1 Nondimensional equations

The nondimensional system of equations we solve is

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \varpi + T\hat{z} + \mathcal{R}^{-1} \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{2}$$

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T + w \boldsymbol{\nabla}_{\text{ad}} = (\text{Pr}\mathcal{R})^{-1} \boldsymbol{\nabla}^2 T' + \boldsymbol{\nabla} \cdot [k \boldsymbol{\nabla} \overline{T}] + Q \tag{3}$$

These are just the Boussinesq equations of motion, with some slight tweaks:

1. There is a nonzero adiabatic temperature gradient, $\partial_z T_{\text{ad}} = -\boldsymbol{\nabla}_{\text{ad}}$. We choose this sign convention ($\boldsymbol{\nabla}_{\text{ad}}$ and similarly $\boldsymbol{\nabla}_{\text{rad}}$ will be positive) to align better with the stellar structure community's intuition. $|\boldsymbol{\nabla} T_0|$ and $\boldsymbol{\nabla}_{\text{ad}}$ may be much larger in magnitude than the convective fluctuations. The background temperature field T_0 does not enter into the momentum equation, as it is automatically canceled by a background $\boldsymbol{\nabla}\varpi$.

2. Convection is driven by internal heating, Q. I've confirmed with some simple tests that $|u|^2 \approx Q$, and therefore the convective frequency is $f_{\text{conv}}^2 \approx Q$. $|u|$ is also a weak function of $\mathcal{R}$, but is a very strong function of Q.

3. The control freefall Reynolds number ($\mathcal{R}$) and Péclet number ($\text{Pr}\mathcal{R}$)—and the diffusivity associated with them—only act on fluctuations, not on the $k_x = 0$ mode. For the $k_x = 0$ mode, we model the radiative flux as $F_{\text{rad}} = -k\boldsymbol{\nabla}\overline{T}$, where k is a nonconstant coefficient, $k(z)$. k may be a larger or smaller diffusivity than Pe_0^{-1}, and its magnitude is set by $\int Q \, dz \approx -k\boldsymbol{\nabla}\overline{T}$.

2 Model Setup

We will study a two-layer model in $z = [0, L_z]$ in which the gradient of the conductivity, $\partial_z k(z)$, is a step function,

$$\frac{\partial k(z)}{\partial z} = \partial_z k_0 \begin{cases} 1 & z < L_{\text{cz}} \\ \mathcal{P}^{-1} & z \geq L_{\text{cz}} \end{cases} \tag{4}$$

Generally, we will set $L_{\rm cz} = 1$. The crucial thing here is that the CZ has a nondimensional slope of order $\partial_z k_0$, but above the place where the Schwarzschild criterion is met ($\nabla_{\rm rad} = \nabla_{\rm ad}$) the slope is larger or smaller than that value according to the magnitude of $\mathcal{P}$. $\mathcal{P}$ therefore sets how quickly the magnitude of $F_{\rm rad}$ conducted along a (constant) $\nabla_{\rm ad}$ varies with height.

Aside from $\mathcal{R}$ and Pr, there will be two important input parameters in this system:

1. The stiffness,
$$\mathcal{S} = \frac{N^2_{\rm RZ}}{f^2_{\rm conv}}, \tag{5}$$
which is the characteristic ratio of the buoyancy frequency (in the stable layer) to the square convective frequency. Since $N^2_{\rm RZ}$ (and $\nabla_{\rm rad}$) will vary with height, we will nondimensionalize using its characteristic value at $z = 2L_{\rm cz}$.

2. $\mathcal{P}$, a parameter which sets the penetration depth. $\mathcal{P}$ determines how steeply the value of $F_{\rm rad}$ (conducted along $\nabla_{\rm ad}$ in the penetration zone) changes as a function of height. As a result, assuming flux conservation with $F_{\rm tot} = F_{\rm rad} + F_{\rm conv}$, $\mathcal{P}$ sets how rapidly the convective flux decreasing (into negative values) as a function of height. We will define $\mathcal{P}$ using an atmosphere where $\boldsymbol{\nabla} = \nabla_{\rm ad}$ everywhere; In this atmosphere,
$$\mathcal{P} \equiv -\frac{F_{\rm conv}|_{\rm cz}}{F_{\rm conv}|_{2L_{\rm cz}}} = -\frac{(F_{\rm tot} - F_{\rm rad})|_{\rm cz}}{(F_{\rm tot} - F_{\rm rad})|_{2L_{\rm cz}}} \tag{6}$$
Our expectation is that we get a lot of convective penetration when $\mathcal{P} \gg 1$ and none when $\mathcal{P} \ll 1$.

