FC equations with Internal Heating, non-dimensionalized

I. FC EQUATIONS

The fully compressible Navier-Stokes equations, which we will solve, are:

$$
\begin{aligned}
\frac{D\ln\rho}{Dt} + \nabla\cdot(\boldsymbol{u}) &= 0 \\
\rho\frac{D\boldsymbol{u}}{Dt} &= -\nabla P + \rho\boldsymbol{g} - \nabla\cdot\bar{\bar{\boldsymbol{\Pi}}} \\
\rho c_V\left(\frac{DT}{Dt} + (\gamma-1)T\nabla\cdot\boldsymbol{u}\right) + \nabla\cdot(-\kappa\nabla T) &= -(\bar{\bar{\boldsymbol{\Pi}}}\cdot\nabla)\cdot\boldsymbol{u} + \kappa H
\end{aligned}
\tag{1}
$$

Which is to say that we've added a term to the energy equation, κH, where κ and H are both constants, and so this is a source term that will internally heat the system.

A. Non-dimensionalization

In order to non-dimensionalize, we need to essentially define new variables in terms of our old variables, where we have removed an important scale factor from each which in some way describes a physical scale of our problem. Thus, I define:

$$
\begin{aligned}
\nabla &= \frac{1}{\bar{L}}\nabla^* \\
t &= \bar{t}t^* \\
T &= \bar{T}T^* \\
u &= \bar{u}u^* = \frac{\bar{L}}{\bar{t}}u^*
\end{aligned}
\tag{2}
$$

Here, variables with over-bars are the dimension-full, characteristic scales of my problem, and starred variables are my non-dimensional, time-evolving variables. Thus far, I have only non-dimensionalized one of my thermodynamic variables (temperature), and I will need to do another one later, but I won't worry about that for now.

B. Non-dimensionalizing the energy equation

1. *New version 10/6*

OK, so I'm going to approach the energy equation in a similar manner here as we generally approach the momentum equation to get a linear form of it. In order to do so, I will break up $T = T_0 + T_1$, and I will assume that *my initial temperature profile is in thermal equilibrium, and it is also invariant over time*, such that

$$
-\kappa\nabla^2 T_0 - \nabla\kappa\cdot\nabla T_0 = \kappa H \qquad \frac{\partial T_0}{\partial t} = 0. \tag{3}
$$

If I assume that $\kappa \neq f(\boldsymbol{x})$, such that it is spatially invariant, then

$$
-\kappa\nabla^2 T_0 = \kappa H
$$

Why is this important? Because under these assumptions, and after dividing through by ρc_V, the energy equation takes the form

$$
\frac{\partial T_1}{\partial t} + \boldsymbol{u}\cdot\nabla(T_0+T_1) + (\gamma-1)(T_0+T_1)\nabla\cdot(\boldsymbol{u}) - \frac{\kappa}{\rho c_V}\nabla^2 T_1 = -\frac{1}{\rho c_V}(\bar{\bar{\boldsymbol{\Pi}}}\cdot\nabla)\cdot\boldsymbol{u} \tag{4}
$$

There are a couple of important things of note, here:

1. The internal heating term is now gone from the equation. Presumably this will make the non-dimensionalization more clear, here.

2. Each term is now linear (or non-linear) with respect to fluctuations in the system. So it's more clear that when we non-dimensionalize T, we're taking $\bar{T}$ out of T_1, not $T = T_0 + T_1$, which will make $\bar{T}$ a vastly different scale based on the choice of ϵ.

Note also that the above equation, Eqn. 4, is also the *complete* equation for constant κ. To make our lives a bit nicer, let's simplify a little. Since

$$\kappa = \chi\rho, \qquad \text{and} \qquad \bar{\bar{\mathbf{\Pi}}} = -\mu\bar{\bar{\boldsymbol{\sigma}}},\ \mu = \nu\rho$$

we can write

$$\boxed{\frac{\partial T_1}{\partial t} + \boldsymbol{u}\cdot\nabla(T_0 + T_1) + (\gamma - 1)(T_0 + T_1)\nabla\cdot(\boldsymbol{u}) - \frac{\chi}{c_V}\nabla^2 T_1 = \frac{\nu}{c_V}(\bar{\bar{\boldsymbol{\sigma}}}\cdot\nabla)\cdot\boldsymbol{u}}. \tag{5}$$

As Ben just pointed out to me, this is now a system that we can do an onset solve on! That's a huge insight. This system responds to perturbations around the mean state, and it's ONSET SOLVABLE.

