Internally heated, stratified, compressible convection

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An abstract will go here eventually

I. INTRODUCTION

1. Importance of internal heating in natural convective systems
2. Shape of internal heating in something like Sun
3. Studies in Rayleigh-Benard
4. Goals of this paper:
   (a) Show how to set up internally heated polytropes, with stable layers (or without)
   (b) Show how the resultant stratification depends on combo of lower flux + internal heating profile choices
   (c) Show how stable layer affects surface power spectrum

II. EXPERIMENT

A. General system info and temperature profile

We choose a parameter, ϵ, which sets the initial scale of gravity,

$$g \equiv m_{ad} + 1 - \epsilon, \tag{1}$$

where $m_{ad} \equiv (\gamma - 1)^{-1}$, and $\gamma = c_P/c_V = 5/3$. The atmosphere has a constant internal heating term, such that the energy equation looks like

$$\frac{\partial}{\partial t}\left(\left[\rho\frac{|\boldsymbol{u}|^2}{2} + c_v T + \phi\right]\right) + \nabla\cdot(\boldsymbol{F}_{\text{conv}} + \boldsymbol{F}_{\text{cond}}) = \kappa H, \tag{2}$$

where H is an internal heating term. In the initial, static, conductive state, the conductive flux must balance the internal heating term,

$$\nabla\cdot(-\kappa\nabla T_c) = \kappa H, \tag{3}$$

A useful form of the initial temperature gradient in such a system is

$$\nabla T_c = -1 + H(L_z - z). \tag{4}$$

This initial conductive solution is *not* the adiabatic solution to this atmosphere,

$$\nabla T_{ad} = -\frac{g}{c_P} = -1 + \frac{\epsilon}{c_P}. \tag{5}$$

If we integrate the conductive solution, our initial temperature profile is

$$T_c(z) = (HL_z - 1)z - \frac{H}{2}z^2 + C$$

and if we want $T_c(L_z) = T_t$,

$$T_c(z) = -\frac{H}{2}z^2 - (1 - HL_z)z + \left(T_t - \frac{H}{2}L_z^2 + L_z\right). \tag{6}$$

So our initial temperature gradient is now quadratic.

The initial *superadiabatic temperature gradient* is an important quantity, as it determines whether or not the atmosphere is stable to convection. This gradient in these systems is

$$\nabla T_c - \nabla T_{ad} = H(L_z - z) - \frac{\epsilon}{c_P}. \tag{7}$$

When this quantity is zero, the atmosphere is carrying the maximum amount of flux that conductivity can carry. When this value is less than zero, the atmosphere requires convection to carry the flux. Solving for a generalized zero-point, we find that

$$z_{\text{cross}} \equiv L_z - \frac{\epsilon}{Hc_P}. \tag{8}$$

In order for the atmosphere to have any region that is unstable to convection, we require that $0 < \epsilon/(Hc_P) < L_z$. In the case that $\epsilon/(Hc_P) < L_z$, the whole atmosphere is unstable to convection. Further, the anticipated length scale of the convecting region of the atmosphere is the full atmosphere above z_{cross}, or

$$d_{\text{conv}} = L_z - z_{\text{cross}} = \frac{\epsilon}{Hc_P} \tag{9}$$

I think there are two natural control parameters of this system:

1. $f \equiv z_{\text{cross}}/L_z$, the fraction of the atmosphere that is stable ($f = 0$ means the full atmosphere convects, $f = 0.5$ means that the upper half convects, etc). This *should* be the parameter that determines the depth of the stable layer.

2. ϵ, the deviation of the initial temperature gradient (at the top of the atmosphere) from the adiabatic temperature gradient. This *should* be the parameter that determines the depth of the stable layer.

By specifying these two parameters, we retrieve the magnitude of the internal heating, $f = 1 - \epsilon/(Hc_PL_z)$

$$H = \frac{\epsilon}{L_z c_P(1-f)}. \tag{10}$$

B. Initial density stratification

The initial density profile is set by hydrostatic equilibrium,

$$\nabla \ln \rho_c = -\frac{g + \nabla T_c}{T_c} = -\nabla \ln T_c - \frac{g}{T_c}. \tag{11}$$

If we integrate over this function, the first term is super easy, and we end up having

$$\ln \rho_c = -\ln T_c - \int \frac{g}{T_c} dz + \xi, \tag{12}$$

where ξ is a constant. The next integral is harder. In general, this integral has the form

$$\int \frac{1}{Az^2 + Bz + C} dz = \frac{2}{\sqrt{4AC - B^2}} \tan^{-1}\left(\frac{2Az + B}{\sqrt{4AC - B}}\right). \tag{13}$$

