BVP Paper

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WOW this is a really long sentence check out this abstract I'll just keep writing words to make this at least one line long so we know what the formatting looks like, ok?

I. INTRODUCTION

In many simple studies of convection, the choice of boundary conditions greatly influences the dynamics of the evolved state. In studies of incompressible, Boussinesq, Rayleigh-Bénard convection (RBC), many studies employ fixed temperature (Dirichlet) boundary conditions at the top and bottom, or fixed heat flux (Neumann) boundary conditions at both plates. The former case represents plates of infinite conductivity, whereas the latter plates of finite conductivity. In both cases, the choice of symmetric boundary conditions maintains overall system symettry, and both have been shown to transport heat in nearly the same manner [1].

Studies of convection in stratified systems which aim to model convection in natural, astrophysical systems, such as that in the outer envelope of low-to-moderate mass stars like the Sun, often employ a mixture of these two types of boundary conditions (references, references). The flux at the lower boundary is fixed, modeling the constant energy generation of the stellar core, while the outer boundary condition is held at a fixed temperature, effectively allowing all heat generated to escape.

While this setup is a useful model for understanding natural systems, simulations which employ this setup often suffer from a long convective transient as the thermodynamic structure of the atmosphere relaxes to the adiabatic profile specified by the fixed temperature upper boundary condition. This long evolution occurs on the "Kelvin-Helmholtz," or thermal diffusion timescale of the atmosphere, $t_{\mathrm{KH}} \approx L_z^2/\chi$, where L_z is the domain depth and χ is the thermal diffusivity. Interesting convection happens as high values of the Rayleigh number, which scales like $\chi^{-1/2}$, such that in the astrophysically interesting regime of high-Ra, highly stratified convection, evolving a simulation for a KH time becomes numerically intractable. As the Rayleigh number increases, the KH time increases while the average timestep required to resolve the more turbulent flows decreases. The net result is that under standard initial conditions of hydrostatic- and thermal- equilibrium, the desired convective solution cannot be obtained and the dynamics of convection there cannot be studied.

Here we present a method for using simple boundary value problems, along with information about the evolved flow fields, to fast-forward the slow thermal evolution of convecting simulations. We present this method in the context of RBC, and then demonstrate that it applies to stratified, compressible convection under a simple modification. These methods allow us to study the convective flows driven by the evolved thermal profile while only requiring initial value problems to run for long enough to resolve the fast dynamical timescales of convection.

II. EXPERIMENT

A. Incompressible, Boussinesq Rayleigh-Bénard Convection

In studies of convection, it is natural to nondimensionalize the flows on the freefall velocity. Under the Boussinesq approximation, where $\rho = \rho_0(1 - \alpha T_1)$ on the gravitational term in the momentum equation and $\rho = \rho_0$ on all other terms, the freefall velocity is $v_{\mathrm{ff}} = \sqrt{\alpha g L^2 (dT/dz)_0}$. where L is the depth of the domain, g is the gravitational acceleration, and $(dT/dz)_0$ is the initial temperature gradient of the atmosphere, which controls the flux through the system. The typical timescale and length scale then relate to each other according to $t_{\mathrm{ff}} = L/v_{\mathrm{ff}}$. Under this nondimensionalization, the Boussinsq, incompressible equations of motion are

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla \varpi + T_1 \hat{z} + \frac{\mathrm{Pr}}{\mathrm{Ra}} \nabla^2 \boldsymbol{u}, \tag{2}$$

$$\frac{\partial T_1}{\partial t} + \boldsymbol{u} \cdot \nabla (T_0 + T_1) = \frac{1}{\mathrm{Pr}\,\mathrm{Ra}} \nabla^2 T_1, \tag{3}$$

where

$$\mathrm{Ra} = \frac{g \alpha L^4 \left(\frac{dT}{dz}\right)_0}{\nu \chi} = \frac{(L\, v_{\mathrm{ff}})^2}{\nu \chi}, \qquad \mathrm{Pr} = \frac{\nu}{\chi} \tag{4}$$

are the nondimensional Rayleigh and Prandtl numbers, where ν is the viscous diffusivity and χ is the thermal diffusivity.

