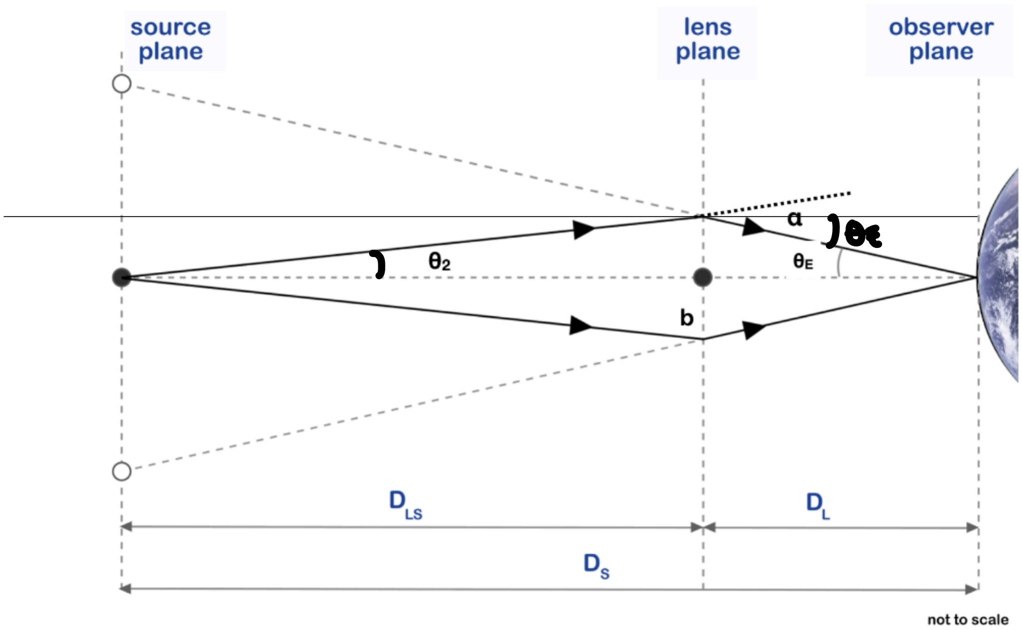
Problem Set #3 Evan Imata

1. Einstein Ring

$$\alpha = \frac{2R_s}{b} = \frac{4GM}{c^2 b}$$



a) Prove that $\theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}}$

given:

$$\alpha = \frac{2R_s}{b} = \frac{4GM}{c^2 b}$$

$$\alpha = \theta_2 + \theta_E$$

Small angle approx.:

$\theta_2 \sim \frac{b}{D_{LS}}$

$\theta_E \sim \frac{b}{D_L}$

$\theta_2 \sim \frac{b}{D_{LS}}$

$$\Rightarrow \frac{4GM}{c^2 b} = \frac{b}{D_L} + \frac{b}{D_{LS}}$$

$$\frac{4GM}{c^2} = b^2\left(\frac{1}{D_L} + \frac{1}{D_{LS}}\right)$$

$$= \theta_E^2 \underbrace{D_L^2\left(\frac{1}{D_L} + \frac{1}{D_{LS}}\right)}$$

$$= D_L^2 \cdot \frac{1}{D_{LS}}\left(\frac{D_{LS}}{D_L} + 1\right)$$

$$= \frac{D_L^2}{D_{LS}}\left(\frac{D_{LS}}{D_L} + \frac{D_L}{D_L}\right)$$

$$= \frac{D_L^2}{D_{LS}}\left(\frac{D_S}{D_L}\right)$$

$$= \frac{D_L D_S}{D_{LS}}$$

$$\Rightarrow \frac{4GM}{c^2} = \theta_E^2\left(\frac{D_L D_S}{D_{LS}}\right)$$

$$\Rightarrow \boxed{\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}} \quad \checkmark$$

2. Constants of Motion and $E=mc^2$

- 1-D Minkowski Spacetime

$$ds^2 = -c^2 dt^2 + dx^2$$

- $d\tau^2 = -\frac{ds^2}{c^2}$
- $d\tau^2 = dt^2 - \frac{dx^2}{c^2}$

a) $d\tau^2 = dt^2 - \frac{dx^2}{c^2}$

$$\frac{d\tau^2}{dt^2} = 1 - \frac{dx^2}{dt^2}\frac{1}{c^2} = 1 - \frac{v^2}{c^2}$$

$$\frac{dt^2}{d\tau^2} = 1/(1 - \frac{v^2}{c^2})$$

$$\Rightarrow \boxed{\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma} \checkmark$$

b) $v \ll c$, $\frac{dt}{d\tau} = (1 - \frac{v^2}{c^2})^{-1/2}$

Binomial expansion: $(1+x)^\alpha \sim 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2$

