

Predicting the Rossby number in convective experiments

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ABSTRACT

The Rossby number is a crucial parameter describing the degree of rotational constraint on the convective dynamics in stars and planets. However, it is not an input to computational models of convection but must be measured ex post facto. Here, we report the discovery of a new quantity, the Predictive Rossby number, which is both tightly correlated with the Rossby number and specified in terms of common inputs to numerical models. The Predictive Rossby number can be specified independent of Rayleigh number, allowing suites of numerical solutions to separate the degree of rotational constraint from the strength of the driving of convection. We examine the scaling of convective transport in terms of the Nusselt number and the degree of turbulence in terms of the Reynolds number of the flow. Finally, we describe the boundary layers as a function of increasing turbulence at constant Rossby number.

Keywords: convection — hydrodynamics — turbulence — dynamo — Sun: rotation

1. INTRODUCTION

Rotation influences the dynamics of convective flows in stellar and planetary atmospheres. Many studies on the fundamental nature of rotating convection in both laboratory and numerical settings have provided great insight into the properties of convection in both the rapidly rotating regime and the transition to the rotationally unconstrained regime (King et al. 2009; Zhong et al. 2009; Schmitz & Tilgner 2009; King et al. 2012; Julien et al. 2012; King et al. 2013; Ecke & Niemela 2014; Stellmach et al. 2014; Cheng et al. 2015; Gastine et al. 2016) [*Explanation of change: re-arranged and added relevant references*] The scaling behavior of heat transport, the nature of convective flow structures, and the importance of boundary layer-bulk interactions in driving dynamics are well known. Yet, we do not know of any simple procedure for predicting the magnitude of vortical flow gradients purely from experimental con-

trol parameters, such as bulk rotation rate and thermal input.

In the astrophysical context, many studies of rotating convection have investigated questions inspired by the solar dynamo (Glatzmaier & Gilman 1982; Busse 2002; Brown et al. 2008, 2010, 2011; Augustson et al. 2012; Guerrero et al. 2013; Käpylä et al. 2014). Even when these simulations nominally rotate at the solar rate, they frequently produce distinctly different behaviors than the true Sun, such as anti-solar differential rotation profiles (Gastine et al. 2014). It seems that these differences occur because the simulations produce less rotationally constrained states than the Sun. The influence of rotation results from the local shear gradients, and these are not direct input parameters. Recent simulations predict significant rotational influence in the deep solar interior, which can drastically affect flows throughout the solar convection zone (Featherstone & Hindman 2016; Greer et al. 2016). In the planetary context, the balance between magnetic and rotational forces likely leads to the observed differences between ice giant and gas giant dynamos in our solar system (Soderlund et al. 2015). In particular, Aurnou & King (2017) suggest that many

studies of planetary systems have over-emphasized the importance of magnetism compared to rotation.

In short, simulations must achieve the proper rotational balance if they are to explain the behavior of astrophysical objects. (Added: In Boussinesq studies, rotational constraint is often measured by comparing dynamical and thermal boundary layers or deviation in heat transport from the non-rotating state (King et al. 2012; Julien et al. 2012; King et al. 2013). Such measurements are not available for astrophysical objects, where) the degree of rotational influence is best assessed by the ratio between nonlinear advection magnitude and the linear Coriolis accelerations. The *Rossby number* is the standard measure of this ratio,

$$\text{Ro} \equiv \frac{|\nabla \times \mathbf{u}|}{2|\mathbf{\Omega}|} \sim \frac{|(\nabla \times \mathbf{u}) \times \mathbf{u}|}{|2\mathbf{\Omega} \times \mathbf{u}|}, \quad (1)$$

where $\mathbf{\Omega}$ denotes the bulk rotation vector. Many proxies for the dynamical Rossby number exist that are based solely on input parameters, most notably the *convective* Rossby number. However, all proxies produce imperfect predictions for the true dynamically relevant quantity.

In this letter, we demonstrate an empirical method of predicting the output Rossby number of convection in a simple stratified system.

In Anders & Brown (2017) (hereafter AB17), we studied non-rotating compressible convection without magnetic fields in polytropic atmospheres. In this work, we extend AB17 to rotationally-influenced, *f*-plane atmospheres (e.g. Brummell et al. 1996, 1998; Calkins et al. 2015). We determine how the input parameters we studied previously, controlling the Mach and Reynolds numbers of the evolved flows, couple with the Taylor number (Ta, Julien et al. 1996), which sets the magnitude of the rotational vector.

In section 2, we describe our experiment and paths through parameter space. In section 3, we present the results of our experiments and in section 4 we offer concluding remarks.

2. EXPERIMENT

We study fully compressible, stratified convection under precisely the same atmospheric model as in AB17, but here we have included rotation. We study polytropic atmospheres with $n_p = 3$ density scale heights and a superadiabatic excess of $\epsilon = 10^{-4}$ such that flows are at low Mach number. We study a domain in which the gravity, $\mathbf{g} = -g\hat{z}$, and rotational vector, $\mathbf{\Omega} = \Omega\hat{z}$, are antiparallel (as in e.g., Julien et al. 1996; Brummell et al. 1996).

We evolve the velocity (\mathbf{u}), temperature (T), and log density ($\ln \rho$) according to the Fully Compressible Navier-Stokes equations in the same form presented in AB17, with the addition of the Coriolis term, $2\mathbf{\Omega} \times \mathbf{u}$, to the left-hand side of the momentum equation. We impose impenetrable, stress-free, fixed-temperature boundary conditions at the top and bottom of the domain.

