Convective heat transport in stratified atmospheres at low and high Mach number

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This is where an abstract defining what we're doing with stratified convection and Nusselt numbers will go.

INTRODUCTION

Convection is ubiquitous among natural systems such as stellar envelopes and planetary atmospheres. Compressible convection in such stratified atmospheres exhibits behavior which is much more complex than what is seen in the widely-studied Rayleigh-Bénard convection. Upflows and downflows are asymmetric, the well-defined Rayleigh Number (Ra, the ratio of buoyant driving to diffusive damping) becomes less defined and depth-dependent, and the Mach number of the flows becomes an additional control parameter. Early studies of stratified convection in two [1–4] and three [5–7] dimensions utilized polytropically stratified atmospheres, in which the temperature gradient, thermal diffusivity, and gravity are constant throughout the depth of the atmosphere and the atmosphere is initially in hydrostatic equilibrium. While often abandoned for atmospheres constructed using more realistic radiative transfer properties (cite some papers), the polytrope is a particularly useful reference state for stratified convection studies.

A canonical value of the "polytropic index" used in the study of convection of monatomic ideal gases is $m = 1$, such that the density and temperature profile are both linear. Recently, [8] argued that as the polytropic index approaches the adiabatic value of $m_{ad} = 1.5$, the convective flux of the system becomes negligible compared to the background flux gradient, and the Nusselt number of the system approaches zero. They argue that, as a result, values of $m \to -1$ are ideal polytropes to study, as the radiative flux approaches zero and the convective flux carries the whole of the system flux in equilibrium.

Here, we argue for The definition of a new Nusselt number, where the radiative flux is defined relative to the *adiabatic* state rather than to a linear temperature profile. We demonstrate that values of m close to adiabatic show a Nusselt number (similar?) to [9] and that such values of m are useful in allowing us to probe low-Mach number convection, such as that deep within the Sun's convection zone (cite a paper for that?)

MODEL & EQUATIONS

We study a fluid whose equation of state is that of an ideal gas, $P = R^*\rho T$ and whose initial stratification is polytropic, where

$$\begin{aligned} \rho_0(z) &= \rho_{00}(z_0 - z)^m \\ T_0(z) &= T_{00}(z_0 - z) \end{aligned} \tag{1}$$

and z increases upwards within the bounds $z = \{0, L_z\}$. The height of the atmosphere is set by specifying the number of density scale heights it spans, n_ρ, which we set to be three throughout this study such that the density at the bottom of the atmosphere is larger than that at the top of the atmosphere by roughly a factor of 20. Variables are nondimensionalized at the top of the atmosphere as $P_0(L_z) = \rho_0(L_z) = T_0(L_z) = 1$, requiring $z_0 \equiv L_z + 1$ and $R^* = T_{00} = \rho_{00} = 1$. The polytropic index is set by the adiabatic index of a monatomic ideal gas, $\gamma = 5/3$ and the control parameter ϵ, such that $m = (\gamma-1)^{-1} - \epsilon = m_{ad} - \epsilon$ and the subsequent entropy gradient at the top of the atmosphere is $\nabla S(L_z) = -\epsilon$.

The primary assumptions utilized in constructing a polytropic atmosphere are that acceleration due to gravity is constant and that the conductive heat flux, $F_{\text{cond}} = -\kappa \nabla T_0$ is constant throughout the atmosphere. To ensure the second condition is met, it is traditional to use a constant value of κ and ∇T_0. The thermal conductivity is controlled by the non-dimensional Rayleigh number,

$$\text{Ra} = \frac{g L_z^3 (\Delta S_0 / c_P)}{\nu \chi}, \tag{2}$$

where ΔS_0 is the entropy jump across the polytropic atmosphere, ν is the kinematic viscosity and κ is the thermal diffusivity. The relationship between the thermal and viscous diffusivities is set by the Prandtl number, Pr$= \nu/\chi$, which we take to be one throughout this study. The thermal and viscous diffusion coefficients are related to the dynamic viscosity, μ and κ by the ratio $\nu \equiv \mu/\rho$ and $\chi \equiv \mu rho$. As a result, $\text{Ra} \propto (\nu\chi)^{-1} \propto \rho^2$, such that for our atmospheres with $n_\rho = 3$, the Rayleigh number increases by a factor of approximately 400 from the top of the domain to the bottom of the domain.

At the constant values of n_ρ and Pr used, the primary control parameters of convection are ϵ and Ra. We decompose our atmosphere into the background polytrope (ρ_0, T_0) and the fluctuations about that background $(\boldsymbol{u}, \rho_1, T_1)$. The scaling of the entropy gradient with ϵ is reflected in the evolved values of these fluctuations, which follow the scaling of $\text{Ma}^{1/2} \propto T_1/T_0 \propto \rho_1/\rho_0 \propto \epsilon$, and which scale weakly with Ra (prove it, we have the data).