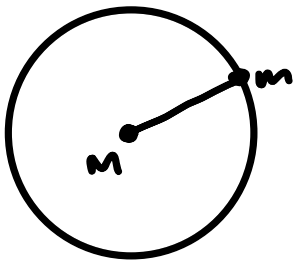
$\Rightarrow$ Through binomial expansion:

$$\frac{dt}{d\tau} \sim 1 + \frac{1}{2}\frac{v^2}{c^2} + \underbrace{\frac{3}{8}\frac{v^4}{c^4}}_{v^2 \ll c}$$

$\Rightarrow$ this and following terms are negligible

$$\Rightarrow \boxed{\frac{dt}{d\tau} \sim 1 + \frac{1}{2}\frac{v^2}{c^2}}$$ ✓

3. Falling into a Black Hole



$d\theta = 0$

Conserved Qtys:

$$E = \underbrace{\frac{1}{2} m r^2}_{\text{rotational}} \left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{GMm}{r}$$

$$L = mr\left(r \frac{d\phi}{dt}\right)$$

$$\rightarrow E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{V_{eff}}$$

$$F = -\frac{du}{dx}$$

a) given:

$d\Omega^2 = 0$, $\quad r_s = \frac{2GM}{c^2}$

Schwarzchild Metric:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{(1-r_s/r)} + r^2 d\Omega^2$$

$d\Omega^2 = 0$

$$d\tau^2 = \left(1-\frac{r_s}{r}\right)dt^2 - \frac{dr^2}{(1-r_s/r)}\cdot\frac{1}{c^2} - \cancel{\frac{r^2}{c^2}d\Omega^2}$$

(Eq. 16): $\frac{E}{m} = \left(1-\frac{r_s}{r}\right)c^2\frac{dt}{d\tau}$

$$1 = \left(1-\frac{r_s}{r}\right)\frac{dt^2}{d\tau^2} - \frac{1}{c^2(1-r_s/r)}\frac{dr^2}{d\tau^2}$$

$$1 = \frac{1}{c^2(1-r_s/r)}\left[\left(1-\frac{r_s}{r}\right)^2 c^2\frac{dt^2}{d\tau^2} - \frac{dr^2}{d\tau^2}\right]$$

mult. both sides by c^2:

$$c^2 = \frac{1}{c^2(1-r_s/r)}\left[\left(1-\frac{r_s}{r}\right)^2 c^4\frac{dt^2}{d\tau^2} - c^2\frac{dr^2}{d\tau^2}\right]$$

$$\Rightarrow c^2 = \frac{1}{c^2(1-r_s/r)}\left[\frac{E^2}{m^2} - c^2\frac{dr^2}{d\tau^2}\right]$$

$$c^2 - \frac{c^2 r_s}{r} = \frac{1}{c^2}\left(\frac{E^2}{m^2} - c^2\frac{dr^2}{d\tau^2}\right)$$

$$\Rightarrow c^2 - \frac{2GM}{r} = \frac{E^2}{m^2c^2} - \frac{dr^2}{d\tau^2}$$

$$\Rightarrow \frac{1}{2}c^2 - \frac{GM}{r} = \frac{E^2}{2m^2c^2} - \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2$$

mult. everything by m

$$\Rightarrow \frac{1}{2}mc^2 - \frac{GMm}{r} = \frac{E^2}{2mc^2} - \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2$$

$$\Rightarrow \underbrace{\frac{E^2}{2mc^2} - \frac{mc^2}{2}}_{\epsilon} = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r}$$

$$\Rightarrow \boxed{\epsilon = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} \quad \checkmark \qquad \text{where } \epsilon \equiv \frac{E^2}{2mc^2} - \frac{mc^2}{2}}$$

b) Show that $\epsilon \equiv \frac{E^2}{2mc^2} - \frac{mc^2}{2} \sim E - mc^2$

$E = E_N + mc^2$

$$\epsilon = \frac{mc^2}{2}\left(\left(\frac{E}{mc^2}\right)^2 - 1\right)$$

$$= \frac{mc^2}{2}\left(\left(\frac{E_N + mc^2}{mc^2}\right)^2 - 1\right)$$

$$= \frac{mc^2}{2}\left(\underbrace{\left(\frac{E_N}{mc^2} + 1\right)^2}_{\sim 1 + \frac{2E_N}{mc^2}} - 1\right)$$

