Lecture #31

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Several and Many Particle Systems

- Consider system of N particles

The i^{th} particle is described by position $\vec{r}_i$, momentum $\vec{p}_i$, and spin $\vec{S}_i$.

- Let q_i be a complete set of commuting observables which describe the system, such as $\vec{r}_i$ and S_{iz}.

System $\Psi(q_1, q_2, \cdots q_N, t)$

These satisfy the SE:

$$i\hbar \frac{\partial}{\partial t} \Psi(q_1, q_2, \cdots q_N, t) = \hat{H} \Psi(q_1, q_2, \cdots q_N, t)$$

For a time indep $\hat{V}$, $\hat{H} = \hat{T} + \hat{V}$

$\hat{H}\Psi_E = E\Psi_E$; can find eigenfunctions

- If wavefunctions overlap, cannot distinguish particles!

$\Longrightarrow$ Hamiltonian of the system must be symmetric in the interchange the of an pair of particles "i" and "j".

We can define an operator $\hat{P}_{ij}$ as an operator with changes q_i and q_j of particles i and j.

$[\hat{P}_{ij}, \hat{H}] = 0$ (Same answer whether you change order first)

• In general, an exact eigenfunction $\Psi(q_1 \ldots q_N)$ has no particular symmetry property under the exchange of variables q_i

<u>Note</u>: However, if $\Psi(q_1, \ldots q_i \ldots q_j \ldots q_N)$ is an eigenfunction of $\hat{H}$ corresponding to the eigenvalue E, then so is $P_{ij}\Psi$

$$\hat{P}_{ij}\,\Psi(q_1 \cdots q_i \cdots q_j \cdots q_N) = \Psi(q_1 \cdots q_j \cdots q_i \cdots$$

$\hat{P}_{ij}^{\,2} = I$, eigenvalues are $\epsilon = \pm 1$

Thus eigenfunctions are given by:

$$\hat{P}_{ij}\,\psi(q_1 \cdots q_i \cdots q_j \cdots q_N) = \psi(q_1 \cdots q_j \cdots q_i \cdots q_N)$$

$$= \psi(q_1 \cdots q_i \cdots q_j \cdots q_N)$$

for $\varepsilon = 1$

Symmetric under exchange.

$$\hat{P}_{ij}\,\psi(q_1 \cdots q_i \cdots q_j \cdots q_N) = \psi(q_1 \cdots q_j \cdots q_i \cdots q_N)$$

$$= -\psi(q_i \cdots q_i \cdots q_j \cdots q_N)$$

$\varepsilon = -1$

Antisymmetric under exchange.

- Note, there are $N!$ different permutations of variables $q_1 \cdots q_N$.

Define $\hat{P}$ as the permutation operator that replaces q_1 by q_{P1}, q_2 by q_{P2}, q_N by q_{PN}.

P can be obtained by successive applications of pairwise exchanges

$$[\hat{P}, \hat{H}] = 0$$

$$\hat{P}\,\psi(q_1 \cdots q_N) = \psi(q_{P1} \cdots q_{PN})$$

* Not all N! permutations $\hat{P}$ commute with themselves. Therefore, the eigenfunctions $\psi(q_1, \cdots q_N)$ are not in general eigenfunctions of all the N! operators $\hat{P}$.

* There are two exceptional states which are eigenstates of $\hat{H}$ and of the N! operators $\hat{P}$

(A) Totally symmetric case $\hat{P}\psi_S(q_1 \cdots q_N)$
$$= \psi_S(q_1 \cdots q_N)$$

which is symmetric for any pairwise exchange P_{ij}.

(B) Totally anti-symmetric case

$$P\,\psi_A(q_1 \cdots q_N) = \begin{cases} \psi_A(q_1 \cdots q_N) & \text{even perm} \\ -\psi_A(q_1 \cdots q_N) & \text{odd permut} \end{cases}$$

* P is a constant of motion so that a wave function of a certain symmetry preserves it.

