Physics 112 - Intro to Statistical and Thermal Physics - Spring 2023
Problem Set 11

Due Friday, April 28 at 11:59 PM (PDT)
Last Update: April 22, 2023

- **Highlighted/Most Relevant Reading for the material on this week's problem set:**
  - Schroeder, Sections 7.3, 7.5
- **Reading for next week:**
  - Schroeder, 7.6, 5.4, 5.5

Problem 11.1 - Debye-bye-bye

The ***Debye model*** is a refinement of the Einstein model of solids. Like the Einstein model we have N lattice sites each of which contains an atom which can oscillate in 3 directions (assuming a 3D solid) giving a total of $3N$ oscillators. The new feature in the Debye model is that the oscillations are now coupled together, allowing waves to form. We can decouple the system by forming ***normal modes***.[1] Excitations of the normal modes are called ***phonons***, similar to how excitations of the modes of the EM field are interpreted as photons. We treat phonons as bosons with chemical potential $\mu = 0$. A phonon for a mode with frequency ω has energy $E = \hbar\omega$.

We will start by re-deriving our Debye results but starting with the idea of phonons as our particles rather than starting with oscillators/normal modes/waves. Recall that for a particle-in-a-box in 1D the quantum number n_x is related to the wave number via $k_x = \pi n_x / L$. For our 3D system a phonon with wave number $\vec{k}$ will have energy $E = \hbar c_s |\vec{k}|$, where c_s is the speed of sound in the solid. You may take the spin-degeneracy to be $g_s = 3$ (in our wave model, these come from the three different wave polarizations). The number of states available for a single phonon is equal to the number of oscillators in our wave model, so the total number of one-phonon states should be $3N$.

(a) Find the density of states $g(E)$ for a gas of phonons in a 3D "box" (our solid) of side-length L (volume $V = L^3$). From this, determine the ***Debye energy*** E_D (the maximum energy of one of our one-phonon states). Show that the result for the Debye frequency ω_D, defined via $E_D = \hbar\omega_D$, matches the result we found in class, $\omega_D = \left(\frac{6\pi^2 N}{V}\right)^{1/3} c_s$ what we found in class.
Hint (highlight to reveal): [The number of one-phonon states in a given eneryg range is found by integrating the density of states over that energy range.]
[*Supplementary Part (Not for Credit): Given the results, also that we can write $g(E) = 9NE^2/E_D^3$. This will be a more convenient expression to use going forward.*]

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[1] In the normal modes, groups of atoms all oscillate in phase together. Based on the pun-title of this problem, you might say they are in sync...

Once we have our Debye energy (or Debye frequency or Debye temperature $T_D \equiv E_D/k_B \equiv \hbar\omega_D/k_B$) we can find all of our other relevant quantities. Since phonons are bosons with chemical potential $\mu = 0$ they are governed by a Planck distribution function,

$$\bar{n}_P(E) = \frac{1}{e^{\beta E} - 1}.$$

At different temperatures our solid will contain different numbers of phonons, $\langle N_{\text{phon}} \rangle$. Note that N_{phon} is our number of phonons which is very distinct from the number of atoms in the solid N.

(b) Find an expression in terms of a dimensionless integral for the number of phonons $\langle N_{\text{phon}} \rangle$ in our solid at temperature T.
Hint (highlight to reveal): [You might want to use the substitution $x = E/k_BT$. Since there is a maximum phonon energy, make sure your integral over E only ranges from 0 to E_D. Be sure to change this upper limit when you do an integral substitution.]
Answer (highlight to reveal): [$\langle N_{\text{phon}} \rangle = 9N \left(\frac{T}{T_D}\right)^3 \int_0^{T_D/T} \frac{x^2}{e^x - 1} dx$]

(c) Evaluate your expression in the low-temperature limit $T \ll T_D$ and the high-temperature limit $T \gg T_D$.
[*Note: In the low-temperature limit your answer will involve a zeta function which you can leave as a zeta function.*]