We set the kinematic viscosity (ν), thermal diffusivity (χ), and strength of rotation (Ω) at the top of the domain by choosing the Rayleigh number (Ra), Prandtl number (Pr), and Taylor number (Ta),

$$\text{Ra} = \frac{gL_z^3 \Delta S / c_P}{\nu \chi}, \quad \text{Pr} = \frac{\nu}{\chi}, \quad \text{Ta} = \left(\frac{2\Omega L_z^2}{\nu} \right)^2, \quad (2)$$

where L_z is the depth of the domain as defined in AB17, $\Delta S \propto \epsilon n_p$ is the specific entropy difference between $z = 0$ and $z = L_z$, and the specific heat at constant pressure is $c_P = \gamma/(\gamma - 1)$ and $\gamma = 5/3$. Throughout this work we set $\text{Pr} = 1$. The Taylor number relates to the often-quoted Ekman number by the equality $\text{Ek} \equiv \text{Ta}^{-1/2}$.

The *convective* Rossby number has provided (e.g., Julien et al. 1996; Brummell et al. 1996) a common proxy (based on input parameters) for the degree of rotational constraint,

$$\text{Ro}_c = \sqrt{\frac{\text{Ra}}{\text{Pr Ta}}} = \frac{1}{2\Omega} \sqrt{\frac{g \Delta S}{c_P L_z}}. \quad (3)$$

This parameter measures the importance of buoyancy relative to rotation without involving dissipation. (Replaced: We show replaced with: In section 2, we empirically demonstrate) that (Added: the) true Rossby number defined in equation 1 depends nonlinearly on Ro_c .

The wavenumber of convective onset increases such that, $k_{\text{crit}} \propto \text{Ta}^{1/6}$ (Chandrasekhar 1961; Calkins et al. 2015). We study horizontally-periodic, 3D Cartesian convective domains with extents of $x, y = [0, 4(2\pi/k_{\text{crit}})]$ and $z = [0, L_z]$. At large values of Ta, these domains are tall and skinny, as in Stellmach et al. (2014). We evolve our simulations using the Dedalus¹ pseudospectral framework, and our numerical methods are identical to those presented in AB17.

The critical value of Ra at which convection onsets also depends on Ta (see the black line in figure 1a); roughly according to $\text{Ra}_{\text{crit}} \sim \text{Ta}^{2/3}$ (Chandrasekhar 1961; Calkins et al. 2015). We have confirmed these

¹ <http://dedalus-project.org/>

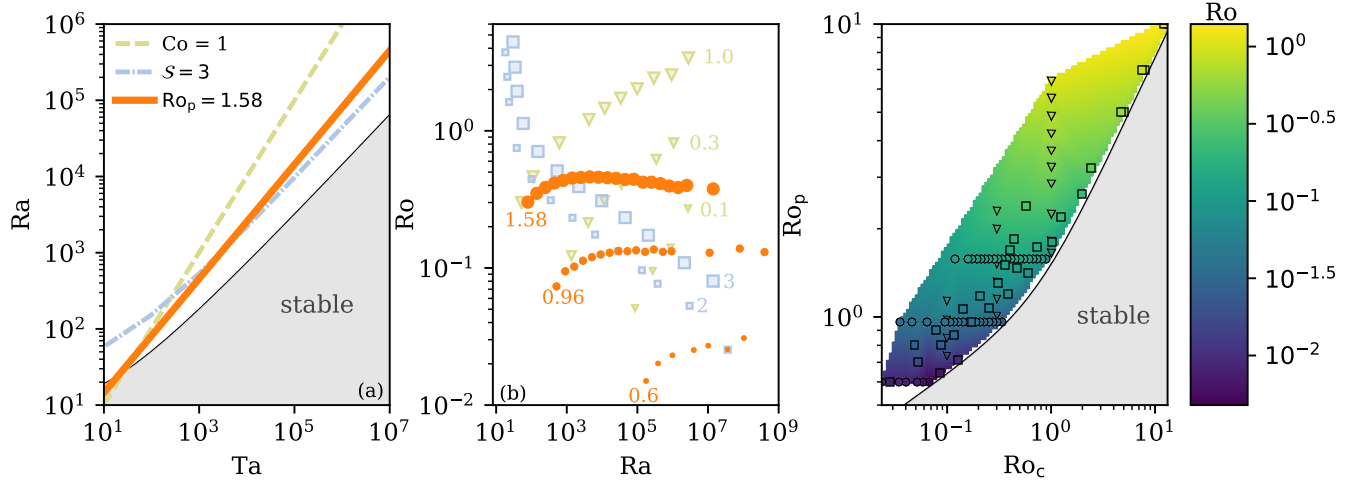


Figure 1. (a) The critical Rayleigh number, as a function of the Taylor number, is plotted as a solid black line. The grey shaded region is subcritical, and rotation suppresses convection there. Paths of constant Convective Rossby number (Ro_c , green dashed line), constant supercriticality (S , blue dash-dot line), and constant Predictive Rossby number (Ro_p , orange solid line) are shown through parameter space. (b) Evolved Ro is plotted vs. Ra along paths of $Ro_p = [1.58, 0.96]$ for [big, small] orange circles. For comparison, paths of constant S (blue squares, $S = [3, 2]$ for [big, small] squares) and constant Ro_c (green triangles, $Ro_c = [1, 0.3, 0.1]$ for [big, medium, small] triangles) are shown. Ro is roughly constant for a constant Ro_p , particularly for the low- Ro , $Ro_p = 0.96$ case, but changes drastically at constant Ro_c or S . (c) The evolved value of Ro is shown as a function of Ro_p and Ro_c . Each of the experiments in (b) is outlined by a black (circle, triangle, square) for points along constant (Ro_p , Ro_c , S) paths. The color inside of the marker represents the exact measured Ro of that experiment, while the colormap outside of markers is a linear interpolation of the data set. A least-squares fit to all experiments with $S \geq 1.5$ and $Ro \geq 0.3$ returns $Ro = 0.2Ro_c^{-0.19}Ro_p^{1.5}$. A similar fit to all experiments with $S \geq 1.5$ and $Ro < 0.3$ returns $Ro = 0.1Ro_c^{-0.21}Ro_p^{3.5}$. In both the high- and low- Ro regime, the measured Rossby number is a strong function of Ro_p and a weak function of Ro_c .