One hallmark of Boussinesq convection is that upflows and downflows are symmetric, and thus the long time- and horizontal- average of the velocity field is exactly zero. Assuming that the velocity field has perfect symmetry over a long time average, the horizontally-averaged, steady state momentum equation is just

$$\frac{\partial}{\partial z} \langle \varpi \rangle = \langle T_1 \rangle \hat{z}, \tag{5}$$

where $\langle A \rangle$ represent the horizontal- and time- average of the quantity A. Thus, in the evolved solution, the vertical profile of the pressure perturbations are directly determined by the temperature field. Under the Boussinesq approximation, $\boldsymbol{u} \cdot \nabla (A) = \nabla \cdot (\boldsymbol{u} A)$, and thus the steady state energy equation is just

$$\frac{\partial}{\partial z} \langle w(T_0 + T_1) \rangle = \frac{1}{\mathrm{PrRa}} \frac{\partial^2}{\partial z^2} \langle T_1 \rangle . \tag{6}$$

Eqns (5) & (6) combine to form a second-order ODE which specifies the mean vertical profile of the temperature and pressure fields in the evolved solution. These fields are completely specified by a set of two thermal boundary conditions and the evolved profile of the convective flux, $F_{\mathrm{conv}} = \langle w(T_0 + T_1) \rangle$.

Under the choice of mixed thermal boundary conditions ($dT_1/dz = 0$ at $z = -1/2$, $T_1 = 0$ at $z = +1/2$), the initial atmosphere starts off with much more thermal energy ($\propto T$) than the evolved adiabatic solution pegged by the top temperature boundary condition. This excess energy must leave the system through the upper (fixed T) boundary during the convective transient, and this results in an asymmetric flux profile during the slow thermal evolution of the atmosphere. Furthermore, the convective flux (and the flux at the upper boundary layer) are O(1) during the convective transient, whereas the flux entering the atmosphere at the lower boundary is $\mathrm{O}(\mathrm{Ra}^{-1/2})$, so the asymmetry of the fluxes becomes increasingly pronounced as Ra is increased into the turbulent realm.

In order to find the evolved temperature profile of the atmosphere, the evolved profile of the convective flux must be known. We know that the evolved solution should be in flux equilibrium, and we know that the evolved flux through the atmosphere is the flux entering through the bottom. Thus, the steady-state profile of the convective flux can be approximated as

$$F_{\mathrm{conv,\ steady}} = F_{\mathrm{bot}} \frac{\langle w(T_0 + T_1) \rangle}{\langle w(T_0 + T_1) - \kappa \partial_z (T_0 + T_1) \rangle} = F_{\mathrm{bot}} \frac{\langle F_{\mathrm{conv,\ IVP}} \rangle}{\langle F_{\mathrm{tot,\ IVP}} \rangle}. \tag{7}$$

Or, put simply, the steady state convective flux is what you get get rid of the asymmetry in the flux profile. This flux defined here is used in a 1D boundary value problem solve of eqns (5) & (6) along with the appropriate boundary values. The solution to that BVP is then used as new initial conditions, convection restarts, and so on.

B. Fully Compressible Convection

We study stratified convection in an ideal gas whose adiabatic index is $\gamma = 5/3$. The initial atmospheric stratification is polytropic [2]. We assume a Newtonian radiative conduction term [3], and solve the fully compressible Navier-Stokes equations of the form

$$\frac{\partial \ln \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \ln \rho + \nabla \cdot \boldsymbol{u} = 0 \tag{8}$$

$$\frac{D\boldsymbol{u}}{Dt} = -T \nabla \ln \rho - \nabla T + \boldsymbol{g} - \nabla \cdot \left(\bar{\bar{\boldsymbol{\Pi}}} \right) \tag{9}$$

$$\frac{DT}{Dt} + (\gamma - 1) T \nabla \cdot \boldsymbol{u} = \frac{1}{\rho c_V} \left(\kappa \nabla^2 T - [\bar{\bar{\boldsymbol{\Pi}}} \cdot \nabla] \cdot \boldsymbol{u} \right), \tag{10}$$

where $D/Dt \equiv \partial/\partial t + \boldsymbol{u}\cdot\nabla$ and the viscous stress tensor is defined as

$$\Pi_{ij} \equiv -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right) \tag{11}$$

and δ_{ij} is the Kronecker delta function.

III. RESULTS & DISCUSSION

Here we talk about how the solutions are different, or similar. This includes:

1. Showing that the flow fields look similar
2. Showing how the temperature / flux profiles look similar/different
3. showing how Nu and Re scale with Ra in BVP / IVP.
4. showing how the PDFs of w, wT, and T change.

Then we need to make some comments about whether this is good or bad

Then we need to mention how the same thing can be done in stratified, just there you don't assume symmetrical boundary layers.

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Appendix A: Table of Boussinesq Runs

Appendix B: Table of stratified runs

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