

# Internally heated, stratified, compressible convection

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An abstract will go here eventually

## I. INTRODUCTION

1. Importance of internal heating in natural convective systems
2. Shape of internal heating in something like Sun
3. Studies in Rayleigh-Benard
4. Goals of this paper:
  - (a) Show how to set up internally heated polytropes, with stable layers (or without)
  - (b) Show how the resultant stratification depends on combo of lower flux + internal heating profile choices
  - (c) Show how stable layer affects surface power spectrum

## II. EXPERIMENT

We study direct numerical simulations of an ideal gas whose equation of state is  $P = \rho T$  and whose adiabatic index is  $\gamma = 5/3$  by evolving the fully compressible Navier-Stokes equations,

$$\frac{\partial \ln \rho}{\partial t} + \nabla \cdot \mathbf{u} = -\mathbf{u} \cdot \nabla \ln \rho, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla T - T \nabla \ln \rho + \mathbf{g} - \nabla \cdot \bar{\bar{\Pi}}, \quad (2)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T + (\gamma - 1)T \nabla \cdot \mathbf{u} + \frac{1}{\rho c_V} \nabla \cdot (-\kappa \nabla T) = -(\bar{\bar{\Pi}} \cdot \nabla) \cdot \mathbf{u} + \kappa H, \quad (3)$$

with the viscous stress tensor given by

$$\Pi_{ij} \equiv -\mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right), \quad (4)$$

where  $\delta_{ij}$  is the Kronecker delta function. In Eq. 3,  $H$  specifies the magnitude of internal heating, and in this study we study  $H$  which is constant in space and time. We assume here that  $\kappa$  and  $\mu$ , the thermal conductivity and dynamic viscosity, are constant in space and time.

The initial atmosphere is constructed under the assumptions of hydrostatic equilibrium and thermal equilibrium in the presence of constant gravity,  $g = (\gamma - 1)^{-1} + 1 - \epsilon$ , where  $\epsilon$  is a control parameter which sets the superadiabaticity and is similar to the superadiabatic excess in polytropic atmospheres [1, 2]. An atmosphere which satisfies these initial conditions takes the form

$$T_0(z) = -\frac{H}{2}z^2 + (HL_z - 1)z + \left(1 - \frac{H}{2}L_z^2 + L_z\right), \quad P_0(z) = \left(\frac{\xi + \nabla T_0}{\xi - \nabla T_0} \cdot \frac{\xi + 1}{\xi - 1}\right)^{g/\xi}, \quad (5)$$

where  $L_z$  is the depth of the atmosphere,  $\xi \equiv \sqrt{1 + 2H}$ ,  $\nabla T_0 = \partial_z T_0(z) = -Hz + (H_L z - 1)$ , and the density profile is  $\rho_0(z) = P_0(z)/T_0(z)$ .

Stratified systems evolve towards a characteristic adiabatic profile. An adiabatically stratified atmosphere composed of an ideal gas in hydrostatic equilibrium has a temperature gradient specified by the gravity,  $\nabla T_{ad} = -\mathbf{g}/c_P$ , where  $c_P = \gamma/(\gamma - 1)$ . In these internally heated systems,

$$\nabla T_0 - \nabla T_{ad} = H(L_z - z) - \frac{\epsilon}{c_P}, \quad (6)$$

and there is a special point in the initial atmosphere,  $z_{\text{cross}} \equiv L_z - \epsilon/Hc_P$ , at which the temperature gradient is exactly the adiabatic temperature gradient. Above that point, the temperature gradient is superadiabatic and unstable to convection. Below that point, the temperature gradient is subadiabatic. Thus, the depth of the region that is convectively unstable is  $d_{\text{conv}} = L_z - z_{\text{cross}} = \epsilon/Hc_P$ . From this, we retrieve the magnitude of the internal heating term,

$$H \equiv \frac{\epsilon}{d_{\text{conv}}c_P}. \quad (7)$$

If  $L_z \leq d_{\text{conv}}$ , the whole atmosphere is unstable or marginally stable. If  $L_z > d_{\text{conv}}$ , there is a stable radiative zone beneath the convective zone. We specify the depth of this radiative zone through a new parameter,  $r \equiv L_z/d_{\text{conv}} - 1$ . We specify the depth of the convective zone by specifying the number of density scale heights,  $n_\rho$ , it spans. To achieve this we use an iterative, root finding algorithm find when  $f(L_z) = \rho_0(z_{\text{cross}})/\rho_0(L_z) - e^{n_\rho}$  is zero.

Diffusivities in the system are specified by choosing a value of the Rayleigh Number and Prandtl number. The thermal diffusivity,  $\chi = \kappa/\rho$  and viscous diffusivity,  $\nu = \mu/\rho$  are constrained by

$$\text{Ra}(z) = \frac{gd_{\text{conv}}^4 \left| \nabla s / c_P \right|}{\nu \chi} = \frac{gd_{\text{conv}}^4 \left| \frac{\nabla s}{c_P}(z) \right| \rho_0^2(z)}{\kappa \mu}, \quad \text{Pr} = \frac{\nu}{\chi}, \quad (8)$$

where

$$\frac{\nabla s}{c_P} = \frac{1}{\gamma} \nabla \ln T - \frac{\gamma - 1}{\gamma} \nabla \ln \rho. \quad (9)$$

We specify the value of Ra at the first moment of the  $\nabla T - \nabla T_{ad} = T \nabla s / c_P$ ,

$$L_{sm1} = \frac{\int_{z_{\text{cross}}}^{L_z} z T \nabla s dz}{\int_{z_{\text{cross}}}^{L_z} T \nabla s dz}. \quad (10)$$

In the limit of classic, polytropic atmospheres, this reduces to the midplane of the atmosphere, which is a commonly chosen location to specify the value of Ra [3]. We choose this location as it minimizes the variation of the critical value of Ra as other parameters ( $n_\rho$ ,  $r$ ,  $\epsilon$ ) change.

