Please note, all Griffiths problems come from our class text, the second edition.

1. **Bloch Sphere**
A particle with magnetic moment $\hat{\mu}$ is placed in a magnetic field $\mathbf{B}$. As a result, it has energy
$$\hat{H} = -\hat{\mu} \cdot \mathbf{B}.$$
The magnetic moment of an electron (which is spin $\frac{1}{2}$) is related to its spin by its gyromagnetic ratio γ (which has units of charge over mass):
$$\hat{\mu} = \gamma \hat{\mathbf{S}}.$$
We will perform this problem in the S_z eigenbasis: $|+z\rangle$ has eigenvalue $+\hbar/2$ and $|-z\rangle$ has eigenvalue $-\hbar/2$.

(a) If we orient the magnetic field such that $\mathbf{B} = B_0\hat{\mathbf{z}}$, (here $\hat{\mathbf{z}}$ is a unit vector not an operator) what are the energy eigenstates and eigenvalues?
Solution: The Hamiltonian becomes
$$\hat{H} = -\gamma B_0 \hat{S}_z,$$
which is just a multiple of $\hat{S}_z$. The eigenstates are the same, and the eigenvalues are multiplied by $-\gamma B_0$. So the eigensates are $|+z\rangle$ with eigenvalue $-\frac{1}{2}\hbar\gamma B_0$ and $|-z\rangle$ with eigenvalue $+\frac{1}{2}\hbar\gamma B_0$.

(b) If we orient the magnetic field in another direction, but we do not change its magnitude, argue why the energy eigenvalues are the same.
Solution: There is nothing physical about the z axis. We could call any direction $\hat{\mathbf{z}}$. So the energy eigenvalues, which are measurable qualities, should not depend on our arbitrary non-physical direction of the z axis.

(c) Let us orient the magnetic field such that $\mathbf{B} = B_0\hat{\mathbf{n}}$, where
$$\hat{\mathbf{n}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}.$$
Again, $\hat{\mathbf{n}}$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors and not operators. Show that the energy eigenstates are given as
$$|+n\rangle = \cos(\theta/2)|+z\rangle + e^{i\phi}\sin(\theta/2)|-z\rangle$$
,
$$|-n\rangle = \sin(\theta/2)|+z\rangle - e^{i\phi}\cos(\theta/2)|-z\rangle.$$
Solution: The Hamiltonian is given by
$$\hat{H} = -\gamma B_0(\sin\theta\cos\phi\hat{S}_x + \sin\theta\sin\phi\hat{S}_y + \cos\theta\hat{S}_z)$$
In the S_z basis, we have the following representation of $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$:
$$\hat{S}_x \to \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{S}_y \to \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_z \to \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So the representation of the Hamiltonian is

$$\hat{H} \to -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\theta + i\sin\phi) & -\cos\theta \end{pmatrix} = -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

The representations of the supposed eigenstates are

$$|{+n}\rangle \to \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix},$$

$$|{-n}\rangle \to \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}.$$

So

$$\hat{H}\,|{+n}\rangle \to -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}.$$

Similarly,

$$\hat{H}\,|{-n}\rangle \to +\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}.$$

So these are the energy eigenstates, and they have eigenvalues $\mp\gamma B_0\hbar/2$, as expected.

(d) The state $|{+n}\rangle$ gives us a way to map from the surface of a sphere, parametrised by (θ, ϕ), to the space of physical states of a two-level quantum mechanical system,

$$|\psi(\theta,\phi)\rangle = \cos(\theta/2)\,|{+z}\rangle + e^{i\phi}\sin(\theta/2)\,|{-z}\rangle\,.$$

Show that this is a bijection between the sphere and space of physical states. That is, show that any normalized state of a two-level system can be uniquely determined by a point on a sphere with coordinates (θ, ϕ). [Hint: Begin by showing that any two-level state is equivalent to a $|{+n}\rangle$. Then show that all coordinate pairs (θ, ϕ) that refer to the same point on a sphere also refer to the same quantum state]. We call the sphere made of these points the Bloch sphere.

