Internally heated, stratified, compressible convection

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An abstract will go here eventually

I. INTRODUCTION

1. Importance of internal heating in natural convective systems
2. Shape of internal heating in something like Sun
3. Studies in Rayleigh-Benard
4. Goals of this paper:
   (a) Show how to set up internally heated polytropes, with stable layers (or without)
   (b) Show how the resultant stratification depends on combo of lower flux + internal heating profile choices
   (c) Show how stable layer affects surface power spectrum

II. EXPERIMENT

A. General system info and temperature profile

We choose a parameter, ϵ, which sets the initial scale of gravity,

$$g \equiv m_{ad} + 1 - \epsilon, \tag{1}$$

where $m_{ad} \equiv (\gamma - 1)^{-1}$, and $\gamma = c_P/c_V = 5/3$. The atmosphere has a constant internal heating term, such that the energy equation looks like

$$\frac{\partial}{\partial t}\left(\left[\rho\frac{|\boldsymbol{u}|^2}{2} + c_v T + \phi\right]\right) + \nabla\cdot(\boldsymbol{F}_{\text{conv}} + \boldsymbol{F}_{\text{cond}}) = \kappa H, \tag{2}$$

where H is an internal heating term. In the initial, static, conductive state, the conductive flux must balance the internal heating term,

$$\nabla\cdot(-\kappa\nabla T_c) = \kappa H, \tag{3}$$

A useful form of the initial temperature gradient in such a system is

$$\nabla T_c = -1 + H(L_z - z). \tag{4}$$

This initial conductive solution is *not* the adiabatic solution to this atmosphere,

$$\nabla T_{ad} = -\frac{g}{c_P} = -1 + \frac{\epsilon}{c_P}. \tag{5}$$

If we integrate the conductive solution, our initial temperature profile is

$$T_c(z) = (HL_z - 1)z - \frac{H}{2}z^2 + C$$

and if we want $T_c(L_z) = T_t$,

$$T_c(z) = -\frac{H}{2}z^2 - (1 - HL_z)z + \left(T_t - \frac{H}{2}L_z^2 + L_z\right). \tag{6}$$

So our initial temperature gradient is now quadratic.

The initial *superadiabatic temperature gradient* is an important quantity, as it determines whether or not the atmosphere is stable to convection. This gradient in these systems is

$$\nabla T_c - \nabla T_{ad} = H(L_z - z) - \frac{\epsilon}{c_P}. \tag{7}$$

When this quantity is zero, the atmosphere is carrying the maximum amount of flux that conductivity can carry. When this value is less than zero, the atmosphere requires convection to carry the flux. Solving for a generalized zero-point, we find that

$$z_{\text{cross}} \equiv L_z - \frac{\epsilon}{Hc_P}. \tag{8}$$

In order for the atmosphere to have any region that is unstable to convection, we require that $0 < \epsilon/(Hc_P) < L_z$. In the case that $\epsilon/(Hc_P) < L_z$, the whole atmosphere is unstable to convection. Further, the anticipated length scale of the convecting region of the atmosphere is the full atmosphere above z_{cross}, or

$$d_{\text{conv}} = L_z - z_{\text{cross}} = \frac{\epsilon}{Hc_P} \tag{9}$$

I think there are two natural control parameters of this system:

1. $f \equiv z_{\text{cross}}/L_z$, the fraction of the atmosphere that is stable ($f = 0$ means the full atmosphere convects, $f = 0.5$ means that the upper half convects, etc). This *should* be the parameter that determines the depth of the stable layer.

2. ϵ, the deviation of the initial temperature gradient (at the top of the atmosphere) from the adiabatic temperature gradient. This *should* be the parameter that determines the depth of the stable layer.

By specifying these two parameters, we retrieve the magnitude of the internal heating, $f = 1 - \epsilon/(Hc_P L_z)$

$$H = \frac{\epsilon}{L_z c_P (1-f)}. \tag{10}$$

B. Initial density stratification

The initial density profile is set by hydrostatic equilibrium,

$$\nabla \ln \rho_c = -\frac{g + \nabla T_c}{T_c}, \tag{11}$$

which we numerically integrate to find $\ln \rho_c$ and ρ_c. By the way, the reason we do this numerically is because, according to wolfram,

$$\int \frac{A + Bz}{C + Dz + Ez^2} = \frac{B}{2E} \ln\left(C + zD + Ez^2\right) - \frac{(BD - 2AE)}{E\sqrt{4CE - D^2}} \tan^{-1}\left(\frac{D + 2Ez}{\sqrt{4CE - D^2}}\right)$$

