

Predicting the Rossby number in convective experiments

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ABSTRACT

The Rossby number is a crucial parameter describing the degree of rotational constraint on the convective dynamics in stars and planets. However, it is not an input to computational models of convection but must be measured ex post facto. Here, we report the discovery of a new quantity, the Predictive Rossby number, which is both tightly correlated with the Rossby number and specified in terms of common inputs to numerical models. The Predictive Rossby number can be specified independent of Rayleigh number, allowing suites of numerical solutions to separate the degree of rotational constraint from the strength of the driving of convection. We examine the scaling of convective transport in terms of the Nusselt number and the degree of turbulence in terms of the Reynolds number of the flow. Finally, we describe the boundary layers as a function of increasing turbulence at constant Rossby number.

Keywords: convection — hydrodynamics — turbulence — dynamo — Sun: rotation

1. INTRODUCTION

Rotation influences the dynamics of convective flows in stellar and planetary atmospheres. Many studies on the fundamental nature of rotating convection in both laboratory and numerical settings have provided great insight into the properties of convection in both the rapidly rotating regime and the transition to the rotationally unconstrained regime (King et al. 2009; Zhong et al. 2009; Schmitz & Tilgner 2009; King et al. 2012; Julien et al. 2012; King et al. 2013; Ecke & Niemela 2014; Stellmach et al. 2014; Cheng et al. 2015; Gastine et al. 2016). The scaling behavior of heat transport, the nature of convective flow structures, and the importance of boundary layer-bulk interactions in driving dynamics are well known. Yet, we do not know of any simple procedure for predicting the magnitude of vortical flow gradients purely from experimental control parameters, such as bulk rotation rate and thermal input.

In the astrophysical context, many studies of rotating convection have investigated questions inspired by the solar dynamo (Glatzmaier & Gilman 1982; Busse 2002; Brown et al. 2008, 2010, 2011; Augustson et al. 2012; Guerrero et al. 2013; Käpylä et al. 2014). Even when these simulations nominally rotate at the solar rate, they frequently produce distinctly different behaviors than the true Sun, such as anti-solar differential rotation profiles (Gastine et al. 2014; Brun et al. 2017). It seems that these differences occur because the simulations produce less rotationally constrained states than the Sun. The influence of rotation results from the local shear gradients, and these are not direct input parameters. Recent simulations predict significant rotational influence in the deep solar interior, which can drastically affect flows throughout the solar convection zone (Featherstone & Hindman 2016; Greer et al. 2016). In the planetary context, the balance between magnetic and rotational forces likely leads to the observed differences between ice giant and gas giant dynamos in our solar system (Soderlund et al. 2015). In particular, Aurnou & King (2017) stress the importance of studying

the proper force balance between lorentz, coriolis, and inertial terms in astrophysical studies.

In short, simulations must achieve the proper rotational balance if they are to explain the behavior of astrophysical objects. In Boussinesq studies, rotational constraint is often measured by comparing dynamical and thermal boundary layers or deviation in heat transport from the non-rotating state (King et al. 2012; Julien et al. 2012; King et al. 2013). Such measurements are not available for astrophysical objects, where the degree of rotational influence is best assessed by the ratio between nonlinear advection magnitude and the linear Coriolis accelerations. The *Rossby number* is the standard measure of this ratio,

$$\text{Ro} \equiv \frac{|\nabla \times \mathbf{u}|}{2|\mathbf{\Omega}|} \sim \frac{|(\nabla \times \mathbf{u}) \times \mathbf{u}|}{|2\mathbf{\Omega} \times \mathbf{u}|}, \quad (1)$$

where $\mathbf{\Omega}$ denotes the bulk rotation vector. Many proxies for the dynamical Rossby number exist that are based solely on input parameters, most notably the *convective* Rossby number. However, all proxies produce imperfect predictions for the true dynamically relevant quantity.

In this letter, we demonstrate an empirical method of predicting the output Rossby number of convection in a simple stratified system.

