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Entrainment of low Mach number thermals in stratified domains

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ABSTRACT

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1. INTRODUCTION

(Para 1) Brief intro to solar convective conundrum.

(Para 2) Entropy rain history and expansion by Axel

(Para 3) Boussinesq thermals and their filling factor vs. depth.

(Para 4) What we're doing here, how it fits into this context.

(Para 5) Standard last paragraph of an intro?

2. THEORY

2.1. *Phenomenological description of thermal evolution*

The initial conditions of the thermals studied in this work are spherical entropy perturbations with zero initial velocity, as depicted in Fig. 1a. Buoyant forces cause these perturbations to fall and develop into vortex rings, as depicted in Figs. 1c-f.

2.2. *Theoretical description of thermal evolution*

The evolution of thermals as buoyant vortex rings has been well described in the unstratified, Boussinesq limit for decades (as early as e.g., CITE, and see Lecoanet & Jeevanjee (2018) for other sources). Buoyant motions in the atmospheres of stars and planets are generally large enough to feel the atmospheric stratification, and therefore a more thorough tratment of the evolution of thermals in stratified domains is required to understand the nature of thermal entrainment in nature.

In this work, we focus on the non-dissipative, low Mach number regime, in which the ideal anelastic equations are a decent approximation to the fully compressible equations. In this regime, the buoyancy is directly proportional to the specific entropy. In the absence of diffusion, or in the limit where diffusivities are sufficiently small, the volume-integrated total entropy is constant,

$$B \equiv \int_{\mathcal{V}} \rho\, s\, dV = \text{const}, \tag{1}$$

where $s = c_V \ln T - R \ln \rho$ is the specific entropy, with c_V the specific heat at constant volume and R the ideal gas constant.

In a stratified domain, the hydrodynamic impulse is defined Shivamoggi (2010),

$$\boldsymbol{I} = \frac{1}{2}\int_{\mathcal{V}} \boldsymbol{x} \times (\nabla \times (\rho \boldsymbol{u}))dV, \tag{2}$$

where $\boldsymbol{x}$ is the position vector, $\boldsymbol{u}$ is vector velocity, ρ is density, and $\mathcal{V}$ is the volume being integrated over. Furthermore, changes in the impulse can be expressed

$$\frac{\partial \boldsymbol{I}}{\partial t} = \int_{\mathcal{V}} \frac{\partial}{\partial t}(\rho \boldsymbol{u})dV = B\hat{z} + S, \tag{3}$$

where S is a combination of surface terms that disappears upon appropriate boundary conditions on $\mathcal{V}$ (Shivamoggi 2010). As we have assumed that the buoyancy, B, is constant, we can straightforwardly integrate

$$I_z = Bt + I_0, \tag{4}$$

for some constant I_0.

(need to work through this section carefully). Assuming an adiabatically stratified atmosphere in hydrostatic equilibrium, the vertical momentum equation is

$$= -\partial_z \varpi + g\frac{S_1}{c_P}, \tag{5}$$

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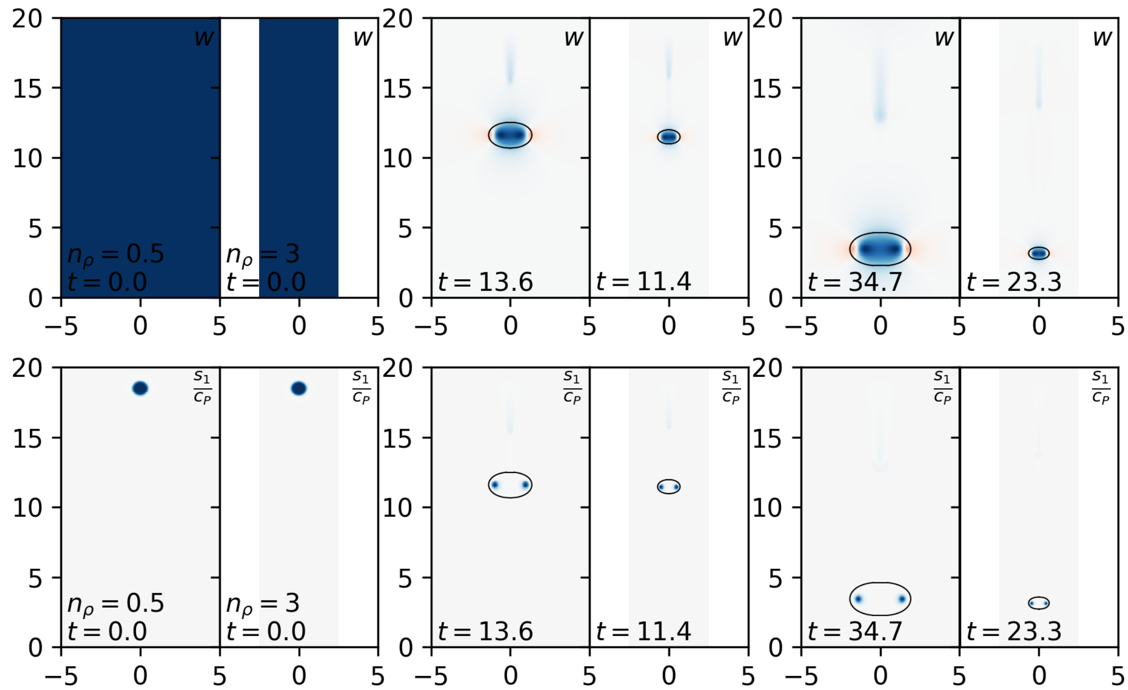