scalings of $Ra_{crit}(Ta)$ and $k_{crit}(Ta)$ in our atmospheres using a linear stability analysis. Even taking account of linear theory, the dependence of the evolved nonlinear fluid flows on the input parameters makes predicting the rotational constraint very challenging. We will explore three paths through Ra - Ta space:

$$Ra = \begin{cases} S Ra_{crit}(Ta), & \text{(I)} \\ (Ro_c)^2 Pr Ta, & \text{(II)} \\ (Ro_p)^2 Pr^{1/2} Ta^{3/4} & \text{(III)}. \end{cases} \quad (4)$$

Paths on constraint I are at constant supercriticality, $S \equiv Ra/Ra_{crit}$ (blue dash-dot line in figure 1a). Paths on constraint II are at a constant value of the classic Ro_c (green dashed line in figure 1a). Paths on constraint III (e.g., orange solid line in figure 1a) set constant a ratio which we call the “Predictive Rossby number,”

$$Ro_p = \sqrt{\frac{Ra}{Pr^{1/2} Ta^{3/4}}} = \frac{1}{(2\Omega)^{3/4}} \sqrt{\frac{g \Delta S}{c_p \chi^{1/2}}} \quad (5)$$

[Explanation of change: Replaced Pr with $Pr^{1/2}$; this makes Ro_p a function of χ , rather than ν , which better fits the boundary layer theory] (Replaced: Paths along constant Ro_c are sensitive to the depth of the

domain, but are blind to changes in diffusivities with increasing Ra and Ta . On the other hand, replaced with: Unlike paths through parameter space which hold Ro_c constant,) paths with constant Ro_p feel changes in diffusivities but not the depth of the domain. To our knowledge, these paths have not been reported in the literature (Added: , although the importance of $Ra/Ta^{3/4} = Ra Ek^{3/2}$ has been independently found by King et al. (2012) using a boundary layer analysis. We compare our results to their theory in Section 4.)

3. RESULTS

(Added: In our stratified domains, for $Ta \geq 10^5$, a linear instability analysis provides $Ra_{crit} = 1.459Ta^{2/3}$ and $k_{crit} = 0.414Ta^{1/6}$.) In figure 1a, the value of Ra_{crit} is shown as a function of Ta . A sample path for each criterion in equation 4 through this parameter space is shown. In figure 1b, we display the evolution of Ro with increasing Ra along various paths through parameter space. We find that Ro increases on constant Ro_c paths, decreases on constant S paths, and remains roughly constant along constant Ro_p paths. In figure 1c, the value of Ro is shown simultaneously as a function of Ro_p and Ro_c for all experiments conducted in this study. We

find a general power-law of the form $\text{Ro} \propto \text{Ro}_c^{-1/5} \text{Ro}_p^\alpha$, where $\alpha \approx 3.5$ in the rotationally constrained, low-Ro regime and $\alpha \approx 1.5$ in the unconstrained, high-Ro regime. In the rotationally constrained regime, Ro is a much stronger function of Ro_p than Ro_c , and the evolved Ro can be approximately determined through specification of Ro_p alone.

In figure 2, sample snapshots of the evolved entropy field in the x - y plane near the top of the domain are shown. In the left panel is a rotationally unconstrained flow at moderately high Ro, and Ro decreases into the rotationally constrained regime from left to right. As Ro decreases, the classic granular structure of convection (see e.g., figure 2 in AB17) gives way to vortical columns of convection, as seen in rapidly rotating Rayleigh-Bénard convection (Stellmach et al. 2014). The select cases displayed in figure 2 each have an evolved volume-averaged $\text{Re} \approx 200$.

We measure the Nusselt number (Nu), which quantifies heat transport in a convective solution, as defined in AB17. In figure 3a, we show how Nu scales as a function of Ra at fixed Ro_p . (Replaced: When $\text{Ro} \approx 0.1$, we find a scaling of $\text{Nu} \propto \text{Ra}^{0.27}$. This is reminiscent of classic scaling laws (e.g., $\text{Ra}^{2/7}$) in non-rotating Rayleigh-Bénard convection (Ahlers et al. 2009). This suggests that changes in heat transport at constant Ro_p are driven by changes in the thermal boundary layer structure with increasing Ra. replaced with: We find that $\text{Nu} \propto \{\text{Ra}^{0.35}, \text{Ra}^{0.30}, \text{Ra}^{0.25}\}$ for $\text{Ro}_p = \{0.6, 0.957, 1.58\}$. As Ro_p , and thus Ro, become smaller, the scaling of Nu becomes closer to a $\text{Ra}^{1/3}$ scaling, which is predicted along paths of constant Ro_p (see King et al. 2012, and section 4).)