In Anders & Brown (2017) (hereafter AB17), we studied non-rotating compressible convection without magnetic fields in polytropic atmospheres. In this work, we extend AB17 to rotationally-influenced, *f*-plane atmospheres (e.g., Brummell et al. 1996, 1998; Calkins et al. 2015). We determine how the input parameters we studied previously, which controlled the Mach and Reynolds numbers of the evolved flows, couple with the Taylor number (Ta, Julien et al. 1996), which sets the magnitude of the rotational vector.

In section 2, we describe our experiment and paths through parameter space. In section 3, we present the results of our experiments and in section 4 we offer concluding remarks.

2. EXPERIMENT

We study fully compressible, stratified convection under precisely the same atmospheric model as in AB17, but here we have included rotation. We study polytropic atmospheres with $n_p = 3$ density scale heights and a superadiabatic excess of $\epsilon = 10^{-4}$ such that flows are at low Mach number. We study a domain in which the gravity, $\mathbf{g} = -g\hat{z}$, and rotational vector, $\mathbf{\Omega} = \Omega\hat{z}$, are antiparallel (as in e.g., Julien et al. 1996; Brummell et al. 1996).

We evolve the velocity (\mathbf{u}), temperature (T), and log density ($\ln \rho$) according to the Fully Compressible Navier-Stokes equations in the same form presented in AB17, with the addition of the Coriolis term, $2\mathbf{\Omega} \times \mathbf{u}$, to the left-hand side of the momentum equation. We impose impenetrable, stress-free, fixed-temperature boundary conditions at the top and bottom of the domain.

We specify the kinematic viscosity (ν), thermal diffusivity (χ), and strength of rotation (Ω) at the top of the domain by choosing the Rayleigh number (Ra), Prandtl number (Pr), and Taylor number (Ta),

$$\text{Ra} = \frac{gL_z^3 \Delta S / c_P}{\nu \chi}, \quad \text{Pr} = \frac{\nu}{\chi}, \quad \text{Ta} = \left(\frac{2\Omega L_z^2}{\nu} \right)^2, \quad (2)$$

where L_z is the depth of the domain as defined in AB17, $\Delta S \propto \epsilon n_p$ is the specific entropy difference between the top and bottom of the atmosphere, and the specific heat at constant pressure is $c_P = \gamma/(\gamma - 1)$ with $\gamma = 5/3$. Throughout this work we set $\text{Pr} = 1$. The Taylor number relates to the often-quoted Ekman number by the equality $\text{Ek} \equiv \text{Ta}^{-1/2}$.

The wavenumber of convective onset increases according to $k_{\text{crit}} \propto \text{Ta}^{1/6}$ (Chandrasekhar 1961; Calkins et al. 2015). We study horizontally-periodic, 3D Cartesian domains with extents of $x, y = [0, 4(2\pi/k_{\text{crit}})]$ and $z = [0, L_z]$. At large values of Ta, these domains are tall and skinny, as in Stellmach et al. (2014). We evolve our simulations using the Dedalus¹ pseudospectral framework, and our numerical methods are identical to those presented in AB17.

The critical value of Ra at which convection onsets also depends on Ta (see the black line in figure 1a), roughly according to $\text{Ra}_{\text{crit}} \sim \text{Ta}^{2/3}$ (Chandrasekhar 1961; Calkins et al. 2015). Even taking account of linear theory, the dependence of the evolved nonlinear fluid flows on the input parameters makes predicting the rotational constraint very challenging. We will explore three paths through Ra-Ta space:

$$\text{Ra} = \begin{cases} \mathcal{S} \text{Ra}_{\text{crit}}(\text{Ta}), & \text{(I)} \\ (\text{Ro}_c)^2 \text{Pr Ta}, & \text{(II)} \\ (\text{Ro}_p)^2 \text{Pr}^{1/2} \text{Ta}^{3/4} & \text{(III).} \end{cases} \quad (3)$$

Paths on constraint I are at constant supercriticality, $\mathcal{S} \equiv \text{Ra}/\text{Ra}_{\text{crit}}$ (blue dash-dot line in figure 1a). Paths on constraint II (green dashed line in figure 1a) are at a constant value of the classic *convective* Rossby number, Ro_c , which has provided (e.g., Julien et al. 1996;