We can similarly find the total energy in the Debye solid via

$$\langle U \rangle = \int_0^{E_D} E g(E) \bar{n}_P(E)\, dE. \tag{1}$$

(d) Using Eq. 1, find an expression for the heat capacity C_V at arbitrary temperatures. You do not have to evaluate the integral but you should to a similar substitution as you did in (c) to leave your expression as a dimensionless integral.
[*Supplementary Part (Not for Credit): In the high-temperature limit, show that your answer reproduces the expected result for C_V based on equipartition.*]
[*Supplementary Part (Not for Credit): Make a graph of C_V/Nk_B-vs-T/T_D.*]

We can also make a Debye model for certain liquids, such as the liquid phase of Helium-4 at very low temperatures. In particular, we will look at temperatures below 0.6 K where Helium-4 is indeed in a liquid phase. At this temperature, the velocity of sound waves is $c_s = 238\,\text{m/s}$ and the density of the liquid is $\rho = 0.145\,\text{gram/cm}^3$. Recall that a single atom of Helium-4 has a mass of approximately 4 u. One important modification that needs to be made for a liquid is that there is *only* the one longitudinal polarization - there are *no* transverse modes, so our spin degeneracy gets reduced from $g_s = 3$ to $g_s = 1$. This means there are only N oscillators in our system (as opposed to $3N$ for our earlier solid). Even with this modification, our formula for the Debye energy and Debye temperature remains the same.[2]

(e) Show that the Debye temperature for this system is $T_D \approx 19.8\,\text{K}$.

(f) Find the phonon contribution to the heat capacity C_V for this system. How does this compare with the experimentally observed result $C_V = Nk_B\,(T/(4.67\,\text{K}))^3$.
Hint (highlight to reveal): [Note that we are in the regime $T \ll T_D$.]

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[2] Try to argue why this should be!

Problem 11.2 - The Degenerate Fermi Gas

Recall that when we have a gas of non-interacting Fermions, we define the ***Fermi energy*** E_F as the chemical potential at $T = 0$, where all states with $E < E_F$ are fully occupied and all states with $E > E_F$ are empty. We can find the Fermi energy for a gas of N particles via

$$N = \int_0^{E_F} g(E)\, dE.$$

For a 3D non-relativistic Fermi gas we found

$$E_F = \frac{\pi^2 \hbar^2}{2m} \left(\frac{3N}{\pi V} \right)^{2/3}.$$

(a) Show that the total energy for this Fermi gas at $T = 0$ is given by

$$\langle U \rangle = \tfrac{3}{5} N E_F.$$

[*Supplementary Part (Not for Credit): Also find the pressure P (called the* ***degeneracy pressure*** *in this context) and bulk modulus $B \equiv -V \left(\frac{\partial P}{\partial V}\right)$.*]

Each atom in a chunk of copper (Cu, $Z = 29$, atomic mass = 63.5 u) contributes one conduction electron. These conduction electrons make up a Fermi gas. At room temperature the density of copper is $\rho_{\text{Cu}} = 8.96\,\text{gram/cm}^3$.

(b) Find the Fermi energy and Fermi temperature for copper. Based on the Fermi temperature you found, is our approximation that the electrons are a degenerate gas appropriate? Is our $T \approx 0$ approximation appropriate?

At low ($T \ll T_F$) but finite temperature we can evaluate the integrals that arise using the ***Sommerfeld expansion***, which first uses an integration-by-parts to re-express an integral with $\bar{n}_F$ in the integrand with an integral with $\frac{\partial \bar{n}_F}{\partial E}$ in the integrand. Since the derivative is non-zero only in a narrow (of width $\sim k_B T$) region of the chemical potential μ, this allows us to do other approximations, like Taylor expanding the rest of the integrand about the point $E = \mu$. We typically do a u-substitution in our integrals to $x = \beta(E - \mu)$, so in this case the $\frac{\partial \bar{n}_F}{\partial E}$ is peaked around $x = 0$ and we Taylor expand the rest of the integrand about $x = 0$. The lower limit is usually lowered to $-\infty$ as part of this approximation. The following integrals may be useful:

$$\int_{-\infty}^{\infty} \frac{x^n e^x}{(e^x + 1)^2} dx = \begin{cases} 0, & n \text{ odd} \\ 1, & n = 0 \\ \pi^2/3 & n = 2 \end{cases}.$$