(Added: In rotating convection, the nature of flows is distinctly different parallel to the rotation vector from perpendicular to the rotation vector. As a result, we use two forms of the RMS Reynolds number which capture different length scales,

$$\text{Re}_\parallel = \frac{|u|L_z}{\nu}, \quad \text{Re}_\perp = \frac{|u|}{\nu} \frac{2\pi}{k_{\text{crit}}}. \quad (6)$$

From our work in AB17, we expect the RMS velocity to scale as $|u| \propto \sqrt{\Delta S}$. By definition, $\nu \propto \sqrt{\text{Ra}/(\text{Pr} \Delta S)}$, and L_z is a constant set by the stratification while $k_{\text{crit}} \propto \text{Ta}^{1/6}$. Along paths of constant Ro_p , we thus expect $\text{Re}_\parallel \propto \text{Ra}^{1/2}$ and $\text{Re}_\perp \propto \text{Ra}^{5/18}$ when Pr is held constant.) In figure 3b, we plot (Replaced: the RMS Reynold's number ($\text{Re} = |u|L_z/\nu$) replaced with: Re_\parallel and Re_\perp) as a function of Ra at fixed Ro_p . (Replaced: , and find that $\text{Re} \propto \text{Ra}^{0.47}$ in the rotationally constrained regime, which is almost precisely the $\text{Re} \propto \text{Ra}^{1/2}$ scaling measured in the

non-rotating regime in AB17. replaced with: We find that $\text{Re}_\parallel \propto \{\text{Ra}^{0.49}, \text{Ra}^{0.47}, \text{Ra}^{0.44}\}$ and $\text{Re}_\perp \propto \{\text{Ra}^{0.26}, \text{Ra}^{0.25}, \text{Ra}^{0.22}\}$ for $\text{Ro}_p = \{0.6, 0.957, 1.58\}$. Furthermore, Re_\perp falls along the same track for all values of Ro_p studied here, while Re_\parallel experiences an offset to larger values as Ro_p shrinks. In general, we find that Re_\perp is a good proxy for the horizontal resolution required to run an experiment, while Re_\parallel probes the vertical resolution required.)

Figure 4 shows time- and horizontally-averaged profiles of Ro and the z-component of the specific entropy gradient, $\partial_z s$. Figures 4a&b show these profiles for $\text{Ro}_p = 0.96$ ($\text{Ro} \approx 0.1$), while Figures 4c&d show these profiles for $\text{Ro}_p = 1.58$ ($\text{Ro} \approx 0.4$). The transition in profile behavior from low Ra (yellow) to high Ra (purple) is denoted by the color of the profile. As Ra increases at a constant value of Ro_p , both the thermal ($\partial_z s$) and dynamical (Ro) boundary layers become thinner. We measure the thickness of the thermal boundary layer (δ_s) at the top of the domain by measuring where a linear fit within the boundary layer crosses through $\partial_z s = 0$. We ensure by-eye for each profile that this is a reasonable measure of the boundary layer thickness. We measure the thickness of the Ro boundary layer (δ_{Ro}) as the height of the peak value of Ro within the upper half of the domain. In figure 4e, we plot $\delta_{\text{Ro}}/\delta_s$, the ratio of the sizes of these two boundary layers. As anticipated, the dynamical boundary layer (δ_{Ro}) becomes relatively thinner with respect to the thermal boundary layer (δ_s) as Ro and Ro_p decrease. At $\text{Ro}_p = 1.58$, both boundary layers are approximately equally thick, and so both rotational and advective effects are equally important. On the other hand, at $\text{Ro}_p = 0.96$, the dynamical boundary layer is only 60% the size of the thermal boundary layer, and rotational effects dominate the dynamics.

4. DISCUSSION

In this letter, we studied low-Mach-number, stratified, compressible convection under the influence of rotation. We examined three paths through Ra-Ta space, and showed that in the rotationally constrained regime at low-Ro, the newly-defined Predictive Rossby number, $\text{Ro}_p = \text{Ra}/(\text{Pr}^{1/2} \text{Ta}^{3/4})$, determines the value of the evolved Ro. (Deleted: While increasing Ra and holding Ro_p constant, we find scaling laws of heat transport (Nu) and turbulence (Re) that are nearly identical to scaling laws seen in nonrotational convection.)

(Added: In this work, we experimentally arrived at the $\text{Ra}/\text{Ta}^{3/4} = \text{RaEk}^{3/2}$ scaling which informs the predictive Rossby number, but this relationship was independently discovered by King et al. (2012). They ar-

