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Circulation in an anelastic system

1. MOMENTUM AND VORTICITY EQUATIONS

The anelastic momentum equation takes the form of eqn 27 in Lecoanet et al. (2014),

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\varpi + \frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right). \tag{1}$$

Taking the curl of Eqn. 1, we can retrieve the vorticity equation in two forms:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times\left[\boldsymbol{u}\times\boldsymbol{\omega} + \frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)\right] \tag{2}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = -\boldsymbol{\omega}\nabla\cdot\boldsymbol{u} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} + \nabla\times\left[\frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)\right] \tag{3}$$

2. INTEGRAL IDENTITIES

As circulation is defined as the path-integral of velocity (or the surface integral of vorticity), it is useful to know a vector identity for each of these types of integrals. For an arbitrary vector field, $\boldsymbol{Q}$, the langrangian derivative ($D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$) of its line integral around contour C is

$$\frac{D}{Dt}\oint_C \boldsymbol{Q}\cdot d\boldsymbol{x} = \oint_C \frac{D\boldsymbol{Q}}{Dt}\cdot d\boldsymbol{x} + \oint_C \boldsymbol{Q}\cdot d\boldsymbol{u}. \tag{4}$$

The lagrangian derivative of a surface integral along a surface A is (Eqn. 4.45 of Choudhuri 1998),

$$\frac{D}{Dt}\iint_A d\boldsymbol{S}\cdot\boldsymbol{Q} = \iint_A d\boldsymbol{S}\cdot\left[\frac{\partial\boldsymbol{Q}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{Q})\right]. \tag{5}$$

3. CIRCULATION FROM THE MOMENTUM EQUATION

Starting with Eqn. 1 and using Eqn. 4 with $\boldsymbol{Q}=\boldsymbol{u}$, we acknowledge that the second term in Eqn. 4 is

$$\oint_C \boldsymbol{u}\cdot d\boldsymbol{u} = \frac{1}{2}\oint_C d|\boldsymbol{u}^2| = 0, \tag{6}$$

as it is a closed loop integral of a perfect differential. Thus, we have

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint_C \boldsymbol{u}\cdot d\boldsymbol{x} = \oint_C \frac{D\boldsymbol{u}}{Dt}\cdot d\boldsymbol{x} = \oint_C\left[-\nabla\varpi + \frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)\right]\cdot d\boldsymbol{x} \tag{7}$$

Since the closed loop integral conservative force (e.g., $-\nabla\varpi$) is zero, we are left with

$$\frac{D\Gamma}{Dt} = \oint_C\left[\frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)\right]\cdot d\boldsymbol{x} \tag{8}$$

4. CIRCULATION FROM THE VORTICITY EQUATION

Starting with Eqn. 2 and using Eqn. 5 with $\boldsymbol{Q}=\boldsymbol{\omega}$, we rather simply retrieve

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\iint_A d\boldsymbol{S}\cdot\boldsymbol{\omega} = \iint_A d\boldsymbol{S}\cdot\left[\frac{\partial\boldsymbol{\omega}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega})\right] = \iint_A d\boldsymbol{S}\cdot\nabla\times\left[\boldsymbol{u}\times\boldsymbol{\omega} + \frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right) - \boldsymbol{u}\times\boldsymbol{\omega}\right]. \tag{9}$$

Or, simplifying,

$$\frac{D\Gamma}{Dt} = \iint_A d\boldsymbol{S}\cdot\nabla\times\left[\frac{g}{c_P}S_1\hat{z} + \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)\right]. \tag{10}$$

Assuming that Stokes' theorem holds for the contour C around area A, the definitions of circulation for Eqns. 8 and 10 are equivalent, and only buoyant and viscous terms should have any effect on the circulation in the thermal.

REFERENCES

Choudhuri, A. R. 1998, The physics of fluids and plasmas : an introduction for astrophysicists /

Lecoanet, D., Brown, B. P., Zweibel, E. G., et al. 2014, ApJ, 797, 94