DRAFT VERSION MARCH 13, 2019
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1. WHAT WE KNOW

From the simulations we've done so far, when viscous heating is neglected, we have the following understanding of thermals:

1. The *total entropy* of the thermal, ρs_1, is conserved in the absence of viscous heating.
2. For a low-ϵ, low-Mach-number thermal, in which the total entropy is conserved and the anelastic approximation is appropriate, we have two more pieces of knowledge:
   (a) From simple momentum equation arguments, the thermal's momentum increases linearly due to the constant buoyant force, $\int \rho w dV \sim \rho w V = \rho s_1 g/c_p t = Bt$.
   (b) From an anelastic-like impulse formulation with $\rho_1/\rho_0 \ll 1$, we also know that the impulse linearly increases, $\rho \pi R^2 \Gamma = Bt$ (Shivamoggi 2010, eqn 36 modified to buoyant vortex ring).
3. The thermal can be approximated as an ellipsoid whose principal semi-axes have lengths $a = b = R$ and $c = h/2$, where R is the radius of the thermal and $h/2$ is half the height of the thermal volume. In other words, its horizontal cross-sections are circles and it's a bit squished, so it's not quite a sphere. We can probably assume that $h \propto R$, and so the volume of the thermal can be written $V = mR^3$ for some constant m.
4. The depth of the thermal follows a power law in time, $d = d_0 t^\alpha$.
5. The circulation in the thermal, Γ, is constant.

2. PUTTING THE PIECES TOGETHER

From piece of knowledge 2 above, we see that $\rho V w \sim \rho R^2 \Gamma$. Plugging in piece of knowledge 3, we find $Rw \sim \Gamma/m$. Since w, the thermal downward velocity, is the derivative of its depth, $w = \partial_t(d_0 t^\alpha) = \alpha d_0 t^{\alpha-1}$, we see that

$$R \sim \frac{\Gamma}{\alpha d_0 m} t^{1-\alpha}. \tag{1}$$

With that figured out, we can go back to piece of knowledge numbers 2 and 3 above, and see that $\rho V w \sim m \rho R^3 w \sim Bt$, or that $(\Gamma/[\alpha d_0])^3 m^{-2} t^{3(1-\alpha)-(1-\alpha)} \rho = Bt$. Dividing both sides by t and rearranging a bit, we find

$$\rho t^{1-2\alpha} = m^2 B \left(\frac{\alpha d_0}{\Gamma} \right)^3 = \text{const.}$$

Taking a time-derivative, and rearranging, we find

$$\frac{\partial \ln \rho}{\partial t} = \frac{2\alpha - 1}{t}. \tag{2}$$

So a simple $1/t$ fit to $\partial \ln \rho / \partial t$, or perhaps even better, a moving average of $t \partial_t \ln \rho$ (not what I'm doing right now), gives us the value of α!

REFERENCES

Shivamoggi, B. K. 2010, Physics Letters A, 374, 4736