MATH 391: Probability
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Final Technical Report
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Poisson Processes

1 Introduction

Spatial statistics is a branch of statistics that applies concepts to understand data in space and time. One fundamental part of spatial statistics are Poisson processes, which fall under a larger category of the theory of point processes, which model event occurrences in space and time. These stochastic processes help statisticians understand and model these possibly unpredictable events in various fields of study, such as physics, biology, etc. Poisson processes are extremely useful and are widely applied to solve problems relating to spaciotemporal randomness.

Take a moment to think about where you might have seen these events in everyday life. Perhaps you are at the grocery store and wonder why it's so darn busy at a certain hour? Or you wonder when a family member will call you next? Or even how earthquakes are disperesed where you live? These can all be modeled by a Poisson process, where these events are thought of as stochastic. Poisson processes are used in many different areas and help model examples like these in the world. However, there is also a lot of theory behind the properties of Poisson processes and how they are constructed.

In this report, we will investigate the relationships between occurrences in Poisson processes and time. In particular, how does the expected arrival of "points" in a Poisson process change through conditioning, superposition, thinning, etc. Not only will we think of Poisson processes as occurrences, but also in terms of interarrival times of events. We will describe Poisson processes in one dimension, as well as generalized in a higher dimension d, and finally relate nonhomogeneous Poisson processes to homogeneous Poisson processes. Not only will we discuss the theory behind Poisson processes, but also build a larger comprehension of the topic while discovering their significance in real world applications and relating them to other Math 391 topics!

2 Math 391 Calculations

We will start by defining Poisson point processes.

Def. 1 (Poisson point process): A sequence of arrivals in a continuous time interval is a *Poisson point process* with rate $\lambda > 0$ if the following conditions hold:

1) The number of arrivals in an interval of length t has a Poisson distribution with rate $\lambda \cdot t$.

2) The number of arrivals in disjoint time intervals are independent.

Poisson point processes are in one dimension. They are commonly used in applications such as counting the number of times an event occurs. We will refer to these as simply Poisson processes in the majority of this section, since the word "point" gets dropped often in the literature. However, we will define the Poisson process as generalizable to any dimension.

Recall that if the random variable $X \sim \text{Pois}(\lambda)$, then by definitions of probability mass function, expectation, and variance, we have

$$\mathbf{p}_X(k) = \begin{cases} e^{-\lambda}\frac{\lambda^k}{k!} & \text{if } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{E}[X] = \sum_k k\, P(X=k) = \sum_k k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda e^{-\lambda}\sum_k \frac{\lambda^{k-1}}{(k-1)!} = \lambda,$$

and

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \left(\sum_k k^2\, P(X=k)\right) - \lambda^2 \\
&= \left(\sum_k k^2 e^{-\lambda}\frac{\lambda^k}{k!}\right) - \lambda^2 \\
&= \left(\sum_k (k^2+0) e^{-\lambda}\frac{\lambda^k}{k!}\right) - \lambda^2 && \text{by additive identity,} \\
&= \left(\sum_k (k^2-k+k) e^{-\lambda}\frac{\lambda^k}{k!}\right) - \lambda^2 \\
&= \left(\sum_k (k(k-1)+k) e^{-\lambda}\frac{\lambda^k}{k!}\right) - \lambda^2 && \text{factor out } k, \\
&= \left(\sum_k (k(k-1)) e^{-\lambda}\frac{\lambda^k}{k!}\right) + \left(\sum_k k e^{-\lambda}\frac{\lambda^k}{k!}\right) - \lambda^2 && \text{distribute and expand,} \\
&= \left(\sum_k e^{-\lambda}\frac{\lambda^k}{(k-2)!}\right) + \mathbb{E}[X] - \lambda^2 \\
&= \left(\lambda^2 e^{-\lambda}\sum_k \frac{\lambda^{k-2}}{(k-2)!}\right) + \lambda - \lambda^2 \\
&= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda - \lambda^2 \\
&= \lambda.
\end{aligned}$$

Now we can explore the concept of conditioning in the context of Poisson processes. For a Poisson process, we know that if we have two disjoint intervals, the number of arrivals are independent. However, how does the expected number of arrivals change when conditioning on the number of arrivals in another interval?

