BVP Paper

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WOW this is a really long sentence check out this abstract I'll just keep writing words to make this at least one line long so we know what the formatting looks like, ok?

I. INTRODUCTION

In numerical studies of convection, the chosen boundary conditions at the upper and lower plates determines key quantities of the dynamics of the evolved state. Studies of incompressible, Boussinesq, Rayleigh-Bénard convection (RBC) often employ fixed temperature (Dirichlet) or fixed heat flux (Neumann) boundary conditions at both plates. Dirichlet conditions represent plates of infinite conductivity, whereas Neumann condition model plates of finite conductivity. In both cases, choosing symmetric boundary conditions maintains overall system symmetry, and despite evolving towards quite different thermal structures, both types of conditions transport heat in the same manner [1].

Studies of convection in stratified systems which aim to model convection in astrophysical systems, such as the outer envelopes of low-to-moderate mass stars like the Sun, often employ a mixture of these two types of boundary conditions [**?** **?**]. The flux at the lower boundary is fixed, modeling the constant energy generation of the stellar core, while the outer boundary condition is held at a fixed temperature, modeling the surface of a star which must output the energy generated internally.

While this setup is a useful model for understanding natural systems, simulations which employ this setup often suffer from a long convective transient as the thermodynamic structure of the atmosphere relaxes to the adiabatic profile specified by the fixed temperature upper boundary condition. This long evolution occurs on the "Kelvin-Helmholtz," or thermal diffusion timescale of the atmosphere, $t_{\mathrm{KH}} \approx L_z^2/\chi$, where L_z is the domain depth and χ is the thermal diffusivity. Interesting convection happens as high values of the Rayleigh number, which scales like $\chi^{-1/2}$, such that in the astrophysically interesting regime of high-Ra, highly stratified convection, evolving a simulation for a KH time becomes numerically intractable. As the Rayleigh number increases, the KH time increases while the average timestep required to resolve the more turbulent flows decreases. The net result is that under standard initial conditions of hydrostatic- and thermal- equilibrium, the desired convective solution cannot be obtained and the dynamics of convection there cannot be studied.

Knowledge of Mixing Length theory and the nature of evolved convection has been used in some previous studies (e.g., [2]) to choose smarter initial conditions than the hydrostatic polytropic states. However, these assumptions work best in convective regions which are bounded by stable layers. In simple atmospheres where boundary layers form to meet boundary conditions at the upper and lower edge of the atmosphere, we cannot know *a priori* what the extent or shape of the boundary layer will be, as that must be chosen by the convective dynamics, and specification of the proper boundary layer is essential for placing the atmosphere along the proper adiabat.

Here we present a method for using simple boundary value problems, along with information about the evolved flow fields, to fast-forward the slow thermal evolution of convecting simulations. We present this method in the context of RBC, and then demonstrate that it applies to stratified, compressible convection under a simple modification. These methods allow us to study the convective flows driven by the evolved thermal profile while only requiring initial value problems to run for long enough to resolve the fast dynamical timescales of convection.

II. EXPERIMENT

A. Incompressible, Boussinesq Rayleigh-Bénard Convection

In studies of convection, it is natural to nondimensionalize the flows on the freefall velocity. Under the Boussinesq approximation, where $\rho = \rho_0(1 - \alpha T_1)$ on the gravitational term in the momentum equation and $\rho = \rho_0$ on all other terms, the freefall velocity is $v_{\mathrm{ff}} = \sqrt{\alpha g L^2 (dT/dz)_0}$. where L is the depth of the domain, g is the gravitational acceleration, and $(dT/dz)_0$ is the initial temperature gradient of the atmosphere, which controls the flux through the system. The typical timescale and length scale then relate to each other according to $t_{\mathrm{ff}} = L/v_{\mathrm{ff}}$. Under this nondimensionalization, the Boussinsq, incompressible equations of motion are