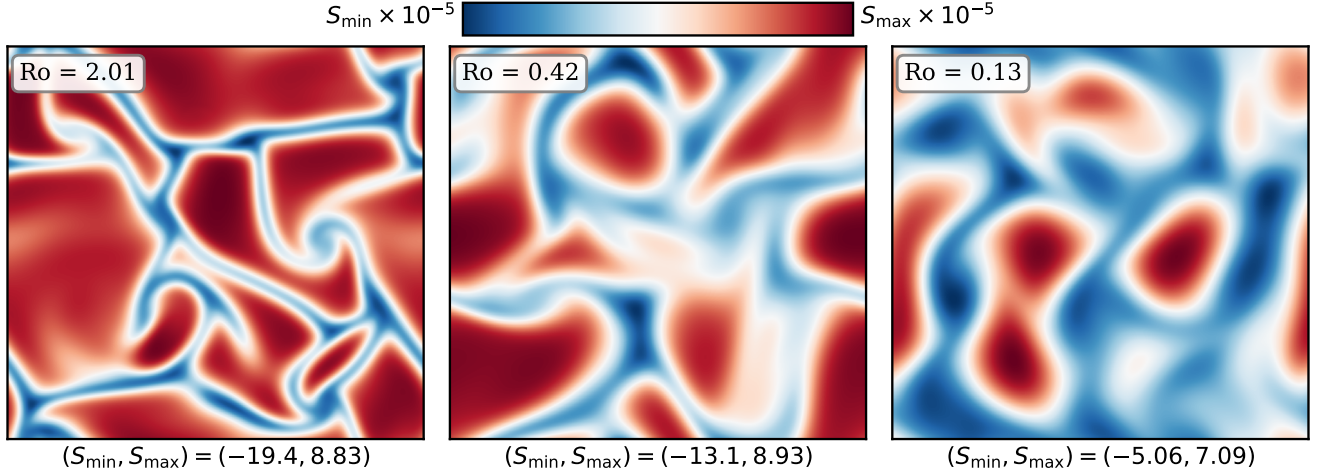


Figure 2. A horizontal slice of the evolved entropy field is plotted at $z = 0.95L_z$ for select simulations. The mean value of entropy at this height has been removed in all cases. All runs displayed here have an evolved volume-averaged $Re \approx 200$. As Ro decreases from $O(1)$ on the left to $O(0.1)$ on the right, and thus the rotational constraint on the flow increases, significant changes in flow morphology are observed. At $Ro = 2.01$ ($Ro_c = 1$), convective dynamics are not hugely dissimilar from the non-rotating case where there are large upflows and narrow, fast downflow lanes (see e.g., AB17). As the rotational constraint increases, the granular convective pattern gives way to vortical columns, as seen at $Ro = 0.13$ ($Ro_p = 0.96$).

gue that the thermal boundary layer should scale as $\delta_S \propto Ra^{-1/3}$, and that rotational Ekman boundary layers should scale as $\delta_{Ro} \propto Ta^{-1/4} = Ek^{1/2}$, and thus when a “transition parameter,” $Ra/Ta^{3/4} \sim 1$, these boundary layers should cross one another, and they demonstrate that the value of $Ra/Ta^{3/4}$ seems to determine whether your flow is rotationally constrained or not. King et al. (2013) continued their work, and showed how numerous convective flow properties scale with this transition parameter, including the ratio of the thermal to viscous boundary layer. In our current work, we find that despite the added complexity of stratification and despite our using stress-free rather than no-slip boundaries, the boundary layer scaling arguments put forth in King et al. (2012) seem to hold up in our systems. This is reminiscent of what we found in AB17, in which convection in these stratified domains, regardless of Mach number, produced boundary-layer dominated scaling laws of Nu that were nearly identical to the Boussinesq scaling laws.)

We note briefly that the scaling $Ra \propto Ta^{3/4}$ is very similar to the theorized boundary between fully rotationally constrained convection and partially constrained convection predicted in Boussinesq theory, of $Ra \propto Ta^{4/5}$ (Julien et al. 2012; Gastine et al. 2016). This $Ta^{4/5}$ scaling arises through arguments of geostrophic balance in the boundary layers, and is a steeper scaling than the $Ta^{3/4}$ scaling present in Ro_p . This suggests that at sufficiently low Ro_p , a suite of simulations across many orders of magnitude of Ra will not only have

the same volume-averaged value of Ro (as in Fig. 1b), but will also maintain proper force balances within the boundary layers.

Our results suggest that by choosing the proper value of Ro_p , experimenters can select the degree of rotational constraint present in their simulations. (Added: We find that $Ro \propto Ro_p^{3.5}$, which is within the error bounds of the estimate in King et al. (2013), who although defining Ro very differently from our vorticity-based definition here, find $Ro \propto Ro_p^{3.84 \pm 0.28}$. We note briefly that they claim that the value of Ro is strongly dependent upon the Prandtl number studied, and that low Ro can be achieved at high Pr without achieving a rotationally constrained flow. We studied only $Pr = 1$ here, and leave it to future authors to determine if the scaling of $Ro_p \propto Pr^{-1/4}$ is the correct scaling to predict the evolved Rossby number.)

(Added: We close by noting that once Ro_p is chosen such that a convective system has the same Rossby number as an astrophysical object of choice,) it is straightforward to increase the turbulent nature of simulations increasing Ra , just as in the non-rotating case. Although all the results reported here are for a Cartesian geometry with antiparallel gravity and rotation, preliminary 3D spherical simulations suggest that Ro_p also specifies Ro in more complex geometries (Brown et al. 2019 in prep).