Thm. 1 (Conditioning): We will use the notation $N(A)$ to denote the number of points inside an interval A. For a Poisson process, the conditional distribution $N(t_1)$ given $N(t_2) = n$ is

$$N(t_1) \mid N(t_2) = n \sim \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right)$$

Proof. The claim is that the number of points in the interval $(0, t_1]$ given that the number of points in the interval $(0, t_2]$ is n will be a binomial distribution with parameters n and t_1/t_2. We know that $N(t_1)$ is independent of $N(t_2 - t_1)$, since $t_2 - t_1$ and t_1 are disjoint. Therefore we have distributions $\mathrm{Pois}(\lambda t_1)$ and $\mathrm{Pois}(\lambda(t_2 - t_1))$, with their sum being the number of arrivals in the interval $(0, t_2]$, which we know has distribution $\mathrm{Pois}(\lambda t_2)$.

Now to find the conditional distribution of X given $X + Y$, we can work out the following. Let $X = \mathrm{Pois}(\lambda t_1)$ and $Y = \mathrm{Pois}(\lambda t_1) + \mathrm{Pois}(\lambda(t_2 - t_1))$, to get the conditional probability mass function, we have

$$P(X=k \mid X+Y=n) = \frac{P(X+Y=n \mid X=k)P(X=k)}{P(X+Y=n)}$$
$$= \frac{P(Y=n-k \mid X=k)P(X=k)}{P(X+Y=n)}.$$

Plugging in what we know about X and Y along with the fact that the sum of two independent Poisson random variables with parameters λ_1 and λ_2 is a Poisson random variable with parameter $\lambda_1 + \lambda_2$, we get

$$P(X=k \mid X+Y=n) = \frac{\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\frac{e^{-\lambda_1}\lambda_1^k}{k!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$
$$= \frac{n!}{(n-k)!\,k!}\frac{e^{-\lambda_2}e^{-\lambda_1}\lambda_2^{n-k}\lambda_1^k}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}$$
$$= \binom{n}{k}\frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k}\frac{\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n-k}}$$
$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

which is $\mathrm{Bin}(n, \lambda_1/(\lambda_1+\lambda_2))$. Therefore, we have

$$N(t_1) \mid N(t_2) = n \sim \mathrm{Bin}\left(n, \frac{\lambda t_1}{\lambda t_1 + (\lambda t_2 - \lambda t_1)}\right) = \mathrm{Bin}\left(n, \frac{t_1}{t_2}\right) \qquad \square$$

Conditioning is very useful when thinking about Poisson point processes. Suppose you are counting the number of customers that enter your store during business hours. If you are given the number of customers that enter during a given interval at some rate λ, then you can use this to find the probability a certain number of customers will arrive within the next hour!

Another key concept of Poisson point processes is the superposition theorem, which states that the sum of independent Poisson processes is also a Poisson process.

Thm. 2 (Superposition): Let η_i for $i \in \mathbb{N}$ be a sequence of independent Poisson processes with intensity measures λ_i. Then

$$\eta := \sum_{i=1}^{\infty} \eta_i$$

is a Poisson process with intensity measure $\lambda = \lambda_1 + \lambda_2 + \ldots$

Proof. We will verify the two properties of a Poisson process.
1) $\forall\, t > 0$, $N_1(t) \sim \mathrm{Pois}(\lambda_1 t)$ and $N_2(t) \sim \mathrm{Pois}(\lambda_2 t)$ independently, so $N_{1,2}(t) \sim \mathrm{Pois}((\lambda_1+\lambda_2)t)$ because we have seen the sum of two indepenent Poisson random variables is a Poisson random variable. This process can be repeated as $n \to \infty$, with the next Poisson processes being $N_{1,2}(t)$ and $N_3(t)$.
2) Arrivals in disjoint intervals are independent in the combined process because of they are independent in the individual processes, which themselves are independent of one another. $\square$

We can use this concept of superposition to verify another key concept called thinning. The theorem below states that if there is a Poisson process with two classifications of events each with a specified intensity, then each type of event forms its own Poisson process.

Thm. 3 (Thinning): Let $N(t)$ where $t > 0$ be a Poisson point process with rate λ. Let the probability of two events ω_1 and ω_2 be p and $1-p$ respectively, where classifying an event as ω_1 or ω_2 are independent of each other and of the arrival times. Then ω_1 events form a Poisson point process with rate λp and ω_2 events form a Poisson point process with rate $\lambda(1-p)$.

