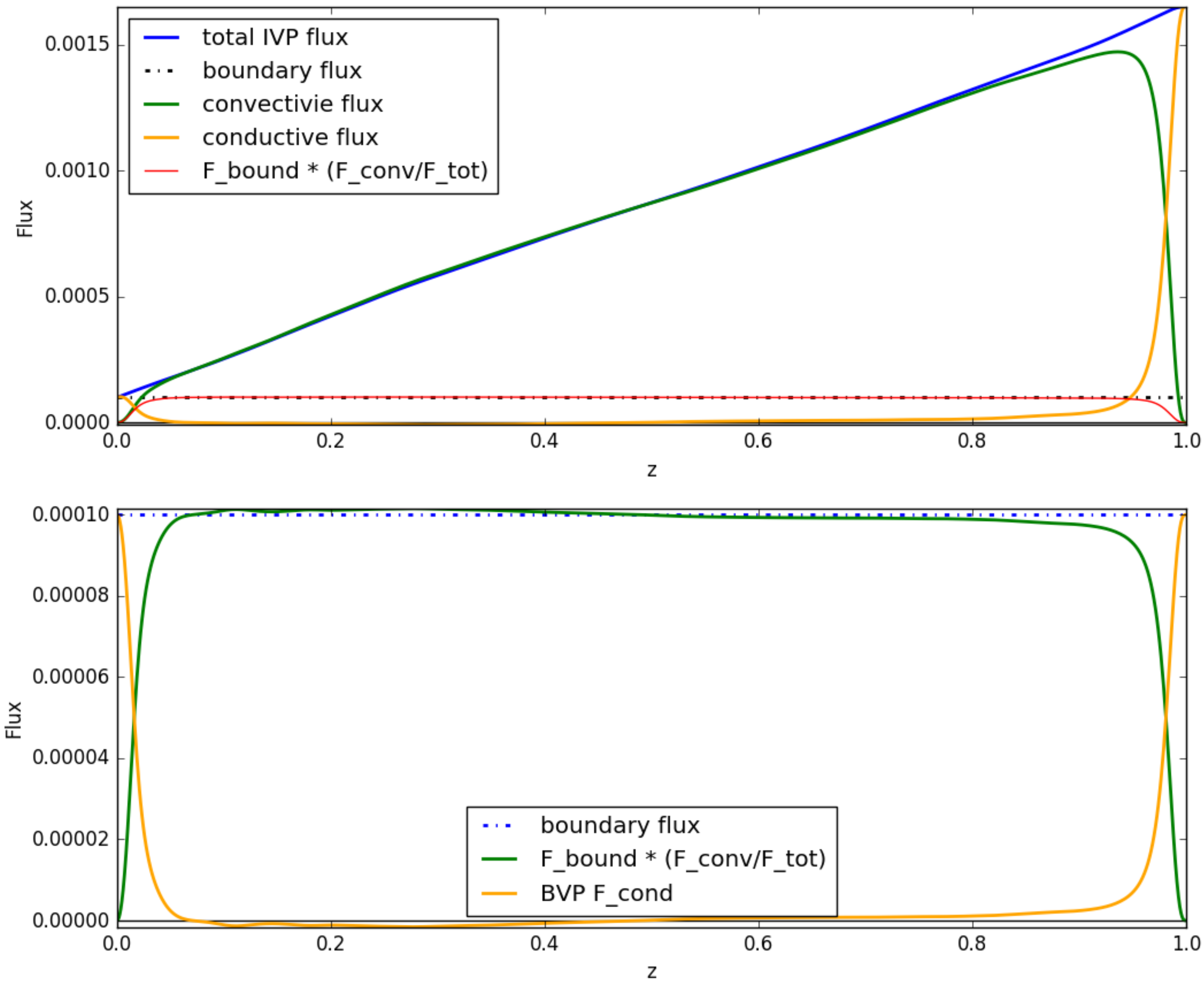

FIG. 1. (top) The evolved fluxes from a 2-D rayleigh-benard convection experiment. The upper (fixed temp) boundary is leaking out much more flux than the bottom boundary (fixed flux) supplies while the atmosphere equilibrates. However, we know that the evolved solution should only carry the amount of flux supplied at the lower plate. In the bottom panel, we show the enthalpy flux in the IVP, divided by the total flux, and then multiplied by the flux at the bottom boundary. This gives us an idea of what the enthalpy flux should be like in the evolved state, including the boundary layers. The corresponding conductive flux that the temperature profile would have to carry is shown in yellow.

which is true of both polytropes and internally heated atmospheres, we find that the time stationary, vertically averaged profile is, once again,

$$\kappa \frac{\partial \langle T_{1z} \rangle}{\partial z} = \frac{\partial}{\partial z}(\langle \boldsymbol{F}_{\rm conv} - \kappa T_{0z} \rangle - \kappa (IH) \tag{2}$$

And, for a properly constructed atmosphere where the initial profile carries the internal heating, the last two terms of this expression perfectly cancel each other out (at least, for constant κ, and for the atmospheres we're worried about right now). So, with that, the full set of BVP equations for fully compressible atmospheres is

$$
\begin{aligned}
\frac{\partial T_1}{\partial z} - T_{1z} &= 0 \\
\kappa \frac{\partial T_{1z}}{\partial z} &= \frac{\partial \boldsymbol{F}_{\text{conv}}}{\partial z} \\
T_0 \partial_z \rho_1 + T_1 \partial_z \rho_0 + \rho_0 T_{1z} + \rho_1 T_{0z} &= -T_1 \partial_z \rho_1 - \rho_1 T_{1z} \\
\frac{\partial M_1}{\partial z} - \rho_1 &= 0.
\end{aligned}
\tag{3}
$$

Here the first two equations are just the energy equation, the third equation is that the fluctuating components must be in hydrostatic equilibrium ($\nabla P = \nabla(\rho T) = \rho \nabla T + T \nabla \rho = 0$, and assuming that the background is in hydrostatic equilibrium such that gravity drops out of the system), and the fourth equation is our mass conservation equation. This requires four boundary conditions: two thermal ones (fixed flux bot, fixed temp top), and two mass ones ($M_1 = 0$ at top and bottom). Once again, by specifying the appropriate profile for the convective flux, the rest of the atmosphere follows. This is also nice, because this formulation should also work with minor tweaks for more complicated forms of κ.