And for this,

$$A = -\left(g - 1 + HLz\right) \qquad B = H \qquad C = \left(T_t - \frac{H}{2}L_z^2 + L_z\right) \qquad D = (HL_z - 1) \qquad E = -\frac{H}{2}$$

which is to say that $E = -B/2$ and $A = -(D + g)$. Simplifying the density profile form a bit,

$$\ln \rho_c = -\ln(T_c) - \frac{-2D - 2A}{\sqrt{4CE - D^2}} \tan^{-1}\left(\frac{D + 2Ez}{\sqrt{4CE - D^2}}\right)$$

So that first term, the $\ln T_c$ term, cancels nicely. the other term is kind of a mess, and to see it more clearly, note that

$$A + D = -g \qquad 4CE - D^2 = -2H\left(T_t - \frac{H}{2}L_z^2 + L_z\right) - (H^2L_z^2 + 1 - 2HL_z) = -2(HT_t + 1),$$

and that's sort of problematic, and also imaginary once we take the square root of it. Note also that $D + 2Ez = HL_z - 1 - Hz = H(L_z - z) - 1$, so the total function is

$$\ln \rho_c = -\ln(T_c) - \frac{2g}{\sqrt{-2(HT_t+1)}} \tan^{-1}\left(\frac{H(L_z - z) - 1}{\sqrt{-2(HT_t+1)}}\right) + const$$

Acknowledging that we're dealing with nasty imaginary numbers, note that $\tan^{-1}(A\sqrt{-1}) = i\tanh^{-1}(A)$, so this function is really

$$\ln \rho_c = -\ln(T_c) - \frac{2g}{\sqrt{2(HT_t+1)}} \tanh^{-1}\left(\frac{H(L_z - z) - 1}{\sqrt{2(HT_t+1)}}\right) + C \tag{12}$$

To make things easier for myself, I will define $\chi \equiv \sqrt{2(HT_t+1)}$, and then I will note $\tanh^{-1}$ has a kinda cool formula,

$$\tanh^{-1}(A) = \frac{1}{2}\left(\ln(A+1) - \ln(1-A)\right) = \frac{1}{2}\ln\left(\frac{A+1}{1-A}\right) = \frac{1}{2}\ln\left(\frac{B+C}{C-B}\right),$$

where $A \equiv B/C$, so

$$\tanh^{-1}\left(\frac{\nabla T_c}{\chi}\right) = \frac{1}{2}\ln\left(\frac{\nabla T_c + \chi}{\chi - \nabla T_c}\right)$$

So our function looks like

$$\ln \rho_c = -\ln(T_c) - \frac{g}{\chi}\ln\left(\frac{\chi + \nabla T_c}{\chi - \nabla T_c}\right) + C \tag{13}$$

At this point, I'm going to *assume that* $\rho_t = 1$, *so* $\ln \rho_c(L_z) = 0$, and while this kills some of our generality, it makes life a bit better. also recall that we have specified $\nabla T_c = -1$ at the top of the atmosphere, so

$$C = \ln(T_t) + \frac{g}{\chi}\ln\left(\frac{\chi - 1}{\chi + 1}\right)$$

And in the end, our initial density profile is specified by

$$\ln \rho_c = -\ln\left(\frac{T_c}{T_t}\right) - \frac{g}{\chi}\ln\left(\frac{\nabla T_c + \chi}{\chi - \nabla T_c}\cdot\frac{\chi+1}{\chi-1}\right) \tag{14}$$

There are still three parameters in these systems that need to be figured out:

1. Density stratification
2. thermal diffusivity
3. viscous diffusvitiy

One of these parameter is free,

$$\mathrm{Pr} = \frac{\nu}{\chi}. \tag{15}$$

We're not going to mess with that.

There will be some type of Rayleigh number in these systems,

$$\mathrm{Ra} = \frac{\text{stuff that comes from the atmosphere}}{\nu\chi}, \tag{16}$$

but I need to think a little more and spend some time with the equations to figure out what this is.

And then as for the stratification....well, I need to figure out what n_ρ I'm specifying. The n_ρ of the corresponding adiabatic polytrope?

III. RESULTS & DISCUSSION