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1. STATE OF THE PROBLEM

In these thermal simulations, as far as I can tell, we have *eight* variables

1. B, the integrated total entropy leading to buoyancy
2. V, the thermal volume
3. R, the radius of the thermal
4. h, the thermal's vertical extent ('radius' in z-direction)
5. w, the thermal bulk vertical velocity
6. z, the height of the thermal
7. Γ, the circulation of the vortex ring
8. ρ, the local density at the height of the thermal.

From the simulations we've done so far, when viscous heating is neglected, we have the following understanding of thermals:

1. $B = \text{const} \propto \rho s_1$
2. $h = AR$, where A is a constant [this assumption may not be true at high stratification].
3. $V = (4/3)\pi h R^2 = (4/3)\pi A R^3 = V_0 R^3$.
4. $w = \frac{\partial z}{\partial t} = -\frac{\partial d}{\partial t}$ (with depth $d \equiv L_z - z$)
5. $\rho = (1 + L_z - z)^{m_{ad}} = (1 + d)^{m_{ad}}$, with $m_{ad} = (\gamma - 1)^{-1} = 1.5$ (for our polytropes)
6. $\rho V w \sim Bt$ (momentum linearly increases)
7. $\rho \pi R^2 \Gamma \sim Bt$ (e.g., eqn 36 of Shivamoggi 2010).

That's 8 unknowns, and 7 equations. There's one piece of the puzzle we're missing: what happens with Γ over time? Ideally, $\Gamma = \text{const}$, but we can't be sure that's the case.

2. LOW STRATIFICATION LIMIT

In the limit of a lightly stratified thermal, we make one more assumption:

1. $\Gamma = \text{const}$ (our eqn # 8)

We also know that, under this assumption, and also in this regime where eqn 2 is valid, from dividing eqn 6 by 7,

$$wR \sim \frac{\pi\Gamma}{V_0}.$$

In general, this suggests that $w \propto t^{-\alpha}$ and $R \propto t^{\alpha}$, such that when multiplied together they return a constant. With this, we return to eqn 6 or 7, and get

$$\rho R^2 = \frac{B}{\pi\Gamma} t \tag{1}$$

With our ansatz of $R \propto t^{\alpha}$, we know that $\rho \sim t^{1-2\alpha}$. But we also know from the stratification of our domain that $\rho = (1 + d)^{m_{ad}} \sim t^{1-2\alpha}$, and that $d \sim -t^{1-\alpha}$ (from theory equation 4, definitionally).

3. WHAT DO WE NEED TO DO IN THE STRATIFIED LIMIT?

Figure out what happens to the circulation.

REFERENCES

Shivamoggi, B. K. 2010, Physics Letters A, 374, 4736