

FC equations with Internal Heating, non-dimensionalized

I. FC EQUATIONS

The fully compressible Navier-Stokes equations, which we will solve, are:

$$\begin{aligned} \frac{D \ln \rho}{Dt} + \nabla \cdot (\mathbf{u}) &= 0 \\ \rho \frac{D \mathbf{u}}{Dt} &= -\nabla P + \rho \mathbf{g} - \nabla \cdot \bar{\bar{\mathbf{\Pi}}} \\ \rho c_V \left(\frac{DT}{Dt} + (\gamma - 1) T \nabla \cdot \mathbf{u} \right) + \nabla \cdot (-\kappa \nabla T) &= -(\bar{\bar{\mathbf{\Pi}}} \cdot \nabla) \cdot \mathbf{u} + \kappa H \end{aligned} \quad (1)$$

Which is to say that we've added a term to the energy equation, κH , where κ and H are both constants, and so this is a source term that will internally heat the system.

A. Non-dimensionalization

In order to non-dimensionalize, we need to essentially define new variables in terms of our old variables, where we have removed an important scale factor from each which in some way describes a physical scale of our problem. Thus, I define:

$$\begin{aligned} \nabla &= \frac{1}{\bar{L}} \nabla^* \\ t &= \bar{t} t^* \\ T &= \bar{T} T^* \\ u &= \bar{u} u^* = \frac{\bar{L}}{\bar{t}} u^* \end{aligned} \quad (2)$$

Here, variables with over-bars are the dimension-full, characteristic scales of my problem, and starred variables are my non-dimensional, time-evolving variables. Thus far, I have only non-dimensionalized one of my thermodynamic variables (temperature), and I will need to do another one later, but I won't worry about that for now.

B. Non-dimensionalizing the energy equation

In the spirit of the last section, I will now replace all of my dimension-filled variables with dimensionless variables. I have also replaced the viscous heating term with a term that has equivalent units ($\mu \nabla^2 \mathbf{u}^2$), to make the nondimensionalizing clearer. For reasons that will become clear, I will leave ρ filled with dimensions:

$$\rho c_V \frac{\bar{T}}{\bar{t}} \left(\frac{DT^*}{Dt^*} + (\gamma - 1) T^* \nabla^* \cdot \mathbf{u}^* \right) - \frac{\kappa \bar{T}}{\bar{L}^2} (\nabla^*)^2 T^* = -\frac{\mu}{\bar{t}^2} (\nabla^*)^2 (u^*)^2 + \kappa H \quad (3)$$

Multiplying the full equation by $\bar{T}/(\bar{t} \rho c_V)$, and then noting that $\kappa/\rho = \chi$ and $\mu/\rho = \nu$ (the corresponding diffusivities), we get

$$\frac{DT^*}{Dt^*} + (\gamma - 1) T^* \nabla^* \cdot \mathbf{u}^* - \frac{\chi \bar{t}}{c_V \bar{L}^2} (\nabla^*)^2 T^* = -\frac{\nu}{c_V \bar{T} \bar{t}} (\nabla^*)^2 (u^*)^2 + \frac{\chi \bar{t}}{c_V \bar{T}} H \quad (4)$$

At this point, I think it's logical to make a standard choice for \bar{t} ,

$$\bar{t} = c_V \frac{\bar{L}^2}{\chi},$$

which is just the thermal diffusion timescale over the characteristic length scale of the problem (which has yet to be specified). With that in mind, and plugging in \bar{t} throughout, we get

$$\frac{DT^*}{Dt^*} + (\gamma - 1) T^* \nabla^* \cdot \mathbf{u}^* - (\nabla^*)^2 T^* = -\frac{\nu \chi}{c_V^2 \bar{T} \bar{L}^2} (\nabla^*)^2 (u^*)^2 + \frac{\bar{L}^2}{\bar{T}} H. \quad (5)$$

At this point, the next step is not entirely obvious to me (I know it must be done to the H term, though). I think there's either two ways to go:

1. The classic choice in Rayleigh-Benard: Non-dimensionalize the internal heating term so that it is 1.
2. A more appropriate one for us (maybe?): non-dimensionalize the internal heating term so that it is ϵ , as this will reflect the size of thermo fluctuations that are produced. I'll try that.

So with choice number 2,

$$\frac{\bar{L}^2}{\bar{T}} H = \epsilon,$$

and

$$\bar{T} = \bar{L}^2 \frac{H}{\epsilon}$$

Then, plugging that in, and dropping the stars because they're annoying, we're left with a dimensionless equation that looks like

$$\boxed{\frac{DT}{Dt} + (\gamma - 1)T \nabla \cdot \mathbf{u} - \nabla^2 T = -\epsilon \frac{\nu \chi}{c_V^2 \bar{L}^4 H} \nabla^2 u^2 + \epsilon.} \quad (6)$$

Two interesting things to note here:

1. It's clearly visible what the scale of the internal heating is, and how it feeds into the total temperature profile (this is some order ϵ deviation from the adiabatic).
2. The term in front of viscous heating is like Ra^{-1} , or something similar, which means that the viscous heating term is shrinking linearly with the Rayleigh number. This will happen without the internal heating, too, it just is mega clear, here, and I've never seen it before, neat.
3. This equation is really quite clean, and from it we have constrained two of our problem's scales: *time* and *temperature*. There are still two remaining free scales that must be specified: *length* and *another thermodynamic variable*, but both of these are bound to the choices that we've made here.