Figure 1.

where w is vertical velocity, S_1 is specific entropy, ϖ is the reduced pressure, $D/Dt = \partial_t + (\boldsymbol{u} \cdot \nabla)$ is the (eulerian? lagrangian?) derivative, g is gravity, and c_P is the specific heat at constant pressure. In the anelastic approximation, the density stratification is constant in time. OH GOD I DON'T KNOW THIS MATH WELL MAGIC MAGIC MAGIC. By doing magic, we arrive at

$$P_z = \beta B t + P_0. \tag{6}$$

Under the assumption that the thermal develops into a thin-core propagating vortex ring whose vortex core is radius r away from the axis of symmetry, the impulse can be approximated as $I_z \approx \pi \rho r^2 \Gamma$, where Γ is the circulation of the thermal vortex, which we assume to be constant. Rearranging, we find our first result,

$$r = \sqrt{\frac{Bt + I_0}{\pi \rho \Gamma}}. \tag{7}$$

In the limit where $\rho \to$ constant, as in the Boussinesq regime, we retrieve the $r \propto \sqrt{t}$ scaling found in the Boussinesq regime by Lecoanet & Jeevanjee (2018). We find that the inclusion of stratification adds the additional complexity of $r \propto \rho^{-1/2}$, such that downward-propogating vortex rings (as studied here) will not entrain to the same degree as boussinesq thermals, and upward-propogating rings will entrain more.

We further assume that the volume-integrated momentum, $P_z \approx \rho V w_{th}$, where w_{th} is the vertical velocity of the thermal as a whole, and the volume of the fluid region propogating with the thermal is a spheroid, $V = V_0 r^3$, for some constant V_0. Plugging Eqn. 7 into this approximation, we find that

$$\rho^{-1/2} w = \left(\frac{(\pi \Gamma)^{3/2}}{V_0} \right) \frac{\beta B t + M_0}{(Bt + I_0)^{3/2}}. \tag{8}$$

Substituting $w = dz/dt$ and $\tau \equiv (Bt + I_0)/\Gamma$, We retrieve

$$\frac{dz}{\sqrt{\rho(z)}} = \left(\frac{\pi^{3/2} \Gamma}{V_0} \right) \left(\beta \Gamma \tau^{-1/2} + [M_0 - \beta I_0] \tau^{-3/2} \right) d\tau. \tag{9}$$

If the atmospheric stratification in which the thermal is falling is known, this result can be straightforwardly integrated with $\rho(z)$ plugged in in order to find the position of the thermal versus time. While we will leave this result general for now, we will plug in our polytropic stratification at the end of section 3, and show that the resulting expression does a good job of explaining the evolution of thermals in these atmospheres in section 4.

3. EXPERIMENT

3.1. *Polytropic Atmospheres*

We study an ideal gas whose equation of state is $P = \rho T$ and whose stratification is a perfectly adiabatic polytrope,

$$T_0 = 1 + (\nabla_{\rm ad})(z - L_z) \tag{10}$$
$$\rho_0 = T_0^{m_{\rm ad}}, \tag{11}$$

where $m_{\rm ad} = (\gamma - 1)^{-1}$, and where the adiabatic temperature gradient is $\nabla_{\rm ad} = -g/c_P$.

3.2. *Anelastic Equations*

The anelastic equations are (Lecoanet et al. 2014),

$$\widetilde{\nabla} \cdot \widetilde{\boldsymbol{u}} = -\widetilde{w}\partial_{\widetilde{z}} \ln \rho_0 \tag{12}$$
$$\frac{D\widetilde{\boldsymbol{u}}}{D\widetilde{t}} = -\widetilde{\nabla}\widetilde{\varpi} + g\frac{\widetilde{S_1}}{c_P}\hat{z} + \frac{1}{\rho_0}\widetilde{\nabla}\cdot\left(\mu\bar{\bar{\widetilde{\boldsymbol{\sigma}}}}\right) \tag{13}$$
$$\frac{D\widetilde{S_1}}{D\widetilde{t}} = \frac{1}{\rho c_P}\widetilde{\nabla}\cdot\left(\kappa T_0 \widetilde{\nabla}\widetilde{S_1}\right) + \frac{\mu}{\rho_0 T_0}\widetilde{\sigma_{ij}}\partial_{\widetilde{x_i}}\widetilde{u_j}. \tag{14}$$

where $D/D\widetilde{t} = \partial/\partial\widetilde{t} + \widetilde{\boldsymbol{u}}\cdot\widetilde{\nabla}$ and the viscous stress tensor is

$$\widetilde{\sigma_{ij}} = \left(\partial_{\widetilde{x_i}}\widetilde{u_j} + \partial_{\widetilde{x_j}}\widetilde{u_i} - \frac{2}{3}\delta_{ij}\widetilde{\nabla}\cdot\widetilde{\boldsymbol{u}}\right). \tag{15}$$