Solution: The most general two level quantum mechanical system (i) must be normalized and (ii) is free up to an overall phase. Imposing (i) means we can write a general state as

$$|\psi\rangle = e^{i\alpha}\cos\gamma\,|{+z}\rangle + e^{i\beta}\sin\gamma\,|{-z}\rangle\,.$$

(ii) allows us to multiply by whatever phase we want. To uniquely identify a state, we will multiply the whole thing by the proper phase to force the $|{+z}\rangle$ coefficient to be real and non-negative. So we can multiple by $e^{-i\alpha}$ and enforce that $\cos\gamma$ is non-negative by choosing $0 \le \gamma \le \pi/2$. Then we have

$$|\phi\rangle = \cos\gamma\,|{+z}\rangle + e^{i(\beta-\alpha)}\sin\gamma\,|{-z}\rangle\,.$$

If we relabel

$$\gamma = \frac{\theta}{2},$$
$$\beta - \alpha = \phi,$$

then we have the restrictions

$$0 \le \theta \le \pi$$
$$0 \le \phi < 2\pi,$$

and our arbitrary state is equivalent to the general $|{+n}\rangle$:

$$|\psi\rangle = \cos(\theta/2)\,|{+z}\rangle + e^{i\phi}\sin(\theta/2)\,|{-z}\rangle\,.$$

Next, we must show that all the coordinate pairs (θ, ϕ) that refer to the same point on a sphere also refer to the same quantum state. On the sphere, when $\theta = 0$, on the positive side of the z axis, the value of ϕ is irrelevant, all values refer to the same point. In the corresponding state, when $\theta = \pi$, we get

$$|\psi\rangle = e^{i\phi}\sin\frac{\theta}{2}\,|{-z}\rangle\,.$$

Here, the $e^{i\phi}$ is a physically meaningless overall phase, so it is also irrelevant. So any normalized state of a two-level system can be uniquely determined by a point on a sphere with coordinates (θ, ϕ)!

(e) Draw the Bloch sphere and place points indicating where the eigenstates of S_x, S_y, and S_z are mapped to. [Hint: the state

$$\cos\frac{\theta}{2}\,|{+z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|{-z}\rangle$$

is at the coordinate (θ, ϕ).]

Solution: To get the spherical coordinates, we are matching with

$$|\phi\rangle = \cos\frac{\theta}{2}\,|{+z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|{-z}\rangle$$

The eigenstates of S_x are

$$|{+x}\rangle = \frac{1}{\sqrt{2}}(|{+z}\rangle + |{-z}\rangle),$$

which has $\theta = \pi/2$ and $\phi = 0$ and

$$|{-x}\rangle = \frac{1}{\sqrt{2}}(|{+z}\rangle - |{-z}\rangle),$$

which has $\theta = \pi/2$ and $\phi = \pi$.
The eigenstates of S_y are

$$|{+y}\rangle = \frac{1}{\sqrt{2}}(|{+z}\rangle + i\,|{-z}\rangle),$$

which has $\theta = \pi/2$, and $\phi = \pi/2$ and

$$|{-y}\rangle = \frac{1}{\sqrt{2}}(|{+z}\rangle - i\,|{-z}\rangle),$$

which has $\theta = \pi/2$ and $\phi = 3\pi/2$.
The eigenstates of S_z are $|{+z}\rangle$, which has $\theta = 0$, and $|{-z}\rangle$, which has $\theta = \pi$ (ϕ is undefined at the poles).
Finally, earlier in the problem, we verified the eigenstates of S_n. Written using physical angles $\tilde{\theta}$ and $\tilde{\phi}$,

$$|{+n}\rangle = \cos\frac{\tilde{\theta}}{2}\,|{+z}\rangle + e^{i\tilde{\phi}}\sin\frac{\tilde{\theta}}{2}\,|{-z}\rangle\,,$$

which has Bloch sphere coordinates $\theta = \tilde{\theta}$ and $\phi = \tilde{\phi}$ and

$$|{-n}\rangle = \sin\frac{\tilde{\theta}}{2}\,|{+z}\rangle - e^{i\tilde{\phi}}\cos\frac{\tilde{\theta}}{2}\,|{-z}\rangle = \cos\left(\frac{\pi}{2} - \frac{\tilde{\theta}}{2}\right)|{+z}\rangle + e^{i(\tilde{\phi}+\pi)}\sin\left(\frac{\pi}{2} - \frac{\tilde{\theta}}{2}\right)|{-z}\rangle\,,$$

which has Bloch sphere coordinates $\theta = \pi - \tilde{\theta}$ and $\phi = \tilde{\phi} + \pi$. This is opposite the $|{+n}\rangle$ state. All of these are plotted below.