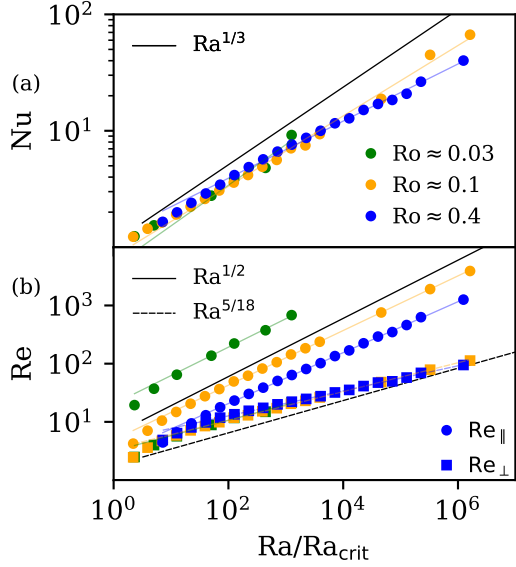


Figure 3. (Replaced: Scaling laws for paths at $Ro_p = 1.58$ ($Ro \approx 0.4$) and $Ro_p = 0.96$ ($Ro \approx 0.1$) are shown. (a) Evolved Nu vs. Ra . The scaling laws here are very reminiscent of classic Rayleigh-Bénard convection theory (Ahlers et al. 2009). (b) Evolved Re vs. Ra . The scaling seen here is nearly identical to scalings in nonrotating convection (AB17). replaced with: Scaling laws for paths at $Ro_p = 1.58$ ($Ro \approx 0.4$), $Ro_p = 0.96$ ($Ro \approx 0.1$), and $Ro_p = 0.6$ ($Ro \approx 0.3$) are shown. Numbers are plotted vs. Ra/Ra_{crit} , where Ra_{crit} is the value at which a given Ro_p path crosses the supercriticality curve (e.g., where the solid orange line crosses the black line in Fig. 1a). (a) The evolved scaling law of Nu approaches $Ra^{1/3}$, as expected from boundary layer theory (King et al. 2012). (b) $Re_{||}$ and Re_{\perp} , as defined in equation 7, are shown. All values of Ro_p trace out similar Re_{\perp} tracks which scale roughly as $Ra^{5/18}$, as expected. $Re_{||}$ tracks experience a constant offset for each different value of Ro_p , but otherwise scale roughly as $Ra^{1/2}$, as anticipated.)

(Added: We thank Jon Aurnou and the anonymous referee who independently directed us to the work of King et al. (2012) during the review process, which greatly helped us understand the results of the experiments here. We further thank the anonymous referee for other helpful and clarifying suggestions.) This work was supported by NASA Headquarters under the NASA Earth and Space Science Fellowship Program – Grant 80NSSC18K1199. EHA further acknowledges the University of Colorado’s George Ellery Hale Graduate Student Fellowship. This work was additionally supported by NASA LWS grant number NNX16AC92G. Computations were conducted with support by the NASA High End Computing (HEC) Program through the NASA Advanced Supercomputing (NAS) Division at Ames Research Center on Pleiades with allocation GID s1647.

REFERENCES

- Ahlers, G., Grossmann, S., & Lohse, D. 2009, *Rev. Mod. Phys.*, 81, 503
- Anders, E. H., & Brown, B. P. 2017, *Physical Review Fluids*, 2, 083501
- Augustson, K. C., Brown, B. P., Brun, A. S., Miesch, M. S., & Toomre, J. 2012, *ApJ*, 756, 169
- Aurnou, J. M., & King, E. M. 2017, *Proceedings of the Royal Society of London Series A*, 473, 20160731
- Brown, B. P., Browning, M. K., Brun, A. S., Miesch, M. S., & Toomre, J. 2008, *ApJ*, 689, 1354
- . 2010, *ApJ*, 711, 424
- Brown, B. P., Miesch, M. S., Browning, M. K., Brun, A. S., & Toomre, J. 2011, *ApJ*, 731, 69
- Brummell, N. H., Hurlburt, N. E., & Toomre, J. 1996, *ApJ*, 473, 494
- . 1998, *ApJ*, 493, 955
- Busse, F. H. 2002, *Physics of Fluids*, 14, 1301
- Calkins, M. A., Julien, K., & Marti, P. 2015, *Geophysical and Astrophysical Fluid Dynamics*, 109, 422
- Chandrasekhar, S. 1961, *Hydrodynamic and hydromagnetic stability*
- Cheng, J. S., Stellmach, S., Ribeiro, A., et al. 2015, *Geophysical Journal International*, 201, 1
- Ecke, R. E., & Niemela, J. J. 2014, *PhRvL*, 113, 114301
- Featherstone, N. A., & Hindman, B. W. 2016, *ApJ*, 830, L15

