

## Physics 112 - Intro to Statistical and Thermal Physics - Spring 2022

### Problem Set 01

Due Thursday, February 2 at 11:59 PM (PST)

*Last Update: January 26, 2023*

- You will submit your work to [Gradescope](#). Please make sure your scans are clear and readable. We also ask that you tag your problems in the Gradescope system.
- You may turn in this assignment late by [Friday, February 3 at 11:59 pm](#). Recall that the first late assignment will be accepted for full credit but subsequent late assignments will receive a 25% late penalty.
- My lecture notes are posted on our course site.
- Grading policy (to ensure that our hard-working GSIs and readers don't go over their time-limits): For problem sets with four problems rather than three, three of the problems will be graded (roughly 30 points each, depending on length) and the fourth (which will be randomly chosen) will merely be checked for completeness for 10 points.
- Throughout the problem sets you will see subparts labeled “**Extra Part** (*Not for Credit*)”. I will designate subparts as extra parts for many reasons, most often to making sure that the problem sets are reasonable length while still being able to explore some really cool things! Other reasons a subpart may be designated as an extra part include the parts being: Extra stepping stones or scaffolding to the problems; deeper and more challenging dives into problems; more examples that are similar to calculations you've already done; computationally challenging without much actual physics; etc. I encourage you to do them, though I don't want you to feel time-pressured on them. Some of them will make good problems to come back to as you prepare for exams. Solutions to extra parts will be given in the solution sets.
- Some problems will come with “spoiler text” ([like this](#)) - text that is intentionally written in a very light color. This allows you to skip hints easily without being visually distracted by them. I will always post a separate document containing the spoiler text in full regular typeface in case you are having trouble reading it on the document itself.

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• **Reading for the material on this week's problem set:**

- Schroeder, Sections 1.1, 1.2, 1.3, 3.1, 3.2.

• **Reading for next week:**

- Schroeder, Sections 1.4, 1.5, 1.6, 4.1, 4.4.
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### Problem 1.1 - Equilibrium

In class we derived the law of atmospheres by using the concept of diffusive equilibrium and our new mysterious friend the chemical potential. Let's re-derive that equation using more familiar concepts. This procedure is inspired by Schroeder Problem 1.16.

Consider a tall cylinder filled with air and in thermal equilibrium with the environment at temperature  $T$ . We will assume our column of air is in mechanical equilibrium as well, although the pressure will *not* be the same throughout due to the additional external force of gravity.

(a) Consider a horizontal thin slab of air whose thickness (height) is  $dz$ . If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for  $dP/dz$ , the variation of pressure with altitude, in terms of the density of air  $\rho$ .

(b) Use the ideal gas law to write the density of air  $\rho$  in terms of pressure  $P$ , temperature  $T$ , and the average mass  $m$  of the air molecules. Use this to show that your result from (a) gives the **barometric equation**,

$$\frac{dP}{dz} = -\frac{mg}{k_B T} P.$$

(c) Ooh, a differential equation! We know what to do now. Solve the barometric equation in terms of the pressure at the bottom of the cylinder  $P(0) = P_0$ . Use the ideal gas law to show that your solution is equivalent to the variation of concentration with height that we found in lecture.

Now let's look at a different system in diffusive equilibrium. For an ideal monoatomic gas (with no external potential energies), the chemical potential is

$$\mu_{IG} = k_B T \ln \left( \frac{n}{n_Q} \right),$$

where  $T$  is the temperature,  $n$  is the concentration (number of particles-per-unit-volume) and  $n_Q$  is the **quantum concentration**.<sup>1</sup> Consider a **centrifuge** - a hollow cylinder of radius  $R$  and length  $L$  that rotates about its axis at angular frequency  $\omega$ . The centrifuge is filled with an ideal gas consisting of particles of mass  $m$  and is at thermal equilibrium at temperature  $T$ . Because of the rotational motion, the chemical potential becomes a function of the radial distance  $r$  from the axis of rotation,

$$\mu = \mu_{IG} - \frac{1}{2} m \omega^2 r^2.$$

After the centrifuge has been spinning for a while the entire gas is in diffusive equilibrium.

(d) Use the fact that gas at radius  $r$  is in diffusive equilibrium with gas along the central axis of the centrifuge to find  $n(r)$ , the concentration as a function of  $r$ , in terms of the concentration on the axis  $n_0$  and the other parameters of this problem (like the temperature  $T$ , the particle mass  $m$ , etc.).