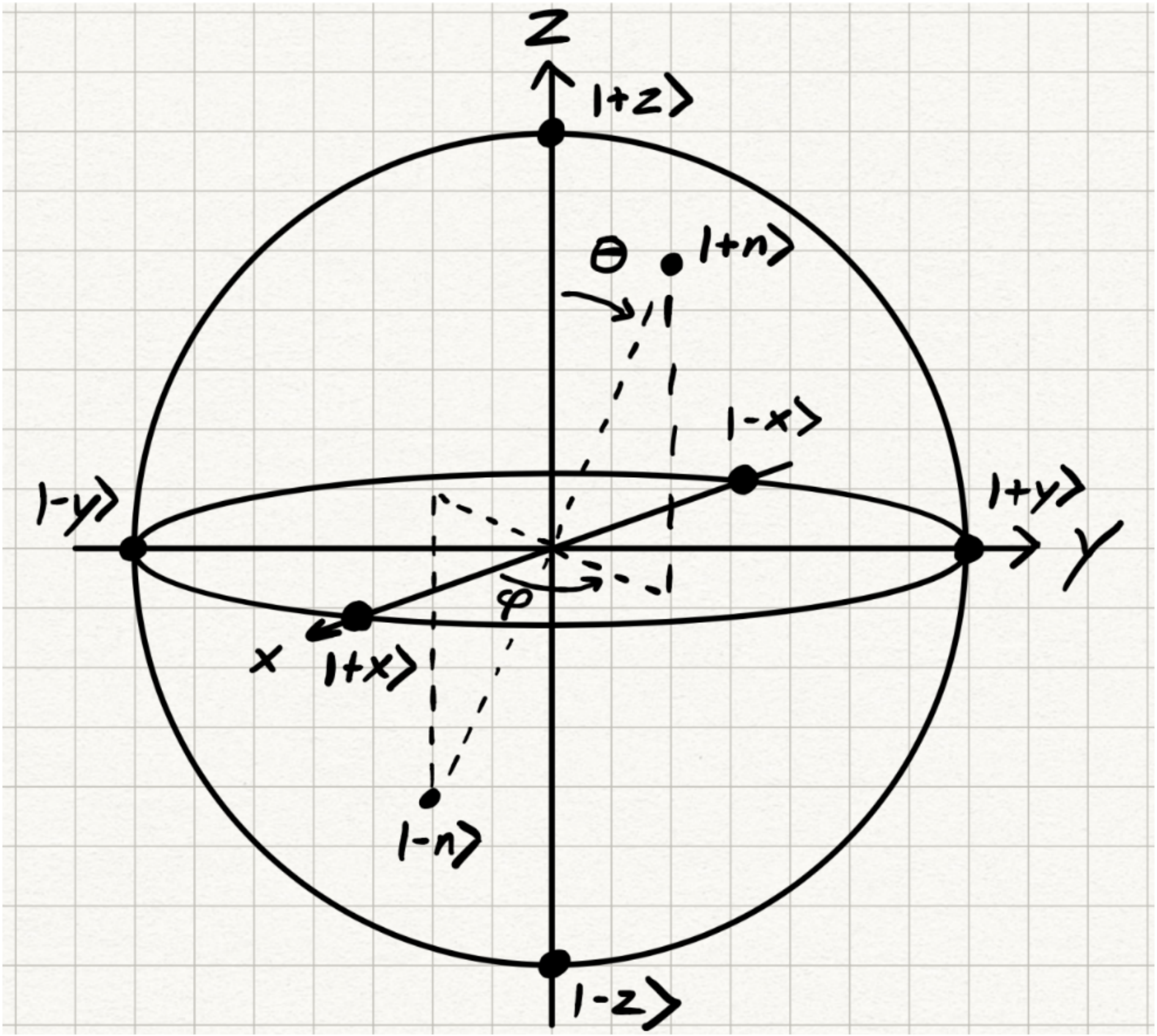

Figure 1: Bloch sphere

(f) Calculate $\langle S_x\rangle$, $\langle S_y\rangle$, and $\langle S_z\rangle$ for the state $|\psi(\theta,\phi)\rangle$.
Solution: The bra corresponding to $|\psi(\theta,\phi)\rangle$ is

