BVPs to assist in convergence of IH atmospheres

I. THE FC BOUNDARY VALUE EQUATIONS

These equations are motivated by stellar structure models (see **??**). One important thing that stellar structure models get right is that they know how to conserve mass in a 1D BVP, so that aspect of these equations is basically stolen right from them. We're dealing with systems where we *know what the flux profile should look in the equilibrated state* a priori, so we can tap into that knowledge. We use a modified hydrostatic equlibrium which accounts for advective and viscous pressure support, and we also don't use a MLT model for how the temperature gradient evolves. Rather, we solve for that using the full fluxes.

So, that being said, our system of five equations (to parallel the system of five equations you have to solve in structure models) becomes

$$\begin{aligned}
\left\langle \frac{dM}{dz} \right\rangle &= \langle \rho \rangle \\
\left\langle \frac{dP}{dz} \right\rangle &= \left\langle -\rho g \hat{z} - \rho(\boldsymbol{u}\cdot\nabla)w - \nabla\cdot\left(\bar{\bar{\boldsymbol{\Pi}}}\right)_z \right\rangle \\
\left\langle \frac{d(\text{Fluxes}_z)}{dz} \right\rangle &= \langle \kappa(\text{IH}) \rangle \\
\langle T_z \rangle &= \left\langle \frac{dT}{dz} \right\rangle \\
\langle P \rangle &= \langle \rho T \rangle\,,
\end{aligned} \tag{1}$$

Here, $\langle A \rangle = \iiint A\, dx\, dy\, dt/(L_x \cdot L_y \cdot T)$ is the time- and horizontal- average of the quantity A, where T is the quantity of time over which we're time-averaging, and "angles" should commute with z-derivatives. All subscript z quantities mean that we are only examining the z-component of the full vector. In these equations, the viscous stress tensor is defined as

$$\Pi_{ij} \equiv -\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - -\frac{2}{3}\delta_{ij}\nabla\cdot(\boldsymbol{u}) \right). \tag{2}$$

So the important thing is to come up with a proper model for which fluxes carry the convection. In general, we don't want to neglect anything, but we also know that *potential energy flux is usually associated with a transient state* and restratification of the atmosphere. Thus, I impose that

$$\langle \rho w \phi \rangle = 0, \tag{3}$$

and we will ignore that flux from here on out. Under that assumption, we still have to worry about the conductive flux, enthalpy flux, viscous flux, and KE flux. Or, in equation form,

$$\langle \text{Fluxes}_z \rangle = \left\langle \rho w \left(c_V T + \frac{P}{\rho} \right) \right\rangle + \left\langle \rho w \frac{|\boldsymbol{u}|^2}{2} \right\rangle + \left\langle -\kappa \frac{dT}{dz} \right\rangle + \left\langle (\boldsymbol{u}\cdot\bar{\bar{\boldsymbol{\Pi}}})_z \right\rangle \tag{4}$$

...and that's the system! It's important for me to note at this point that the variables that I solve for in the BVPs are T, dT/dz, and ρ (and M). In other words, I solve for the *thermodynamic structure* of the atmosphere that is in flux equilibrium and is *no longer evolving* (no PE flux) given the current velocity field fed in from an IVP.

Once we plug the EOS into the other equations, we're left with four equations in our systems. The boundary conditions I will specify are:

1. left(M) = 0, right(M) = integrated mass of initial atmosphere. The BVP is not allowed to add mass to the problem. This is two boundary conditions.
2. right(T_1) = 0, fixed temperature top boundary, just like in my IVP.
3. left($dz(T_1)$) = 0, fixed flux bottom boundary, which specifies exactly the shape that the full flux profile will take (and the full amount of flux at any point in the system).

It's also important for me to say that these equations work well for constant values of κ and μ (in both time and space). I haven't thought about the problem for variable κ/μ, and while it will still work, it will be more complicated.

A. Implementing the Momentum equation

There's really two parts of the momentum equation: the part that we normally think of as hydrostatic balance, and the velocity parts. Let's look at the former, first. We have

$$\left\langle \frac{dP}{dz} \right\rangle + \langle \rho g \rangle = \left\langle \frac{d}{dz}(\rho T) \right\rangle + \langle \rho g \rangle = \langle T d_z \rho + \rho d_z T \rangle + \langle \rho g \rangle \tag{5}$$

Temperature and density are broken up such that

$$T \equiv T_0 + T_{IVP} + T_1; \qquad \rho \equiv \rho_0 + \rho_{IVP} + \rho_1. \tag{6}$$

Here, T_{IVP} is just the temperature fluctuations from the IVP, and $\rho_{IVP} = \rho_0(e^{\ln \rho_1} - 1)$ of the IVP. The "subscript 1" variables here are the fluctuations that will be solved for in the BVP. With that in mind, my previous HS balance equation is:

$$\langle T d_z \rho + \rho d_z T \rangle + \langle \rho g \rangle = \langle (T_0 + T_{IVP} + T_1) d_z(\rho_0 + \rho_{IVP} + \rho_1) + (\rho_0 + \rho_{IVP} + \rho_1) d_z(T_0 + T_{IVP} + T_1) + (\rho_0 + \rho_{IVP} + \rho_1) g \rangle \tag{7}$$