1. Sanity check: entropy equation

As a sanity check, I want to do the same thing as I did above, but with the entropy form of the energy equation from [1]. This equation takes the form:

$$\rho T \frac{DS}{Dt} = -\nabla \cdot \left(-\kappa \nabla \frac{S}{c_P} \right) - \mu \nabla^2 u^2 + \kappa H \quad (7)$$

where here, $\kappa = \chi_s \rho T$.

C. Getting to a linearized momentum equation

OK, so in order to do anything meaningful with the momentum equation (or at least, anything that is both easy and meaningful), we need to change its form. The first step in this process is to decompose all thermodynamic variables into background and fluctuating components, e.g.

$$\begin{aligned} \rho &= \rho_0 + \rho_1 \\ T &= T_0 + T_1 \\ P &= P_0 + P_1 \end{aligned} \quad (8)$$

Next, I will make the assumption that my background (0) state is in hydrostatic equilibrium, such that $-\nabla P_0 - \rho g \hat{z} = 0$, and in making that assumption I will remove it from the momentum equation. This leaves me with

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P_1 - \rho_1 g \hat{z} - \nabla \cdot \bar{\Pi}. \quad (9)$$

Next, I divide by ρ , and I put the stress tensor term into a more familiar form $-\nabla \cdot \bar{\mathbf{\Pi}} \sim -\mu \nabla^2 \mathbf{u}$, so that I can more clearly see its units and non-dimensionalize. I also note that $\nu = \mu/\rho$. Thus, my momentum equation becomes

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P_1}{\rho} + \frac{\rho_1}{\rho} \mathbf{g} - \nu \nabla^2 \mathbf{u}. \quad (10)$$

At this point, aside from the sort of contrived form of the viscous term, this equation is still general to perturbations of any size. But...now I'm going to make an assumption. All thermodynamic perturbations are small compared to the background ($\rho_1 \ll \rho_0$, and so on, such that $\rho = \rho_0 + \rho_1 \approx \rho_0$).

Under this approximation, we must examine the equation of state,

$$\frac{\nabla S}{c_P} = \frac{1}{\gamma} \nabla \ln(P_0 + P_1) - \nabla \ln(\rho_0 + \rho_1). \quad (11)$$

To make this prettier, I will do a couple of things. First, note that

$$\ln(A + B) = \ln\left(A \left[1 + \frac{B}{A}\right]\right) = \ln(A) + \ln\left(1 + \frac{B}{A}\right).$$

Thus, we write the equation of state (after trivially integrating away the dels):

$$\frac{S_0}{c_P} + \frac{S_1}{c_P} = \frac{1}{\gamma} \ln(P_0) + \frac{1}{\gamma} \ln\left(1 + \frac{P_1}{P_0}\right) - \ln(\rho_0) - \ln\left(1 + \frac{\rho_1}{\rho_0}\right). \quad (12)$$

By definition, the “0” terms on both sides of the equation are equal to one another, so they drop out. Then, I use the (Taylor expansion) relation that

$$\ln(1 + x) \approx x$$

for $x \ll 1$, and I find the linearized equation of state,

$$\boxed{\frac{S_1}{c_P} = \frac{1}{\gamma} \frac{P_1}{P_0} - \frac{\rho_1}{\rho_0}}. \quad (13)$$

Now this term can go into the $\rho_1/\rho_0 \mathbf{g} \approx \rho_1/\rho_0 \mathbf{g}$ term of the momentum equation, and I find

$$\frac{\rho_1}{\rho_0} \mathbf{g} = \frac{1}{\gamma} \frac{P_1}{P_0} \mathbf{g} - \frac{S_1}{c_P} \mathbf{g}.$$

Now, I play math tricks, and note that:

$$\frac{1}{\gamma} \frac{P_1}{P_0} \frac{\rho_0}{\rho_0} \mathbf{g} = \frac{1}{\gamma} \frac{P_1}{P_0} \nabla \ln P_0 = \frac{P_1}{\rho_0} \left(\frac{\nabla S_0}{c_P} + \nabla \ln \rho_0 \right)$$

where I've used the hydrostatic equilibrium of the initial conditions to cancel terms ($\nabla \ln P_0 = \rho_0 \mathbf{g}/P_0$), and then I've used the initial conditions and their entropy EOS to do the next term ($\nabla \ln P_0/\gamma = \nabla S_0/c_P + \nabla \ln \rho_0$). Putting this into the momentum equation,

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla P_1}{\rho_0} + \frac{P_1}{\rho_0} \nabla \ln \rho_0 + \frac{P_1}{\rho_0} \frac{\nabla S_0}{c_P} - \frac{S_1}{c_P} \mathbf{g} - \nu \nabla^2 \mathbf{u} \quad (14)$$

The first two terms on the RHS condense, because

$$\nabla \left(\frac{P_1}{\rho_0} \right) = \frac{\nabla P_1}{\rho_0} - \frac{P_1}{\rho_0} \nabla \ln \rho_0$$

and then I define $\varpi \equiv P_1/\rho_0$, such that the momentum equation is

$$\boxed{\frac{D\mathbf{u}}{Dt} = -\nabla \varpi + \varpi \frac{\nabla S_0}{c_P} - \frac{S_1}{c_P} \mathbf{g} - \nu \nabla^2 \mathbf{u}}. \quad (15)$$

This form of the equation is ripe to be non-dimensionalized, but it is *not* the full equation, as the viscous term has been butchered so that it's dimensionality is clear, and we have assumed that the fluctuations are small.