Proof. Let $\lambda_1 = \lambda p$ and $\lambda_2 = \lambda(1-p)$ such that $\lambda = \lambda_1 + \lambda_2$. Then from theorem 2,

$$\begin{aligned} N(t) \sim \text{Pois}(\lambda t) &= \text{Pois}((\lambda_1 + \lambda_2)t) \\ &= \text{Pois}((\lambda p + \lambda(1-p))t) \\ &= \text{Pois}(\lambda pt) + \text{Pois}(\lambda(1-p)t) \end{aligned}$$

which are two independent Poisson processes $N_1(t) \sim \text{Pois}(\lambda_1 pt)$ and $N_2(t) \sim \text{Pois}(\lambda_2(1-p)t)$. $\square$

Note that the previous theorem can be generalized for some finite set of "colors" C, each with intensity $\lambda_i \; \forall\, i \in C = \{1, \ldots, c\}$. This is referred to as coloring.

Now that we have discussed some important properties of Poisson processes, we can shift to thinking about them not only in terms of the number of "arrivals" or "points", but in terms of random variables defined by interarrival times. This way of defining Poisson processes is useful when considering times in between occurences of events.

Thm. 5 (Interarrival Times): Let $0 < T_1 < T_2 < \ldots$ where $T_i \in \mathbb{R}$. If we have a Poisson point process with intensity $\lambda > 0$, then the random variables $T_1, T_2 - T_1, T_3 - T_2, \ldots, T_n - T_{n-1}$, $\ldots$ are i.i.d. with distribution $\text{Exp}(\lambda)$.

Proof. We will begin with T_1, the arrival of the first point of a Poisson process. Notice

$$P(N(t) = 0) = P(T_1 > t)$$

or equivalently, the probability that there are no points in the interval $(0, t]$ is equal to the probability that the first event arrives at some T_1 greater than t. For a Poisson process, this is equal to

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}.$$

We can then find the cumulative distribution function of T_1 which is

$$P(T_1 \leq t) = 1 - e^{-\lambda t}.$$

This result is the same cumulative distribution function of an Exponential random variable. For any n, we also see that

$$P(T_n > t) = P(N(T_{n-1} + t) - N(T_{n-1}) = 0) = P(N(t) = 0) = e^{-\lambda t}.$$

Therefore,

$$P(T_n \leq t) = 1 - e^{-\lambda t}$$

which shows that each interarrival time has an Exponential distribution with parameter λ. $\square$

Def. 2 (Gamma Distribution): Let $r, \lambda > 0$. A random variable X has the gamma distribution with parameters (r, λ) if X is nonnegative and has probability density function

$$f_X(x) = \frac{\lambda^r x^{r-1}}{\Gamma(r)} e^{-\lambda x}$$

for $x \geq 0$, where $\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx$.

Thm. 6 (Sum of Exponential Random Variables): The sum of n independent $\text{Exp}(\lambda)$ random variables has the $\text{Gamma}(n, \lambda)$ distribution.

Proof. We will prove this fact using moment generating functions. For an Exponential distribution X_i for $i \in \{1, \dots, n\}$, its moment generating function is defined by

$$M_{X_i}(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t}$$

for $t < \lambda$ from example 5.9 in ASV. We also know that the sum of n mutually independent random variables is the product of their moment generating functions. So to determine the distribution of $Z = X_1 + \cdots + X_n$, we can find its moment generating function. We have

$$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \frac{\lambda}{\lambda - t} = \left(\frac{\lambda}{\lambda - t}\right)^n = \left(\frac{\lambda - t}{\lambda}\right)^{-n} = \left(1 - \frac{1}{\lambda}t\right)^{-n}.$$

This gives the moment generating function of a Gamma distribution with parameters n and λ. Therefore, Z has the Gamma distribution with these parameters as desired. $\square$

Therefore, a Poisson point process can be dually described as points that follow a Poisson distribution, or interarrival times that follow an Exponential distribution!

Since the distances between points in a Poisson point process (defined as $T_i - T_{i-1}$ for some interval $(i-1, i])$ have an exponential distribution, the Poisson point process has a *memoryless* property. This is similar to a Markov property—where a Markov process in the future depends only on the present state and not its history—but has to do with arrival time intervals between points instead.

