I. WORK ON UNDERSTANDING KRAMERS OPACITY ATMOSPHERES (E.G., [1])

In this work, we have a two-layer atmosphere: an unstable layer lying above a stable layer. Let's basicallly try to make the exact same experiment as Kapyla did, so let's assume that the atmosphere has polytropic stratification, or

$$T = \begin{cases} T_b + \nabla T_{RZ} z, & z <= L_{RZ} \\ T_m + \nabla T_{CZ}(z - L_{RZ}), & z > L_{RZ} \end{cases}, \qquad \rho = \begin{cases} \rho_b \left(1 + \frac{\nabla T_{RZ}}{T_b} z\right)^{m_{RZ}}, & z <= L_{RZ} \\ \rho_m \left(1 + \frac{\nabla T_{CZ}}{T_m}(z - L_{RZ})\right)^{m_{CZ}}, & z > L_{RZ} \end{cases}. \tag{1}$$

There's a lot of different variables in there, so let's break them down a bit. T_b and T_m are the temperature values at the bottom (b) and match (m) points in the atmosphere. In other words, they're the temperatures at the bottom of the RZ and the bottom of the CZ, respectively. ρ_b and ρ_m are the density analogues of those two. L_{RZ} is the geometric depth of the radiative zone. ∇T_{RZ} and ∇T_{CZ} are the temperature gradients of the two polytropes. m_{RZ} and m_{CZ} are the polytropic indices of the two layers. Thanks to polytropes having some really easy-to-understand properties, here's some definitions on some of the variables we have kicking around:

$$\begin{aligned}
T_m &= T_b e^{-n_{\rho,RZ}/m_{RZ}} \\
\rho_m &= \rho_b e^{-n_{\rho,RZ}} \\
L_{RZ} &= \frac{T_b}{|\nabla T_{RZ}|}\left(e^{-n_{\rho,RZ}/m_{RZ}} - 1\right) \\
L_{CZ} &= \frac{T_b}{|\nabla T_{CZ}|} e^{-n_{\rho,RZ}/m_{RZ}}\left(e^{-n_{\rho,CZ}/m_{CZ}} - 1\right) \\
\nabla T_{CZ} &= \frac{\boldsymbol{g}}{R(1+m_{CZ})} \\
\nabla T_{RZ} &= \frac{\boldsymbol{g}}{R(1+m_{RZ})} \\
\nabla T_{ad} &= \boldsymbol{g}/c_P.
\end{aligned} \tag{2}$$

OK, so that takes care of a lot of things, and if we specify the number of density scale heights in both parts of the atmosphere, the polytropic indicies, the temperature and density at the bottom of the atmosphere, and the profile of gravity as a function of height, then we end up with a fully specified stitched polytrope.

The interesting new thing that makes setting up this problem a mess is the radiative conductivity profile,

$$\kappa(z) = \kappa_0 \rho^{-(1+a)} T^{3-b}, \tag{3}$$

where $a = 1$ and $b = -7/2$ for free-free interactions. Note that if we have a constant gravity $\boldsymbol{g} = -\text{constant} \cdot \hat{z}$, then setting $m = (3-b)/(1+a)$ will produce a stratification that has a constant conductivity. This is a logical m to set for the RZ, so it turns out that $\kappa \nabla T_{RZ}$ is constant in the RZ so long as gravity is a constant. This means that if we can get this atmosphere functioning for a single polytrope in the CZ, then we can easily attach on the stable layer down below.

A. Figuring out the non-dimensionalization of a single unstable layer

OK, so with the conductivity described in Eqn. (3), we're going to figure out a single, unstable layer that carries flux convectively, and in which we can control the mach number and the Reynolds number through two separate control knobs. If this atmosphere is polytropic, we have the following stratification:

$$T = T_{top} e^{n_\rho/m} + (\nabla T) z, \qquad \rho = \rho_{top} e^{n_\rho}\left[1 + \frac{\nabla T}{T_{top}} e^{-n_\rho/m} z\right]^m \tag{4}$$