### A. Stability

We decompose thermodynamic variables such that  $\ln \rho = (\ln \rho)_0 + (\ln \rho)_1$  and  $T = T_0 + T_1$ . We assume that the background terms are constant with respect to time, and this allows us to subtract out the background thermal equilibrium and hydrostatic equilibrium. The *linearized* equations of motion are then

$$\begin{aligned} \frac{\partial (\ln \rho)_1}{\partial t} + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla (\ln \rho)_0 &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \nabla T_1 + T_1 \nabla (\ln \rho)_0 + T_0 \nabla (\ln \rho)_1 + \nabla \cdot \bar{\Pi} &= 0 \\ \frac{\partial T_1}{\partial t} + \mathbf{u} \cdot \nabla T_0 + (\gamma - 1) T_0 \nabla \cdot \mathbf{u} - \kappa e^{-(\ln \rho)_0} \nabla^2 T_1 &= 0 \end{aligned} \quad (11)$$

We assume that all fluctuations  $\{T_1, (\ln \rho)_1, \mathbf{u}\} = f(z)g(x, y)e^{i\omega t}$ , and we use Dedalus to solve eigenvalue problems to determine when  $\omega = 0$ . From this, we find Fig. 1.

## III. RESULTS & DISCUSSION

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  - [2] E. Graham, “Numerical simulation of two-dimensional compressible convection,” *J. Fluid Mech.* **70**, 689–703 (1975).
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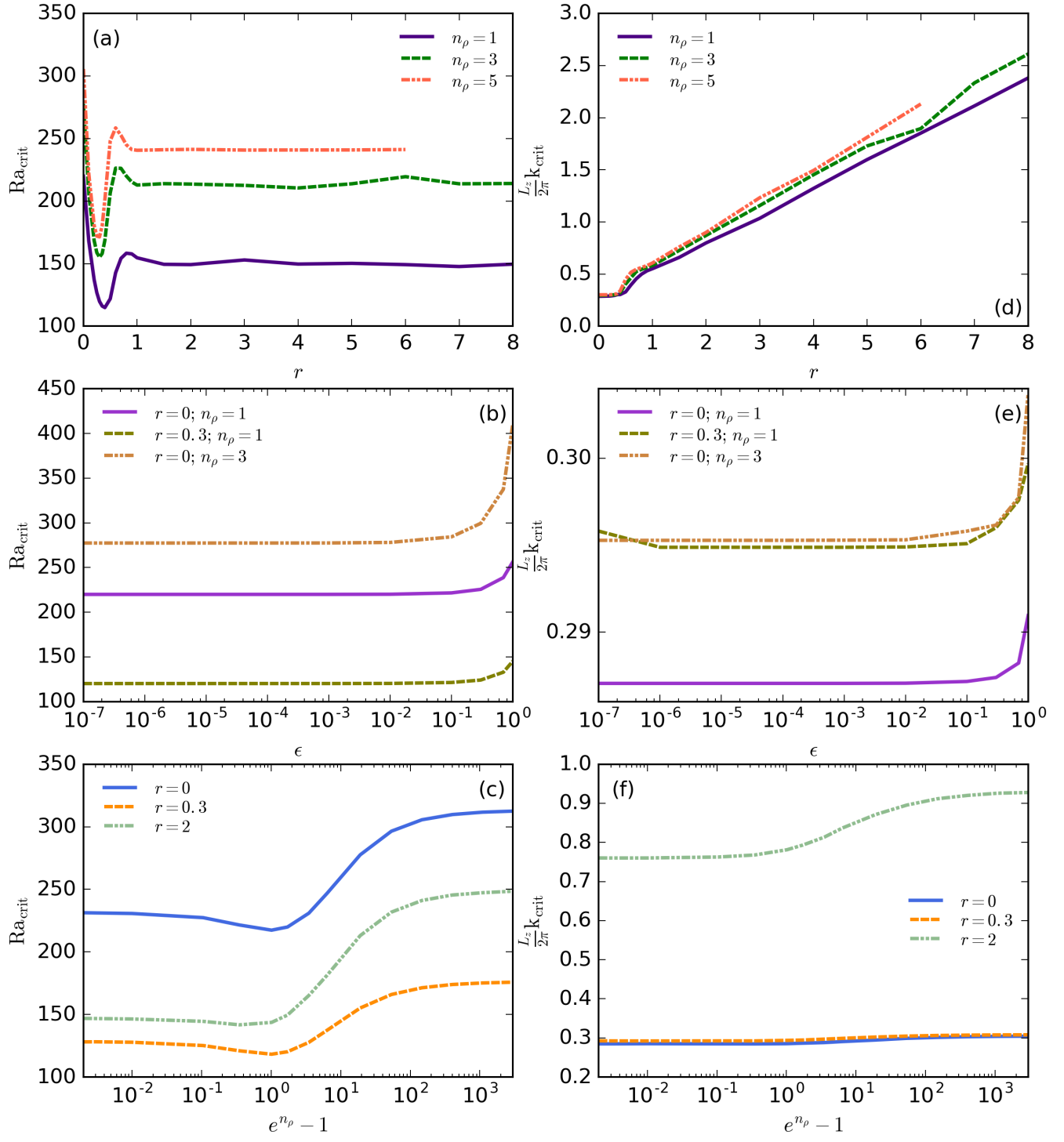


FIG. 1. The value of the critical Rayleigh number and normalized critical wavenumber as the control parameters of the problem are varied.