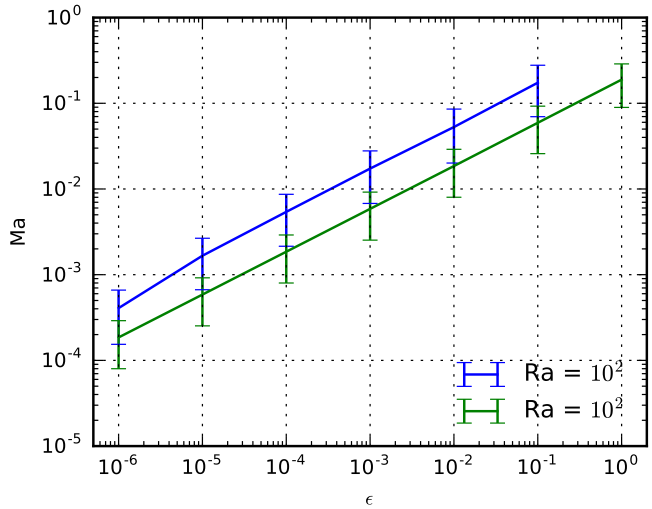

FIG. 1. Shown are characteristic mach numbers from IVPs spanning six decades of ϵ, from very low Mach number to near Mach one. A very strong Ma $\propto \epsilon^{1/2}$ relation is clear. (need to actually do linear regression)

We start with initial conditions of random small perturbations compared to ϵ in the temperature field. We evolve the Fully Compressible Navier-Stokes equations with an energy-conserving energy equation, which take the form:

$$\frac{D\ln\rho}{Dt} + \nabla\cdot(\boldsymbol{u}) = 0 \tag{3}$$

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla P + \rho\boldsymbol{g} - \nabla\cdot\bar{\bar{\boldsymbol{\Pi}}} \tag{4}$$

$$\rho c_V\left(\frac{DT}{Dt} + (\gamma-1)T\nabla\cdot(\boldsymbol{u})\right) + \nabla\cdot(-\kappa\nabla T) = -\left(\bar{\bar{\boldsymbol{\Pi}}}\cdot\nabla\right)\cdot\boldsymbol{u} \tag{5}$$

where $D/Dt \equiv \partial_t + \boldsymbol{u}\cdot\nabla$ and the viscous stress tensor is defined as

$$\Pi_{ij} \equiv -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot(\boldsymbol{u})\right). \tag{6}$$

In such stratified systems, the total convective flux can be defined as

$$\boldsymbol{F}_{\text{conv}} = \boldsymbol{F}_{\text{enth}} + \boldsymbol{F}_{\text{KE}} + \boldsymbol{F}_{\text{PE}} + \boldsymbol{F}_{\text{visc}}, \tag{7}$$

where $\boldsymbol{F}_{\text{enth}} \equiv \rho\boldsymbol{u}(c_V T + P/\rho)$ is the enthalpy flux, $\boldsymbol{F}_{\text{KE}} \equiv \rho|\boldsymbol{u}|^2\boldsymbol{u}$ is the kinetic energy flux, $\boldsymbol{F}_{\text{PE}} \equiv \rho\boldsymbol{u}\phi$ is the potential energy flux (with $\phi \equiv -gz$), and $\boldsymbol{F}_{\text{visc}} \equiv \boldsymbol{u}\cdot\bar{\bar{\boldsymbol{\Pi}}}$ is the viscous flux. Dotting Eq. 4 with $\boldsymbol{u}$ and adding it to Eq. 5, we retrieve the full energy equation in conservation form,

$$\frac{\partial}{\partial t}\left(\rho\left[\frac{|\boldsymbol{u}|^2}{2} + c_V T + \phi\right]\right) + \nabla\cdot(\boldsymbol{F}_{\text{conv}} + \boldsymbol{F}_{\text{rad}}) = 0 \tag{8}$$

where $\boldsymbol{F}_{\text{rad}} = -\kappa\nabla T$.