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\varpi + T_1\hat{z} + \frac{\mathrm{Pr}}{\mathrm{Ra}}\nabla^2\boldsymbol{u}, \tag{2}$$

$$\frac{\partial T_1}{\partial t} + \boldsymbol{u}\cdot\nabla(T_0+T_1) = \frac{1}{\mathrm{Pr}\,\mathrm{Ra}}\nabla^2 T_1, \tag{3}$$

where

$$\mathrm{Ra} = \frac{g\alpha L^4\left(\frac{dT}{dz}\right)_0}{\nu\chi} = \frac{(L\,v_{\mathrm{ff}})^2}{\nu\chi}, \qquad \mathrm{Pr} = \frac{\nu}{\chi} \tag{4}$$

are the nondimensional Rayleigh and Prandtl numbers, where ν is the viscous diffusivity and χ is the thermal diffusivity.

One hallmark of Boussinesq convection is that upflows and downflows are symmetric, and thus the long time- and horizontal- average of the velocity field is exactly zero. Assuming that the velocity field has perfect symmetry over a long time average, the horizontally-averaged, steady state momentum equation is just

$$\frac{\partial}{\partial z}\langle\varpi\rangle = \langle T_1\rangle\,\hat{z}, \tag{5}$$

where $\langle A\rangle$ represent the horizontal- and time- average of the quantity A. Thus, in the evolved solution, the vertical profile of the pressure perturbations are directly determined by the temperature field. Under the Boussinesq approximation, $\boldsymbol{u}\cdot\nabla(A) = \nabla\cdot(\boldsymbol{u}A)$, and thus the steady state energy equation is just

$$\frac{\partial}{\partial z}\langle w(T_0+T_1)\rangle = \frac{1}{\mathrm{PrRa}}\frac{\partial^2}{\partial z^2}\langle T_1\rangle\,. \tag{6}$$

Eqns (5) & (6) combine to form a second-order ODE which specifies the mean vertical profile of the temperature and pressure fields in the evolved solution. These fields are completely specified by a set of two thermal boundary conditions and the evolved profile of the convective flux, $F_{\mathrm{conv}} = \langle w(T_0+T_1)\rangle$.

Under the choice of mixed thermal boundary conditions ($dT_1/dz = 0$ at $z=-1/2$, $T_1 = 0$ at $z=+1/2$), the initial atmosphere starts off with much more thermal energy ($\propto T$) than the evolved adiabatic solution pegged by the top temperature boundary condition. This excess energy must leave the system through the upper (fixed T) boundary during the convective transient, and this results in an asymmetric flux profile during the slow thermal evolution of the atmosphere. Furthermore, the convective flux (and the flux at the upper boundary layer) are O(1) during the convective transient, whereas the flux entering the atmosphere at the lower boundary is $\mathrm{O}(\mathrm{Ra}^{-1/2})$, so the asymmetry of the fluxes becomes increasingly pronounced as Ra is increased into the turbulent realm.

In order to find the evolved temperature profile of the atmosphere, the evolved profile of the convective flux must be known. We know that the evolved solution should be in flux equilibrium, and we know that the evolved flux through the atmosphere is the flux entering through the bottom. Thus, the steady-state profile of the convective flux can be approximated as

$$F_{\mathrm{conv,\ steady}} = F_{\mathrm{bot}}\frac{\langle w(T_0+T_1)\rangle}{\langle w(T_0+T_1) - \kappa\partial_z(T_0+T_1)\rangle} = F_{\mathrm{bot}}\frac{\langle F_{\mathrm{conv,\ IVP}}\rangle}{\langle F_{\mathrm{tot,\ IVP}}\rangle}. \tag{7}$$

Or, put simply, the steady state convective flux is what you get get rid of the asymmetry in the flux profile. This flux defined here is used in a 1D boundary value problem solve of eqns (5) & (6) along with the appropriate boundary values. The solution to that BVP is then used as new initial conditions, convection restarts, and so on.