**Commentary:** We can do some fun stuff with this (it's a bit much for now but we might come back to it)! For example, if you know how much total gas you put in the centrifuge you can use the result from (d) to determine the concentration along the center of the axis as a function of mass. By putting two different species of particles in the centrifuge you get different resulting concentrations along the axis and this can be used to separate gases according to mass, even if the differences are tiny (like with isotopes of Uranium).<sup>2</sup>

<sup>1</sup>For those who are curious, the quantum concentration is given by  $n_Q = \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2}$ , though you won't need that for this problem!

<sup>2</sup>Great, I couldn't make it through even one problem in stat mech without raising national security concerns...

## Problem 1.2 - The van der Waals Equation

The first equation of state we met was the ideal gas law. The *van der Waals equation of state* is another equation of state which accounts for the finite size of the particles and intermolecular interactions,

$$\left(P + \frac{aN^2}{V^2}\right)(V - bN) = Nk_B T,$$

where  $a$  and  $b$  are certain empirically-determined parameters that relate to the strength of the intermolecular interactions the finite size of the particles, respectively.<sup>3</sup> We can determine these parameters by finding the *critical point*<sup>4</sup>. The critical point of the van der Waals equation of state is defined as the state where both  $\left(\frac{\partial P}{\partial V}\right)_{N,T} = 0$  and  $\left(\frac{\partial^2 P}{\partial V^2}\right)_{N,T} = 0$ .

(a) Determine the pressure  $P_c$ , volume  $V_c$ , and temperature  $T_c$  at the critical point.

*Hint (highlight to reveal):* [Don't try to explicitly solve for the derivatives. Rather, take the derivatives with respect to  $V$  on both sides of the van der Waals equation.]

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**Non-dimensionalization** Rather than using standardized units (like the SI system) for dimensionful quantities<sup>5</sup>, a system will often come with its own natural scales that are intrinsic to the system. For example, in our van der Waals gas the critical point naturally gives us a pressure, volume, and temperature scale based on the  $P_c$ ,  $V_c$ , and  $T_c$ . One way to use these natural scales is to create *reduced variables* which are dimensionless versions of our dimensionful quantities,

$$p \equiv \frac{P}{P_c}, \quad v \equiv \frac{V}{V_c}, \quad t \equiv \frac{T}{T_c}.$$

Spoilers! You should have found in part (c) that...

$$P_c = \frac{a}{27b^2}, \quad V_c = 3Nb, \quad T_c = \frac{8a}{27k_B b}.$$

The process of determining scales and rewriting equations in terms of reduced variables is an important one in physics and is known as *non-dimensionalization*.

(b) “Non-dimensionalize” the van der Waals equation (that is, re-express the van der Waals equation using the reduced variables).

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Using the reduced variables, we find that the parameters  $a$ ,  $b$ , and  $N$  cancel out of the equation! Since we no longer care about these parameters, analysis of the non-dimensionalized equation will give us some universal features of van der Waals substances. That is, we can analyze *all* van der Waals systems in general without having to specify the parameters. It also lets us easily plot and analyze data on a computer.

(c) Create a  $pv$  diagram (a graph of  $p$ -vs- $v$ ) and graph some isotherms for  $t = 0.8, 0.9, 1.0, 1.1, 1.2$ . You may use a graphing program, a Jupyter notebook, etc. Is there anything that strikes you as wrong or suspect about any of your graphs?

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<sup>3</sup>Note that when  $a$  and  $b$  go to zero we recover the Ideal Gas Equation of State!

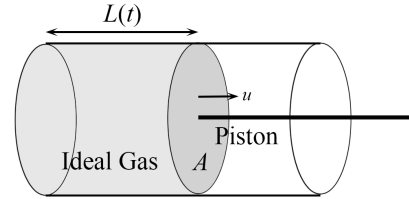
<sup>4</sup>Late in the semester we will see how this equation of state allows for a phase transition. The critical point will be where the properties of the liquid and gas phases become identical

<sup>5</sup>*Dimension* here referring to *physical dimension* - [https://en.wikipedia.org/wiki/Dimensional\\_analysis](https://en.wikipedia.org/wiki/Dimensional_analysis)

## Problem 1.3 - Adiabatic Transformations of an Ideal Gas

In this problem we will use the model we developed for the kinetic theory of gases to explore how an ideal gas transforms during **reversible adiabatic compressions and expansions**. Recall that in this model, the pressure  $P$  arises from elastic collisions between the gas molecules and the piston and we will assume that the temperature  $T$  is related to the internal energy of the system  $U$  via equipartition. Let the gas molecules have  $f$  degrees of freedom that are active (so, for example, in nitrogen at room temperature we would have  $f = 5$ ).

Our system, shown to the right, consists of an ideal gas in a cylindrical chamber of cross-sectional area  $A$ . The length of the chamber is determined by the piston and at time  $t_0$  the length is  $L(t_0)$  and the volume is  $V(t_0)$ . The chamber is sealed so the number of particles in the gas remains constant at  $N$ . We will assume all walls of the chamber are perfectly insulating so that the only way of transferring energy between the system and the environment is through the motion of the piston.



