Meeting Notes

Looked at Since Last Meeting

- **Big Update to Linkage Renderer**

  The linkage rendering has been accomplished, and it has grown in scope fairly substantially. Namely, the broken 4-gon from before now renders nicely going from this:

  C:/Users/evan/Documents/configuration-spaces/images/image-2020021

  Figure 1:

  To this; fixed, with points highlighted and antialiasing enabled:

  C:/Users/evan/Documents/configuration-spaces/images/image-2020022

  Figure 2:

  Obviously this is now functional, but I also have added a 'show alternate switch option' so we can see this:

  C:/Users/evan/Documents/configuration-spaces/images/image-2020022

  Figure 3:

  And also the coordinates and lengths can be edited with this user interface:

  This is all good, but the problem is that the lengths and coordinates can be altered to provide configurations that don't fulfill this equation: $\sum_{k=i}^{j} \boldsymbol{\alpha}_k \ell_k = 0$ that ensures that the polygon connects at the end.

C:/Users/evan/Documents/configuration-spaces/images/image-2020022

Figure 4:

- **Thinking of Range Limiting and it's Topological Consequences**

  Leading on from the last point, I believe it is possible to calculate analytically (geometrically) the range of each linkage, and using a conjecture from Kevin Walker about how the order of linkages (need to check this) doesn't matter, essentially as addition in $\mathbb{R}^2$ is commutative.

Current Problems

- Describing linkages for orientable 2-manifolds.
- Work through the ideas behind the 'range limiting' of the linkage renderer. Could this lead to a way to iterate different manifolds?

Current Tasks

- Add range limiters to the coordinate selector in the linkage renderer so the linkage can't be put into unattainable positions.
- Fix the dynamic size choice in the linkage renderer
- Add angle helper class to the linkage renderer

Links

Kapovich and Millson
https://arxiv.org/pdf/math/9803150.pdf

Robert Ghrist
https://www.math.upenn.edu/~ghrist/EAT/EATchapter1.pdf

Gaiane Panina
http://amj.math.stonybrook.edu/pdf-Springer-final/017-0070.pdf

Kevin Walker
https://canyon23.net/math/1985thesis.pdf

"On the Conjecture of Kevin Walker" unread
https://arxiv.org/pdf/0708.2995.pdf

Some Lecture Notes from a Course on this, talks about chambers in a way that sounds good unread
http://homepages.warwick.ac.uk/~maskas/courses/schuetz1.pdf

Project Notes

Thinking about the Fundamental Group, Free Products and the $(C_4, (a, b, a, b))$ Looking at this manifold:

`C:/Users/evan/Documents/configuration-spaces/images/image-2020021`

Figure 5:

We can find a homotopy between this and the $\bigwedge_3 \mathbb{S}^1$ as follows: somehow I think I might want to redraw this

`C:/Users/evan/Documents/configuration-spaces/images/image-2020022`

Figure 6:

This then means we can apply the Siefert-van Kampen Theorem and get that $\Pi_1(M((C_4, (a, b, a, b)))) \simeq *_3\mathbb{Z}$.

This is interesting if we remember the meaning of the configuration space and the fundamental group.

We first have to follow our fixed point of intersection (red in the diagram above) and work out what that means in our linkage. This is a position where the switch has only one state. I think without loss of generality, depending on how we manipulate our manifold, this will be the position where the coordinate of the first linkage is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and the structure follows a straight line.

Given the free product and the fundamental group, this leads to an idea of a 'loop' in our linkage where we end up back at this fixed point, and the fact it's an $\bigwedge_3 \mathbb{S}^1$ implies that there is 3 unique ways of 'looping' back to this position.

To outline a potential form these three loops might take:

non complete

Describing the Homotopy between the 1-arm, 2-arm and n-arm and the $\mathbb{T}^n$ work in progress__

Describing the Homotopy between a $(C4, (a,b,a,b))$andtheIntersecting\mathbb S^1$'s__ I believe this holds for all $(C_4, (a, b, a, b))$ with $a, b \in \mathbb{R}^+$ and $a \neq b$.