And at this point, we're too long for one line, so

$$\begin{aligned}
&\langle (T_0 + T_{IVP} + T_1) d_z(\rho_0 + \rho_{IVP} + \rho_1) \rangle = \langle (T_0 + T_{IVP}) d_z(\rho_0 + \rho_{IVP}) \rangle + \langle (T_0 + T_{IVP}) \rangle d_z(\rho_1) + T_1 \langle d_z(\rho_0 + \rho_{IVP}) \rangle + T_1 d_z(\rho_1) \\
&\langle (\rho_0 + \rho_{IVP} + \rho_1) d_z(T_0 + T_{IVP} + T_1) \rangle = \langle (\rho_0 + \rho_{IVP}) d_z(T_0 + T_{IVP}) \rangle + \langle (\rho_0 + \rho_{IVP}) \rangle d_z(T_1) + \rho_1 \langle d_z(T_0 + T_{IVP}) \rangle + \rho_1 d_z(T_1) \\
&\langle (\rho_0 + \rho_{IVP} + \rho_1) g \rangle = \langle (\rho_0 + \rho_{IVP}) \rangle g + \rho_1 g,
\end{aligned} \tag{8}$$

where in the RHS expressions I have taken ρ_1 and T_1 out of the angles, because they are definitionally vertical profiles with no time- or horizontal- variance (that's what it means for them to be the solution to the BVP). All of the yucky terms which have angles around them are terms that I should *directly find a time- and horizontal- average of* if I want to have precisely the right BVP.

The rest of the momentum equation is fairly easy:

$$\left\langle -\rho \boldsymbol{u} \cdot \nabla w - \nabla \cdot \left(\bar{\bar{\boldsymbol{\Pi}}}\right)_z \right\rangle = -\langle (\rho_0 + \rho_{IVP}) \boldsymbol{u} \cdot \nabla w \rangle - \rho_1 \langle \boldsymbol{u} \cdot \nabla w \rangle - \left\langle \nabla \cdot \left(\bar{\bar{\boldsymbol{\Pi}}}\right)_z \right\rangle. \tag{9}$$

Here, the stress tensor term depends only on velocity and κ, and since I am solving systems with constant κ, I don't need to worry about breaking it up any more. That means that I need the following profiles going into my BVP for the momentum equation:

1. `T0_full` = $\langle (T_0 + T_{IVP}) \rangle$
2. `T0_z_full` = $\langle d_z(T_0 + T_{IVP}) \rangle$ (honestly, this should just be dz(above))
3. `rho0_full` = $\langle (\rho_0 + \rho_{IVP}) \rangle$
4. `rho0_z_full` = $\langle d_z(\rho_0 + \rho_{IVP}) \rangle$ (honestly, this should just be dz(above))
5. `T_grad_rho` = $\langle (T_0 + T_{IVP}) d_z(\rho_0 + \rho_{IVP}) \rangle$
6. `rho_grad_T` = $\langle (\rho_0 + \rho_{IVP}) d_z(T_0 + T_{IVP}) \rangle$
7. `rho_uDotGradw` = $\langle (\rho_0 + \rho_{IVP}) \boldsymbol{u} \cdot \nabla w \rangle$
8. `uDotGradw` = $\langle \boldsymbol{u} \cdot \nabla w \rangle$
9. `visc_w` = $\left\langle \nabla \cdot \left(\bar{\bar{\boldsymbol{\Pi}}}\right)_z \right\rangle$

B. Energy Equation

So this equation is $\nabla \cdot (\text{fluxes}) = \kappa IH$. The RHS is already just a constant of the system (or a constant profile in time if we want to add that complexity later). The LHS needs some love. Let's examine each flux individually.

$$\langle \text{conductive flux} \rangle = \langle -\kappa d_z(T_0 + T_{IVP} + T_1) \rangle = -\kappa(\langle d_z(T_0 + T_{IVP}) \rangle + d_z T_1), \tag{10}$$

so...yeah, that one's super simple for the constant kappa case.

$$\langle \text{viscous flux} \rangle = \left\langle (\boldsymbol{u} \cdot \bar{\bar{\boldsymbol{\Pi}}})_z \right\rangle, \tag{11}$$

This is just a function of κ and $\boldsymbol{u}$, so once again...here, we have it easy.

$$\langle \text{KE flux} \rangle = \left\langle \rho w (\boldsymbol{u})^2/2 \right\rangle = \left\langle (\rho_0 + \rho_{IVP}) w (\boldsymbol{u})^2/2 \right\rangle + \rho_1 \left\langle w(\boldsymbol{u})^2/2 \right\rangle, \tag{12}$$

which is slightly more rough, but not bad. Then there's

$$\langle \text{enthalpy flux} \rangle = \langle \rho w T (C_v + 1) \rangle = (C_v{+}1) \left\{ \langle (\rho_0 + \rho_{IVP})(T_0 + T_{IVP}) w \rangle + \langle (\rho_0 + \rho_{IVP}) w \rangle T_1 + \rho_1 \langle w (T_0 + T_{IVP}) \rangle + \rho_1 T_1 \langle w \rangle \right\} \tag{13}$$

and we're ignoring PE flux, by choice.