A transformation in such a thermally isolated system is called **adiabatic**. If we carry out the transformation reversibly, then the transformation may be called **isentropic**. Let the piston be pulled out at a constant speed  $u$ . We want to model a **reversible** process so we need to ensure that our transformation is **quasi-static**. That is, changes to the system are very slow compared to the relaxation time of the system to reach thermodynamic equilibrium.

(a) If the system is in thermal equilibrium at temperature  $T$ , determine the rms velocity in the  $x$ -direction  $v_{x,\text{rms}} \equiv \sqrt{\langle v_x^2 \rangle}$ .

*Hint (highlight to reveal):* [We basically did this in lecture.]

We want to ensure lots of collisions between the piston and each gas molecule so our first limitation on the piston speed will be that  $u \ll v_{x,\text{rms}}$ . Now suppose a particle with  $x$ -component of velocity  $v_x$  is incident on the piston and collides elastically. You may very safely assume that the mass of the piston is much much greater than the mass of the gas particle.

(b) What is the  $x$ -component of velocity of the particle *after* the collision? What is the change in translational kinetic energy  $\Delta K$ ? Since  $u \ll v_x$  you need only give your result over-order in the small parameter  $u/v_x$ .

*Hint (highlight to reveal):* [You may want to think about what happens in the frame of reference of the piston first.]

Let's make another simplifying assumption and assume all of our molecules all have the same magnitude of  $x$ -component of velocity, so  $v_x^2 = v_{x,\text{rms}}^2$  (we will be able to refine this analysis once we know the actual distribution of velocities). Consider an infinitesimal time interval  $dt$ . We want to calculate how many collisions happen between the gas molecules and the piston in this time.

(c) What infinitesimal length along the  $x$ -direction can our particles travel in this time? What infinitesimal distance does the piston travel in this time? What is the infinitesimal change in volume  $dV$  of the chamber? What infinitesimal volume  $dV_{\text{collide}}$  must a particle be within to collide with the piston some time between time  $t$  and  $t + dt$ ?

[Note: Careful! While the last two prompts are both asking for infinitesimal volumes, the first is the change in the volume of the full chamber due to the motion of the piston and the second is the small volume within the chamber that is close enough to the piston to allow particles to collide with the piston in the time  $dt$ . Try drawing some pictures to help!]

(d) Assuming uniform particle density  $N/V(t)$  (as is statistically the case assuming our system is in an equilibrium state), how many collisions will occur in this time frame?

*Answer (highlight to reveal):* [The number of collisions should be  $\frac{N}{2} \frac{(v_1 - u) A dt}{V(t)}$  ]

*Hint (highlight to reveal):* [Be careful! Missing a factor of 2? A particle in the relevant volume but moving to the left will not be able to collide with the piston!]

(e) Given your answer to (d), determine the infinitesimal change in internal energy  $dU$  and thus the infinitesimal change in temperature  $dT$ .

(f) Manipulate the answer in (e) so that you get a differential equation relating the variables  $V$  and  $T$ .

*Hint (highlight to reveal):* [Recall that in part (c) we found  $dV$  in terms of  $dt$ .]

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If you've done everything correct up to this point, your differential equation should read

$$\frac{dT}{dV} = -\frac{2}{f} \frac{T}{V}.$$

Note that we are keeping  $N$  constant and using the equation of state (the ideal gas law) to specify  $P$  as a function of  $T$  and  $V$  so we are justified in treating the right hand side as an actual derivative (even though we formed it from a ratio of infinitesimals to many mathematicians' chagrin). Suppose we start at volume  $V_1$  and temperature  $T_1$  and change the volume of the gas isentropically to volume  $V_2$ .

(g) Solve the differential equation to find  $T_2$ .

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For an ideal gas, an isentropic transformation is a ***polytropic process*** - a transformation where the pressure and volume are related by

$$PV^n = \text{const.}$$

In this equation, the exponent  $n$  is some constant that depends on the type of transformation.<sup>6</sup> For an isentropic transformation,  $n = \gamma$ , the ratio of the specific heat at constant pressure to the specific heat at constant volume (which we will explore next week).

(h) Use the ideal gas law and your result from (g) to find  $\gamma$ ! Your answer should only depend on the number of degrees of freedom  $f$ .

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<sup>6</sup> $n = 0$  for an isobaric transformation;  $n = 1$  for an isothermal transformation;  $n = \gamma$  for a reversible adiabatic/isentropic transformation, and  $n = \infty$  for an isochoric/isovolumetric transformation.

