1 Nondimensional equations

The nondimensional system of equations we solve is

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \varpi + T\hat{z} + \mathrm{Re}_0^{-1} \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{2}$$

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T + w \boldsymbol{\nabla}_{\mathrm{ad}} = \mathrm{Pe}_0^{-1} \boldsymbol{\nabla}^2 \mathrm{T}' + \boldsymbol{\nabla} \cdot [\mathrm{k} \boldsymbol{\nabla} \overline{\mathrm{T}}] + \mathrm{Q} \tag{3}$$

These are just the Boussinesq equations of motion, with some slight tweaks:

1. There is a nonzero adiabatic temperature gradient, $\partial_z T_{\mathrm{ad}} = -\boldsymbol{\nabla}_{\mathrm{ad}}$. We choose this sign convention ($\boldsymbol{\nabla}_{\mathrm{ad}}$ and similarly $\boldsymbol{\nabla}_{\mathrm{rad}}$ will be positive) to align better with the stellar structure community's intuition. $|\boldsymbol{\nabla} T_0|$ and $\boldsymbol{\nabla}_{\mathrm{ad}}$ may be much larger in magnitude than the convective fluctuations. The background temperature field T_0 does not enter into the momentum equation, as it is automatically canceled by a background $\boldsymbol{\nabla}\varpi$.

2. Convection is driven by internal heating, Q. I've confirmed with some simple tests that $|u|^2 \approx Q$, and therefore the convective frequency is $f_{\mathrm{conv}}^2 \approx Q$.

3. The control freefall Péclet number, Pe_0, and the diffusivity associated with it, only acts on fluctuations, not on the $k_x = 0$ mode. For the $k_x = 0$ mode, we model the radiative flux as $F_{\mathrm{rad}} = -k\boldsymbol{\nabla}\overline{T}$, where k is a nonconstant coefficient, $k(z)$. k may be a larger or smaller diffusivity than Pe_0^{-1}, and its magnitude is set by $\int Q \, dz \approx -k\boldsymbol{\nabla}\overline{T}$.

2 Model Setup

We will study a two-layer model in $z = [0, L_z]$ in which the conductivity, $k(z)$ is a step function such that

$$k(z) = \begin{cases} k_b & z < L_{\mathrm{cz}} \\ k_t & z \geq L_{\mathrm{cz}} \end{cases} \tag{4}$$

where subscript "t" refers to the top layer and "b" refers to the bottom. Generally, we will set $L_{\rm cz} = 1$. (These won't be true discontinuities, but will be smooth step-like functions achieved with erfs.)

There will be two important input parameters in this system:

1. The stiffness,
$$\mathcal{S} = \frac{N_t^2}{f_{\rm conv}^2}, \tag{5}$$
which is the ratio of the buoyancy frequency (in the stable layer) to the square convective frequency.

2. $\mathcal{P}$, a parameter which sets the penetration depth. $\mathcal{P}$ determines what the magnitude of the *negative* convective flux would be in an adiabatic penetrative layer with $k = k_t$. We will define $\mathcal{P} = -F_{\rm conv}/F_{\rm conv,rz}$, and our expectation is that we get a lot of convective penetration when $\mathcal{P} \gg 1$ and none when $\mathcal{P} \ll 1$.

In addition to these parameters, the magnitude of the heating, Q and the vertical extent of the heating layer, δ_H, must be specified as they set $F_{\rm conv} = Q\delta_H$. We will choose $Q = 1$ and $\delta_H = 0.2$ so that $F_{\rm conv} = 0.2$; we will furthermore choose and $F_{\rm bot} = \zeta F_{\rm conv}$, where ζ is some arbitrary scalar that sets how much smaller k is in the cz than the RZ; we'll set $\zeta = 10^{-3}$ for now.

Under these choices, the system is determined by five equations,

1. At some point, the adiabatic gradient is equal to the radiative gradient,
$$k_{\rm ad}\boldsymbol{\nabla}_{\rm ad} = F_{\rm tot}. \tag{6}$$

2. at $z = 0$, the base of the CZ, the adiabatic temperature gradient is the radiative gradient, $\boldsymbol{\nabla}_{\rm ad} = \boldsymbol{\nabla}_{\rm rad}$, and carries $F_{\rm bot}$,
$$k_b\boldsymbol{\nabla}_{\rm ad} = F_{\rm bot} \tag{7}$$

3. The radiative gradient can carry the flux in the RZ,
$$k_t\boldsymbol{\nabla}_{\rm rad,t} = F_{\rm tot} \tag{8}$$

4. Assuming $\boldsymbol{\nabla} T = -\boldsymbol{\nabla}_{\text{ad}}$ everywhere, the convective flux in the top layer is $F_{\text{conv,t}} = F_{\text{conv}} - \Delta k \boldsymbol{\nabla}_{\text{ad}}$, where $F_{\text{conv}} = Q\delta_H$ and $\Delta k = k_t - k_b$. Thus, definitionally,

$$\mathcal{P} = -\frac{F_{\text{conv}}}{F_{\text{conv,t}}} = \left(\frac{\Delta k \boldsymbol{\nabla}_{\text{ad}}}{Q\delta_H} - 1\right)^{-1}. \tag{9}$$

5. Defining $f_{\text{conv}}^2 \approx Q$ and $N^2 \approx -(\boldsymbol{\nabla}_{\text{rad,t}} - \boldsymbol{\nabla}_{\text{ad}})$, the stiffness gives

$$\mathcal{S} = \frac{N^2}{f_{\text{conv}}^2} = \frac{-(\boldsymbol{\nabla}_{\text{rad,t}} - \boldsymbol{\nabla}_{\text{ad}})}{Q} = \frac{F_{\text{tot}}}{Q}(k_{\text{ad}}^{-1} - k_t^{-1}) \tag{10}$$

Solving this system of equations gives

$$k_t = \frac{\delta_H}{\mathcal{S}\mathcal{P}}, \qquad k_b = k_t \frac{\zeta}{1 + \zeta + \mathcal{P}^{-1}}, \qquad k_{\text{ad}} = k_t \frac{1+\zeta}{1 + \zeta + \mathcal{P}^{-1}}, \tag{11}$$

$$\boldsymbol{\nabla}_{\text{ad}} = Q\mathcal{S}\mathcal{P}(1 + \zeta + \mathcal{P}^{-1}), \qquad \boldsymbol{\nabla}_{\text{rad}} = Q\mathcal{S}\mathcal{P}(1 + \zeta). \tag{12}$$

Just to be safe, I'm going to offset the internal heating from the bottom boundary. We wouldn't need to do this if we didn't have a bottom, impenetrable boundary condition with a viscous boundary layer. But, since $w \to 0$ at the bottom boundary, I don't want to inject heating into the domain somewhere where it must be carried by a superadiabatic temperature gradient (I explicitly don't want to study boundary-driven convection). So

$$Q(z) = \begin{cases} 0 & z < 0.1 \\ Q & 0.1 \leq z < 0.1 + \delta_H, \\ 0 & \text{elsewhere} \end{cases}. \tag{13}$$

We then choose T_0 to satisfy

$$\partial_z T_0 = -\begin{cases} \boldsymbol{\nabla}_{\text{ad}} & z \leq 1 \\ \boldsymbol{\nabla}_{\text{rad}} & z > 1 \end{cases} \tag{14}$$

(again, perhaps not perfectly this, but with smooth transitions). And we're off to the races.