¹ <http://dedalus-project.org/>

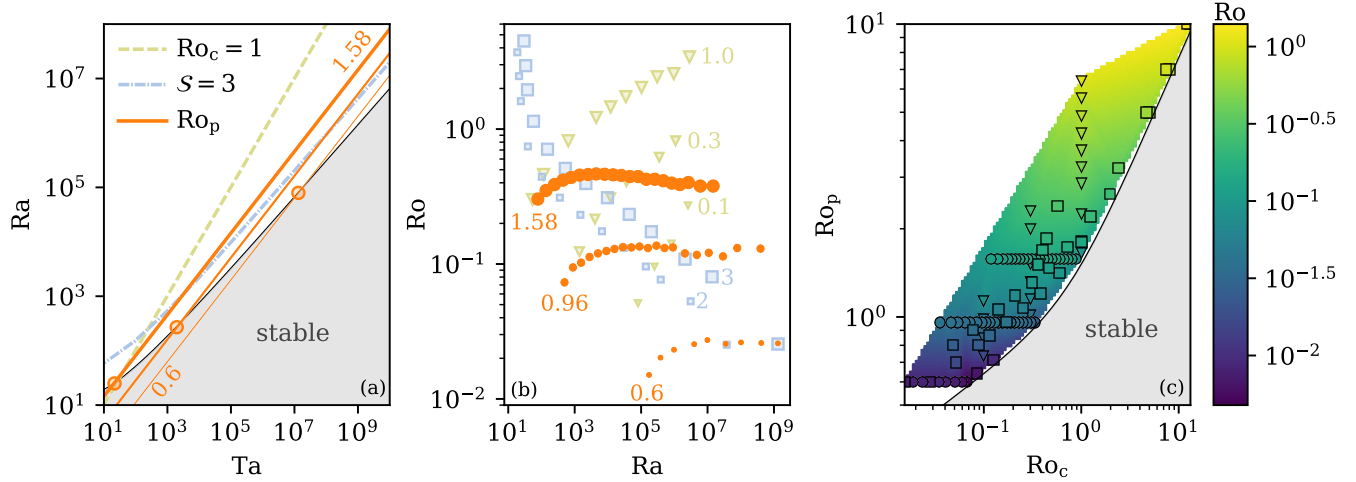


Figure 1. (a) The critical Rayleigh number, as a function of the Taylor number, is plotted as a solid black line. The grey shaded region is subcritical, and rotation suppresses convection there. Paths of constant Convective Rossby number (Ro_c , green dashed line), constant supercriticality (S , blue dash-dot line), and constant Predictive Rossby number (Ro_p , orange solid lines) are shown. From thickest to thinnest, paths with $Ro_p = [1.58, 0.96, 0.6]$ are plotted, and the value of (Ta_{crit}, Ra_{crit}) for each path is denoted by a circular marker. (b) Evolved Ro is plotted vs. Ra along paths of $Ro_p = [1.58, 0.96, 0.6]$ for [big, medium, small] orange circles. For comparison, paths of constant $S = [3, 2]$ for [big, small] squares and constant $Ro_c = [1, 0.3, 0.1]$ for [big, medium, small] triangles are shown. (c) The evolved value of Ro is shown as a function of Ro_p and Ro_c . Each of the experiments in (b) is outlined by a black (circle, triangle, square) for points along constant (Ro_p, Ro_c, S) paths. The color inside of the marker represents the exact measured Ro of that experiment, while the colormap outside of markers is a linear interpolation of the data set.