The efficiency of convection is defined by the Nusselt number. While the Nusselt number is well-defined in Rayleigh-Bénard convection as the amount of total flux divided by the steady-state background conductive flux [9, 10], a well-defined Nusselt number is more elusive in stratified convection. Convection generally works to drive the overall *entropy* gradient to zero. In the case of the Boussinesq approximation of Rayleigh-Bénard convection, $\nabla S = 0$ implies that $\nabla T = 0$ because density is no longer a factor in determining the stratification. However, once compressibility and stratification are considered, convection works to cause $\nabla S = 0$ but *not* necessarily $\nabla T = 0$. As such, even perfectly efficient convection will still have some amount of adiabatic conductive flux, $F_A \equiv -\kappa\nabla T_{ad}$. In the case of an initially superadiabatic polytrope, assuming that hydrostatic equilibrium is conserved to first order and that the system evolves towards a new, adiabatic polytropic state such that the temperature gradient takes the form $\nabla T_{\text{ad}} = -g/c_P = \nabla T_0(1-\epsilon(m_{ad}+1))$. In order to obtain a meaningful definition of the conductive flux, this constant adiabatic component must be subtracted, which yields $\boldsymbol{F}_{\text{rad}} - \boldsymbol{F}_{\text{ad}} = -\kappa\nabla T_1 - \kappa\nabla T_0\epsilon(m_{ad}+1)$. Unlike the full radiative flux, which is $O(\kappa\nabla T)$, this is $O(\kappa\nabla T_1) \approx O(\epsilon)$, and it is thus comparable to the other fluxes in the system, as would be expected. Acknowledging this, we define the Nusselt number in the same form as [3], who wrote it as

$$N \equiv \frac{F_{\text{conv, z}} + F_{\text{rad, z}} - F_A}{F - F_A}, \tag{9}$$

where $F = \Delta T/Lz$ is the radiative flux carried by a linear temperature profile from the top boundary value to the bottom boundary value. It is important to note that such a definition explains the results of [8], who noted that as $m \to -1$, $F_{\text{rad}} \to 0$ and F_{conv} becomes large. As $m \to -1$, $g \to 0$, such that the adiabatic flux become zero and the Nusselt number definition reverts to a form more similar to that of Rayleigh-Bénard convection.

WE USE DEDALUS, EXPLAIN WHAT IT ARE

EXPLAIN BOUNDARY CONDITIONS

EXPLAIN WHAT THE CHARACTERISTIC TIME SCALE, THE BUOYANCY TIME, IS

RESULTS

We ran initial value problems for a few hundreds of buoyancy times past the convective transients from Rayleigh numbers around R_{crit} up to Rayleigh numbers of $\approx 10^7 R_{crit}$. While bulk thermodynamic structures are similar between low and high ϵ, high ϵ runs start to exhibit shock fronts propagating away from downflow chan-

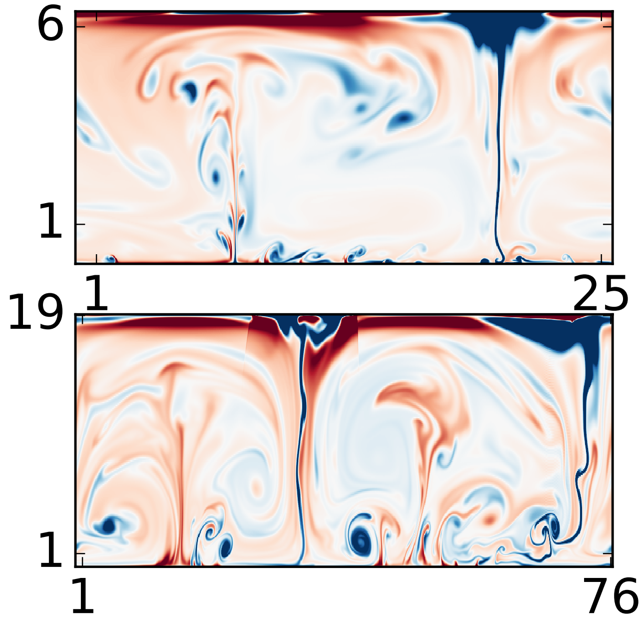

FIG. 2. Two characteristic snapshots after about 200 buoyancy times are shown at $\epsilon = 10^{-4}$ (top) and $\epsilon = 0.5$ (bottom) at Ra $= 10^6$.

nels, such as those in Fig. 2, as reported in (cite Juri's paper from 1990 that has this).

