Midterm 2 review

Here are some problems that will be useful to go over in the review session on Wednesday April 6 at 6pm in Physics 251. There is no need to do all of them, please read through them first and decide what order to attempt them in. And keep in mind that these questions are longer (and maybe harder) than what one can reasonably expect from a 50-minute exam.

1 Fundamentals

1. True or false?
   (a) The states $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ can be distinguished from each other.
   (b) The expectation values of all observables in a stationary state are constant.
   (c) For a potential $V(r)$ that depends only on distance from the origin, if a system starts off in an eigenspace of $\hat{L}_x$, it stays in that eigenspace, i.e. $\hat{L}_x|\psi(0)\rangle = \lambda|\psi(0)\rangle \implies \forall t, \hat{L}_x|\psi(t)\rangle = \lambda|\psi(t)\rangle$.
   (d) For a potential $V(r)$ that depends only on distance from the origin, the Hamiltonian must be degenerate (i.e. there is at least one energy eigenvalue with more than one eigenstate).
   (e) If we measure the energy of a system repeatedly (without doing anything else in between), we will get the same result each time.
   (f) If the Hamiltonian commutes with operators $\hat{A}$ and $\hat{B}$, there is an energy eigenbasis in which both $\hat{A}$ and $\hat{B}$ are diagonal.
2. For the following operators, are they Hermitian and/or unitary?
   (a) the harmonic oscillator Hamiltonian $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$
   (b) the time-evolution operator $\hat{\mathcal{T}}(t) = e^{-i\hat{H}t/\hbar}$ that maps $|\psi(t_1)\rangle$ to $|\psi(t_1+t)\rangle$
   (c) the harmonic oscillator ladder operators $\hat{a}_\pm$
   (d) $\hat{L}_n$, an arbitrary component of (orbital) angular momentum
   (e) $\hat{L}_\pm$, the angular momentum ladder operators
   (f) $\hat{\mathbf{L}} \cdot \hat{\mathbf{r}} = \hat{L}_x\hat{x} + \hat{L}_y\hat{y} + \hat{L}_z\hat{z}$
3. What are the following commutators?
   (a) $[\hat{H}, \hat{p}]$ for the harmonic oscillator
   (b) $[\hat{H}, \hat{a}_\pm]$ for the harmonic oscillator
   (c) $[\hat{p}_x, \hat{x}]$
   (d) $[\hat{p}_z, \hat{L}_z]$
   (e) $[\hat{\mathbf{L}}^2, \hat{L}_\pm]$
   (f) $[\hat{L}_z, \hat{L}_\pm]$
4. Suppose we have an operator $\hat{A}$ such that $[\hat{H}, \hat{A}] = \lambda\hat{A}$. Given an energy eigenstate $|\psi\rangle$ with energy E, use $\hat{A}$ to construct a normalised eigenstate with energy $E+\lambda$. And use $\hat{A}^\dagger$ to construct a normalised eigenstate with energy $E-\lambda$. (Show that your constructions work.)
   (Bonus: can λ be complex? Can $\hat{A}$ be Hermitian and/or unitary?)

2 Harmonic oscillator

1. Show that $\psi(x) = Ae^{\frac{m\omega}{2\hbar}x^2}$ is an energy eigenstate. Find its energy. Is there anything dubious about this wavefunction?

2. Recall that the classical period of the harmonic oscillator is $T = \frac{2\pi}{\omega}$. Show that for any starting state $|\psi(0)\rangle$, we return to the same state after time T, i.e. $|\psi(T)\rangle$ and $|\psi(0)\rangle$ describe the same physical state.

3. Suppose we start off with a wavefunction $\Psi(x, t = 0) = Ax^3 e^{-\frac{m\omega}{2\hbar}x^2}$.
   (a) Write the wavefunction as a normalised linear combination of energy eigenstates.
   (b) Find an expression for $\Psi(x, t)$.
   (c) At time t, if you measured the energy, what possible values could you obtain, and with what probabilities?
   (d) What are $\langle x\rangle$ and $\langle p\rangle$ at time t?
   (e) What are $\sigma(x)$ and $\sigma(p)$ at time t?
   (f) At time t, what is the probability of finding the particle in the right half, $x > 0$?

4. Suppose we are in the ground state of the harmonic oscillator and the potential is instantaneously quadrupled without changing the wavefunction. (Define ω to be the original one.)
   (a) Write down the wavefunction and the original energy. What is the new ground state and what is its energy?
   (b) How much energy did it cost to change the potential (on average)?
   (c) If we measure the energy, what is the probability that we obtain that of the new ground state? If we do, what is the state of the system after measurement?

3 Angular momentum

1. Griffiths 4.24

Problem 4.24 Two particles of mass m are attached to the ends of a massless rigid rod of length a. The system is free to rotate in three dimensions about the center (but the center point itself is fixed).

(a) Show that the allowed energies of this **rigid rotor** are

$$E_n = \frac{\hbar^2 n(n+1)}{ma^2}, \qquad \text{for} \quad n = 0,\ 1,\ 2, \ldots$$

Hint: First express the (classical) energy in terms of the total angular momentum.

(b) What are the normalized eigenfunctions for this system? What is the degeneracy of the nth energy level?

2. Suppose we have a system that we know is in the $\ell = 1$ eigenspace of $\hat{\mathbf{L}}^2$.
   (a) What are the possible values you could obtain upon measuring $\mathbf{L}^2$?
   (b) What are the possible values you could obtain upon measuring L_x? L_y? L_z?
   (c) Since this is a finite subspace, we can represent states as vectors and operators as matrices. What dimension is this space? Find a matrix representation of $\hat{L}_z$, in its own eigenbasis. (Order the basis vectors from high m to low m.)
   (d) What is the vector representation of the state $\psi(\mathbf{r}) = \sqrt{\frac{3}{8\pi}} f(r)(\cos\theta + \sin\theta \sin\phi)$?
   (e) What is the expected value of L_z in this state? What is its standard deviation?
   (f) Find a matrix representation of $\hat{L}_\pm$, and hence of $\hat{L}_x$ and $\hat{L}_y$.

3. Suppose we have a system described by the wavefunction $\psi(\mathbf{r}) = Af(r)\cos\phi$.
   (a) Write down the condition for $f(r)$ to be normalised. (Assume it is normalised.)
   (b) Find the normalisation constant A.
   (c) If we measured L_z, what are the possible values we could obtain, and with what probability? For each outcome, what is the state of the system after measurement?
   (d) If we measured $\mathbf{L}^2$, what is the lowest possible value we could obtain, and what is its probability? If we obtain this value, what is the state of the system after measurement?
   (e) Suppose we measure L_z and then $\mathbf{L}^2$. Would our results follow the same probability distribution as if we measured $\mathbf{L}^2$ first and then L_z?

4. Let us denote $\hat{\mathbf{L}}^2, \hat{L}_z$ eigenstates as $|\ell, m\rangle$, and ignore the radial part (or consider the rigid rotor). Suppose we know that we start off in the state $|\psi\rangle = \sqrt{\frac{2}{3}}\,|2,1\rangle + \frac{i}{\sqrt{6}}\,|3,0\rangle - \frac{1+i}{\sqrt{12}}\,|3,1\rangle$.
   (a) Check that the state is normalised
   (b) If we measure L_z, what are the outcomes?
   (c) If we instead measure $\mathbf{L}^2$, what are the outcomes and probabilities?
   (d) If we measure L_z, obtain the result $\hbar$, and then measure $\mathbf{L}^2$, what are the outcomes and probabilities?
   (e) What is the expected value of L_x?