Brummell et al. 1996) a common proxy for the degree of rotational constraint,

$$Ro_c = \sqrt{\frac{Ra}{Pr Ta}} = \frac{1}{2\Omega} \sqrt{\frac{g \Delta S}{c_p L_z}}. \quad (4)$$

This parameter measures the importance of buoyancy relative to rotation without involving dissipation. Paths on constraint III (e.g., orange solid lines in figure 1a) set constant a ratio which we call the “Predictive Rossby number,”

$$Ro_p = \sqrt{\frac{Ra}{Pr^{1/2} Ta^{3/4}}} = \frac{1}{(2\Omega)^{3/4}} \sqrt{\frac{g \Delta S}{c_p \chi^{1/2}}} \quad (5)$$

Unlike paths through parameter space which hold Ro_c constant, paths with constant Ro_p feel changes in diffusivities but not the depth of the domain. To our knowledge, these paths have not been reported in the literature, although the importance of $Ra/Ta^{3/4} = Ra Ek^{3/2}$ has been independently found by King et al. (2012) using a boundary layer analysis. We compare our results to their theory in Section 4.

3. RESULTS

In our stratified domains, for $Ta \geq 10^5$, a best-fit to results from a linear stability analysis provides $Ra_{crit} = 1.459Ta^{2/3}$ and $k_{crit} = 0.414Ta^{1/6}$ for direct onset of

Table 1. Values of the critical Ra and Ta for each Ro_p track are reported, as well as the maximal values of Ra and Ta studied on each track. All values reported are for the top of the atmosphere, and Ra and Ta both scale like ρ^2 . Thus, at the midplane, Ra and ta are larger by a factor of ~ 70 , and at the bottom of the atmosphere they are greater by a factor of ~ 400 .

Ro_p	(Ra_{crit}, Ta_{crit})	(Ra_{max}, Ta_{max})
1.58	$(10^{1.39}, 10^{1.33})$	$(10^{7.14}, 10^{8.99})$
0.96	$(10^{2.44}, 10^{3.30})$	$(10^{8.58}, 10^{11.49})$
0.60	$(10^{4.88}, 10^{7.10})$	$(10^{9.09}, 10^{12.72})$

convection. In figure 1a, the value of $Ra_{crit}(Ta)$ is shown. Sample paths for each criterion in equation 3 through this parameter space are also shown. In figure 1b, we display the evolution of Ro with increasing Ra along various paths through parameter space. We find that Ro increases on constant Ro_c paths, decreases on constant S paths, and remains roughly constant along constant Ro_p paths. In figure 1c, the value of Ro is shown simultaneously as a function of Ro_p and Ro_c for all experiments conducted in this study. We find a general power-law of the form $Ro = C Ro_c^\alpha Ro_p^\beta$. In the rotationally-dominated regime where $Ro < 0.2$ and $Re_\perp > 5$ (see Eqn. 6), we find $\alpha = -0.02$, and Ro can be said to

be a function of Ro_p alone. Under this assumption, we report a scaling of $Ro = (0.148 \pm 0.003)Ro_p^{3.34 \pm 0.07}$. In the less rotationally dominated regime of $Ro > 0.2$ and $Re_\perp > 5$, we find $\{C, \alpha, \beta\} = \{0.2, -0.19, 1.5\}$.

In figure 2, sample snapshots of the evolved entropy field in the x - y plane near the top and at the middle of the domain are shown. In the left column, flows are at $Ro \sim 1$ and resemble the classic granular structure of nonrotating convection (see e.g., figure 2 in AB17), where intense narrow downflow lanes punctuate broad upwellings. Ro decreases from left to right into the rotationally constrained regime ($Ro \sim 0.03$), in which we observe dynamically persistent, warm upflow columns surrounded by bulk weak downflow regions. These dynamics are similar to those seen in rapidly rotating Rayleigh-Bénard convection (e.g., Stellmach et al. 2014). The select cases displayed in figure 2 each have an evolved volume-averaged $Re_\perp \approx 32$ (defined below in equation 6).

We measure the Nusselt number (Nu), which quantifies heat transport in a convective solution, as defined in AB17. In figure 3a, we plot Nu as a function of Ra/Ra_{crit} at fixed Ro_p . We find that $Nu \propto \{Ra^{0.29 \pm 0.01}, Ra^{0.29 \pm 0.01}, Ra^{0.24}\}$ for $Ro_p = \{0.6, 0.957, 1.58\}$. These scalings are in line with classic $2/7$ power law scalings in Rayleigh-Bénard convection (Ahlers et al. 2009), and approach a $Ra^{1/3}$ scaling, which has been predicted when Ro_p is held constant (see King et al. 2012, and section 4).