So in the end, this energy equation requires the following *new* things that we didn't have from the momentum equation

1. `visc_flux` = $\left\langle (\boldsymbol{u} \cdot \bar{\bar{\boldsymbol{\Pi}}})_w \right\rangle$
2. `KE_flux_IVP` = $\left\langle (\rho_0 + \rho_{IVP}) w (\boldsymbol{u})^2/2 \right\rangle$
3. `w_vel_squared` = $\left\langle w(\boldsymbol{u})^2/2 \right\rangle$
4. `Enth_flux_IVP` = $\langle (C_v + 1)(\rho_0 + \rho_{IVP})(T_0 + T_{IVP}) w \rangle$
5. `rho_w_IVP` = $\langle (\rho_0 + \rho_{IVP}) w \rangle$
6. `T_w_IVP` = $\langle (T_0 + T_{IVP}) w \rangle$
7. `w_IVP` = $\langle w \rangle$

C. Implementation in Dedalus

With the substitutions from the above sections, my four equations that I implement in dedalus are

1. `dz(M1) - rho1 = 0`
2. `dz(T1) - T1_z = 0`
3. `dz(`$-\kappa$`T1_z + rho1 * (w_vel_squared + T_w_IVP * (Cv + 1)) + T1 * rho_w_IVP * (Cv + 1)) =`
   `-dz(`$-\kappa$`T0_z_full + visc_flux + KE_flux_IVP + Enth_flux_IVP + (Cv+1) * rho1 * T1 * w_IVP) +`κ`(IH)`
4. `T0_full*dz(rho1) + T1*rho0_z_full + rho0_full*dz(T1) + rho1 * T0_z_full + rho1*g + rho1 * uDotGradw`
   `= -T_grad_rho - T1*dz(rho1) - rho_grad_T - rho1*dz(T1) - rho0_full * g - rho_uDotGradw - visc_w`

The boundary conditions of this system are then

1. `left(M1) = 0`
2. `right(M1) = 0`
3. `left(T1_z) = 0`
4. `right(T1) = 0`

The last two of these conditions are just the standard thermal boundary conditions used in these simulations. The first two conditions ensure that no mass is added to the system. The structure of the dz(fluxes) = IH equation ensures that flux equilibrium is met throughout the atmosphere, and the dz(P) equation ensures that there are no $m = 0$ pressure imbalances in the atmosphere.

Sweetness.

II. STELLAR STRUCTURE MODELS

So in the spirit of Steve's stellar structures class (and from his notes found online, http://lasp.colorado.edu/~cranmer/ASTR_5700_2016/index.html) I am going to draw inspiration from stellar structure models to solve BVPs which make the thermal state of my solutions converge more rapidly.

Stellar structure models essentially have five equations:

$$\begin{aligned}
\frac{dM_r}{dr} &= 4\pi r^2 \rho \qquad \text{(mass conservation)} \\
\frac{dP}{dr} &= -\frac{GM_r}{r^2}\rho \qquad \text{(Hydrostatic balance)} \\
\frac{dL_r}{dr} &= 4\pi r^2 \rho \epsilon \qquad \text{(Conservation of energy)} \\
\frac{dT}{dr} &= \begin{cases} \left(\frac{dT}{dr}\right)_{\text{rad}} & \text{, if convectively stable} \\ \left(\frac{dT}{dr}\right)_{\text{ad}} - \Delta\nabla T & \text{, if convectively unstable} \end{cases} \qquad \text{(Basically where all of the model-dependent stuff comes in)} \\
P &= P(\rho, T, \mu) \qquad \text{(equation of state)}
\end{aligned} \tag{14}$$

where ϵ is the energy generation rate (erg / g / s), and μ is the mean atomic weight, or something of the sort.

Basically, you have to solve a boundary value problem in order to find out more about the problem. In general, in stellar structure models, there are technically six variables,

1. Position: $r = [0, R_*]$
2. Mass: $M_r = [0, M_*]$
3. Mass density: $\rho = [\rho_c, \rho_{photo}]$
4. Pressure: $P = [P_c, P_{photo}]$
5. Temperature: $T = [T_c, T_{eff}]$
6. Luminosity: $L_r = [0, L_*]$

Generally, in stellar structure models, we're interested in 6 things: $R_*, M_*, \rho_c, P_c, T_c, L_*$. That's six variables for five equations, so usually M_* is specified and then the rest are determined based on a boundary value problem.

...that's basically what we want to do in our problems.