

1. PARAMETERS AND SOME DEFINITIONS

We're going to study penetrative convection in a compressible ideal gas. To start with, let's make some definitions.

1. The Brunt-Vaisala frequency is $N^2 = g\nabla s/c_P$, where g is the gravitational acceleration, ∇s is the entropy gradient, and c_p is the specific heat at constant pressure.
2. the convective frequency is $f_{\text{conv}} = 1/\tau_{\text{conv}}$, where τ_{conv} is a characteristic convective timescale.
3. We will assume the characteristic convective timescale is the timescale of internal heating, $\tau_{\text{conv}} = (\rho L^2/Q)^{1/3}$, where Q is a heating term with units [energy / time / volume], ρ is the density, and L is the characteristic convective length scale.
4. The system will be heated at a constant rate in a layer of depth δ_H so that the flux carried by convection is $F_{\text{conv}} = Q\delta_H$.
5. The mean radiative conductivity varies from a small value κ_{CZ} in the convection zone to a large value κ_{RZ} in the radiative zone.
6. Since there is some flux $F_{\text{bound}} = -\kappa_{\text{CZ}}\nabla T_{\text{ad}}$ conducted along the convective boundary, there is a total flux $F_{\text{tot}} = F_{\text{conv}} + F_{\text{bound}}$ carried in the system. We will define a parameter $\mu = F_{\text{bound}}/F_{\text{conv}}$ so that $F_{\text{tot}} = (1 + \mu)F_{\text{conv}}$.
7. In the radiative zone, all of the flux is carried by radiation, $F_{\text{rad}}^{\text{RZ}} = F_{\text{tot}}$.
8. In the hypothetical adiabatic penetrative zone, the flux is $F_{\text{tot}} = F_{\text{rad}}^{\text{PZ}} + F_{\text{conv}}^{\text{PZ}}$. We define the penetration parameter:

$$\mathcal{P} \equiv -\frac{F_{\text{conv}}}{F_{\text{conv}}^{\text{PZ}}}. \quad (1)$$

Furthermore we note that $F_{\text{rad}}^{\text{PZ}} = F_{\text{rad}}^{\text{RZ}}(\nabla T_{\text{ad}}/\nabla T_{\text{RZ}})$. Since $F_{\text{rad}}^{\text{RZ}} = F_{\text{tot}}$, we get

$$F_{\text{tot}} \left(1 - \frac{\nabla T_{\text{ad}}}{\nabla T_{\text{RZ}}} \right) = -\frac{F_{\text{conv}}^{\text{CZ}}}{\mathcal{P}}. \quad (2)$$

Rearranging, we can define \mathcal{P} as

$$\mathcal{P} = \left[(1 + \mu) \left(\frac{\nabla T_{\text{ad}}}{\nabla T_{\text{RZ}}} - 1 \right) \right]^{-1}. \quad (3)$$

We expect the size of a hypothetical penetration zone to get large when this gets large, and vice versa. Note that it will also be useful to express the ratio of the temperature gradients here:

$$\frac{\nabla T_{\text{ad}}}{\nabla T_{\text{RZ}}} = 1 + [\mathcal{P}(1 + \mu)]^{-1}, \quad (4)$$

so $|\nabla T_{\text{ad}}| > |\nabla T_{\text{RZ}}|$.

9. It is useful to define the stiffness,

$$\mathcal{S} \equiv \frac{N_{\text{RZ}}^2}{f_{\text{conv}}}^2 = \left(\frac{\rho L^2}{Q} \right)^{2/3} \left(\frac{g\nabla s_{\text{RZ}}}{c_P} \right). \quad (5)$$

From the ideal gas equation of state, we know

$$\frac{\nabla s_{\text{RZ}}}{c_P} = \nabla \left[\frac{1}{\gamma} \ln T - \frac{\gamma-1}{\gamma} \ln \rho \right] \rightarrow \frac{1}{T} [\nabla T - \nabla T_{\text{ad}}], \quad (6)$$

where we have assumed that hydrostatic equilibrium applies, $T \nabla \ln \rho = -g \hat{z}/R - \nabla T$ and $\nabla T_{\text{ad}} = -g/c_P \hat{z}$ and $c_P = R\gamma/(\gamma-1)$. So

$$S = \left(\frac{\rho L^2}{Q} \right)^{2/3} \frac{g}{T} (\nabla T_{\text{RZ}} - \nabla T_{\text{ad}}) = \left(\frac{\rho L^2}{Q} \right)^{2/3} \frac{g \nabla T_{\text{ad}}}{T} \left(\frac{1}{1 + [\mathcal{P}(1 + \mu)]^{-1}} - 1 \right). \quad (7)$$

Defining the adiabatic gradient $\nabla_{\text{ad}} = (d \ln T / d \ln P)|_S$, and then $\nabla T_{\text{ad}} = \nabla_{\text{ad}} h^{-1} T$ where $h = (d \ln P / d \ln z)^{-1}$ is the pressure scale height, we get

$$S = \nabla_{\text{ad}} \frac{g \tau_{\text{conv}}^2}{h} \left(\frac{1}{1 + [\mathcal{P}(1 + \mu)]^{-1}} - 1 \right). \quad (8)$$

In hydrostatic equilibrium, $gh = P/\rho = c_s^2/\gamma$, and if we define the convective velocity $u_{\text{conv}} = L/\tau_{\text{conv}}$, we find

$$\frac{g \tau^2}{h} = \frac{1}{\gamma} \frac{c_s^2}{u_{\text{conv}}^2} \left(\frac{L}{h} \right)^2. \quad (9)$$

Defining the mach number of convection $\mathcal{M} = u_{\text{conv}}/c_s$, we find

$$\boxed{S = \mathcal{M}^{-2} \frac{\nabla_{\text{ad}}}{\gamma} \left(\frac{L}{h} \right)^2 \left(\frac{1}{1 + [\mathcal{P}(1 + \mu)]^{-1}} - 1 \right)}. \quad (10)$$