Flows are distinctly different parallel to and perpendicular from the rotation vector, which aligns with gravity and stratification. We measure two forms of the RMS Reynolds number,

$$Re_\parallel = \frac{|u|L_z}{\nu}, \quad Re_\perp = \frac{|u|}{\nu} \frac{2\pi}{k_{crit}}, \quad (6)$$

where the length scale in Re_\perp is the wavelength of convective onset, and is related to the horizontal extent of our domain (see section 2). From our work in AB17, we expect the RMS velocity to scale as $|u| \propto \sqrt{\Delta S}$. By definition, $\nu \propto \sqrt{Ra/(Pr \Delta S)}$, and L_z is a constant set by the stratification while $k_{crit} \propto Ta^{1/6}$. Along paths of constant Ro_p , we thus expect $Re_\parallel \propto Ra^{1/2}$ and $Re_\perp \propto Ra^{5/18}$ when Pr is held constant.

In figure 3b, we plot Re_\parallel and Re_\perp as a function of Ra/Ra_{crit} at fixed Ro_p . We find that $Re_\parallel \propto \{Ra^{0.44 \pm 0.01}, Ra^{0.45 \pm 0.01}, Ra^{0.44}\}$ and $Re_\perp \propto \{Ra^{0.22 \pm 0.01}, Ra^{0.23 \pm 0.01}, Ra^{0.21}\}$ for $Ro_p = \{0.6, 0.957, 1.58\}$. These scalings are similar to but slightly weaker than our predictions in all cases. Furthermore, Re_\perp collapses for each Ro_p track, while Re_\parallel experiences an offset to larger values as Ro_p shrinks. The offset in Re_\parallel is un-

surprising, because more rotationally constrained flows result in smaller boundary layers relative to the vertical extent of our stratified domain. The horizontal extent of our domain scales with the strength of rotation, and so regardless of Ro_p , flows perpendicular to the rotational and buoyant direction are comparably turbulent at the same Ra/Ra_{crit} . We find Re_\perp and Re_\parallel are, respectively, good proxies for the horizontal and perpendicular resolution required to resolve an experiment.

Figure 4 shows time- and horizontally-averaged profiles of Ro and the standard deviation of the entropy, σ_s . Figures 4a&b show these profiles for $Ro_p = 1.58$ ($Ro \approx 0.4$), while Figures 4c&d show these profiles for $Ro_p = 0.96$ ($Ro \approx 0.1$). The transition in profile behavior from low Ra (yellow) to high Ra (purple) is denoted by the color of the profile. As Ra increases at a constant value of Ro_p , both the thermal (σ_s) and dynamical (Ro) boundary layers become thinner. We measure the thickness of the thermal boundary layer (δ_s) at the top of the domain by finding the location of the first maxima of σ_s away from the boundary. We measure the thickness of the Ro boundary layer (δ_{Ro}) in the same manner. In figure 4e, we plot δ_{Ro}/δ_s , the ratio of the sizes of these two boundary layers. As anticipated, the dynamical boundary layer (δ_{Ro}) becomes relatively thinner with respect to the thermal boundary layer (δ_s) as Ro and Ro_p decrease. However, the precise scaling of this boundary layer ratio with Ro_p and Ra is unclear, and we cannot immediately compare these ratios to similar measures from the Rayleigh-Bénard convection literature, such as Fig. 5 of King et al. (2013). They measure the dynamical boundary layer thickness as the peak location of the horizontal velocities, but our horizontal velocities are subject to stress-free boundary conditions, and we find that the maxima of horizontal velocities occur precisely at the boundaries. In figure 4f, we plot δ_s in units of the density scale height at the top of the atmosphere, and we plot vertical lines when this crosses 1. We find no systematic change in behavior when δ_s is smaller than the local density scale height.

