1 Dimensional equations

The dimensional Boussinesq equations are

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla} p + \frac{\rho'}{\rho_0}\boldsymbol{g} + \nu \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{2}$$

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T + w \boldsymbol{\nabla}_{\mathrm{ad}} = \chi \boldsymbol{\nabla}^2 T' + \boldsymbol{\nabla} \cdot [k \boldsymbol{\nabla} \overline{T}] + Q \tag{3}$$

$$\frac{\rho'}{\rho_0} = -|\alpha| T \tag{4}$$

Where ρ is the density, T is the temperature, $\boldsymbol{g}$ is the gravitational acceleration, α is the coefficient of thermal expansion, ν and χ are the viscous and thermal diffusivity, k is a radiative diffusivity, and Q is a heating term. We have baked in an assumption that $\alpha < 0$ to make sign conventions more straightforward after substituting Eqn. 4 into Eqn. 2.

2 Convective Penetration argument

First some definitions:

1. $z = L_s$ is the the top of the convection zone according to the Schwarzschild criterion; it's the height where $\boldsymbol{\nabla}_{\mathrm{ad}} = \boldsymbol{\nabla}_{\mathrm{rad}}$.

2. L_{cz} is the top of the convection zone; it's roughly the top of the region where convection flattens $\boldsymbol{\nabla} \to \boldsymbol{\nabla}_{\mathrm{ad}}$. We generally get $L_{\mathrm{cz}} > L_s$.

3. The penetration depth is $\delta_p = L_{\mathrm{cz}} - L_s$.

4. The flux carried by convection for $z < L_s$ is $F_{\mathrm{conv,cz}} = Q\delta_H$, where Q is the magnitude of the internal heating and δ_H is the depth of the heating layer.

5. $\overline{w}$ is the vertical profile of the characteristic (vertical) convective velocity, which is a constant w_{cz} for $z \leq L_{\mathrm{cz}}$.

6. Similarly, $\overline{\delta T}$ is the vertical profile of the characteristic temperature perturbation.

And some key assumptions.

1. Convection flattens $\boldsymbol{\nabla} \rightarrow \boldsymbol{\nabla}_{\mathrm{ad}}$ for $z \leq L_{\mathrm{cz}}$ (baked into our definitions).

2. We assume a system in thermal equilibrium, at least in (adiabatic) convection zone with $z \leq L_{\mathrm{cz}}$. Therefore $F_{\mathrm{conv}}(z) = F_{\mathrm{tot}}(z) - F_{\mathrm{rad,ad}}(z)$, where F is a flux and $F_{\mathrm{rad,ad}}$ is the radiative flux along the adiabatic gradient.

3. We assume $F_{\mathrm{conv}} = \overline{wT} \approx \overline{w}\overline{\delta T}$. Combined with our previous assumption, we get
$$\overline{\delta T} \approx \frac{F_{\mathrm{tot}}(z) - F_{\mathrm{rad,ad}}(z)}{\overline{w}}. \tag{5}$$

As we increase in height, the conductivity and thus $F_{\mathrm{rad,ad}}(z)$ also increases. This means that $\overline{\delta T}$ has the opposite sign of $\overline{w}$ for $z > L_s$.

We presume that buoyancy breaking is the dominant mechanism which brings convective motions to a stop in this adiabatic layer. In other words, we assume that, if we drop the nonlinear, pressure, and viscous terms from Eqn. 2, we describe the dynamics reasonably well,

$$\frac{d\overline{w}}{dt} = \frac{\overline{\rho'}}{\rho_0}(-g\hat{z}) = \alpha g\, \overline{\delta T} = \alpha g\, \frac{F_{\mathrm{tot}}(z) - F_{\mathrm{rad,ad}}(z)}{\overline{w}}. \tag{6}$$

If we multiply both sides by $\overline{w}$ and absorb it into the the derivative, then apply the chain rule with $d/dt = d/dz(dz/dt) = d/dz\overline{w}$, we retrieve,

$$\frac{1}{2} d\overline{w}^3 = \alpha g\, \left[F_{\mathrm{tot}}(z) - F_{\mathrm{rad,ad}}(z)\right] dz. \tag{7}$$

We now replace $F_{\mathrm{rad,ad}}(z) = k_0(z)\boldsymbol{\nabla}_{\mathrm{ad}}$, and we assume that $k(z)$ instantaneously jumps from a low value k_{cz} to a high value k_{rz} at $z = L_s$. Per this assumption, the convective flux is a negative constant for $z \geq L_s$. We integrate from $z = L_s$ with $\overline{w} = w_{\mathrm{cz}}$ to $z = L_s$ with $\overline{w} = 0$, and get

$$-\frac{w_{\mathrm{cz}}^3}{2} = \alpha g\, F_{\mathrm{conv,p}}\, \delta_p. \tag{8}$$

By definition, the convective flux in the penetrative layer with $z > L_s$ is related to $\mathcal{P}$,

$$F_{\text{conv,p}} = -\frac{F_{\text{conv,cz}}}{\mathcal{P}}, \tag{9}$$

so we retrieve

$$\delta_p = \frac{w_{\text{cz}}^3}{2\alpha g\, F_{\text{conv,cz}}}\mathcal{P}, \tag{10}$$

and the penetration depth scales with the cube of the velocity and linearly with $\mathcal{P}$. This prediction does not exactly line up with the simulation results (this overpredicts the magnitude of δ_p). The important thing we need to determine is if it scales with w_{cz}, $\mathcal{P}$, and $F_{\text{conv,cz}}$ in the appropriate way. So far, it seems to scale linearly with $\mathcal{P}$ at fixed other parameters.

Finally, we note that for an ideal gas with $P = \mathcal{R}\rho T$,

$$\alpha = \frac{\partial \ln \rho}{\partial T} = \frac{1}{\rho}\frac{\partial \rho}{\partial T} \approx -T^{-1}. \tag{11}$$

If we had assumed that the convective flux had the form $\rho c_p w \delta T$ during Eqn. 5, we would have retrieved

$$\delta_p = \frac{\rho\, c_p\, w_{\text{cz}}^3}{2\alpha g\, F_{\text{conv,cz}}}\mathcal{P} \qquad \Rightarrow \qquad \frac{\delta_p}{H_P} = \frac{\rho\, w_{\text{cz}}^3}{2F_{\text{conv,cz}}}\mathcal{P}, \tag{12}$$

with $H_P = c_P T/g$ the rough pressure scale height.