In class we used the Sommerfeld expansion to find the chemical potential as a function of temperature for low temperatures,

$$\mu \approx E_F \left(1 - \tfrac{\pi^2}{12} \left(\tfrac{k_B T}{E_F} \right)^2 \right). \tag{2}$$

We also claimed that if we used this expansion to find the energy we would get

$$\langle U \rangle \approx \tfrac{3}{5} N E_F + \tfrac{\pi^2}{4} N \frac{(k_B T)^2}{E_F}. \tag{3}$$

(c) **Extra Part** (*Not for Credit*) Graph $\frac{\partial \bar{n}_F}{\partial E}$ as a function of $x = (E-\mu)/k_B T$. Show or argue that this is a symmetric function and determine the width $\sigma_x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. Letting $\sigma_x = \sigma_E / k_B T$, what is σ_E, the scale of the width of the energy range of interest?

(d) Use the Sommerfeld expansion to find the energy of a Fermi gas at finite temperature out to the order $(T/T_F)^2$ term. First leave your answer in terms of μ and then plug in Eq. 2 to (hopefully) get Eq. 3.
Hint (highlight to reveal): [Recall that $(1+\epsilon)^p \approx 1 + p\epsilon + \frac{1}{2}p(p-1)\epsilon^2$ out to second order.]

(e) Use your result (or Eq. 3) to find an expression for the entropy of a Fermi gas at small but finite temperatures as a function of T. You may assume that $S = 0$ at $T = 0$.
Hint (highlight to reveal): [Knowing $\langle U \rangle$ we can find the heat capacity C_V and from there... well, we did something similar in the previous problem set!]

Problem 11.3 - White Dwarf Stars and Neutron Stars

A ***white dwarf*** is a dense stellar core that tends to be the end-result of stellar evolution for main-sequence stars of mass between roughly 0.1 and 10 solar masses. Unlike in stars where the energy produced by fusion provides stability against the gravitational forces, fusion does not occur in white dwarf stars. What keeps a white dwarf from undergoing gravitational collapse is the degeneracy pressure from the electrons in the star. We will treat the electrons in the star as a Fermi gas.

Consider a model where a white dwarf is a sphere of mass M and radius R with *uniform* mass density throughout. In such a setup the gravitational potential energy of the star is given by

$$U_{\text{grav}} = -\frac{3}{5}\frac{GM^2}{R},$$

where $G = 6.67 \times 10^{-11}\,\text{J}\cdot\text{m/kg}^2$ is the gravitational constant. We start by making the following assumptions:

- The electrons form a non-relativistic non-interacting Fermi gas in 3D.
- There is roughly 1 electron per proton in the white dwarf and we have equal numbers of protons and neutrons. Since the electron mass m_e is much much less than the proton mass $m_p \approx 1\,\text{u}$, this means the white dwarf's mass is essentially all from nucleons and can be used to determine the number of electrons present.

The first discovered white dwarf star (and also the closest and one of the most massive known) is Sirius B, which has a mass of $M = 1.02 M_\odot$, radius[3] $R = 0.0084 R_\odot$, and a temperature[4] of $T = 2.5 \times 10^4\,\text{K}$.

(a) Find the Fermi energy E_F and the total energy $\langle U \rangle_F$ (ignoring gravitational effects) of this electron gas as functions of the radius R and mass M. What is the Fermi temperature T_F for Sirius B? Given this temperature scale and the known temperature of Sirius B, is the approximation $T \approx 0$ valid?
[*Note: You don't have to derive these from scratch - just look at the previous problem!*]

[3]Sirius B is a just a little bit smaller than the earth.
[4]We can tell the temperature by the spectrum! What is the peak wavelength for this temperature? Using your code from last week, determine the "color" will this appear.

Hint (highlight to reveal): [The number of electrons to use is $N = M/2m_p$ (since M/m_p is the total number of nucleons in the star, the number of protons is half the number of nucleons, and the number of electrons is equal to the number of protons).]