$$\langle\psi(\theta,\phi)| = \cos(\theta/2)\,\langle +z| + e^{-i\phi}\sin(\theta/2)\,\langle -z|\,.$$

All these expectation values can be computed in the S_z basis.

$$\begin{aligned}\langle S_x\rangle &= \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\\ &= \frac{\hbar}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\phi}+e^{-i\phi}\right)\\ &= \frac{\hbar}{2}\sin\theta\cos\phi\end{aligned}$$

$$\begin{aligned}\langle S_y\rangle &= \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix}\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}\\ &= \frac{\hbar}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(-ie^{i\phi}+ie^{-i\phi}\right)\\ &= \frac{\hbar}{2}\sin\theta\sin\phi\end{aligned}$$

$$\begin{aligned}
\langle S_z \rangle &= \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) \\
&= \frac{\hbar}{2}\cos\theta
\end{aligned}$$

2. **Rotation Matrices**
Do this problem after problem 1. It will make more sense if you do it that way

(a) If we rotate the $|+z\rangle$ state around the $\hat{\mathbf{x}}$-axis by angle φ, what will it become? What about $|-z\rangle$? [Hint: Make use of the physical connection from the previous problem. If a state was in the $|+z\rangle$ state, it would have a specific energy when a magnetic field is pointing in the $\hat{\mathbf{z}}$ direction. If we rotate out (unphysical) coordinates so that the magnetic field points in the $\hat{\mathbf{n}}$ direction, the state must still have the same energy, and the state that has that energy is $|+n\rangle$. So if we rotate $\hat{\mathbf{z}}$ to $\hat{\mathbf{n}}$, then $|+z\rangle$ becomes $|+n\rangle$.]
Solution: If we rotate a vector pointing in the $+\hat{\mathbf{z}}$ direction by an angle φ around the $\hat{\mathbf{x}}$-axis, then it becomes a vector pointing in the $(\theta = \varphi, \phi = 3\pi/2)$ direction. The state $|+z\rangle$ becomes

$$\hat{R}(\varphi\hat{\mathbf{x}})\,|+z\rangle = \cos\frac{\varphi}{2}\,|+z\rangle - i\sin\frac{\varphi}{2}\,|-z\rangle\,,$$

where we have used $e^{i3\pi/2} = -i$.
Similarly, a vector pointing in the $-\hat{\mathbf{z}}$ direction gets rotated to a vector in the $(\theta = \pi - \varphi, \phi = \pi/2)$ direction. The state $|-z\rangle$ becomes

$$\hat{R}(\varphi\hat{\mathbf{x}})\,|-z\rangle = \sin\frac{\varphi}{2}\,|+z\rangle + i\cos\frac{\varphi}{2}\,|-z\rangle\,,$$

where we've used the identity $e^{i\pi/2} = i$ and the trig identities

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x, \quad \sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

(b) Use the above result to find the matrix representation (in the S_z eigenbasis) of the operator $\hat{R}(\varphi\hat{\mathbf{x}})$ that rotates a spin state around the $\hat{\mathbf{x}}$-axis by an angle φ.
Solution: The matrix must map

$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\cos\frac{\varphi}{2}\\ -i\sin\frac{\varphi}{2}\end{pmatrix} e^{i\alpha}$$

and

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}\sin\frac{\varphi}{2}\\ i\cos\frac{\varphi}{2}\end{pmatrix} e^{i\beta},$$

so we know that $\hat{R}(\varphi\hat{\mathbf{x}})$ is represented as

$$\hat{R}(\varphi\hat{\mathbf{x}}) \to \begin{pmatrix}\cos\frac{\varphi}{2}e^{i\alpha} & \sin\frac{\varphi}{2}e^{i\beta}\\ -i\sin\frac{\varphi}{2}e^{i\alpha} & i\cos\frac{\phi}{2}e^{i\beta}\end{pmatrix}.$$

