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1. PARAMETERS AND SOME DEFINITIONS

We're going to study penetrative convection in a compressible ideal gas. To start with, let's make some definitions.

1. The Brunt-Vaisaila frequency is $N^2 = g\nabla s/c_P$, where g is the gravitational acceleration, ∇s is the entropy gradient, and c_p is the specific heat at constant pressure.

2. the convective frequency is $f_{\rm conv} = 1/\tau_{\rm conv}$, where $\tau_{\rm conv}$ is a characteristic convective timescale.

3. We will assume the characteristic convective timescale is the timescale of internal heating, $\tau_{\rm conv} = (\rho L^2/Q)^{1/3}$, where Q is a heating term with units [energy / time / volume], ρ is the density, and L is the characteristic convective length scale.

4. The system will be heated at a constant rate in a layer of depth δ_H so that the flux carried by convection is $F_{\rm conv}^{\rm CZ} = Q\delta_H$.

5. The mean radiative conductivity varies from a small value $\kappa_{\rm CZ}$ in the convection zone to a large vale $\kappa_{\rm RZ}$ in the radiative zone.

6. Since there is some flux $F_{\rm bound} = -\kappa_{\rm CZ}\nabla T_{\rm ad}$ conducted along the convective boundary, there is a total flux $F_{\rm tot} = F_{\rm conv}^{\rm CZ} + F_{\rm bound}$ carried in the system. We will define a parameter $\mu = F_{\rm bound}/F_{\rm conv}^{\rm CZ}$ so that $F_{\rm tot} = (1+\mu)F_{\rm conv}^{\rm CZ}$.

7. In the radiative zone, all of the flux is carried by radiation, $F_{\rm rad}^{\rm RZ} = F_{\rm tot}$.

8. In the hypothetical adiabatic penetrative zone, the flux is $F_{\rm tot} = F_{\rm rad}^{\rm PZ} + F_{\rm conv}^{\rm PZ}$. We define the penetration parameter:

$$\mathcal{P} \equiv -\frac{F_{\rm conv}^{\rm CZ}}{F_{\rm conv}^{\rm PZ}}. \tag{1}$$

   Furthermore we note that $F_{\rm rad}^{\rm PZ} = F_{\rm rad}^{\rm RZ}(\nabla T_{\rm ad}/\nabla T_{\rm RZ})$. Since $F_{\rm rad}^{\rm RZ} = F_{\rm tot}$, we get

$$F_{\rm tot}\left(1 - \frac{\nabla T_{\rm ad}}{\nabla T_{\rm RZ}}\right) = -\frac{F_{\rm conv}^{\rm CZ}}{\mathcal{P}}. \tag{2}$$

   Rearranging, we can define $\mathcal{P}$ as

$$\boxed{\mathcal{P} = \left[(1+\mu)\left(\frac{\nabla T_{\rm ad}}{\nabla T_{\rm RZ}} - 1\right)\right]^{-1}}. \tag{3}$$

   We expect the size of a hypothetical penetration zone to get large when this gets large, and vice versa. Note that it will also be useful to express the ratio of the temperature gradients here:

$$\frac{\nabla T_{\rm ad}}{\nabla T_{\rm RZ}} = 1 + [\mathcal{P}(1+\mu)]^{-1}, \tag{4}$$

   so $|\nabla T_{\rm ad}| > |\nabla T_{\rm RZ}|$.

9. It is useful to define the stiffness,

$$\mathcal{S} \equiv \frac{N_{\rm RZ}^2}{f_{\rm conv}^2} = \left(\frac{\rho L^2}{Q}\right)^{2/3}\left(\frac{g\nabla s_{\rm RZ}}{c_P}\right). \tag{5}$$

From the ideal gas equation of state, we know

$$\frac{\nabla s_{\mathrm{RZ}}}{c_P} = \nabla \left[\frac{1}{\gamma} \ln T - \frac{\gamma - 1}{\gamma} \ln \rho \right]_{\mathrm{RZ}} \to \frac{1}{T_{\mathrm{RZ}}} \left[\nabla T_{\mathrm{RZ}} - \nabla T_{\mathrm{ad}} \right], \tag{6}$$

where we have assumed that hydrostatic equilibrium applies, $T \nabla \ln \rho = -g\hat{z}/R - \nabla T$ and $\nabla T_{\mathrm{ad}} = -g/c_P \hat{z}$ and $c_P = R\gamma/(\gamma - 1)$. So

$$\mathcal{S} = \left(\frac{\rho L^2}{Q} \right)^{2/3} \frac{g}{T_{\mathrm{RZ}}} \left(\nabla T_{\mathrm{RZ}} - \nabla T_{\mathrm{ad}} \right) = \left(\frac{\rho L^2}{Q} \right)^{2/3} \frac{g \nabla T_{\mathrm{ad}}}{T_{\mathrm{RZ}}} \left(\frac{1}{1 + [\mathcal{P}(1+\mu)]^{-1}} - 1 \right). \tag{7}$$

Defining the adiabatic gradient $\nabla_{\mathrm{ad}} = (d \ln T / d \ln P)|_{\mathcal{S}}$, and then $\nabla T_{\mathrm{ad}} = \nabla_{\mathrm{ad}} h_{\mathrm{RZ}}^{-1} T_{\mathrm{RZ}}$ where $h = (d \ln P / d \ln z)^{-1}$ is the pressure scale height, and simplifying the parenthetical, we get

$$\boxed{\mathcal{S} = \nabla_{\mathrm{ad}} \frac{(g \tau_{\mathrm{conv}}^2)/h}{1 + \mathcal{P}(1+\mu)}}. \tag{8}$$

Going further, in hydrostatic equilibrium, $g h_{\mathrm{RZ}} = (P/\rho)_{\mathrm{RZ}} = c_{s,\mathrm{RZ}}^2/\gamma$, and if we define the convective velocity $u_{\mathrm{conv}} = L/\tau_{\mathrm{conv}}$, we find

$$\frac{g \tau^2}{h_{\mathrm{RZ}}} = \frac{1}{\gamma} \frac{c_{s,\mathrm{RZ}}^2}{u_{\mathrm{conv}}^2} \left(\frac{L}{h_{\mathrm{RZ}}} \right)^2. \tag{9}$$

Defining the mach number of convection $\mathcal{M} = u_{\mathrm{conv}}/c_{s,\mathrm{RZ}}$ (evaluating 'RZ' terms right at the radiative-convective boundary), we find

$$\boxed{\mathcal{S} = \mathcal{M}^{-2} \frac{\nabla_{\mathrm{ad}}}{\gamma} \left(\frac{L}{h_{\mathrm{RZ}}} \right)^2 \left(\frac{1}{1 + \mathcal{P}(1+\mu)} \right)}. \tag{10}$$