4. DISCUSSION

In this letter, we studied low-Mach-number, stratified, compressible convection under the influence of rotation. We examined three paths through Ra - Ta space, and showed that the newly-defined Predictive Rossby number, $Ro_p = Ra/(Pr^{1/2}Ta^{3/4})$, determines the value of the evolved Rossby number.

In this work, we experimentally arrived at the $Ra/Ta^{3/4} = RaEk^{3/2}$ scaling in Ro_p , but this relationship was independently discovered by King et al. (2012). Arguing that the thermal boundary layers should scale

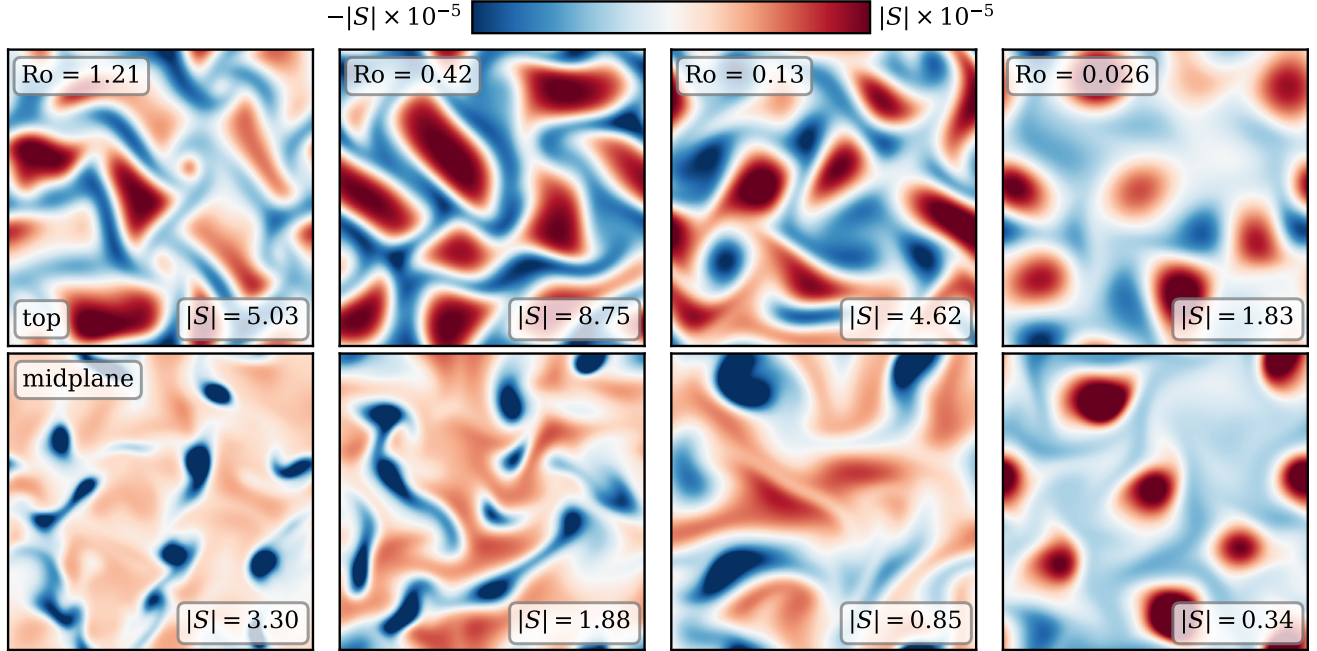


Figure 2. Horizontal slices of the evolved entropy deviation from the mean at $z = 0.95L_z$ (top row) and $z = 0.5L_z$ (bottom row) are shown for select simulations. All runs displayed here have an evolved volume-averaged $Re_\perp \approx 32$. As Ro decreases from $O(1)$ on the left to $O(0.03)$ on the right, and thus the rotational constraint on the flow increases, significant changes in flow morphology are observed. As Ro decreases, Coriolis forces more effectively deflect the convective flows, and the classic granular convective pattern gives way to vortical columns.