In addition to these parameters, the magnitude of the heating, Q and the vertical extent of the heating layer, δ_H, must be specified as they set $F_{\rm conv} = Q\delta_H$. We will choose $Q = 1$ and $\delta_H = 0.2$ so that $F_{\rm conv} = 0.2$; we will furthermore choose and $F_{\rm bot} = \zeta F_{\rm conv}$, where ζ is some arbitrary scalar that sets how much smaller k is in the cz than the RZ; we'll set $\zeta = 10^{-3}$ for now. This choice sets a boundary condition on the integral of k_0 so that $k_0(z = 0) = F_{\rm bot}/\nabla_{\rm ad}$.

Under these choices, the system is determined by five equations,

1. At $z = L_{\rm cz}$, the adiabatic gradient is equal to the radiative gradient,

$$k_{\rm ad}\boldsymbol{\nabla}_{\rm ad} = F_{\rm tot}. \tag{7}$$

2. at $z = 0$, the base of the CZ, the adiabatic temperature gradient is the radiative gradient, $\boldsymbol{\nabla}_{\rm ad} = \boldsymbol{\nabla}_{\rm rad}$, and carries $F_{\rm bot}$,

$$k_0(z = 0)\boldsymbol{\nabla}_{\rm ad} = F_{\rm bot} \tag{8}$$

3. The radiative gradient can carry the flux in the RZ,

$$k(z > L_{\rm cz})\boldsymbol{\nabla}_{\rm rad} = (k_{\rm ad} + \partial_z k_0 \mathcal{P}^{-1}(z - L_{\rm cz}))\nabla_{\rm rad} = F_{\rm tot} \tag{9}$$

4. The magnitude of $\mathcal{P}$ is based on the radiative flux conducted along the adiabat at $z = 2L_{\rm cz}$, where $k(z = 2L_{\rm cz}) = k_{\rm ad} + \partial_z k_0 L_{\rm cz}\mathcal{P}^{-1}$. Thus, with $F_{\rm conv}|_{\rm cz} = Q\delta_H$,

$$\mathcal{P} = -\frac{F_{\rm conv}|_{\rm cz}}{F_{\rm conv}|_{2L_{\rm cz}}} = -\frac{Q\delta_H}{\partial_z k_0 L_{\rm cz}\mathcal{P}^{-1}\nabla_{\rm ad}} \Rightarrow \mathcal{P}^2 = \frac{Q\delta_H}{\nabla_{\rm ad}\partial_z k_0 L_{\rm cz}} \tag{10}$$

5. Defining $f_{\rm conv}^2 \approx Q$ and $N^2 \approx -(\boldsymbol{\nabla}_{\rm rad}|_{2L_{\rm cz}} - \boldsymbol{\nabla}_{\rm ad})$, the stiffness gives

$$\mathcal{S} = \frac{N^2}{f_{\rm conv}^2} = \frac{-(\boldsymbol{\nabla}_{\rm rad}|_{2L_{\rm cz}} - \boldsymbol{\nabla}_{\rm ad})}{Q} = \frac{F_{\rm tot}}{Q}(k_{\rm ad}^{-1} - [k_{\rm ad} + \partial_z k_0 \mathcal{P}^{-1}L_{\rm cz}]^{-1}) \tag{11}$$

Solving this system of equations gives

$$\partial_z k_0 = \frac{\delta_H}{L\mathcal{P}^2\mathcal{S}}\frac{1}{\xi}, \qquad k_b = \frac{\delta}{\mathcal{S}}\frac{\zeta}{\xi}, \qquad k_{\rm ad} = \frac{\delta}{\mathcal{S}}\frac{1+\zeta}{\xi}, \qquad \nabla_{\rm ad} = Q\mathcal{S}\xi \tag{12}$$

with $\xi \equiv 1 + \mathcal{P}^3(1+\zeta)$ and $\nabla_{\rm rad} = F_{\rm tot}/k(z)$.

Just to be safe, I'm going to offset the internal heating from the bottom boundary. We wouldn't need to do this if we didn't have a bottom, impenetrable boundary condition with a viscous boundary layer. But, since $w \to 0$

at the bottom boundary, I don't want to inject heating into the domain somewhere where it must be carried by a superadiabatic temperature gradient (I explicitly don't want to study boundary-driven convection). So

$$Q(z) = \begin{cases} 0 & z < 0.1 \\ Q & 0.1 \leq z < 0.1 + \delta_H, \\ 0 & \text{elsewhere} \end{cases}. \tag{13}$$

We then choose T_0 to satisfy

$$\partial_z T_0 = -\begin{cases} \boldsymbol{\nabla}_{\text{ad}} & z \leq 1 \\ \boldsymbol{\nabla}_{\text{rad}} & z > 1 \end{cases} \tag{14}$$

(again, perhaps not perfectly this, but with smooth transitions). And we're off to the races.