Lecture 16 137A Simple Harmonic oscillator: $\hat{a}_+, \hat{a}_-$

Recall the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Introduce:

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} \mp i\frac{\hat{p}_x}{(m\hbar\omega)^{1/2}}\right.$$

Note: $\hat{x}, \hat{p}_x$ are Hermitian, so $\boxed{\begin{aligned} a_+ &= a_-^\dagger \\ a_- &= a_+^\dagger \end{aligned}}$

$$[\hat{a}_-, \hat{a}_+] = 1$$

$\longrightarrow$ Can rewrite $\hat{H}$ as:

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-)$$

$$= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

$$= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

$$= \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

tells you which state of the H.O. is occupied

- $[\hat{H}, \hat{a}_\pm] = \pm \hbar\omega \hat{a}_\pm$

Let's try to see what $\hat{a}_\pm$ do to a state:

Consider energy eigenstates $|E\rangle$

$E_n = (n+\frac{1}{2})\hbar\omega$

To calculate $\hat{a}_\pm |E\rangle$, let's act on this quantity with $\hat{H}$.

- $\hat{H}\, \hat{a}_\pm |E\rangle = (\underbrace{\hat{a}_\pm \hat{H} \pm \hbar\omega \hat{a}_\pm}_{\text{we've used the commutator above.}}) |E\rangle$

$= (E \pm \hbar\omega)\, \hat{a}_\pm |E\rangle$

where we have used $\hat{H}|E\rangle = E|E\rangle$

$\Rightarrow$ ** Thus, $\hat{a}_\pm |E\rangle$ is <u>also</u> an energy eigenstate with energy <u>differ</u> by one level.

$\Rightarrow$ $\hat{a}_+$ "raising" operators.
$\hat{a}_-$ "lowering"

We further define the action of $\hat{a}_-$ on the ground state $(n=0)$.

$$\hat{a}_- |E_0\rangle = 0$$

Now, lets act with $\hbar\omega \hat{a}_+$ to the left of this relation:

$$\hbar\omega \hat{a}_+ \hat{a}_- |E_0\rangle = \hbar\omega \hat{N} |E_0\rangle$$

$$= (\hat{H} - \tfrac{1}{2}\hbar\omega) |E_0\rangle$$

$$\Rightarrow \hat{H} |E_0\rangle = \underbrace{\tfrac{1}{2}\hbar\omega}_{= E_0 \,!} |E_0\rangle$$

- $\hat{a}_+ |E_0\rangle$ must thus have energy eigenvalue $\frac{3}{2}\hbar\omega \ldots$

$$\Rightarrow |E_{n+1}\rangle = C_{n+1} \hat{a}_+ |E_n\rangle$$

Require $\langle E_{n+1} | E_{n+1}\rangle = 1$

$$|C_{n+1}|^2 \langle E_n | \underbrace{\hat{a}_-}_{\hat{a}_+^\dagger} \hat{a}_+ |E_n\rangle$$

, now use $\hat{a}_- \hat{a}_+ = \frac{\hat{H}}{\hbar\omega} + \frac{1}{2}$

$$|C_{n+1}|^2 \langle E_n | E_n\rangle [n+1] = 1$$

$$\longrightarrow |C_{n+1}|^2 = (n+1)^{-1}$$

$$\longrightarrow C_{n+1} = 1/\sqrt{n+1}$$

$$\boxed{\hat{a}_+ |E_n\rangle = \sqrt{n+1}\, |E_{n+1}\rangle} \;\; **$$

- If we start from the ground state:

$$|E_n\rangle = \frac{1}{\sqrt{n!}}\hat{a}_+^n |E_0\rangle$$

- Similarly, we can find the action of $\hat{a}_-$

$$|E_n\rangle = \underbrace{C_n}_{\frac{1}{\sqrt{n}}} \hat{a}_+ |E_{n-1}\rangle$$