Now that we have discussed Poisson processes in one dimension, we can define a Poisson process generalized to any dimension d.

Def. 3 (Poisson process): Events in $\mathbb{R}^d$ are a *Poisson process* with intensity $\lambda > 0$ if the following conditions hold:

1) The number of arrivals in a region $A \subset \mathbb{R}^d$ has a Poisson distribution with rate $\lambda \cdot |A|$.

2) The number of events in disjoint regions are independent.

For example, events in the plane $\mathbb{R}^2$ are called a *spatial Poisson process* where $|A| = \text{area}(A)$.

The Poisson processes that have been defined up to this point are called *homogeneous point processes*, where the intensity is the constant rate λ. We will now introduce a different kind of Poisson process, defined below.

Def. 4 (Inhomogeneous Poisson process): An *inhomogeneous Poisson process* (or nonhomogeneous Poisson process) is a Poisson process with intensity parameter set as a positive locally integrable function $\lambda : B \to [0, \infty)$ where $B \subset \mathbb{R}^d$ defined by

$$\Lambda(B) = \int_B \lambda(x)\, dx.$$

where the location-dependent function $\Lambda(B)$ is the expectation $\mathbb{E}[N(B)]$.

Inhomogeneous Poisson processes still satisfy the assumption of independence of disjoint regions and having Poisson joint distributions, just with the use of a different function (as opposed to a constant function) for the intensity rate.

We will conclude this section by solving a fun example, which involves a Poisson process in three dimensions.

Ex. 1 (Stars): Suppose you live in space and stars around you are distributed according to a 3D Poisson process with intensity λ. What is the distribution of the distance from you to the nearest star?

Solution. Since we have a 3D Poisson process, we know that the number of events (stars) in a given space $S \subset \mathbb{R}^3$ is Poisson with expectation $\lambda \cdot \text{Vol}(S)$. Define R to be the distance between you and the nearest star and r to be the radius of an arbitrary sphere. Therefore, for the event $R > r$ to occur, there must be no stars within the sphere of radius r around you.

Now for any $r > 0$, let N_r be the number of stars with radius r near you such that $N_r \sim \text{Pois}(\lambda \cdot \frac{4}{3}\pi r^3)$ by the definition for volume of a sphere. Notice that the two events $R > r$ and $N_r = 0$ are the same event. Therefore, we have

$$P(R > r) = P(N_r = 0) = e^{-\frac{4}{3}\lambda\pi r^3} \frac{(-\frac{4}{3}\lambda\pi r^3)^0}{0!} = e^{-\frac{4}{3}\lambda\pi r^3}.$$

Finally, this gives the cumulative distribution function of R is

$$P(R \leq r) = 1 - e^{-\frac{4}{3}\lambda\pi r^3}$$

for $r > 0$. This solves the example! As a side note, this distribution is called a Weibull, so $R \sim \text{Wei}(\frac{4\pi\lambda}{3}, 3)$.

3 Conclusion

Exploring Poisson processes in depth has been a fascinating time. There is a whole world of Poisson process writing that I never knew existed before researching this topic. This final project was a great opportunity to learn more about Poisson processes, yet it still only forms an introduction to this vast area of spatial statistics. I have come across many books that I have put on my reading list, since I didn't have enough time to read all of them or relate them to this project in a concise way.

As a summary of my findings, Poisson processes are defined to model occurrences with a specified intensity (or intensity function) λ, where the number of occurrences have a Poisson distribution and are independent in disjoint intervals/regions. By superposing and thinning Poisson processes, we are left with a new Poisson process and by conditioning, we are left with a Binomial distribution. We can dually describe Poisson processes in terms of interarrival times, where the intervals between each arrival are distributed i.i.d. exponentially, and their sums being gamma distributed. Poisson processes are also used in more than one dimension, where independence and a Poisson distribution still apply. Finally, we have inhomogeneous Poisson processes, which use an intensity function λ rather than a constant λ rate.

Wrapping up this project, I will be taking this knowledge of Poisson processes and starting to recognize applications where they might be useful. Not only are they helpful to capture spatial randomness in real life events, but also the vast theory behind them explains why they are so useful in all of spatial statistics. I hope to explore them further in the future and read more into the many books that discuss all kinds of point processes, rather than just Poisson processes.

4 References

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