For our specific implementation, we get our constants from the temperature profile,

$$A = -H/2 \qquad B = (HL_z - 1) \qquad C = \left(T_t - \frac{H}{2}L_z^2 + L_z\right),$$

and note that $2Az + B = -Hz + (HL_z - 1) = \nabla T_c$. Note also that $4AC - B^2 = -2H(T_t - (H/2)L_z^2 + L_z) - (H^2L_z^2 - 2HL_z + 1) = -2HT_t - 1$, so $\sqrt{4AC - B^2} = i\chi$, where $i = \sqrt{-1}$ and $\chi = \sqrt{1 + 2HT_t}$. With these definitions in hand, we find that

$$\ln \rho_c = -\ln T_c + \frac{2gi}{\chi} \tan^{-1}\left(-i\frac{\nabla T_c}{\chi}\right) \tag{14}$$

Now it's important to know that there's a neat identity: $\tan^{-1}(-iA) = -i\tanh^{-1}(A)$, and plugging this in we get

$$\ln\rho_c = -\ln T_c + \frac{2g}{\chi}\tanh^{-1}\left(\frac{\nabla T_c}{\chi}\right). \tag{15}$$

At this point, there's another useful identity: $\tanh^{-1}(A) = (1/2)\ln([1+A]/[1-A])$, so the density profile is

$$\ln\rho_c = -\ln T_c + \frac{g}{\chi}\ln\left(\frac{\chi+\nabla T_c}{\chi-\nabla T_c}\right) + \xi. \tag{16}$$

In general, this profile is valid for any value of ρ_t, such that

$$\xi = \ln\rho_t + \ln T_t - \frac{g}{\chi}\ln\left(\frac{\chi-1}{\chi+1}\right), \tag{17}$$

where $\nabla T_c(L_z) = -1$ has already been previously specified. This means that the general, full form of the log density is

$$\ln\left(\frac{\rho_c}{\rho_t}\right) = -\ln\left(\frac{T_c}{T_t}\right) + \frac{g}{\chi}\ln\left(\frac{\chi+\nabla T_c}{\chi-\nabla T_c}\cdot\frac{\chi+1}{\chi-1}\right) \tag{18}$$

1. *Specifying the depth of the atmosphere*

I still want n_ρ to be the parameter that specifies the depth of the parameter, and I specifically want it to specify the number of density scale heights in the part of the atmosphere that carries a superadiabatic flux.

To get started, I need to consider the value of $\ln\rho_c$ at the bottom of the atmosphere,

$$\ln\rho_c(z=0) = n_\rho = -\ln\left(\frac{T_c(z=0)}{T_t}\right) + \frac{g}{\chi}\ln\left(\frac{(\chi+\nabla T_c(z=0))(\chi+1)}{(\chi-\nabla T_c(z=0))(\chi-1)}\right)$$

And we have a problem here, right? The stuff on the RHS is a complex function of ϵ and L_z, and we're trying to specify L_z using n_ρ, not the other way around. I'm going to consistently move all of the terms from the RHS to the LHS, then non-dimensionalize and assume $T_t = \rho_t = 1$, making the function

$$f(L_z) = n_\rho + \ln\left(T_t - \frac{H}{2}L_z^2 + L_z\right) - \frac{g}{\chi}\ln\left(\frac{\chi^2-1+HL_z(\chi+1)}{\chi^2-1-HL_z(\chi-1)}\right). \tag{19}$$

When this function is zero, L_z appropriately captures the right number of density scale heights.

Since we have set up all of our non-dimensionalizations at the *top* of the atmosphere, and we are building atmospheres which have CZs above RZs, we can actually really simply specify the density stratification of the CZ. The CZ is an atmosphere with $f = 0$ whose $z = 0$ starts at z_{cross}. If we set $n_\rho = n_{\rho,cz}$ and set $f = 0$, and then find the appropriate L_z from $f(L_z)$, that will give us L_{z-cz}. And since that is a *depth* from the top of the atmosphere, we can use that depth to determine how big the rest of the atmosphere should be for our "true" value of f. That is,

$$L_z = \frac{L_{z-cz}}{1-f}. \tag{20}$$

This way, by adding the RZ, we're actually extending the atmosphere captured down into the RZ, rather than gobbling up part of the CZ with the RZ. In order to find L_{z-cz}, a numerical root-finding algorithm is the logical choice.

C. Musings

There are still three parameters in these systems that need to be figured out:

1. thermal diffusivity
2. viscous diffusvitiy

One of these parameter is free,

$$\mathrm{Pr} = \frac{\nu}{\chi}. \tag{21}$$

We're not going to mess with that.

There will be some type of Rayleigh number in these systems,

$$\mathrm{Ra} = \frac{\text{stuff that comes from the atmosphere}}{\nu\chi}, \tag{22}$$

but I need to think a little more and spend some time with the equations to figure out what this is.

And then as for the stratification....well, I need to figure out what n_ρ I'm specifying. The n_ρ of the corresponding adiabatic polytrope?

III. RESULTS & DISCUSSION