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1. DESCRIPTION OF STRATIFIED THERMALS

In the thermal simulations, we have nine variables, although many of them are geometric and likely intrinsically linked. These variables are:

1. B, the integrated total entropy leading to buoyancy
2. V, the thermal volume
3. R, the radius of the thermal
4. r, the radius (from axis of symmetry) of the thermal's vortex torus.
5. R_z, the thermal's radius in the z-direction (we assume it is an oblate spheroid)
6. w, the thermal bulk vertical velocity
7. z, the height of the thermal
8. Γ, the circulation of the vortex ring
9. ρ, the local density at the height of the thermal.

From the simulations we've done so far, when viscous heating and detrainment are neglected, and the diffusivities are sufficiently low (Re is sufficiently high) such that diffusive terms are not dominant, the following equations seem to govern the thermal evolution (note that these equations are azimuthally averaged, so the quantities here are divided by 2π compared to their full "true" volume averaged quantities):

$$B = \int \rho s_1 dV = \text{const}, \tag{1}$$

$$R_z = R/A, \qquad \text{where } A \text{ is a constant aspect ratio }, \tag{2}$$

$$r = fR, \qquad \text{where } f \text{ is a constant fraction }, \tag{3}$$

$$V = (2/3)hR^2 = (2/3)R^3/A = V_0 R^3 = (V_0/f^3)r^3, \tag{4}$$

$$w = \frac{\partial z}{\partial t}, \tag{5}$$

$$\rho = [1 + (\nabla T)_{ad}(z - F^{-1})]^{m_{ad}} = T^{m_{ad}}, (\text{where } F = L_{thermal}/L_{box}) \tag{6}$$

$$\rho V w = \beta B t + M_0, \tag{7}$$

$$\frac{1}{2}\rho r^2 \Gamma = Bt + I_0, \tag{8}$$

$$\Gamma \sim \text{const}. \tag{9}$$

Note that all of these equations are azimuthal averages, I think. You'd probably have to multiply by 2π to retrieve the "full" values in a 3D sim.

2. SOLUTION FOR TEMPERATURE, HEIGHT, AND VELOCITY

One thing that we have here that we didn't have a month ago is the constant offsets in the impulse and momentum: I_0 and M_0. They make it so that plugging the two together is less straightforward that we had once thought, but it's still possible. From Eqn. 8, we solve for

$$r = \sqrt{2\frac{Bt + I_0}{\rho\Gamma}}. \tag{10}$$

Plugging this in to Eqn. 7, we get

$$\rho V w = \left(\frac{V_0}{f^3}\right)\rho r^3 w = \left(\frac{V_0}{f^3}\right)\rho w\left(2\frac{Bt+I_0}{\rho\Gamma}\right)^{3/2} = \beta Bt + M_0, \tag{11}$$

or, with $w = dz/dt$,

$$\rho^{-1/2}dz = \left(\frac{f^3\Gamma^{3/2}}{2^{3/2}V_0}\right)\frac{\beta Bt + M_0}{(Bt+I_0)^{3/2}}dt \tag{12}$$

But we know that $\rho = T^{m_{ad}}$, with T the temperature, and $dT = (\nabla T)_{ad}dz$, and we can define $\tau = (Bt + I_0)/\Gamma$, with $d\tau = (B/\Gamma)dt$ (these choice of constants ensure that τ is positive definite and monotonically increases). Substituting these in, we retrieve

$$T^{-m_{ad}/2}dT = \left(\frac{f^3\Gamma(\nabla T)_{ad}}{2^{3/2}V_0 B}\right)\frac{\beta\Gamma\tau + M_0 - \beta I_0}{\tau^{3/2}}d\tau. \tag{13}$$

Defining a constant $C \equiv [f^3\Gamma(\nabla T)_{ad}]/[2^{3/2}V_0 B]$, we can write this more simply,

$$T^{-m_{ad}/2}dT = C\left(\beta\Gamma\tau^{-1/2}d\tau + (M_0 - \beta I_0)\tau^{-3/2}d\tau\right). \tag{14}$$