This provides an interesting configuration space $M((C_4, (a, b, a, b)))$, which we'll shorten to M for this exercise, that is composed of a two $\mathbb{S}^1$'s which are disjoint besides two points (call this manifold X for now). This looks like so:

`C:/Users/evan/Documents/configuration-spaces/images/image-2020021`

Figure 7:

To prove this we have to find a pair of maps:

$$f : MX g : XM s.t. f \circ g \simeq Id_X \, and \, g \circ f \simeq Id_M$$

This is not too hard, using the definition of the Moduli Space that I have outlined, we can see that any position of the manifold can be indexed with two values, as we have 2 degrees of freedom.

The first value is the angle subtended between $\hat{e}_y$ and the direction of the link connected to the base on the left. add a diagram here using `Polygon` with angle helpers

The second value can be expressed with a Boolean, as we have a switch formed with the remaining two links, however, the two possible values are equivalent if the first value is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. This happens because the first two links line up (add diagram) meaning that the angle between the 3rd and 4th link is either 0 or π meaning that flipping it doesn't produce a new state.

Hence:

1. Any point (a_1, a_2) in $M((C_4, (2, 1, 2, 1)))$ is also in $\mathbb{S}^1 \times \mathbb{S}^0$ but fixing $(0, 1) = (0, 0)$ and $(\pi, 0) = (\pi, 1)$.
2. Any point $\boldsymbol{x}$ in X is on one of the $\mathbb{S}^{\mathbb{1}}$'s that X is composed of, at an angle we'll call θ.

Without loss of generality, as the manifold is the same if you rotate it $\frac{\pi}{2}$ about the y-axis, we can specify the $\mathbb{S}^1$'s that make up X and call them B and Y (blue and yellow matching the diagram).

Then define:

$$f : MXf := f((a_1, a_2)) = \begin{cases} \theta = a_1 \text{ around } Y \text{ if } a_2 = 0 \\ \theta = a_1 \text{ around } B \text{ if } a_2 = 1 \\ \theta = a_1 \text{ in } B \cap Y \text{ if } a_1 = 0 \text{ or } a_1 = \pi \end{cases}$$

and:

$$g : XMg := g(\boldsymbol{x}) = (a_1, a_2)\text{where } a_1 \text{ is } \theta \text{ and } a_2 = \begin{cases} 0 \text{ if } \boldsymbol{x} \in Y \\ 1 \text{ if } \boldsymbol{x} \in B/Y \end{cases}$$

Now evaluating:

$$(g \circ f)((a_1, a_2)) = g\left(\boldsymbol{x} := \begin{cases} \theta = a_1 \text{ around } Y \text{ if } a_2 = 0 \\ \theta = a_1 \text{ around } B \text{ if } a_2 = 1 \\ \theta = a_1 \text{ in } B \cap Y \text{ if } a_1 = 0 \text{ or } a_1 = \pi \end{cases}\right)$$

$=$

$\left(\theta, \begin{cases} 0 \text{ if } \boldsymbol{x} \in Y \\ 1 \text{ if } \boldsymbol{x} \in B \end{cases}\right)$

$= (\mathrm{a}_1, a_2)$

$=> \mathrm{g} \quad \circ f \simeq Id_M$

and:

$$(f \circ g)(\boldsymbol{x}) = f\left(\left(\theta, \begin{cases} 0 \text{ if } \boldsymbol{x} \in Y \\ 1 \text{ if } \boldsymbol{x} \in B/Y \end{cases}\right)\right) = \boldsymbol{x} => f \circ g \simeq Id_X$$

Proving that $M \simeq X$.

Describing the homotopy between the switch and $\mathbb{S}^0$ work in progress

Definition and Intuition of Nash Isomorphisms A Nash function f on a set U satisfies there exists a polynomial P in x and $f(x)$ such that $\forall x \in U, \quad P(x, f(x)) = 0$

A Nash Manifold is an example of a U from the above definition that also has the structure of a manifold.

From Wikipedia, this result looks like it could be helpful:

"More generally, a smooth manifold admits a Nash manifold structure if and only if it is diffeomorphic to the interior of some compact smooth manifold possibly with boundary."

https://en.wikipedia.org/wiki/Nash_functions

Working with the 5-Cycle Linkage to get 2-Manifolds **work in progress**

The Classification Theorem for 2-Manifolds We can describe all 2-manifolds with a genus and an orientability. Probably write more and some drawings here if relevant, delete in a couple weeks if not going to be used.

Current Working Definition of a Linkage, Configuration Space and the State of a Linkage A linkage is composed of a SCaF graph L paired with a length function on the edges (ℓ).