as $\delta_S \propto Ra^{-1/3}$ and rotational Ekman boundary layers should scale as $\delta_{Ro} \propto Ta^{-1/4} = Ek^{1/2}$, they expect these boundary layers to be equal in size when $Ra/Ta^{3/4} \sim 1$. They demonstrate that when $2 \lesssim Ra/Ta^{3/4} \lesssim 20$ flows are in the transitional regime, and for $Ra/Ta^{3/4} \lesssim 2$, flows are rotationally constrained. We note that Boussinesq values of Ra and Ta are not the same as their values in our stratified domains here, as diffusivities change with depth (see AB17). Taking into account this change with depth, our simulations fall in King et al. (2012)’s rotationally constrained ($Ro_p = 0.6$) and near-constrained transitional regime ($Ro_p = \{0.957, 1.58\}$). The measured values of Ro in Fig. 1b and the observed dynamics in Fig. 2 agree with this interpretation.

We note briefly that the scaling $Ra \propto Ta^{3/4}$ is very similar to another theorized boundary between fully rotationally constrained convection and partially constrained convection predicted in Boussinesq theory, of $Ra \propto Ta^{4/5}$ (Julien et al. 2012; Gastine et al. 2016). This $Ta^{4/5}$ scaling also arises through arguments of geostrophic balance in the boundary layers, and is a steeper scaling than the $Ta^{3/4}$ scaling present in Ro_p . This suggests that at sufficiently low Ro_p , a suite of simulations across many orders of magnitude of Ra will not only have the same volume-averaged value of Ro (as

in Fig. 1b), but will also maintain proper force balances within the boundary layers.

Our results suggest that by choosing the desired value of Ro_p , experimenters can select the degree of rotational constraint present in their simulations. We find that $Ro \propto Ro_p^{3.34 \pm 0.07}$, which is within 2σ of the estimate in King et al. (2013), who although defining Ro very differently from our vorticity-based definition here, find $Ro \propto Ro_p^{3.84 \pm 0.28}$. We note briefly that they claim that the value of Ro is strongly dependent upon the Prandtl number studied, and that low Ro can be achieved at high Pr without achieving a rotationally constrained flow. We studied only $Pr = 1$ here, and leave it to future work to determine if the scaling of $Ro_p \propto Pr^{-1/4}$ is the correct scaling to predict the evolved Rossby number.

Despite the added complexity of stratification and despite our using stress-free rather than no-slip boundaries, the boundary layer scaling arguments put forth in King et al. (2012) seem to hold up in our systems. This is reminiscent of what we found in AB17, in which convection in stratified domains, regardless of Mach number, produced boundary-layer dominated scaling laws of Nu that were nearly identical to the scaling laws found in Boussinesq Rayleigh-Bénard convection.

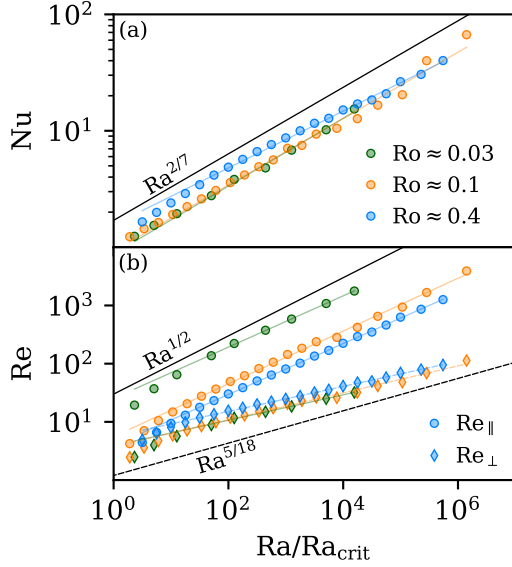


Figure 3. Scaling laws for paths at $Ro_p = 1.58$ ($Ro \approx 0.4$), $Ro_p = 0.96$ ($Ro \approx 0.1$), and $Ro_p = 0.6$ ($Ro \approx 0.03$) are shown. Numbers are plotted vs. Ra/Ra_{crit} , where Ra_{crit} is the value at which a given Ro_p path crosses the supercriticality curve (e.g., orange circles in Fig. 1a). (a) Nu , as defined in AB17, is shown. (b) Re_{\parallel} and Re_{\perp} , as defined in equation 6, are shown. All values of Ro_p trace out similar Nu and Re_{\perp} tracks, whereas Re_{\parallel} tracks shift upwards as Ro decreases.