We nondimensionalize these equations as in Lecoanet & Jeevanjee (2018) such that the length scale is the diameter of the thermal and the velocity scale is the freefall velocity,

$$\begin{aligned} \widetilde{\nabla} &\to (\widetilde{L}_{th}^{-1})\nabla, \qquad \widetilde{S}_1 \to (\Delta\widetilde{S})S_1, \\ \widetilde{\boldsymbol{u}} &\to (\widetilde{u}_{th})\boldsymbol{u}, \qquad \widetilde{\varpi} \to (\widetilde{u}_{th}^2)\varpi, \\ \partial_{\widetilde{t}} &\to (\widetilde{u}_{th}/\widetilde{L}_{th})\partial_t, \end{aligned} \tag{16}$$

with

$$\tilde{u}_{th}^2 = \frac{g\tilde{L}_{th}\Delta\tilde{s}}{c_P}, \qquad {\rm Re_{ff}} = \frac{\tilde{u}_{th}\tilde{L}_{th}}{\nu}, \qquad {\rm Pr_{ff}} = \frac{\tilde{u}_{th}\tilde{L}_{th}}{\chi}. \tag{17}$$

The resulting equations are very similar to the boussinesq equations,

$$\nabla\cdot\boldsymbol{u} = -w\partial_z \ln\rho_0, \tag{18}$$
$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\varpi + S_1\hat{z} + \frac{1}{\rm Re_{ff}}\left[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})\right] \tag{19}$$
$$\partial_t S_1 + \boldsymbol{u}\cdot\nabla S_1 = \frac{1}{\rm Re_{ff}}\left(\frac{1}{{\rm Pr_{ff}}\rho_0 c_P}[\nabla^2 S_1 + \partial_z \ln T_0\cdot\partial_z S_1] + \frac{-(\nabla_{ad})}{\rho_0 T_0}\sigma_{ij}\partial_{x_i}u_j\right) \tag{20}$$

3.3. *Fully Compressible Equations*

The fully compressible equations, under the same assumptions as above, are

$$\frac{D\ln\rho}{Dt} + \nabla\cdot\boldsymbol{u} = 0 \tag{21}$$
$$\frac{D\boldsymbol{u}}{Dt} = -\nabla T - T\nabla\ln\rho - g\hat{z} + \frac{1}{\rho}\nabla\cdot(\mu\bar{\bar{\boldsymbol{\sigma}}}) \tag{22}$$
$$\frac{DT}{Dt} + (\gamma-1)T\nabla\cdot\boldsymbol{u} = \frac{1}{\rho c_V}\nabla\cdot(\kappa\nabla T) + \frac{\mu}{\rho c_V}\sigma_{ij}\partial_{x_i}u_j. \tag{23}$$

3.4. *Initial Conditions*

We set the initial specific entropy perturbation as

$$S_1 = -A(1 - {\rm erf}\left(\frac{r - r_{th}}{\delta}\right))/2, \tag{24}$$

where in the FC cases $A = 10^{-4}$ and in the nondimensionalized AN cases, $A = 1$. Here, $r = \sqrt{x^2 + y^2 + (z - z_0)^2}$, where $z_0 = Lz - 3 * r_{th}$, and $r_{th} = Lz/40$ is the radius of the thermal.

In the fully compressible equations, it is essential that the initial conditions be in pressure equilibrium such that the low mach number nature of the thermal can be studied. In order to achieve this pressure balance, we set

$$\ln\rho_1 = S_1/c_P, \qquad T_1 = T_0(e^{-\ln\rho_1} - 1). \tag{25}$$

3.5. *Solution for thermal evolution in a Polytrope*

blah blah blah T(t) and derivative blah blah blah.

3.6. *Verification of 2D Anelastic approximation*

blah blah blah comparison of 2D and 3D blah blah blah. z v t, r v z, w v z. Also side-by-side pictoral comparison with diff.

4. RESULTS

Picture of a big old grid of z v t, r v z, w v z.

Pretty picture showing thermal evolution (comparing low and high stratification).

Verification of theory by simulations

Table of found parameters for the fits

Two regimes: stalling and falling.

5. DISCUSSION

Wild speculation about extensions to the solar regime. Do things on the sun shrink to the point where they viscously dissipate?

Talk about what would happen if we were to study up-thermals.

Extensions, and the fact that we trust these results will likely hold in the solar regime.

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APPENDIX

A. THERMAL TRACKING

B. TABLE OF SIMULATIONS

REFERENCES

Lecoanet, D., Brown, B. P., Zweibel, E. G., et al. 2014, ApJ, 797, 94

Lecoanet, D., & Jeevanjee, N. 2018, arXiv e-prints, arXiv:1804.09326

Shivamoggi, B. K. 2010, Physics Letters A, 374, 4736