The protons and neutrons in the white dwarf also each (individually) form a Fermi gas but we often leave them out of the discussion of white dwarves. Why? Let's find out. Assume for this part that *all* nucleons in the white dwarf are protons and again ignore the interactions between the protons.

(b) Extra Part (*Not for Credit*) How does the degeneracy pressure of the proton gas compare to the degeneracy pressure of the electron gas? How does the energy of the proton gas compare to the energy of the electron gas? Will the proton gas be a significant contribution to our study of the white dwarf?

The equilibrium size of the white dwarf will be when the total energy - the sum of the gravitational potential energy and the energy of the Fermi gas we found in part (a) - is minimized. This is equivalent to saying the degeneracy pressure of the electron gas is balancing the inward force due to gravity. Spoilers! You should have found

$$\langle U \rangle_F = \frac{3}{5}\left(\frac{9\pi}{4}\right)^{2/3}\left(\frac{M}{2m_p}\right)^{5/3}\frac{\hbar^2}{2m_e R^2}.$$

(c) For a given M, find the radius that minimizes $U = U_{\text{grav}} + \langle U \rangle_F$. How well does your prediction match the data for Sirius B?
[*Note: You should find* $R \propto M^{-1/3}$.]

From the previous part you should have found that the Sirius B data didn't fit well to our prediction. That's okay - we've made a lot of approximations. The $M^{-1/3}$ scaling is what's really important and does fit much better with the data. One feature we should consider though is that as the mass gets larger and larger the Fermi energy becomes comparable to and then passes the rest mass energy of the electron. If we want to properly treat this case we need to look at relativistic effects! First let's go way overboard and consider the *hyper-relativistic* case $E_F \gg m_e c^2$? Now the dispersion relation relating energy to momentum is $E = |\vec{p}|\, c$, similar to what we found with photons. In a previous problem set we found that the density of states for such a relativistic dispersion relation in 3D is

$$g(E) = \frac{4\pi g_s V E^2}{h^3 c^3} = \frac{V E^2}{\pi^2 \hbar^3 c^3},$$

where we have used $g_s = 2$ since we are dealing with spin-1/2 fermions.

(d) Compute the Fermi energy E_F and total energy $\langle U \rangle_F$ for this density of states for a gas of total mass M and radius R.
Answer (highlight to reveal): [In terms of the number of electrons N and the volume V you should find $E_F = \hbar c(3\pi^2 N/V)^{1/3}$ and $\langle U_F \rangle = \frac{3}{4} N E_F$.]

(e) Show that in this case our star is ***unstable*** - there *is* no finite R that minimizes the total energy.

(f) Using the non-relativistic results, compute the mass (in terms of solar masses) where the Fermi energy equals the rest-mass energy of the electron.

A (Neutron) Star Is Born The mass you found in part (f) can be refined in the fully relativistic case to determine the mass above which the electrons are too relativistic, causing instability and gravitational collapse. This is known as the ***Chandrasekhar limit***. The currently accepted value is $M \approx 1.4M_{\odot}$. As the mass grows beyond the Chandrasekhar limit the white dwarf begins to collapse again - the gravitational forces overwhelm the degeneracy pressure. In this case electrons wind up combining with protons to form a very dense soup of neutrons. Since neutrons are roughly 1800 times more massive than electrons, they remain non-relativistic for much higher Fermi energies. Thus we can again reach a stable configuration known as a ***neutron star***.[5]

(g) Modify your earlier results for a non-relativistic gas of neutrons (which mainly amounts to replacing electron masses with nucleon masses) to find the mass-radius relation of a neutron star and compute the Fermi energy, radius, and density of a neutron star with $M = 1M_{\odot}$.

(h) **Extra Part** (*Not for Credit*) Using the non-relativistic results, compute the mass (in terms of solar masses) where the Fermi energy equals the rest-mass energy of the neutron and thus our neutron star becomes unstable. It is at this point where gravity ultimately wins and our star collapses to form a ***black hole***.

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[5]Book recommendation! *Dragon's Egg* is a science fiction book by author and physicist Robert L. Forward and imagines an encounter between humans and exotic life born on the surface of a neutron star.