And because

$$\rho = \rho_{top} e^{n_\rho} T^m,$$

the profile for the conductivity is

$$\kappa(z) = \kappa_0 \rho_{top}^{-(1+a)} e^{-(1+a)n_\rho} T^{3-b-m(1+a)} = K_0 T^{3-b-m(1+a)}, \tag{5}$$

where $K_0 = \kappa_0 \rho_{top}^{-(1+a)} e^{-(1+a)n_\rho}$ is a constant. The change in the conductivity over the depth of the atmosphere is

$$\frac{\kappa(0)}{\kappa(L_z)} = e^{(3-b)n_\rho/m - n_\rho(1+a)}.$$

This is large, for example when $m = 1.5$, $b = -7/2$, $a = 1$, and $n_\rho = 3$, this fraction is $\mathrm{O}(10^3)$. That means that only one part in 10^3 of the flux carried by conduction at the bottom of the atmosphere can be conducted at the top of the atmosphere. Put simply, for a polytrope with constant ∇T,

$$-\kappa(z=0)\nabla T = \left(1 - \frac{\kappa(z=L_z)}{\kappa(z=0)}\right) F_{\text{conv, top}}. \tag{6}$$

For highly stratified atmospheres, the term that is "1 - a ratio" will be roughly equal to 1. However, it is important to keep this term for atmospheres without much stratification, or for atmospheres where a and b are different.

If we think about the polytropic analogue, there we had

$$F_{cond} = -\kappa \nabla T_0,$$

where $\nabla T_{ad} = -g/c_P = \nabla T_0(-1 + \epsilon/c_P)$ (for $R = 1$), and so

$$F_{cond,avail} = -\kappa(\nabla T_0 - -\nabla T_0(\epsilon/c_P - 1)) = -\kappa \nabla T_0 \epsilon/c_P.$$

So there, we found that the flux that convection had to carry was $\mathrm{O}(\epsilon)$, and that the mach number was $\mathrm{O}(\epsilon^{1/2})$. Thus, it's probably a safe assumption for us to assume that

$$-\kappa(z=0)\nabla T = \left(1 - \frac{\kappa(z=L_z)}{\kappa(z=0)}\right) F_c \epsilon, \tag{7}$$

where F_c is just some constant that we can use to make sure that we don't encounter problems with machine precision, etc. I now define

$$\mathscr{F} \equiv \left(1 - \frac{\kappa(z=L_z)}{\kappa(z=0)}\right),$$

and also

$$\chi_{top} \equiv \frac{K_0 T_{top}^{(3-b)-m(1+a)}}{\rho_{top} c_P}, \qquad K_0 \equiv \frac{\chi_{top}\rho_{top}c_P}{T_{top}^{(3-b)-m(1+a)}},$$

such that the flux balance becomes

$$\frac{g}{R(1+m)} \chi_{top}\rho_{top}c_P e^{(n_\rho/m)\{(3-b)-m(1+a)\}} = \mathscr{F} F_c \epsilon.$$

Assuming that χ_{top} is set as

$$\chi_{top} = \sqrt{\frac{g L_z^3 \epsilon}{\mathrm{Ra} \cdot \mathrm{Pr}}}, \tag{8}$$

and where

$$L_z = \frac{T_{top}}{\nabla T}(e^{n_\rho/m} - 1).$$

Solving for gravity, we find

$$g = \left[\frac{R(m+1)\mathscr{F} F_c}{c_p \rho_{top}}\right]^{2/3} \frac{(\mathrm{Ra}\,\mathrm{Pr}\,\epsilon)^{1/3}}{L_z} e^{-\frac{2}{3}\frac{n_\rho}{m}(3-b-m[1+a])}. \tag{9}$$

This means that we have three things left to specify: F_c, ρ_{top}, and L_z (which is set by T_{top}). Now we start making assumptions.

First, I think it's reasonable to set

$$F_c \equiv (\mathrm{Ra\ Pr})^{-1/2}.$$

This means that the total flux in the system is $\mathscr{F}\epsilon/\sqrt{\mathrm{Ra\ Pr}}$. This scales nicely as I would expect. If we find that the flux is too small for numerical stability, we can always bump this up by a factor of 10, etc. I will also arbitrarily set $T_{top} = \rho_{top} = 1$, as we have done in previous systems. In making these choices, I find that

$$g = \left[\frac{R(m+1)\mathscr{F}}{c_p}\right]^{2/3} \frac{\epsilon^{1/3}\nabla T}{e^{n_\rho/m} - 1} e^{-\frac{2}{3}\frac{n_\rho}{m}(3-b-m[1+a])}. \tag{10}$$

[1] Petri J. Käpylä, Matthias Rheinhardt, Axel Brandenburg, Rainer Arlt, Maarit J. Käpylä, Andreas Lagg, Nigul Olspert, and Jörn Warnecke, "Extended Subadiabatic Layer in Simulations of Overshooting Convection," Astrophys. J. **845** (2017), 10.3847/2041-8213/aa83ab.