## Problem 1.4 - Some Order-of-Magnitude Estimations

An important skill to develop is how to quickly estimate the scales of quantities of interest. Occasionally throughout the semester we will use some problems that were developed as part of the Berkeley Discover grant to the physics department that will exercise our estimation and order-of-magnitude muscles! The point here is not to get exact answers but to get within a factor of 10! A lot of details will be unnecessary and we will still be able to make some meaningful predictions.

You may refer to Edward Purcell's "cheat sheet" of useful orders of magnitude that he published in *Physics Today* (this is the same Purcell who taught at Berkeley and who wrote the textbook *Electricity and Magnetism*).

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In Problem 1.1 we re-derived the law of atmospheres which suggested a natural length scale for pressure variations of  $k_B T / mg$ .

(a) Estimate this length scale at room temperature for air. From this estimate the pressure at the top of Mt. Whitney, California (4420 m elevation) assuming  $P = 1$  atm at sea level.

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For a solid, the *linear* thermal expansion coefficient  $\alpha$  is defined as the fractional increase in *length* per unit temperature,

$$\alpha \equiv \frac{1}{L} \frac{dL}{dT} \approx \frac{1}{L} \frac{\Delta L}{\Delta T}.$$

The linear expansion coefficient of steel is of the scale  $10^{-5} \text{ K}^{-1}$ .

(b) Estimate the total variation in length of the San Francisco-Oakland Bay Bridge over the course of a typical day.

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Consider a gas of molecular hydrogen  $\text{H}_2$ . We want to estimate the temperature above which the rotational transitions of  $\text{H}_2$  become excited. This is related to the energy scale for the quantized energy levels of molecular hydrogen due to rotations. Luckily we don't have to do the full quantization or deal with much quantum mechanics! The key quantum fact here is that the **angular momentum**  $L$  is quantized and the scale of angular momentum can be given by Planck's constant  $\hbar$ . We will walk you through solving this problem with the following steps:

(c)

- What is the formula for the energy of a system with angular momentum  $L$  and moment of inertia  $I$ ?
- What is the formula for the moment of inertia  $I$  for a "barbell" with two equal masses of mass  $m$  separated by a length  $\ell$ ?
- What is the length scale of a hydrogen atom (essentially the Bohr radius, which serves as an estimate of the length of the bond)?
- What is the mass scale of a hydrogen atom?
- What is the scale for the moment of inertia  $I$  of  $\text{H}_2$ ?
- What is the scale for the energy levels of molecular hydrogen?
- What is the associated temperature scale?

See! That wasn't so bad!

[Note: Again, Purcell's "cheat sheet" is a good source of estimates of numbers for some of these parts!]

Now let's answer the same questions but for the vibrational modes of molecular hydrogen. Note that we can estimate the effective "spring constant"  $k$  for a diatomic molecule like hydrogen by imagining the spring is stretched to the verge of breaking which is roughly at the same energy as the binding energy of the molecule. The "length scale" at this breaking will be roughly the same scale as the size of the molecule or length of the bond, which we already estimated in part (c).

**(d) Extra Part (Not for Credit)**

- What is the formula for the angular frequency  $\omega$  for an oscillator with mass  $m$  and "spring constant"  $k$ ?
- What is a reasonable estimate for the binding energy of molecular hydrogen?<sup>7</sup>
- What is the scale for the effective spring constant  $k$ ?
- What is the scale for the angular frequency  $\omega$ ?
- What is the scale for the vibrational energy levels of molecular hydrogen?
- What is the associated temperature scale?

Once you are done you can check your answers against Figure 1.13 from Schroeder:

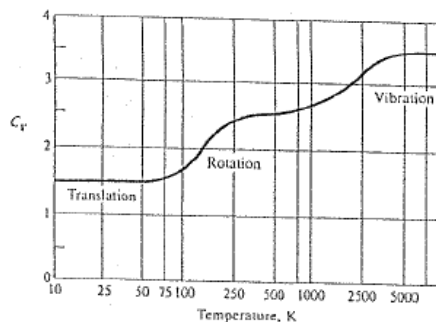


Figure 3.9 Heat capacity at constant volume of one molecule of  $H_2$  in the gas phase. The vertical scale is in fundamental units; to obtain a value in conventional units, multiply by  $k_B$ . The contribution from the three translational degrees of freedom is  $\frac{3}{2}$ ; the contribution at high temperatures from the two rotational degrees of freedom is 1; and the contribution from the potential and kinetic energy of the vibrational motion in the high temperature limit is 1. The classical limits are attained when  $\tau \gg$  relevant energy level separations.



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<sup>7</sup>You can look this up but try to estimate first!