An overall multiplication by a complex phase is unphysical, so we can set $\alpha \to 0$ (and redefine β accordingly). To determine β, we will see how a linear combination gets rotated. If we rotate a vector pointing in the $\hat{y}$ direction by an angle φ around the $\hat{\mathbf{x}}$-axis, we get a vector pointing in the $(\theta = \pi/2 - \varphi, \phi = \pi/2)$ direction, so the matrix must also map

$$\frac{1}{\sqrt{2}}\begin{pmatrix}i\\i\end{pmatrix} \to \begin{pmatrix}\cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\\ i\sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right)\end{pmatrix} e^{i\gamma} = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\\ i(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2})\end{pmatrix} e^{i\gamma}.$$

If we act the above representation of $\hat{R}(\varphi\hat{\mathbf{x}})$ with α set to zero on the representation of $|+y\rangle$, we get

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\frac{\varphi}{2} + i\sin\frac{\varphi}{2}e^{i\beta}\\ -\cos\frac{\varphi}{2}e^{i\beta} - i\sin\frac{\varphi}{2}\end{pmatrix}.$$

To make this agree with the above mapping with the γ, we need $\gamma = 0$ and $\beta = 3\pi/2$. So the correct matrix representation of $\hat{R}(\varphi\hat{\mathbf{x}})$ is

$$\hat{R}(\varphi\hat{\mathbf{x}}) \to \begin{pmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}.$$

[NOTE FOR GRADING: As the importance of the phases α and β were not obvious, we will not take off points if the answer was written with $\beta = 0$, which is not correct.]

3. **Griffiths 4.2**

***Problem 4.2** Use separation of variables in *cartesian* coordinates to solve the infinite *cubical* well (or "particle in a box"):

$$V(x, y, z) = \begin{cases} 0, & \text{if } x, y, z \text{ are all between 0 and } a; \\ \infty, & \text{otherwise.} \end{cases}$$

(a) Find the stationary states, and the corresponding energies.

(b) Call the distinct energies $E_1, E_2, E_3, \ldots$, in order of increasing energy. Find E_1, E_2, E_3, E_4, E_5, and E_6. Determine their degeneracies (that is, the number of different states that share the same energy). *Comment:* In *one* dimension degenerate bound states do not occur (see Problem 2.45), but in three dimensions they are very common.

(c) What is the degeneracy of E_{14}, and why is this case interesting?

Solution:

(a) Inside the box the Shrodinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)=E\psi.$$

We assume a separable solution $\psi(x, y, z) = X(x)Y(y)Z(z)$. We can plug this into the SE and divide by ψ:

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=-\frac{2m}{\hbar^2}E.$$

Since each term on the left is a function of a different variable, the solution is that each term must be equal to a constant. We can call these constants k_x^2,k_y^2, and k_z^2.

$$\frac{d^2X}{dx^2}=-k_x^2X;\quad \frac{d^2Y}{dy^2}=-k_y^2Y;\quad \frac{d^2Z}{dz^2}=-k_z^2Z,\quad \text{where } E=\frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2)$$

The solutions for X, Y, and Z are the same as the 1D solution to the infinite square well:

$$X(x) = A_x\sin(k_xx)$$
$$Y(y) = A_x\sin(k_yy)$$
$$Z(z) = A_z\sin(k_zz)$$

where $k_x = n_x\pi/a$, $k_y = n_y\pi/a$, and $k_z = n_z\pi/a$ to satisfy the boundary conditions, with the n's being positive nonzero integers. The normalization constants are $A_x = A_y = A_z = \left(\frac{2}{a}\right)^{1/2}$.
Putting all the pieces together the stationary states are

$$\psi(x,y,z)=\left(\frac{2}{a}\right)^{2/3}\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right),$$

and the corresponding energies are

$$E=\frac{\pi^2\hbar^2}{2ma^2}(n_x^2+n_y^2+n_z^2)\quad \text{with } n_x,n_y,n_z=1,2,3,...$$

(b) First let's list the possible n values

n_x	n_y	n_z	$(n_x^2 + n_y^2 + n_z^2)$
1	1	1	3
1	1	2	6
1	2	1	6
2	1	1	6
1	2	2	9
2	1	2	9
2	2	1	9
1	1	3	11
1	3	1	11
3	1	1	11
2	2	2	12
1	2	3	14
1	3	2	14
2	1	3	14
2	3	1	14
3	1	2	14
3	2	1	14

Now we can see the first 6 energies and their degeneracies.

