Derivation of Roxburgh's Integral Constraint

1. NAVIER-STOKES EQUATIONS & ENERGY EQUATIONS

Roxburgh (1989) writes the equations of fluid dynamics as

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \rho\mathbf{g} + \frac{\partial\eta_{ij}}{\partial x_j}, \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

$$\rho T\frac{\partial s}{\partial t} + \rho T\mathbf{u}\cdot\nabla s = -\nabla\cdot(\mathbf{F}_{\rm rad}) + \rho\epsilon + \eta_{ij}\frac{\partial u_i}{\partial x_j}, \tag{3}$$

with the viscous stress tensor defined as

$$\eta_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right). \tag{4}$$

Here, ρ is density, T is temperature, P is pressure, s is specific entropy, $\mathbf{u}$ is velocity, μ is the dynamic viscosity, ϵ is energy generation per unit mass, $\mathbf{F}_{\rm rad}$ is radiative flux, and $\mathbf{g}$ is gravity. It is convenient to define

$$\nabla\cdot\mathbf{F}_{\rm tot} = \rho\epsilon \qquad \rightarrow \qquad \mathbf{F}_{\rm tot} = \hat{z}\left(\int\rho\epsilon dz + F_{\rm bot}\right). \tag{5}$$

Here, $\mathbf{F}_{\rm tot}$ is the total energy flux going through the system and $F_{\rm bot}$ is the vertical energy flux imposed at the bottom boundary of the system. In the notation of Roxburgh (1989), $\mathbf{F}_{\rm tot} = \mathbf{\Gamma}$. We will also define the viscous dissipation per unit volume

$$\Phi = \eta_{ij}\frac{\partial u_i}{\partial x_j} \geq 0. \tag{6}$$

Upon using the continuity Eqn. 2 on the energy Eqn. 10, we get

$$T\frac{\partial\rho s}{\partial t} + T\nabla\cdot(\rho s\mathbf{u}) = \nabla\cdot(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad}) + \Phi. \tag{7}$$

Noting that $dE = T\,ds - p\,dV$, where E is the internal energy and $V = 1/\rho$, we can recast the LHS of Eqn. 7 (using continuity a couple times) as

$$\frac{\partial\rho E}{\partial t} + \nabla\cdot(\rho E\mathbf{u}) + P\nabla\cdot\mathbf{u} = \nabla\cdot(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad}) + \Phi. \tag{8}$$

Likewise dotting $\mathbf{u}$ into the momentum Eqn. 1 and using continiuity we get

$$\frac{\partial\mathcal{K}}{\partial t} + \nabla\cdot(\mathbf{u}[\mathcal{K} + P] - \mathbf{u}\cdot\bar{\bar{\eta}}) = P\nabla\cdot\mathbf{u} + \rho\mathbf{u}\cdot\mathbf{g} - \Phi, \tag{9}$$

where $\mathcal{K} = \rho\mathbf{u}\cdot\mathbf{u}/2$ is the kinetic energy. Combining Eqns. 8 and 9, we retrieve the full energy equation,

$$\frac{\partial}{\partial t}(\rho E + \mathcal{K}) + \nabla\cdot(\mathbf{u}[\rho h + \mathcal{K}] - \mathbf{u}\cdot\bar{\bar{\eta}} + \mathbf{F}_{\rm rad} - \mathbf{F}_{\rm tot}) = \rho\mathbf{u}\cdot\mathbf{g}, \tag{10}$$

where $h = E + P/\rho$ is the enthalpy. The various flux terms which arise here are discussed around equation 8 of Anders & Brown (2017).

2. CONSTRAINTS

We will now use Eqn. 7 & 10 to derive some constraints on convecting regions. I won't handle Eqn. 10 too carefully here (see Roxburgh (1989) eqns 11-13). Suffice to say that if you integrate over a convecting region V with surface area Σ, and you assume statistically stationary convection and $\mathbf{u} = 0$ at the boundaries of the convecting region, you find

$$\mathbf{F}_{\rm rad} = \mathbf{F}_{\rm tot} \text{ on } \Sigma. \tag{11}$$

This is the simple statement that energy is conserved, and the flux is carried radiatively if there are no convective motions.

With this in mind, we examine Eqn. 7 in more detail. We divide through by T and integrate over a volume V to find eqn 10 in Roxburgh (1989),

$$\frac{\partial}{\partial t}\int_V \rho s \, dV = \int_V \frac{1}{T}\nabla\cdot(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\, dV + \int_V \frac{1}{T}\Phi\, dV. \tag{12}$$

Note

$$\begin{aligned}\int_V \frac{1}{T}\nabla\cdot(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\, dV &= \int_V \nabla\cdot\left(\frac{1}{T}[\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad}]\right) dV - \int_V (\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\nabla\frac{1}{T}\, dV \\ &= \int_\Sigma \frac{1}{T}(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\, dA + \int_V \frac{1}{T^2}(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\cdot\nabla T\, dV.\end{aligned}$$

From the above constraint (Eqn. 11), the first term is zero, and we can write

$$\frac{\partial}{\partial t}\int_V \rho s\, dV = \int_V \frac{1}{T^2}(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\cdot\nabla T\, dV + \int_V \frac{1}{T}\Phi\, dV. \tag{13}$$

In a statistically-stationary state, the first term (the time derivative) is also zero, and so we can rearrange to arrive at the general form of Roxburgh's integral constraint,

$$\boxed{-\int_V \frac{1}{T^2}(\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad})\cdot\nabla T\, dV = \int_V \frac{1}{T}\Phi\, dV}. \tag{14}$$

Note that $\mathbf{F}_{\rm tot} - \mathbf{F}_{\rm rad} = \mathbf{F}_{\rm conv}$ (including all contributions to the convective flux, like enthalpy flux, kinetic energy flux, viscous flux). Taking a horizontal average of the constraint (and assuming that there is no net horizontal flux), we can write the constraint

$$\overline{-\int_V \frac{1}{T}F_{\rm conv}\frac{\partial \ln T}{\partial z}\, dV} = \overline{\int_V \frac{1}{T}\Phi\, dV} \tag{15}$$

defining $\nabla \equiv d\ln T/d\ln P$ and $h = dz/d\ln P$ we can rewrite this as

$$\overline{-\int_V \frac{\nabla}{hT}F_{\rm conv}\, dV} = \overline{\int_V \frac{1}{T}\Phi\, dV}, \tag{16}$$

where presumably $\nabla = \nabla_{\rm ad}$, a constant, in the convection zone. Both h and T increase with depth, so these integrals are weighted in such a way that they have large contributions in the less dense, upper regions of a convection zone.

REFERENCES

Anders, E. H., & Brown, B. P. 2017, Physical Review Fluids, 2, 083501, doi: 10.1103/PhysRevFluids.2.083501

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