Act with $\hat{a}_-$ on both sides:

$$\hat{a}_- |E_n\rangle = \frac{1}{\sqrt{n}} \hat{a}_- \hat{a}_+ |E_{n-1}\rangle$$

$$\boxed{\hat{a}_- |E_n = \sqrt{n}\, |E_{n-1}\rangle} \quad **.$$

- We can use $\hat{a}_\pm$ to calculate other properties of the system.

eg. $$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (\hat{a}_+ + \hat{a}_-)$$

Calculate $\langle E_0 | \hat{x}^4 | E_0 \rangle$

$$= \frac{\hbar^2}{4m^2\omega^2} \langle E_0 | a_+^4 + a_+^3 a_- + a_+^2 a_- a_- \cdots a_-^4 | E_0 \rangle$$

5

$$= \frac{\hbar^2}{4m^2\omega^2} \langle E_0 | \underbrace{a_- a_+ a_- a_+}_{|E_0\rangle} + \underbrace{a_-^2 a_+^2}_{2|E_0\rangle} | E_0 \rangle$$

since we must only keep terms with the same # of a_+, a_-

$$= \frac{3\hbar^2}{4m^2\omega^2}$$

Matrix Representation

- Use $\{|E_n\rangle\}$ as a basis

$$[H] = \hbar\omega \begin{pmatrix} 1/2 & 0 & \cdots & 0 & \cdots \\ 0 & 3/2 & \cdots & & \\ & & 5/2 & \cdots & \\ \vdots & & & \ddots & \end{pmatrix}$$

Since

$$[H] = \begin{array}{c|ccc} & |E_0\rangle & |E_1\rangle & |E_2\rangle \cdots \\ \hline \langle E_0| & \langle E_0|\hat{H}|E_0\rangle & \langle E_0|\hat{H}|E_1\rangle & \langle E_0|H|E_2\rangle \\ \langle E_1| & \langle E_1|H|E_0\rangle & \langle E_1|H|E_1\rangle & \cdots \\ \langle E_2| & & \vdots & \ddots \\ \vdots & & & \end{array}$$

$$[N] = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 & \ddots \end{pmatrix}$$

Thus,

$$\langle E_k | E_{n+1} \rangle = C_{n+1} \langle E_k | \hat{a}_+ | E_n \rangle$$
$$= \delta_{k,n+1}$$

$$\rightarrow \quad [a_+]_{kn} = (n+1)^{1/2}\, \delta_{k,n+1}$$

$$[a_-]_{kn} = (k+1)^{1/2}\, \delta_{k+1,n}$$

$$[a_+] = \begin{pmatrix} 0 & & & \\ \sqrt{1} & \ddots & & \\ & \sqrt{2} & 0 & \\ & & \sqrt{3} & \ddots \end{pmatrix} \qquad [a_-] = \begin{pmatrix} 0 & \sqrt{1} & & \\ & \ddots & \sqrt{2} & \\ & & 0 & \sqrt{3} \\ & & & \ddots \\ & & & & 0 & \ddots \end{pmatrix}$$

Can also define $[x]$, $[p]$ in terms of $[\hat{a}_\pm]$

Return back to position representation.

<u>Recall</u>: $$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left[\left(\frac{m\omega}{\hbar}\right)^{1/2}\hat{x} \mp \left(\frac{\hbar}{m\omega}\right)^{1/2}\frac{d}{dx}\right]$$

let $$\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x = \alpha x$$

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\xi \mp \frac{d}{d\xi}\right)$$

$$\hat{a}_- |E_0\rangle = 0$$

$$\left(\xi + \frac{d}{d\xi}\right)\psi_0(\xi) = 0$$

solution is $\psi_0(\xi) = N_0\, e^{-\xi^2/2}$

$$N_0 \downarrow \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}$$

Thus $\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} e^{-(m\omega/\hbar)x^2/2}$

and $\psi_n(\xi) = (n!)^{-1/2}\left[\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right]^n \psi_0(\xi)$

(Recall generating function!)

This is equivalent to

$$\boxed{\psi_n(\xi) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{1/2} e^{-\xi^2/2} H_n(\xi)}$$