This is big.

2. Brute Force

In the spirit of the last section, I will now replace all of my dimension-filled variables with dimensionless variables. I have also replaced the viscous heating term with a term that has equivalent units ($\mu\nabla^2\boldsymbol{u}^2$), to make the nondimensionalizing clearer. For reasons that will become clear, I will leave ρ filled with dimensions:

$$\rho c_v \frac{\bar{T}}{\bar{t}}\left(\frac{DT^*}{Dt^*} + (\gamma - 1)T^*\nabla^*\cdot\boldsymbol{u}^*\right) - \frac{\kappa\bar{T}}{\bar{L}^2}(\nabla^*)^2 T^* = -\frac{\mu}{\bar{t}^2}(\nabla^*)^2(u^*)^2 + \kappa H \tag{6}$$

Multiplying the full equation by $\bar{T}/(\bar{t}\rho c_V)$, and then noting that $\kappa/\rho = \chi$ and $\mu/\rho = \nu$ (the corresponding diffusivities), we get

$$\frac{DT^*}{Dt^*} + (\gamma - 1)T^*\nabla\cdot\boldsymbol{u}^* - \frac{\chi\bar{t}}{c_V\bar{L}^2}(\nabla^*)^2 T^* = -\frac{\nu}{c_V\bar{T}\bar{t}}(\nabla^*)^2(u^*)^2 + \frac{\chi\bar{t}}{c_V\bar{T}}H \tag{7}$$

At this point, I think it's logical to make a standard choice for $\bar{t}$,

$$\bar{t} = c_V\frac{\bar{L}^2}{\chi},$$

which is just the thermal diffusion timescale over the characteristic length scale of the problem (which has yet to be specified). With that in mind, and plugging in $\bar{t}$ throughout, we get

$$\frac{DT^*}{Dt^*} + (\gamma - 1)T^*\nabla^*\cdot\boldsymbol{u}^* - (\nabla^*)^2 T^* = -\frac{\nu\chi}{c_V^2\bar{T}\bar{L}^2}(\nabla^*)^2(u^*)^2 + \frac{\bar{L}^2}{\bar{T}}H. \tag{8}$$

At this point, the next step is not entirely obvious to me (I know it must be done to the H term, though). I think there's either two ways to go:

1. The classic choice in Rayleigh-Benard: Non-dimensionalize the internal heating term so that it is 1.

2. A more appropriate one for us (maybe?): non-dimensionalize the internal heating term so that it is ϵ, as this will reflect the size of thermo fluctuations that are produced. I'll try that.

So with choice number 2,

$$\frac{\bar{L}^2}{\bar{T}}H = \epsilon,$$

and

$$\bar{T} = \bar{L}^2 \frac{H}{\epsilon}$$

Then, plugging that in, and dropping the stars because they're annoying, we're left with a dimensionless equation that looks like

$$\boxed{\frac{DT}{Dt} + (\gamma - 1)T\nabla \cdot \boldsymbol{u} - \nabla^2 T = -\epsilon \frac{\nu\chi}{c_V^2 \bar{L}^4 H}\nabla^2 u^2 + \epsilon}. \tag{9}$$

Two interesting things to note here:

1. It's clearly visible what the scale of the internal heating is, and how it feeds into the total temperature profile (this is some order ϵ deviation from the adiabatic).
2. The term in front of viscous heating is like Ra^{-1}, or something similar, which means that the viscous heating term is shrinking linearly with the Rayleigh number. This will happen without the internal heating, too, it just is mega clear, here, and I've never seen it before, neat.
3. This equation is really quite clean, and from it we have constrained two of our problem's scales: *time* and *temperature*. There are still two remaining free scales that must be specified: *length* and *another thermodynamic variable*, but both of these are bound to the choices that we've made here.