Despite diferent thermodynamic structures, the fluxes look fairly similar at low and high mach number (sort of? I don't think the ones I have are averaged over enough time, especially the eps=0.5 one, but I Pleiades is a bit slammed right now). See Fig 3

At low Rayleigh number, diffusivities are high and the

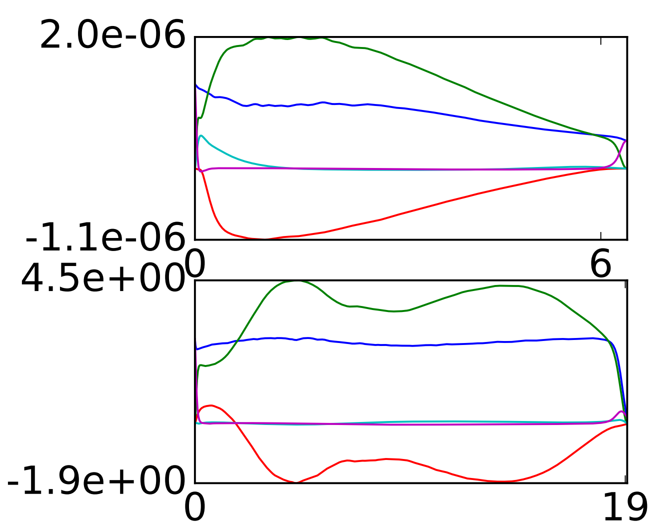

FIG. 3. Flux profiles for low (top) and high (bottom) Mach number flows.

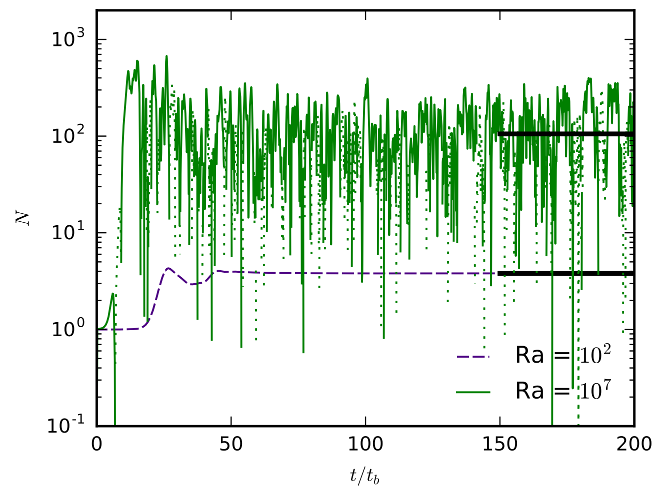

FIG. 4. The evolution of the Nusselt number is shown at low (10^2) and high (10^7) Rayleigh number and at $\epsilon = 10^{-4}$. At low Rayleigh number, the system is able to settle into a time invariant solution. As the Rayleigh number is raised, the thermodynamic structure become increasingly complex and time variant, and a time-average of the Nusselt number is required to obtain a sensible mean.

flows are very laminar. Such flows often achieve a steady state and have a well-defined Nusselt number which is independent of time. However, as the Rayleigh number increases, the flows become increasingly time-dependent. Even steady structures such as solid "rolls" like those pictured in Fig. 2 have highly time-dependent Nusselt numbers. This is, in part, due to the fact that cold downdrafts floating to the bottom of the domain can be entrained by upflows, or warm risen parcels can be entrainedi n the intense cold downdrafts. Such events reverse the preferred direction of flux in the system, and even let the Nusselt number become negative for short periods of time. See Fig. 4. As a result, it is necessary to take a long time average of the fluxes before calculating the Nusselt number at higher Rayleigh number.

The evolution of the Nusselt number as the Rayleigh number is increased is shown for both high and low ϵ in Fig. 5. Below convective onset, the Nusselt number is perfectly one. Just above onset, there is a brief range of highly inflated scaling between N and Ra. From about $10R_{crit}$ to roughly $10^{4-5}R_{crit}$, Ra and N follow the relationship: (put a power law here) Above about 10^5R_{crit}, the Nusselt number flattens out as Ra is increased – perhaps this is some Featherstone 2016 shennanigans.

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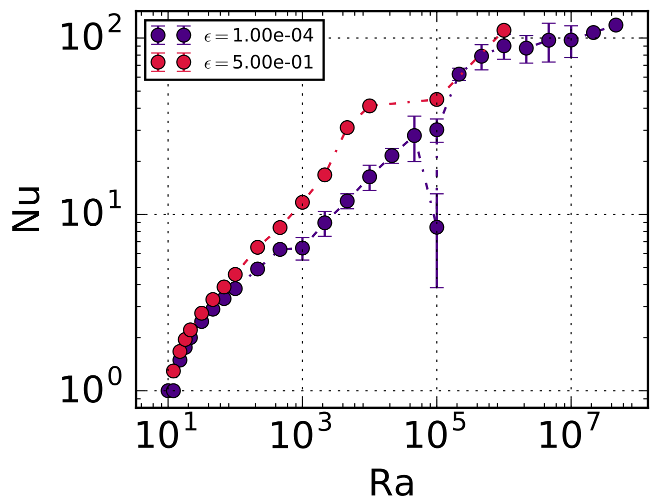

FIG. 5. Variation of the Nusselt number as a function of the Rayleigh number is shown.