B. Fully Compressible Convection

We study stratified convection in an ideal gas whose adiabatic index is $\gamma = 5/3$. The initial atmospheric stratification is polytropic [3]. We assume a Newtonian radiative conduction term [4], and solve the fully compressible Navier-Stokes equations of the form

$$\frac{\partial\ln\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\ln\rho + \nabla\cdot\boldsymbol{u} = 0 \tag{8}$$

$$\frac{D\boldsymbol{u}}{Dt} = -T\nabla\ln\rho - \nabla T + \boldsymbol{g} - \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right) \tag{9}$$

$$\frac{DT}{Dt} + (\gamma-1)T\nabla\cdot\boldsymbol{u} = \frac{1}{\rho c_V}\left(\kappa\nabla^2 T - [\bar{\bar{\boldsymbol{\Pi}}}\cdot\nabla]\cdot\boldsymbol{u}\right), \tag{10}$$

where $D/Dt \equiv \partial/\partial t + \boldsymbol{u}\cdot\nabla$ and the viscous stress tensor is defined as

$$\Pi_{ij} \equiv -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right) \tag{11}$$

and δ_{ij} is the Kronecker delta function.

1. *The Boundary Value Equations*

In studies of fully compressible convection, the flux carried by the adiabatic temperature gradient is not available for convection. Thus, only the flux *in excess* of the adiabat will drive convection and be carried by convection. As such, this is the only portion of the flux which must be examined to determine if the solution is in a converged state. In a system with a fixed flux boundary condition, the available superadiabatic flux is

$$F_{\text{avail}} = -\kappa(\nabla T_0 - \nabla T_{\text{ad}}) = \kappa\frac{\epsilon}{c_P}\nabla T_0, \tag{12}$$

which is small when ϵ is small and only requires low Mach number convective flows to carry it. In a perfectly evolved solution, there will be thin boundary layers in which conduction carries this flux in addition to the adiabatic flux, but in an efficient convective interior, convective fluxes must carry this full amount.

The BVP equations are inspired by equations of stellar modeling [5] but adapted to these simulations of fully compressible convection. Here, rather than parameterizing convection, we can get the convective fluxes directly from our simulation adn use them to solve for the appropriate structure of the atmosphere. The FC BVP equations are

$$\frac{dM_1}{dz} = \rho_1 \tag{13}$$

$$T_0\nabla\rho_1 + T_1\nabla\rho_0 + \rho_1 g = -T_0\nabla\rho_0 - T_1\nabla\rho_1 - \rho_0 g \tag{14}$$

$$\kappa\frac{d^2T_1}{dz^2} = -\frac{d}{dz}F_{\text{conv, z}}, \tag{15}$$

which ensure mass conservation, thermal equilibrium, and that the atmosphere is, on average, in hydrostatic equilibrium. We couple these equations with four boundary conditions (mixed flux / temperature boundary conditions, as well as setting $M_1 = 0$ at the top and bottom of the atmosphere).

III. RESULTS & DISCUSSION

Here we talk about how the solutions are different, or similar. This includes:

1. Showing that the flow fields look similar
2. Showing how the temperature / flux profiles look similar/different
3. showing how Nu and Re scale with Ra in BVP / IVP.
4. showing how the PDFs of w, wT, and T change.

Then we need to make some comments about whether this is good or bad

Then we need to mention how the same thing can be done in stratified, just there you don't assume symmetrical boundary layers.

ACKNOWLEDGMENTS

EHA acknowledges the support of the University of Colorado's George Ellery Hale Graduate Student Fellowship. This work was additionally supported by NASA LWS grant number NNX16AC92G. Computations were conducted with support by the NASA High End Computing (HEC) Program through the NASA Advanced Supercomputing (NAS) Division at Ames Research Center on Pleiades with allocations GID s1647 and GID g26133.

Appendix A: Table of Boussinesq Runs

Appendix B: Table of stratified runs

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