Bosons & Fermions

→ ψ_S and ψ_A are sufficient to describe all known particles

- Symmetric wave functions are Bosons → Zero or integer spin

egs. - All mesons (π, K's $s=0$) (ρ's $s=1$)
- Photons ($S=1$)
- Intermediate vector Bosons (W, Z $S=1$)
- Higgs Boson ($S=0$)

- Anti-symmetric wave functions are Fermions → half odd integer spin

egs. All leptons (e^-, μ, ν) $S=\frac{1}{2}$

. Baryons ($p, n, ..$)

Constructing symmetrized wave functions:

eg. Two identical particles
$\psi(q_1, q_2)$

$$\psi_S(q_1, q_2) = \frac{1}{\sqrt{2}}\left[\psi(q_1, q_2) + \psi(q_2, q_1)\right]$$

$$\psi_A(q_1, q_2) = \frac{1}{\sqrt{2}}\left[\psi(q_1, q_2) - \psi(q_2, q_1)\right]$$

For Bosons use ψ_S, Fermions ψ_A.

For $N = 3$:

$$\psi_S(q_1 q_2 q_3) = \frac{1}{\sqrt{6}}\left[\psi(q_1 q_2 q_3) + \psi(q_2 q_1\right.$$
$$+ \psi(q_2, q_3, q_1) + \psi(q_3 q_2 q_1) +$$
$$\left.\psi(q_3 q_1 q_2) + \psi(q_1 q_3 q_2)\right]$$

$$\psi_A(q_1, q_2, q_3) = \frac{1}{\sqrt{6}}\left[\psi(q_1 q_2 q_3) - \psi(q_2, q_1, q_3)\right.$$
$$+ \psi(q_2 q_3 q_1) - \psi(q_3 q_2 q_1) + \psi(q_3 q_1 q_2)$$
$$\left. - \psi(q_1 q_3 q_2)\right]$$

- Consider special case of $\hat{H}$ is the sum of N single particle Hamiltonians h_i i.e. <u>no interactions</u>.

If we had no spin symmetry

$$h_i u_\lambda(q_i) = E_\lambda u_\lambda(q_i)$$

Then solution is a <u>product</u>:

$$\psi(q_1 q_2 \cdots q_N) = u_\alpha(q_1)\, u_\beta(q_2) \cdots u_\lambda(q_i) \cdots u_\nu(q_N)$$

where $\alpha, \beta, \delta \ldots \nu$ represent quantum #s for the one particle states.

$$E = E_\alpha + E_\beta + E_\delta \cdots + E_\lambda + E_\nu$$

If you have to include spin symmetry: for identical particles

Consider first $N=2$:

$$\psi_S(q_1, q_2) = \frac{1}{\sqrt{2}}\left[u_\alpha(q_1)\, u_\beta(q_2) + u_\alpha(q_2)\, u_\beta(q_1)\right]$$

$$\psi_A(q_1, q_2) = \frac{1}{\sqrt{2}}\left[u_\alpha(q_1)\, u_\beta(q_2) - u_\alpha(q_2)\, u_\beta(q_1)\right]$$

Not product states → entangled!
Eigenstates of energy $E = E_\alpha + E_\beta$
Neither of the particles are in an (1 or 2)

energy eigenstate, being partly in the state α and partly in state β.

** • A measurement of the energy of one of the particles will produce E_α or E_β
50% of the time.

For N particles, the totally antisymmetric wavefunction can be constructed from a **Slater** determinant:

quantum number for n_α — particle #

$$\psi_A(q_1 q_2 \cdots q_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_\alpha(q_1) & u_\beta(q_1) & \cdots & u_\nu(q_1) \\ u_\alpha(q_2) & u_\beta(q_2) & \cdots & u_\nu(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ u_\alpha(q_N) & u_\beta(q_N) & \cdots & u_\nu(q_N) \end{vmatrix}$$

** If two or more sets of individual quantum numbers $\alpha, \beta, \ldots$ are identical → wavefunction vanishes

→ Only one Fermion can occupy a single quantum state

→ W. Pauli 1925

"Pauli Exclusion Principle"