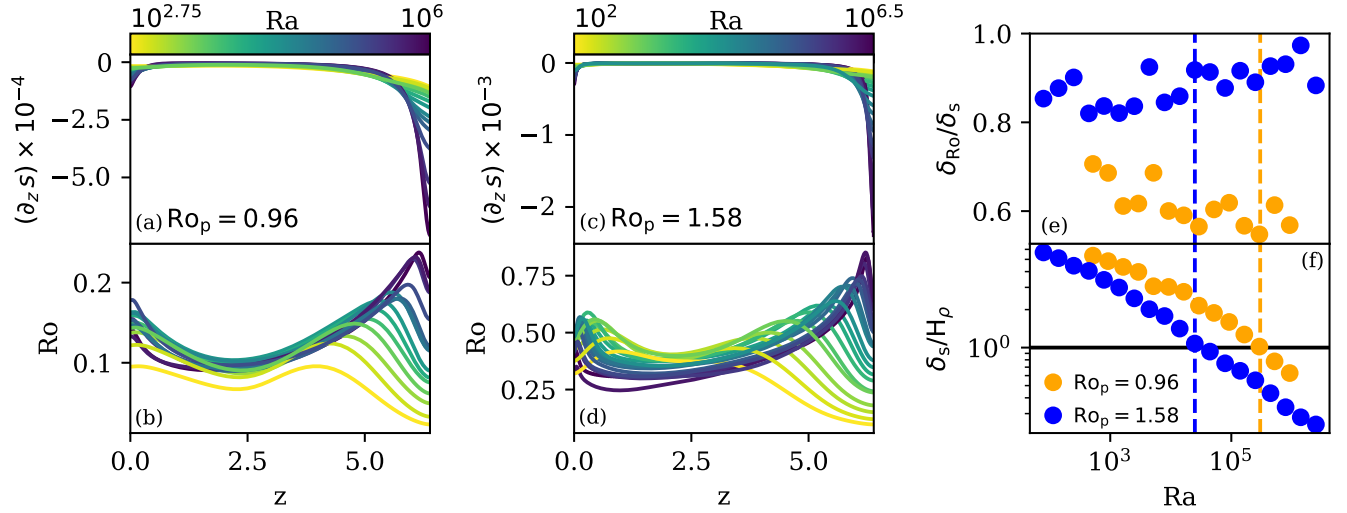


Figure 4. Horizontally-averaged profiles of the z -derivative of the specific entropy profile ($\partial_z s$, a) and Rossby number (Ro , b) are shown vs. height for $Ro_p = 0.96$ ($Ro \approx 0.1$). Similar profiles are shown in (c) and (d) for $Ro_p = 1.58$ ($Ro \approx 0.4$). The color of the profiles denotes the value of Ra , with yellow profiles being at very low Ra and purple at the highest values of Ra studied here. (e) The ratio of the thicknesses of the dynamical (Ro) boundary layers and thermal ($\partial_z s$) boundary layers is shown for both values of Ro_p at each value of Ra . This ratio, and thus the relative importance of both thermal and rotational dynamics, remains roughly constant across orders of magnitude of Ra . (Added: (f) We plot the size of the entropy boundary layer divided by the size of the density scale height at the top of the atmosphere. Vertical lines denote when these two length scales become equal, and we find that the size of the boundary layer compared to an important dynamical length scale does not predict any measurable change in system behavior.)

Gastine, T., Wicht, J., & Aubert, J. 2016, *Journal of Fluid Mechanics*, 808, 690

Gastine, T., Yadav, R. K., Morin, J., Reiners, A., & Wicht, J. 2014, *MNRAS*, 438, L76

Glatzmaier, G. A., & Gilman, P. A. 1982, *ApJ*, 256, 316

Greer, B. J., Hindman, B. W., & Toomre, J. 2016, *ApJ*, 824, 4

Guerrero, G., Smolarkiewicz, P. K., Kosovichev, A. G., & Mansour, N. N. 2013, *ApJ*, 779, 176

Julien, K., Knobloch, E., Rubio, A. M., & Vasil, G. M. 2012, *Physical Review Letters*, 109, 254503

Julien, K., Legg, S., McWilliams, J., & Werne, J. 1996, *Journal of Fluid Mechanics*, 322, 243

Käpylä, P. J., Käpylä, M. J., & Brandenburg, A. 2014, *A&A*, 570, A43

King, E. M., Stellmach, S., & Aurnou, J. M. 2012, *Journal of Fluid Mechanics*, 691, 568

King, E. M., Stellmach, S., & Buffett, B. 2013, *Journal of Fluid Mechanics*, 717, 449

King, E. M., Stellmach, S., Noir, J., Hansen, U., & Aurnou, J. M. 2009, *Nature*, 457, 301

Schmitz, S., & Tilgner, A. 2009, *PhRvE*, 80, 015305

Soderlund, K. M., Sheyko, A., King, E. M., & Aurnou, J. M. 2015, *Progress in Earth and Planetary Science*, 2, 24

Stellmach, S., Lischper, M., Julien, K., et al. 2014, *PhRvL*, 113, 254501

Zhong, J.-Q., Stevens, R. J. A. M., Clercx, H. J. H., et al. 2009, *Physical Review Letters*, 102, 044502

List of Changes

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Replaced: ~~We show~~ replaced with: In section 2, we empirically demonstrate, on page 2.

Added: ~~the~~, on page 2.

Replaced: ~~Paths along constant Ro_c are sensitive to the depth of the domain, but are blind to changes in diffusivities with increasing Ra and Ta . On the other hand,~~ replaced with: Unlike paths through parameter space which hold Ro_c constant,, on page 3.

Added: , although the importance of $Ra/Ta^{3/4} = Ra Ek^{3/2}$ has been independently found by King et al. (2012) using a boundary layer analysis. We compare our results to their theory in Section 4., on page 3.

Added: In our stratified domains, for $Ta \geq 10^5$, a linear instability analysis provides $Ra_{crit} = 1.459Ta^{2/3}$ and $k_{crit} = 0.414Ta^{1/6}$., on page 3.

Replaced: ~~Scaling laws for paths at $Ro_p = 1.58$ ($Ro \approx 0.4$) and $Ro_p = 0.96$ ($Ro \approx 0.1$) are shown. (a) Evolved Nu vs. Ra . The scaling laws here are very reminiscent of classic Rayleigh-Bénard convection theory (Ahlers et al. 2009). (b) Evolved Re vs. Ra . The scaling seen here is nearly identical to scalings in nonrotating convection (AB17).~~
replaced with: Scaling laws for paths at $Ro_p = 1.58$ ($Ro \approx 0.4$), $Ro_p = 0.96$ ($Ro \approx 0.1$), and $Ro_p = 0.6$ ($Ro \approx 0.3$) are shown. Numbers are plotted vs. Ra/Ra_{crit} , where Ra_{crit} is the value at which a given Ro_p path crosses the supercriticality curve (e.g., where the solid orange line crosses the black line in Fig. 1a). (a) The evolved scaling law of Nu approaches $Ra^{1/3}$, as expected from boundary layer theory (King et al. 2012). (b) $Re_{||}$ and Re_{\perp} , as defined in equation 7, are shown. All values of Ro_p trace out similar Re_{\perp} tracks which scale roughly as $Ra^{5/18}$, as expected. $Re_{||}$ tracks experience a constant offset for each different value of Ro_p , but otherwise scale roughly as $Ra^{1/2}$, as anticipated.
, on page ??.

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, on page 6.