These are all just power-law integrals, and so long as we avoid the special case where $m_{ad} = 2$ (the left-hand integral is a log then), we retrieve

$$\frac{1}{1 - m_{ad}/2}T^{1-m_{ad}/2}\bigg|_{T_0}^{T} = 2C\left(\beta\Gamma\tau^{1/2} - (M_0 - \beta I_0)\tau^{-1/2}\right)\bigg|_{\tau_0}^{\tau} \tag{15}$$

For simplicity of writing, I define $\alpha^{-1} = 1 - m_{ad}/2$ and

$$\xi(\tau) = \beta\Gamma\tau^{1/2} - (M_0 - \beta I_0)\tau^{-1/2}.$$

Plugging things in and rearranging, our final equation for the evolution of temperature with time is

$$T(t) = \left[\frac{2C}{\alpha}(\xi[\tau] - \xi[\tau_0]) + T_0^{1/\alpha}\right]^{\alpha}, \tag{16}$$

where the thermal velocity and height can be simply retrieved as just

$$w = (\nabla T)_{ad}^{-1}\frac{\partial T}{\partial t}, \qquad z = (\nabla T)_{ad}^{-1}(T - 1) + F^{-1} \tag{17}$$

where, for completeness, we solve out

$$\frac{\partial T}{\partial t} = C\frac{B}{\Gamma}(T^{m_{ad}/2})\left(\beta\Gamma\tau^{-1/2} + (M_0 - \beta I_0)\tau^{-3/2}\right). \tag{18}$$

In our simulations where $m_{ad} = 3/2$, we get the solution where $\alpha = 4$.

3. DENSITY SOLUTION

It's pretty straight-forward to find the solution for density now that we have the solution for temperature. Returning to Eqn. 14, we substitute $T = \rho^{1/m_{ad}}$ and $dT = (1/m_{ad})\rho^{m_{ad}^{-1}-1}d\rho$ to get

$$\frac{1}{m_{ad}}\rho^{-3/2+m_{ad}^{-1}}d\rho = C\left(\beta\Gamma\tau^{-1/2}d\tau + (M_0 - \beta I_0)\tau^{-3/2}d\tau\right). \tag{19}$$

In the limiting case where $m_{ad} = 2$, the LHS is ρ^{-1} and is a log integral. For cases where $m_{ad} < 2$, We integrate to get

$$(m_{ad}^{-1} - 0.5)^{-1}m_{ad}\rho^{m_{ad}^{-1}-1/2}\bigg|_{\rho_0}^{\rho} = 2C\xi(\tau)\bigg|_{\tau_0}^{\tau} \tag{20}$$

I'll now define $\chi^{-1} = m_{ad}^{-1} - 0.5$, rearrange, and find

$$\rho = \left[\frac{2C}{m_{ad}\chi}(\xi[\tau] - \xi[\tau_0]) + \rho_0^{1/\chi}\right]^{\chi}, \tag{21}$$

and $\chi = 6$ for our $m_{ad} = 3/2$ solutions.

4. LEADING ORDER EVOLUTION

What we eventually find from all of our derivations here is that, to leading order in τ,

$$\rho = \left(\frac{2C\beta\Gamma}{m_{ad}\chi}\right)^{\chi} \tau^{\chi/2} \tag{22}$$

$$T = \left(\frac{2C\beta\Gamma}{\alpha}\right)^{\alpha} \tau^{\alpha/2} \tag{23}$$

$$w = \frac{CB\beta}{(\nabla T)_{ad}} \left(\frac{2C}{\alpha}\right)^{\alpha-1} \tau^{\alpha/2-1}, \tag{24}$$

where $\alpha = 2$ and $\chi = 3$ for $m_{ad} = 3/2$.