3. *Sanity check: entropy equation*

As a sanity check, I want to do the same thing as I did above, but with the entropy form of the energy equation from [1]. This equation takes the form:

$$\rho T \frac{DS}{Dt} = -\nabla \cdot (-\kappa \nabla \frac{S}{c_P}) - \mu \nabla^2 u^2 + \kappa H \tag{10}$$

where here, $\kappa = \chi_s \rho T$. Note that $\mu = \rho\nu$. The alternative form of this equation is one in which the diffusion is $-\kappa\nabla T$, where $\kappa = \chi_T \rho$.

If we assume constant kappa/mu, then divide by ρT, we get

$$\frac{DS}{Dt} = \chi_S \nabla^2 \frac{S}{c_P} - \frac{\nu}{T}\nabla^2 u^2 + \chi_s H \tag{11}$$

Doing the thing where we pull dimensions out of everything,

$$\frac{\bar{S}}{\bar{t}}\frac{DS^*}{Dt^*} = \chi_S \frac{\bar{S}}{c_P \bar{L}^2}(\nabla^*)^2 S^* - \frac{\nu}{\bar{T}\bar{t}^2}\frac{1}{T^*}(\nabla^*)^2 (u^*)^2 + \chi_s H$$

Then, multiplying by $\bar{t}/\bar{S}$,

$$\frac{DS^*}{Dt^*} = \frac{\chi_S \bar{t}}{c_P \bar{L}^2}(\nabla^*)^2 S^* - \frac{\nu}{\bar{T}\bar{S}\bar{t}}(\nabla^*)^2 (u^*)^2 + \frac{\chi_s \bar{t}}{\bar{S}} H \tag{12}$$

So it's pretty clear that, again, we're going to non-dimensionalize time on the entropy diffusion timescale,

$$\bar{t} = \frac{\bar{L}^2}{\chi_S} c_P.$$

And with that choice, it seems that the appropriate non-dimensionalization of entropy is

$$\bar{S} = \frac{\bar{t}\chi_S H}{\epsilon} = \frac{\bar{L}^2 H c_P}{\epsilon}$$

Under these choices, the entropy energy equation with entropy diffusion becomes (dropping the annoying stars because at this point all important things are non-dimensionalized),

$$\frac{DS}{Dt} = \nabla^2 S - \left(\frac{\nu\chi_S\epsilon}{\bar{T}\bar{L}^4 H c_P}\right)\nabla^2 u^2 + \epsilon \tag{13}$$

Once again, viscous heating has a 1/Ra type term in front of it, with a few extra constants thrown in.

C. Getting to a linearized momentum equation

OK, so in order to do anything meaningful with the momentum equation (or at least, anything that is both easy and meaningful), we need to change its form. The first step in this process is to decompose all thermodynamic variables into background and fluctuating components, e.g.

$$\begin{aligned} \rho &= \rho_0 + \rho_1 \\ T &= T_0 + T_1 \\ P &= P_0 + P_1 \end{aligned} \tag{14}$$

Next, I will make the assumption that my background (0) state is in hydrostatic equilibrium, such that $-\nabla P_0 - \rho g \hat{z} = 0$, and in making that assumption I will remove it from the momentum equation. This leaves me with

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla P_1 - \rho_1 g \hat{z} - \nabla \cdot \bar{\bar{\boldsymbol{\Pi}}}. \tag{15}$$

Next, I divide by ρ, and I put the stress tensor term into a more familiar form $-\nabla \cdot \bar{\bar{\boldsymbol{\Pi}}} \sim -\mu \nabla^2 \boldsymbol{u}$, so that I can more clearly see its units and non-dimensionalize. I also note that $\nu = \mu/\rho$. Thus, my momentum equation becomes

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{\nabla P_1}{\rho} + \frac{\rho_1}{\rho}\boldsymbol{g} - \nu \nabla^2 \boldsymbol{u}. \tag{16}$$

At this point, aside from the sort of contrived form of the viscous term, this equation is still general to perturbations of any size. But...now I'm going to make an assumption. All thermodynamic perturbations are small compared to the background ($\rho_1 \ll \rho_0$, and so on, such that $\rho = \rho_0 + \rho_1 \approx \rho_0$).

