

1. HYDRODYNAMIC IMPULSE

The theory that we're going to use to describe thermals relies on hydrodynamic impulse, and since we happen to be looking at *stratified* domains, we're lucky that [Shivamoggi \(2010\)](#) went through the work of deriving impulse in anelastic atmospheres. He finds that for an anelastic fluid (e.g., one where $\nabla \cdot (\rho_0 \mathbf{u}) = 0$ for a constant $\rho_0(z)$), the impulse is

$$\mathbf{I} = \frac{1}{2} \int_{\mathcal{V}} \mathbf{r} \times (\nabla \times (\rho_0 \mathbf{u})) d\mathbf{r}, \quad (1)$$

where $\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector and \mathcal{V} is a closed volume. Basically our theory will consist of two parts:

1. An expectation for the time-derivative of the impulse of the thermal based on its density excess (and/or other thermodynamic properties)
2. The time-derivative of an approximate expression for the thermal's impulse when it is in its evolved vortex-ring state.

Our expectation is that these two things will be approximately equal, and will tell us something about the size of the thermal as it descends and thus its entrainment.

1.1. Volumetric & Surface terms of the impulse

For the first part of our theory we'll need to find the time-derivative of the impulse. Before we do that, we need to break apart the impulse expression into surface terms and volumetric terms (as Daniel did in an email on 9/4/2018). First, the component of the impulse in the i th direction for i in (x, y, z) is

$$[\mathbf{r} \times (\nabla \times (\rho_0 \mathbf{u}))]_i = \nabla_i (\rho_0 \mathbf{u}) \cdot \mathbf{r} - (\mathbf{r} \cdot \nabla) (\rho_0 u)_i = r_j \frac{\partial}{\partial x_i} (\rho_0 u)_j - r_j \frac{\partial}{\partial x_j} (\rho_0 u)_i.$$

By definition, each of the RHS terms can be further expanded as

$$r_j \frac{\partial}{\partial x_i} (\rho_0 u)_j = \frac{\partial}{\partial x_i} (r_j (\rho_0 u)_j) - \delta_{ij} (\rho_0 u)_j, \quad \text{and} \quad r_j \frac{\partial}{\partial x_j} (\rho_0 u)_i = \frac{\partial}{\partial x_j} (r_j (\rho_0 u)_i) - 3(\rho_0 u)_i.$$

Plugging these expanded expressions in, we find that

$$[\mathbf{r} \times (\nabla \times (\rho_0 \mathbf{u}))]_i = \frac{\partial}{\partial x_i} (r_j (\rho_0 u)_j) - \frac{\partial}{\partial x_j} (r_j (\rho_0 u)_i) + 2(\rho_0 u)_i.$$

Plugging back in to the full impulse defn in Eqn. 1, we find

$$\mathbf{I} = \int_{\mathcal{V}} (\rho_0 \mathbf{u}) d\mathbf{r} + \frac{1}{2} \int_{\mathcal{V}} [\nabla(\mathbf{r} \cdot (\rho_0 \mathbf{u})) - \nabla \cdot (\mathbf{r}(\rho_0 \mathbf{u}))] d\mathbf{r}. \quad (2)$$

As this integral is being done over a closed volume, the last integral can be expressed in terms of surface terms,

$$\mathbf{I} = \int_{\mathcal{V}} (\rho_0 \mathbf{u}) d\mathbf{r} + \frac{1}{2} \int_{\mathcal{S}} d\mathbf{S} (\mathbf{r} \cdot (\rho_0 \mathbf{u})) - d\mathbf{S} \cdot (\mathbf{r}(\rho_0 \mathbf{u})). \quad (3)$$

2. TIME DERIVATIVE OF THE IMPULSE

The time derivative of the impulse is found from taking a time derivative of Eqn. 3 and assuming that the fluid we're studying is unbounded so that the surface terms don't change in time (as in [Shivamoggi 2010](#)),

$$\frac{d\mathbf{I}}{dt} = \int_{\mathcal{V}} \frac{\partial}{\partial t} (\rho_0 \mathbf{u}) d\mathbf{r}. \quad (4)$$

(Aside: the unbounded assumption / the assumption that the surface integrals don't matter seems a little sketchy to me for our sims, but it's common practice. Let's try it. Also, I'm not entirely sure why $d/dt \rightarrow \partial/\partial t$ inside of the derivative, but oh well).

By combining the non-viscous momentum and anelastic continuity equations,

$$\begin{aligned}\nabla \cdot (\rho_0 \mathbf{u}) &= 0, \\ \rho_0 \frac{\partial \mathbf{u}}{\partial t} + \rho_0 (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla P - \rho g \hat{z},\end{aligned}\tag{5}$$

where here I'm keeping the full ρ on the buoyant term, we can find an equation for the expression in our impulse time derivative,

$$\frac{\partial}{\partial t} \rho_0 \mathbf{u} + \nabla \cdot (\rho_0 \mathbf{u} \mathbf{u}) = -\nabla P - (\rho_0 + \rho_1) g \hat{z}.$$

Subtracting out hydrostatic balance, we have something like

$$\frac{\partial}{\partial t} \rho_0 \mathbf{u} = -\nabla \cdot (\rho_0 \mathbf{u} \mathbf{u}) - \nabla P_1 - \rho_1 g \hat{z}.$$

and thus the change in impulse over time is

$$\frac{d\mathbf{I}}{dt} = - \int_V (\nabla \cdot (\rho_0 \mathbf{u} \mathbf{u}) + \nabla P_1 + \rho_1 g \hat{z}) d\mathbf{r} = - \int_V \rho_1 g \hat{z} d\mathbf{r} - \int_S [d\mathbf{S} \cdot (\rho_0 \mathbf{u} \mathbf{u}) + dS P_1].\tag{6}$$

Generally (in [Shivamoggi 2010](#), and I think in other thermal work), the surface terms here are said to be zero (at least, under appropriate boundary conditions), and so we're left with

$$\boxed{\frac{d\mathbf{I}}{dt} = -g M_1 \hat{z}, \quad M_1 \equiv \int_V \rho_1 dV}.\tag{7}$$

Or, your change in impulse over time is directly proportional to the excess mass in the thermal, or the integrated buoyant force over your thermal.