Added: (f) We plot the size of the entropy boundary layer divided by the size of the density scale height at the top of the atmosphere. Vertical lines denote when these two length scales become equal, and we find that the size of the boundary layer compared to an important dynamical length scale does not predict any measurable change in system behavior. , on page 4.

Added: (f) We plot the size of the entropy boundary layer divided by the size of the density scale height at the top of the atmosphere. Vertical lines denote when these two length scales become equal, and we find that the size of the boundary layer compared to an important dynamical length scale does not predict any measurable change in system behavior. , on page 4.

Replaced: ~~When $Ro \approx 0.1$, we find a scaling of $Nu \propto Ra^{0.27}$. This is reminiscent of classic scaling laws (e.g., $Ra^{2/7}$) in non-rotating Rayleigh-Bénard convection (Ahlers et al. 2009). This suggests that changes in heat transport at constant Ro_p are driven by changes in the thermal boundary layer structure with increasing Ra .~~ replaced with: We find that $Nu \propto \{Ra^{0.35}, Ra^{0.30}, Ra^{0.25}\}$ for $Ro_p = \{0.6, 0.957, 1.58\}$. As Ro_p , and thus Ro , become smaller, the scaling

of Nu becomes closer to a $Ra^{1/3}$ scaling, which is predicted along paths of constant Ro_p (see King et al. 2012, and section 4)., on page 4.

Added: In rotating convection, the nature of flows is distinctly different parallel to the rotation vector from perpendicular to the rotation vector. As a result, we use two forms of the RMS Reynolds number which capture different length scales,

$$Re_{\parallel} = \frac{|u|L_z}{\nu}, \quad Re_{\perp} = \frac{|u|}{\nu} \frac{2\pi}{k_{\text{crit}}}. \quad (7)$$

From our work in AB17, we expect the RMS velocity to scale as $|u| \propto \sqrt{\Delta S}$. By definition, $\nu \propto \sqrt{Ra/(Pr \Delta S)}$, and L_z is a constant set by the stratification while $k_{\text{crit}} \propto Ta^{1/6}$. Along paths of constant Ro_p , we thus expect $Re_{\parallel} \propto Ra^{1/2}$ and $Re_{\perp} \propto Ra^{5/18}$ when Pr is held constant., on page 4.

Replaced: the RMS Reynold's number ($Re = |u|L_z/\nu$) replaced with: Re_{\parallel} and Re_{\perp} , on page 4.

Replaced: ~~and find that $Re \propto Ra^{0.47}$ in the rotationally constrained regime, which is almost precisely the $Re \propto Ra^{1/2}$ scaling measured in the non-rotating regime in AB17.~~ replaced with: We find that $Re_{\parallel} \propto \{Ra^{0.49}, Ra^{0.47}, Ra^{0.44}\}$ and $Re_{\perp} \propto \{Ra^{0.26}, Ra^{0.25}, Ra^{0.22}\}$ for $Ro_p = \{0.6, 0.957, 1.58\}$. Furthermore, Re_{\perp} falls along the same track for all values of Ro_p studied here, while Re_{\parallel} experiences an offset to larger values as Ro_p shrinks. In general, we find that Re_{\perp} is a good proxy for the horizontal resolution required to run an experiment, while Re_{\parallel} probes the vertical resolution required., on page 4.

Deleted: While increasing Ra and holding Ro_p constant, we find scaling laws of heat transport (Nu) and turbulence (Re) that are nearly identical to scaling laws seen in nonrotational convection. on page 5.

Added: In this work, we experimentally arrived at the $Ra/Ta^{3/4} = RaEk^{3/2}$ scaling which informs the predictive Rossby number, but this relationship was independently discovered by King et al. (2012). They argue that the thermal boundary layer should scale as $\delta_S \propto Ra^{-1/3}$, and that rotational Ekman boundary layers should scale as $\delta_{Ro} \propto Ta^{-1/4} = Ek^{1/2}$, and thus when a “transition parameter,” $Ra/Ta^{3/4} \sim 1$, these boundary layers should cross one another, and they demonstrate that the value of $Ra/Ta^{3/4}$ seems to determine whether your flow is rotationally constrained or not. King et al. (2013) continued their work, and showed how numerous convective flow properties scale with this transition parameter, including the ratio of the thermal to viscous boundary layer. In our current work, we find that despite the added complexity of stratification and despite our using stress-free rather than no-slip boundaries, the boundary layer scaling arguments put forth in King et al. (2012) seem to hold up in our systems. This is reminiscent of what we found in AB17, in which convection in these stratified domains, regardless of Mach number, produced boundary-layer dominated scaling laws of Nu that were nearly identical to the Boussinesq scaling laws., on page 5.

Added: We find that $Ro \propto Ro_p^{3.5}$, which is within the error bounds of the estimate in King et al. (2013), who although defining Ro very differently from our vorticity-based definition here, find $Ro \propto Ro_p^{3.84 \pm 0.28}$. We note briefly that they claim that the value of Ro is strongly dependent upon the Prandtl number studied, and that low Ro can be achieved at high Pr without achieving a rotationally constrained flow. We studied only $Pr = 1$ here, and leave it to future authors to determine if the scaling of $Ro_p \propto Pr^{-1/4}$ is the correct scaling to predict the evolved Rossby number., on page 5.

Added: We close by noting that once Ro_p is chosen such that a convective system has the same Rossby number as an astrophysical object of choice., on page ??.

Added: We thank Jon Aurnou and the anonymous referee who independently directed us to the work of King et al. (2012) during the review process, which greatly helped us understand the results of the experiments here. We further thank the anonymous referee for other helpful and clarifying suggestions., on page ??.