A configuration or moduli space $C((L,\ell))$ is defined like so:
$M(L,\ell) := \{(\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n) \in (\mathbb{S}^1)^n | for\, any\, cycle\, l_i, ..., l_j\, in\, L : \sum_{k=i}^{j} \boldsymbol{\alpha}_k \ell_k = 0\}$ modulo distance preserving isometries of the plane.

When we work with a linkage we need a way to consistently talk about the position that the links are in. The way I will work with this is to consider the fixed link to be oriented in the x-axis and then measure the angle that each link makes to the y-axis clockwise, working with the links in chain from the left of the base.

For now I am going to ignore forks, as they would require some more complex thought to decide how to work with their state.

If this is a polygon we will end up working round to the other side of the base link. Note that; unless the 3rd to last link ends up in the same position as the right of the base, we will be able to index the last two links using a single Boolean value as it will form a switch (two fixed points with two links between them).

If this is not a polygon then we have either an n-arm or nothing connected to both sides of the base link, and we can define these working with the left one, followed by the right one looking at the angles and working up through each arm in turn.

Examples of Manifolds and the Linkages that Display them as Configuration Spaces $\mathbb{S}^1$: A manifold that leads to a $\mathbb{S}^1$ configuration space is easy to find, as it is the configuration space of a single linkage at the origin.

$\mathbb{S}^2$?

$\mathbb{S}^1 \bigcup \mathbb{S}^1 = T^2$: A manifold with the configuration space two disjoint spheres is also pretty okay. This is the linkage represented by a pair of rigid links connected by a linkage that pivots freely around any angle.

What Happens to the Configuration Space as one modifies the Linkage "Adding to the Chain"

When we add another link onto the end of a simple chain of length n we go from a configuration space of $\bigcup_1^n \mathbb{S}^1 = T^n$ to one of $\bigcup_1^{n+1} \mathbb{S}^1 = T^{n+1}$, in fact, it seems that if we add another link to a system with a configuration space M we will get a configuration space of $M \bigcup S^1$. (prove this?)

Is any Manifold Diffeomorphic to the Configuration Space of a well defined Linkage? Robert Ghrist states on page 12 of his book *Elementary*

`C:/Users/evan/Documents/configuration-spaces/images/image-2020020252237`

Applied Topology that:

https://www.math.upenn.edu/~ghrist/EAT/EATchapter1.pdf

However, Kapovich and Millson disagree with this in their paper below:

`C:/Users/evan/Documents/configuration-spaces/images/image-2020020`

Figure 8:

https://arxiv.org/pdf/math/9803150.pdf

At first glace I believe this is essentially telling us that not every manifold can be expressed as the configuration of some linkage as these configuration spaces can be transformed in some way (corresponding to moving the linkage in the plane in which it resides, like flipping it perhaps) and thus for any manifold that can't admit this transformation, it can't be diffeomorphic to a configuration space.

I think the best way to work out what is going on here is to work with the definitions of configuration space of a linkage, and decide if this relates to the differing theorem. It seems unlikely that Ghrist is stating his theorem more strongly without reason, given the cited source is Kapovich and Millson's paper.

Robert Ghrist Defines a Configuration Space: Robert Ghrist defines a configuration space as follows (also page 12 of his book):

C:/Users/evan/Documents/configuration-spaces/images/image-2020020254030088 pm.png

https://www.math.upenn.edu/~ghrist/EAT/EATchapter1.pdf

This has some points to take away, namely we **are** assigning a distinct point to each layout of the linkage, but we **are not** considering rotations and reflections of the plane as different configurations.

Also, the point *'almost always a manifold'* should also be addressed, this can be narrowed down by assuming that the linkages we look at can never fit into a straight line; this can be found here http://amj.math.stonybrook.edu/pdf-Springer-final/017-0070.pdf in Gaiane Panina's paper, but should probably trace the reference to find it's original source, likely Kapovich and Millson.

Kapovich and Millson Define a Moduli Space Firstly, we should address the difference in terminology used by the authors here, it seems that in this context, looking at linkages the Moduli Space and the Configuration Space are the same thing.

This is strongly suggested in this paper by Gaiane Panina on the Moduli Space of Planar Linkages here:

C:/Users/evan/Documents/configuration-spaces/images/image-2020020

Figure 9:

http://amj.math.stonybrook.edu/pdf-Springer-final/017-0070.pdf

However, Kapovich and Millson definitely make a point about the