Energy	Degeneracy
$E_1 = 3\frac{\pi^2\hbar^2}{2ma^2}$	1
$E_2 = 6\frac{\pi^2\hbar^2}{2ma^2}$	3
$E_3 = 9\frac{\pi^2\hbar^2}{2ma^2}$	3
$E_4 = 11\frac{\pi^2\hbar^2}{2ma^2}$	3
$E_5 = 12\frac{\pi^2\hbar^2}{2ma^2}$	1
$E_6 = 14\frac{\pi^2\hbar^2}{2ma^2}$	6

(c) The next combinations before we get to E_{14} are $E_7(322)$, $E_8(411)$, $E_9(331)$, $E_{10}(421)$, $E_{11}(332)$, $E_{12}(422)$, and $E_{13}(431)$. We get E_{14} by two combinations pf n values: 333 and 511. This is what makes this case interesting. To get the degeneracy we need the number of ways of ordering each combination. There is one way to order 333 and 3 ways of ordering 511. So the degeneracy of E_{14} is $d = 4$.

4. **Griffiths 4.38**

∗Problem 4.38 Consider the **three-dimensional harmonic oscillator**, for which the potential is

$$V(r) = \frac{1}{2}m\omega^2 r^2. \qquad [4.188]$$

(a) Show that separation of variables in cartesian coordinates turns this into three one-dimensional oscillators, and exploit your knowledge of the latter to determine the allowed energies. *Answer:*

$$E_n = (n + 3/2)\hbar\omega. \qquad [4.189]$$

(b) Determine the degeneracy $d(n)$ of E_n.

Solution:

(a) The first part of this problem is very similar to the last problem. Let's first write the Shrodinger equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)\psi = E\psi$$

We again assume a separable solution $\psi(x, y, z) = X(x)Y(y)Z(z)$, plug this in, and divide by psi

$$\left(-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{2}m\omega^2x^2\right)+\left(-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{2}m\omega^2y^2\right)+\left(-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega^2z^2\right) = E$$

Each of the terms on the left hand side will be equal to a constant. We can call these constants E_x, E_y, and E_z.

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + \frac{1}{2}m\omega^2x^2X &= E_xX \\
-\frac{\hbar^2}{2m}\frac{d^2Y}{dy^2} + \frac{1}{2}m\omega^2y^2Y &= E_yY \\
-\frac{\hbar^2}{2m}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega^2z^2Z &= E_zZ.
\end{aligned}$$

Now we just have three 1-D harmonic oscillators, so

$$E_x = \left(n_x + \frac{1}{2}\right)\hbar\omega;_y = \left(n_y + \frac{1}{2}\right)\hbar\omega;_z = \left(n_z + \frac{1}{2}\right)\hbar\omega; \quad \text{where } n_x, n_y, n_z = 0, 1, 2, 3...$$

So the allowed energies are

$$\begin{aligned}
E &= E_x + E_y + E_z = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega \\
E &= \left(n + \frac{3}{2}\right)\hbar\omega, \quad \text{with } n \equiv n_x + n_y + n_z.
\end{aligned}$$

(b) We need to figure out how many ways we can add up n_x, n_y, and n_z to get a total of n.
$n_x = n$ with $n_y = n_z = 0$; one way
$n_x = n - 1$ with $n_y = 1, n_z = 0$ or $n_y = 0, n_z = 1$; two ways
$n_x = n - 2$ with $n_y = 2, n_z = 0$, or $n_y = 1, n_z = 1$, or $n_y = 0, n_z = 2$; three ways
If we continue this pattern we find

$$d(n) = 1 + 2 + 3 + ... + (n + 1)$$
$$d(n) = \frac{(n + 1)(n + 2)}{2}$$