We close by noting that once Ro_p is chosen such that a convective system has the same Rossby number as an astrophysical object of choice, it is straightforward to increase the turbulent nature of simulations by increasing Ra , just as in the non-rotating case. Although all the results reported here are for a Cartesian geometry with antiparallel gravity and rotation, preliminary 3D spherical simulations suggest that Ro_p also specifies Ro in more complex geometries (Brown et al. 2019 in prep).

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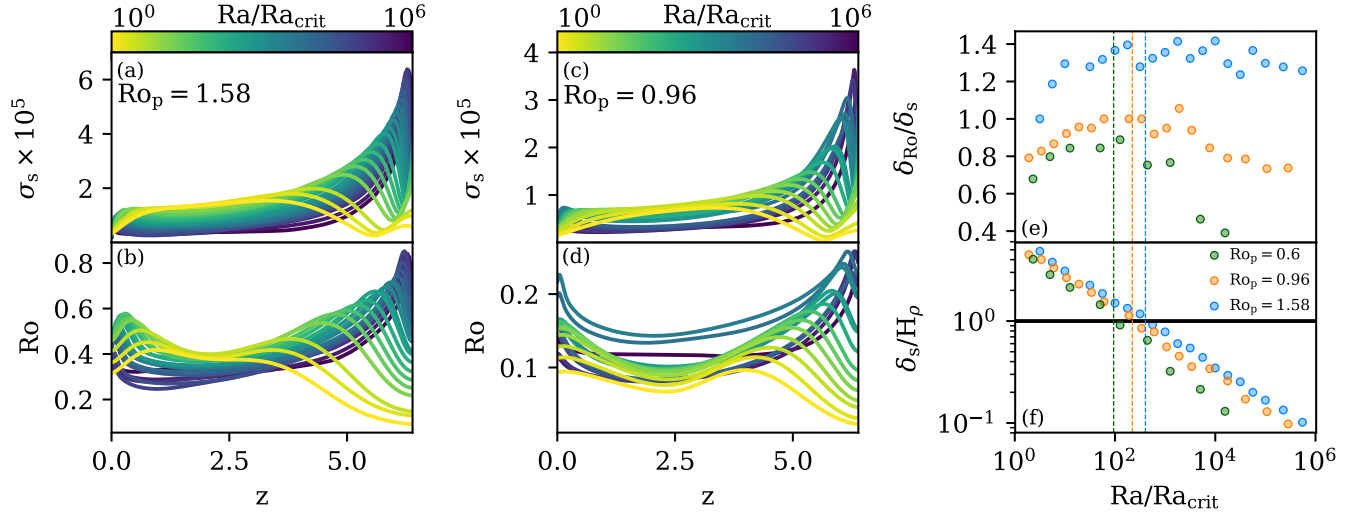


Figure 4. Horizontally-averaged profiles of the standard deviation of entropy (σ_s , a) and Rossby number (Ro , b) are shown vs. height for $Ro_p = 1.58$ ($Ro \approx 0.4$). Similar profiles are shown in (c) and (d) for $Ro_p = 0.96$ ($Ro \approx 0.1$). The color of the profiles denotes the value of Ra/Ra_{crit} , where Ra_{crit} is denoted by an orange circle for each Ro_p path in Fig. 1a. (e) The ratio of the thicknesses of the dynamical boundary layers (δ_{Ro}) and thermal boundary layers (δ_s) is shown vs. Ra/Ra_{crit} for fixed Ro_p . (f) δ_s is plotted vs. Ra/Ra_{crit} in units of the density scale height at the top of the atmosphere (H_p). Vertical lines denote when $\delta_s/H_p = 1$ for each value of Ro_p .

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