Under this approximation, we must examine the equation of state,

$$\frac{\nabla S}{c_P} = \frac{1}{\gamma} \nabla \ln(P_0 + P_1) - \nabla \ln(\rho_0 + \rho_1). \tag{17}$$

To make this prettier, I will do a couple of things. First, note that

$$\ln(A + B) = \ln\left(A\left[1 + \frac{B}{A}\right]\right) = \ln(A) + \ln\left(1 + \frac{B}{A}\right).$$

Thus, we write the equation of state (after trivially integrating away the dels):

$$\frac{S_0}{c_P} + \frac{S_1}{c_P} = \frac{1}{\gamma}\ln(P_0) + \frac{1}{\gamma}\ln\left(1 + \frac{P_1}{P_0}\right) - \ln(\rho_0) - \ln\left(1 + \frac{\rho_1}{\rho_0}\right). \tag{18}$$

By definition, the "0" terms on both sides of the equation are equal to one another, so they drop out. Then, I use the (taylor expansion) relation that

$$\ln(1 + x) \approx x$$

for $x \ll 1$, and I find the linearized equation of state,

$$\boxed{\frac{S_1}{c_P} = \frac{1}{\gamma}\frac{P_1}{P_0} - \frac{\rho_1}{\rho_0}}. \tag{19}$$

Now this term can go into the $\rho_1/\rho \boldsymbol{g} \approx \rho_1/\rho_0 \boldsymbol{g}$ term of the momentum equation, and I find

$$\frac{\rho_1}{\rho_0}\boldsymbol{g} = \frac{1}{\gamma}\frac{P_1}{P_0}\boldsymbol{g} - \frac{S_1}{c_p}\boldsymbol{g}.$$

Now, I play math tricks, and note that:

$$\frac{1}{\gamma}\frac{P_1}{P_0}\frac{\rho_0}{\rho_0}\boldsymbol{g} = \frac{1}{\gamma}\frac{P_1}{\rho_0}\nabla \ln P_0 = \frac{P_1}{\rho_0}\left(\frac{\nabla S_0}{c_P} + \nabla \ln \rho_0\right)$$

where I've used the hydrostatic equilibrium of the initial conditions to cancel terms ($\nabla \ln P_0 = \rho_0 \boldsymbol{g}/P_0$), and then I've used the initial conditions and their entropy EOS to do the next term ($\nabla \ln P_0/\gamma = \nabla S_0/c_P + \nabla \ln \rho_0$). Putting this into the momentum equation,

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{\nabla P_1}{\rho_0} + \frac{P_1}{\rho_0}\nabla \ln \rho_0 + \frac{P_1}{\rho_0}\frac{\nabla S_0}{c_P} - \frac{S_1}{c_P}\boldsymbol{g} - \nu\nabla^2\boldsymbol{u} \tag{20}$$

The first two terms on the RHS condense, because

$$\nabla\left(\frac{P_1}{\rho_0}\right) = \frac{\nabla P_1}{\rho_0} - \frac{P_1}{\rho_0}\nabla \ln \rho_0$$

and then I define $\varpi \equiv P_1/\rho_0$, such that the momentum equation is

$$\boxed{\frac{D\boldsymbol{u}}{Dt} = -\nabla\varpi + \varpi\frac{\nabla S_0}{c_P} - \frac{S_1}{c_P}\boldsymbol{g} - \nu\nabla^2\boldsymbol{u}.} \tag{21}$$

This form of the equation is ripe to be non-dimensionalized, but it is *not* the full equation, as the viscous term has been butchered so that it's dimensionality is clear, and we have assumed that the fluctuations are small.

D. Non-dimensional, linear momentum equation

Taking the linearized momentum equation and pulling out dimensions from all terms except those that have ϖ,

$$\frac{\bar{L}}{\bar{t}^2}\frac{D\boldsymbol{u}^*}{Dt^*} = \left(-\nabla\varpi + \varpi\frac{\nabla S_0}{c_P}\right) + \frac{\bar{S}g}{c_P}S_1^*\hat{z} - \frac{\nu}{\bar{L}\bar{t}}(\nabla^*)^2\boldsymbol{u}^*.$$

If we multiply through by $\bar{t}^2/\bar{L}$, we get

$$\frac{D\boldsymbol{u}^*}{Dt^*} = \frac{\bar{t}^2}{\bar{L}}\left(-\nabla\varpi + \varpi\frac{\nabla S_0}{c_P}\right) + \frac{\bar{S}g\bar{t}^2}{\bar{L}c_P}S_1^*\hat{z} - \frac{\nu\bar{t}}{\bar{L}^2}(\nabla^*)^2\boldsymbol{u}^*.$$