$$\sim \frac{mc^2}{2} \cdot \frac{2E_N}{mc^2} = E_N$$

$$\Rightarrow \boxed{\epsilon = E_N} \checkmark$$

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\frac{r_s}{r} + \left[\frac{E^2}{(mc^2)^2} - 1\right]c^2$$

c) $\left(\frac{dr}{d\tau}\right)^2 = \frac{2E}{m} + \underbrace{\frac{2GM}{r}}_{=c^2\frac{r_s}{r}}$

$$= \cancel{\frac{2(E-mc^2)}{m}}^{\,E=mc^2 \to 0} + c^2\frac{r_s}{r}$$

$$\frac{dr}{d\tau} = \pm\left(c^2\frac{r_s}{r}\right)^{1/2} = \pm c\left(\frac{r_s}{r}\right)^{1/2}$$

$$\int_{r_1}^{r_2}\left(\frac{r}{r_s}\right)^{1/2}dr = \int_0^{\tau} c\,d\tau = c\tau$$

$$= \left(\frac{1}{r_s}\right)^{1/2}\underbrace{\int_{r_1}^{r_2} r^{1/2}dr}$$

$$= \frac{2}{3}r^{3/2}\Big|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^{3/2} - r_1^{3/2}\right)$$

$$\left(\frac{1}{r_s}\right)^{1/2} = \left(\frac{1}{r_s}\right)^{3/2}\cdot\left(\frac{1}{r_s}\right)^{-1}$$

$$\Rightarrow \int_{r_1}^{r_2}\left(\frac{r}{r_s}\right)^{1/2}dr = \frac{2}{3}r_s\left[\left(\frac{r_2}{r_s}\right)^{3/2} - \left(\frac{r_1}{r_s}\right)^{3/2}\right] = c\tau$$

$$\Rightarrow \boxed{\tau = \frac{2}{3}\frac{r_s}{c}\left[\left(\frac{r_2}{r_s}\right)^{3/2} - \left(\frac{r_1}{r_s}\right)^{3/2}\right]}\ \checkmark$$

d) $\tau = \frac{2}{3}\frac{r_s}{c}\left[\left(\frac{r_2}{r_s}\right)^{3/2} - \left(\frac{r_1}{r_s}\right)^{3/2}\right]$

$\Rightarrow r_1 = 0, \quad r_2 = r_s$

$\Rightarrow \tau = \frac{2}{3}\frac{r_s}{c}\left[1 - 0\right]$

$$\boxed{\tau = \frac{2}{3}\frac{r_s}{c}}$$

e) $\frac{E}{m} = \left(1 - \frac{r_s}{r}\right)c^2 \frac{dt}{d\tau}$, $g_{tt} = \left(1 - \frac{r_s}{r}\right)c^2 \frac{dt}{d\tau}$

$\frac{p_r}{m} = \frac{1}{\left(1-\frac{r_s}{r}\right)} \frac{dr}{d\tau}$ $\quad \Rightarrow \frac{d\tau}{dt} = \frac{m}{E}\cdot\left(1-\frac{r_s}{r}\right)c^2$

$\frac{dr}{d\tau} = \pm c\sqrt{\frac{r_s}{r}} \Rightarrow \left|\frac{dr}{d\tau}\right| = c\sqrt{\frac{r_s}{r}}$

$\frac{dr}{d\tau}\cdot\frac{d\tau}{dt} = \pm c\sqrt{\frac{r_s}{r}} \cdot \overbrace{\frac{m}{E}\cdot\left(1-\frac{r_s}{r}\right)c^2}^{d\tau/dt}$

$E = mc^2$

$$\Rightarrow \boxed{\left|\frac{dr}{dt}\right| = c\sqrt{\frac{r_s}{r}}\cdot\left(1 - \frac{r_s}{r}\right)}$$ ✓

4. Schwarzschild Orbits

$\theta = \pi/2, \quad d\theta = 0$

Energy for orbiting object:

$$E = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 - \frac{GMm}{r}$$

Schwarzschild Metric:

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{dr^2}{(1-r_s/r)} + r^2d\phi^2$$

C.O.M.:

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} \qquad\qquad r_s = \frac{2GM}{c^2}$$

$$\frac{E}{m} = \left(1-\frac{r_s}{r}\right)c^2\frac{dt}{d\tau}$$

a) $ds^2 = -c^2d\tau^2$

$$\Rightarrow \cancel{c^2}d\tau^2 = \left(1-\frac{r_s}{r}\right)\cancel{c^2}dt^2 - \frac{dr^2}{(1-r_s/r)} - r^2d\phi^2$$

$$d\tau^2 = \left(1-\frac{r_s}{r}\right)dt^2 - \frac{dr^2}{c^2(1-r_s/r)} - \frac{r^2}{c^2}d\phi^2$$

$$1 = \left(1-\frac{r_s}{r}\right)\frac{dt^2}{d\tau^2} - \frac{1}{c^2(1-r_s/r)}\frac{dr^2}{d\tau^2} - \frac{r^2}{c^2}\frac{d\phi^2}{d\tau^2}$$

$$c^4\left(1-\frac{r_s}{r}\right) = \left(1-\frac{r_s}{r}\right)^2c^4\frac{dt^2}{d\tau^2} - c^2\frac{dr^2}{d\tau^2} - c^2\left(1-\frac{r_s}{r}\right)r^2\frac{d\phi^2}{d\tau^2}$$

$$\underbrace{c^4 r^2\left(1-\frac{r_s}{r}\right)} = r^2\frac{E^2}{m^2} - r^2c^2\frac{dr^2}{d\tau^2} - c^2\left(1-\frac{r_s}{r}\right)\frac{L^2}{m^2}$$

$$c^4r^2 - 2GMrc^2 = r^2\frac{E^2}{m^2} - r^2c^2\frac{dr^2}{d\tau^2} - c^2\frac{L^2}{m^2} + \frac{2GML^2}{m^2 r}$$

Divide all by c^2r^2:

$$c^2 - \frac{2GM}{r} = \frac{E^2}{m^2c^2} - \frac{dr^2}{d\tau^2} - \frac{L^2}{m^2r^2} + \frac{2GML^2}{m^2c^2r^3}$$

mult. all by $\frac{1}{2}m$:

$$\frac{1}{2}mc^2 - \frac{GMm}{r} = \frac{E^2}{2mc^2} - \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{2mr^2} + \frac{L^2GM}{mc^2r^3}$$

$$\Rightarrow \underbrace{\frac{E^2}{2mc^2} - \frac{1}{2}mc^2} = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3}$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 - \frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3}} \checkmark$$

where $\mathcal{E} = \frac{E^2}{2mc^2} - \frac{1}{2}mc^2$

b) $\frac{dV_{eff}}{dr} = 0$

$$V_{eff} = -\frac{GMm}{r} + \frac{L^2}{2mr^2} - \frac{L^2GM}{mc^2r^3}$$

$$\frac{dV_{eff}}{dr} = \frac{GMm}{r^2} - \frac{L^2}{mr^3} + \frac{3L^2GM}{mc^2r^4} = 0$$

$$\Rightarrow \frac{3L^2GM}{mc^2r^4} - \frac{L^2}{mr^3} = -\frac{GMm}{r^2}$$

Mult. all by r^4

$$\Rightarrow \frac{3L^2GM}{mc^2} - \frac{L^2 r}{m} = -GMmr^2$$

$$\Rightarrow -GMmr^2 + \frac{L^2}{m}r - \frac{3L^2GM}{mc^2} = 0$$

$$r_c = \frac{-\frac{L^2}{m} \pm \sqrt{\frac{L^4}{m^2} - 4(GMm)\left(\frac{3L^2GM}{mc^2}\right)}}{-2GMm}$$

$$\boxed{r_c = \frac{\frac{L^2}{m} + \sqrt{\frac{L^4}{m^2} - \frac{12G^2M^2L^2}{c^2}}}{2GMm}}$$

c) $\frac{L^4}{m^2} - \frac{12G^2M^2L^2}{c^2} \geq 0$ this is the piece inside the rational from part b

$$\frac{L^4}{m^2} \geq \frac{12G^2M^2L^2}{c^2}$$

$$\frac{L^2}{m^2} \geq \frac{12G^2M^2}{c^2} \Rightarrow L \geq \frac{GMm}{c}\cdot\sqrt{12}$$

$$r_{isco} = \frac{\frac{12G^2M^2m}{c^2} + \sqrt{\overbrace{\frac{G^4M^4m^2\cdot 144}{c^4} - \frac{144\,G^4M^4m^2}{c^4}}^{0}}}{2GMm}$$

$$r_{isco} = 6\frac{GM}{c^2}, \quad r_s = 2\frac{GM}{c^2}$$

$$\Rightarrow \boxed{r_{isco} = 3r_s} \checkmark$$