Rough Estimate of M_1 size—For a thermal whose entropy perturbation is ϵ , the temperature perturbation is $T_{1,therm} = T_0(e^{\epsilon/c_P} - 1) \approx T_0\epsilon/c_P$, and the associated density perturbation to maintain pressure equilibrium is $\rho_1 = P_0/(T_0 + T_{1,therm}) - \rho_0$ where $\rho = \rho_0 + \rho_1$. For $P_0 = \rho_0 T_0$, this is $\rho_1 = \rho_0([1 + T_{1,therm}/T_0]^{-1} - 1) \approx -\rho_0\epsilon/c_P$. So for positive ϵ (a hot thermal), we have a low density region, and for a falling cold thermal (with negative ϵ), we have a density excess. The total mass excess in the thermal to start with is something like $M_1 \approx -\pi r_{therm}^2 \epsilon \rho_{0,therm}/c_P$, assuming the density is fairly constant across the thermal. I'll do this calculation more exactly at some point in the future, but this is a good rough estimate.

3. APPROXIMATE EXPRESSION FOR THE IMPULSE

Going back to Eqn. 1, we can see that the impulse can be simply expanded into two expressions:

$$\mathbf{I} = \frac{1}{2} \int_V \mathbf{r} \times (\nabla \times (\rho_0 \mathbf{u})) d\mathbf{r} = \frac{1}{2} \int_V (\rho_0 \mathbf{r} \times \boldsymbol{\omega} + \mathbf{r} \times (\nabla \rho_0 \times \mathbf{u})) d\mathbf{r},\tag{8}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. The first term is identical to the term that shows up in boussinesq flows, and *under the assumption that ρ is constant across the thermal's depth* becomes ([Saffman 1970](#))

$$\frac{1}{2} \int_V \rho_0 \mathbf{r} \times \boldsymbol{\omega} d\mathbf{r} = \rho_0 c R^2 \Gamma \hat{z},$$

where R is the radius of the vortex ring, Γ is its circulation, and c is some constant describing its aspect ratio. The second term here is

$$\mathbf{r} \times (\nabla \rho_0 \times \mathbf{u}) = (\mathbf{r} \cdot \mathbf{u}) \nabla \rho_0 - (\mathbf{r} \cdot \nabla \rho_0) \mathbf{u}.$$

Assuming that $\nabla \rho_0 = \partial_z \rho_0 \hat{z}$, this becomes

$$\mathbf{r} \times (\nabla \rho_0 \times \mathbf{u}) = (-uz\hat{x} - vz\hat{y} + (xu + yv)\hat{z}).$$

We don't care about the x and y terms here, because the main impulse of the thermal is vertical. We're essentially left with

$$I_z = \rho_0 c R^2 \Gamma + \frac{1}{2} \int_V u_\perp r_\perp \frac{\partial \rho_0}{\partial z} d\mathbf{r}.$$

If the thermal is small, as we assumed in evaluating the first part, then $\partial \rho_0 / \partial z$ is constant across the thermal, and the last term drops out ($u_\perp r_\perp$ at the bottom of the thermal cancels out with $u_\perp r_\perp$ at the top of the thermal where it's oppositely signed). If the thermal is large and the stratification of the background varies across the thermal's size, then we need to worry about that second term.

For now, dropping the last term, we can say that for a small thermal, the time derivative of the impulse should be roughly (assuming constant c)

$$\frac{dI_z}{dt} = \frac{d}{dt} (\rho_0 c R^2 \Gamma) = c \left(R^2 \Gamma \frac{d\rho_0}{dt} + 2\Gamma \rho_0 R \frac{dR}{dt} + \rho_0 R^2 \frac{d\Gamma}{dt} \right)$$

Note that $d\rho_0/dt = d\rho_0/dz \cdot dz/dt = w d\rho_0/dz$, and so

$$\boxed{\frac{dI_z}{dt} = c \left(\Gamma R^2 w \frac{\partial \rho_0}{\partial z} \Big|_{z=z_{th}} + 2\rho_0 \Gamma R \frac{dR}{dt} + \rho_0 R^2 \frac{d\Gamma}{dt} \right)} \quad (9)$$

4. PUTTING IT TOGETHER: THEORY FOR A SMALL, ANELASTIC THERMAL

Basically the theory comes down to this: If this impulse framework is good for understanding thermals, then Eqns. 7 & 9 should be equal to each other. In other words, we expect

$$-gM_1 \approx c \left(\Gamma R^2 w \frac{\partial \rho_0}{\partial z} \Big|_{z=z_{th}} + 2\rho_0 \Gamma R \frac{dR}{dt} + \rho_0 R^2 \frac{d\Gamma}{dt} \right). \quad (10)$$

This expression is slightly different from our first take at this theory. On the left, we have mass instead of entropy. On the right, we're not assuming that the circulation is constant.

REFERENCES

- Saffman, P. G. 1970, *Studies in Applied Mathematics*, 49, 371
 Shivamoggi, B. K. 2010, *Physics Letters A*, 374, 4736