The last two terms are the ones we're interested in now. On the last term, after plugging in the definition of $\bar{t}$ from the energy equation, we get $\nu c_V/\chi$, which is basically Pr. On the second to last term, we get $\bar{S}\bar{L}^3 c_V^2 g/c_P\chi^2$, which is basically RaPr. So, under this set of equations,

$$\mathrm{Pr} \equiv c_V\frac{\nu}{\chi} \qquad \mathrm{Ra} = c_V\frac{g\bar{L}^3(\bar{S}/c_P)}{\chi\nu} \tag{22}$$

Which is to say, these are the standard definitions of Ra and Pr, but with c_V's in front, which means either I've messed up here, or I've been careless in the past.

Either way, the important thing to do now is figure out $\bar{L}$ and $\bar{S}$.

1. *Argument for $\bar{L}$*

I think this one is simple. The length scale we're interested in is the length scale of the system in which a superadiabatic flux is being carried – this is the scale over which we *know* we need convection, and for our setup, as I explained in our "paper,"

$$\bar{L} = d_{\mathrm{conv}} = \frac{\epsilon}{Hc_P},$$

where the heating term's magnitude is

$$H = \frac{\epsilon}{L_z c_P(1-f)},$$

which means that in terms of control parameters, as we would anticipate,

$$\bar{L} = (1-f)L_z.$$

2. Argument for $\bar{S}$

This one's trickier. I think it's important to remember that an underlying assumption of this whole form of the rayleigh number and prandtl number is that we used the linearized equation of state to get here. So in a consistently scaled system,

$$\frac{\bar{S}}{c_P} \sim \frac{1}{\gamma}\frac{\bar{T}}{T_{0,scale}} - \frac{\gamma-1}{\gamma}\frac{\bar{\rho}}{\rho_{0,scale}}.$$

Or something of the sort. So here, I'm explicitly stating that $\bar{T}$ is non-dimensionalizing the *non-background* motions, and the same goes for $\bar{S}$, and the same goes for $\bar{\rho}$. This is actually an important distinction. I think I need to use something like eqn (14) of [2] along with this assumption to get a reasonable scale for S_0 (and I'll need to bring in my assumptions about $\bar{t}$ and $\bar{L}$ to get there.)

The linearized momentum equation [2] is

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{u}\cdot\nabla\rho_0 = -\rho_0\nabla\cdot\boldsymbol{u} \tag{23}$$

and I just realized that this is going to give us just $\rho_1/\rho_0 =$ whatever I set it to be, so oops.

But still, let's just say that T_0, ρ_0 are O(1), then

$$\frac{\bar{S}}{c_P} \sim \frac{1}{\gamma}\bar{T} - \frac{\bar{\rho}}{c_P}.$$

We have $\bar{T}$ from the energy equation,

$$\bar{T} = \bar{L}^2 H,$$

so ok.

It's a bit reckless, but for now, a *good approximation* is that

$$\bar{S} = c_P(\bar{L}^2 H). \tag{24}$$

E. Making a Rayleigh Number

So in the end, our initial Rayleigh number (to get some first cut simulations up and running) is going to be

$$\boxed{\mathrm{Ra} = \frac{g\bar{L}^3(\bar{S}/c_P)}{\nu\chi} = \frac{g(1-f)^5 L_z^5 H}{\nu\chi} = \frac{g(1-f)^4 L_z^4 \epsilon}{\nu\chi}} \tag{25}$$

[1] D. Lecoanet, B. P. Brown, E. G. Zweibel, K. J. Burns, J. S. Oishi, and G. M. Vasil, "Conduction in Low Mach Number Flows. I. Linear and Weakly Nonlinear Regimes," Astrophys. J. **797**, 94 (2014), arXiv:1410.5424 [astro-ph.SR].

[2] B. P. Brown, G. M. Vasil, and E. G. Zweibel, "Energy Conservation and Gravity Waves in Sound-proof Treatments of Stellar Interiors. Part I. Anelastic Approximations," Astrophys. J. **756**, 109 (2012), arXiv:1207.2804 [astro-ph.SR].