1. THE ANELASTIC EQUATIONS

We write the anelastic equations under a constant dynamic diffusivity formulation ($\mu = \rho\nu = \text{const}$, $\kappa = \rho\chi = \text{const}$) as in Lecoanet et al. (2014), eqns. 27-29,

$$\nabla \cdot \boldsymbol{u} = -w\partial_z \ln \rho = \frac{w}{H_\rho(z)} \tag{1}$$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\varpi = -\boldsymbol{g}\frac{S_1}{c_P} + \frac{\mu}{\rho_0}\left[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})\right] \tag{2}$$

$$\partial_t S_1 + \boldsymbol{u}\cdot\nabla S_1 = +\frac{\kappa}{\rho c_V}\left[\nabla^2 S_1 + \partial_z \ln T_0 \cdot \partial_z S_1\right] + \frac{\mu}{\rho T}\sigma_{ij}\partial_{x_i}u_j, \tag{3}$$

with the modified stress tensor

$$\sigma_{ij} = \left(\partial_{x_i}u_j + \partial_{x_j}u_i - \frac{2}{3}\delta_{ij}\nabla\cdot\boldsymbol{u}\right). \tag{4}$$

Here, the c_V^{-1} shows up in the diffusion term because we're aiming for as close of a one-to-one comparison with the FC temperature equation as possible. There, we had $F = -\kappa\nabla T_1 \approx -\kappa c_v^{-1} T_0 \nabla S_1$ in the anelastic limit.

1.1. *Nondimensional anelastic equations*

We then nondimensionalize as in Lecoanet & Jeevanjee (2018), (where all of the terms from the previous, dimensionful equations now have $\sim$ over them):

$$\begin{aligned} \tilde{\nabla} &\rightarrow (\tilde{L}_{th}^{-1})\nabla, & \tilde{S}_1 &\rightarrow (\Delta\tilde{S})S_1, \\ \tilde{\boldsymbol{u}} &\rightarrow (\tilde{u}_{th})\boldsymbol{u}, & \tilde{\varpi} &\rightarrow (\tilde{u}_{th}^2)\varpi, \\ \partial_{\tilde{t}} &\rightarrow (\tilde{u}_{th}/\tilde{L}_{th})\partial_t, \end{aligned} \tag{5}$$

with

$$\tilde{u}_{th}^2 = \frac{g\tilde{L}_{th}\Delta\tilde{s}}{c_P}, \qquad \mathrm{Re}_{\mathrm{ff}} = \frac{\tilde{u}_{th}\tilde{L}_{th}}{\nu}, \qquad \mathrm{Pr}_{\mathrm{ff}} = \frac{\tilde{u}_{th}\tilde{L}_{th}}{\chi}. \tag{6}$$

Here we acknowledge that $\mu = \nu$ and $\chi = \kappa$ when $\rho = 1$ (at the top of the domain), so these values are specified at the upper boundary, but will increase with depth.

Under these assumptions, eqns. 2 & 3 become (with $\boldsymbol{g} = -g\hat{z}$),

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla\varpi = S_1\hat{z} + \frac{1}{\mathrm{Re}_{\mathrm{ff}}}\left[\nabla^2\boldsymbol{u} + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{u})\right] \tag{7}$$

$$\partial_t S_1 + \boldsymbol{u}\cdot\nabla S_1 = \frac{1}{\mathrm{Re}_{\mathrm{ff}}}\left(\frac{1}{\mathrm{Pr}_{\mathrm{ff}}\rho_0 c_V}[\nabla^2 S_1 + \partial_z \ln T_0 \cdot \partial_z S_1] + \frac{g\tilde{L}_{th}}{\rho_0 T_0 c_P}\sigma_{ij}\partial_{x_i}u_j\right), \tag{8}$$

The viscous heating term is a bit ugly, but it's truly not too bad. This formulation is great in that we can set up experiments in boxes which are exactly the same size as in Lecoanet & Jeevanjee (2018) (e.g., 10 x 20), with a thermal whose diameter is 1 length unit, and then we can evolve for some tens of time units to get it to the bottom of the domain.

1.2. *Nondimensional atmosphere*

In order to make these equations work, it's important that the atmosphere also be scaled appropriately. In a normal polytrope, we have

$$T_0 = (1 + \tilde{L}_z - \tilde{z}), \tag{9}$$

$$\rho_0 = T^m, \tag{10}$$

with $\tilde{L}_z = e^{n_\rho/m} - 1$ and the adiabatic temperature gradient $\partial_{\tilde{z}} T_{ad} = -g/c_P = -1$, so $g = c_P$. We want to maintain the stratification of these fields, as that's already nondimensionalized ($\rho_0 = T_0 = 1$ at the top of the atmosphere), but we need to rescale the length scales. Thus, let's acknowledge that the true stratification here is $T_0 = 1 + (\partial_{\tilde{z}} T_{ad})(\tilde{z} - \tilde{L}_z)$. If we take out the dimensionality of $\tilde{z} = \tilde{L}_{th} z$ and $\tilde{L}_z = \tilde{L}_{th} L_z$, then we get $T_0 = 1 + (\tilde{L}_{th}\partial_{\tilde{z}} T_{ad})(z - Lz)$. Thus, in these domains, $\partial_z T_{ad} = (\tilde{L}_{th}\partial_{\tilde{z}} T_{ad})$.

So, what is $\tilde{L}_{th}$? Well, if we have a dimensional polytrope whose depth is $\tilde{L}_z = (e^{n_\rho/m} - 1)$, and we want 20 thermals to fit in that depth, then $\tilde{L}_{th} = \tilde{L}_z/20$, or since 20 is the nondimensional L_z, $\tilde{L}_{th} = ((e^{n_\rho/m} - 1))/L_z$.

The resulting stratification is just a more careful specification of where we started,

$$T_0 = 1 + \partial_z T_{ad}(z - L_{exp}) \tag{11}$$

$$\rho_0 = T^m, \tag{12}$$

with $\partial_z T_{ad} = (\tilde{L}_{th}\partial_{\tilde{z}} T_{ad})$ and $\tilde{L}_{th} = ((e^{n_\rho/m} - 1))/L_z$. These stratified terms enter the equations mostly in log form,

$$\begin{aligned} \partial_z \ln T_0 &= \frac{\partial_z T_{ad}}{T} \\ \partial_z \ln \rho_0 &= \frac{m\partial_z T_{ad}}{T} \end{aligned} \tag{13}$$

In the case of low stratification, $\tilde{L}_{th} \to 0$, so $\partial_z T_{ad} \to 0$ meanwhile $\rho_0 \to T_0 \to 1$, and our equations take on their boussinesq form.

REFERENCES

Lecoanet, D., Brown, B. P., Zweibel, E. G., et al. 2014, ApJ, 797, 94

Lecoanet, D., & Jeevanjee, N. 2018, arXiv e-prints, arXiv:1804.09326