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1. HYDRODYNAMIC IMPULSE

The theory that we're going to use to describe thermals relies on hydrodynamic impulse, and since we happen to be looking at *stratified* domains, we're lucky that Shivamoggi (2010) went through the work of deriving impulse in anelastic atmospheres. He finds that for an anelastic fluid (e.g., one where $\nabla \cdot (\rho_0 \boldsymbol{u}) = 0$ for a constant $\rho_0(z)$), the impulse is

$$\boldsymbol{I} = \frac{1}{2}\int_{\mathcal{V}} \boldsymbol{r} \times (\nabla \times (\rho_0 \boldsymbol{u})) d\boldsymbol{r}, \tag{1}$$

where $\boldsymbol{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector and $\mathcal{V}$ is a closed volume. Basically our theory will consist of two parts:

1. An expectation for the time-derivative of the impulse of the thermal based on its density excess (and/or other thermodynamic properties)
2. The time-derivative of an approximate expression for the thermal's impulse when it is in its evolved vortex-ring state.

Our expectation is that these two things will be approximately equal, and will tell us something about the size of the thermal as it descends and thus its entrainment.

1.1. *Volumetric & Surface terms of the impulse*

For the first part of our theory we'll need to find the time-derivative of the impulse. Before we do that, we need to break apart the impulse expression into surface terms and volumetric terms (as Daniel did in an email on 9/4/2018). First, the component of the impulse in the ith direction for i in (x, y, z) is

$$[\boldsymbol{r} \times (\nabla \times (\rho_0 \boldsymbol{u}))]_i = \nabla_i (\rho_0 \boldsymbol{u}) \cdot \boldsymbol{r} - (\boldsymbol{r} \cdot \nabla)(\rho_0 u)_i = r_j \frac{\partial}{\partial x_i}(\rho_0 u)_j - r_j \frac{\partial}{\partial x_j}(\rho_0 u)_i.$$

By definition, each of the RHS terms can be further expanded as

$$r_j \frac{\partial}{\partial x_i}(\rho_0 u)_j = \frac{\partial}{\partial x_i}(r_j (\rho_0 u)_j) - \delta_{ij}(\rho_0 u)_j, \qquad \text{and} \qquad r_j \frac{\partial}{\partial x_j}(\rho_0 u)_i = \frac{\partial}{\partial x_j}(r_j(\rho_0 u)_i) - 3(\rho_0 u)_i.$$

Plugging these expanded expressions in, we find that

$$[\boldsymbol{r} \times (\nabla \times (\rho_0 \boldsymbol{u}))]_i = \frac{\partial}{\partial x_i}(r_j(\rho_0 u)_j) - \frac{\partial}{\partial x_j}(r_j(\rho_0 u)_i) + 2(\rho_0 u)_i.$$

Plugging back in to the full impulse defn in Eqn. 1, we find

$$\boldsymbol{I} = \int_{\mathcal{V}} (\rho_0 \boldsymbol{u}) d\boldsymbol{r} + \frac{1}{2}\int_{\mathcal{V}} [\nabla(\boldsymbol{r} \cdot (\rho_0 \boldsymbol{u})) - \nabla \cdot (\boldsymbol{r}(\rho_0 \boldsymbol{u}))] d\boldsymbol{r}. \tag{2}$$

As this integral is being done over a closed volume, the last integral can be expressed in terms of surface terms,

$$\boldsymbol{I} = \int_{\mathcal{V}} (\rho_0 \boldsymbol{u}) d\boldsymbol{r} + \frac{1}{2}\int_{\mathcal{S}} d\boldsymbol{S}(\boldsymbol{r} \cdot (\rho_0 \boldsymbol{u})) - d\boldsymbol{S} \cdot (\boldsymbol{r}(\rho_0 \boldsymbol{u})). \tag{3}$$

REFERENCES

Shivamoggi, B. K. 2010, Physics Letters A, 374, 4736