Postulates of Quantum Mechanics

Postulate #1: To an ensemble of physical systems, one can assign (in certain cases), a wavefunction which contains all the information that can be known about the ensemble.

This function is in general complex!

eg. For N particles: $\psi(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_N)$

$\psi^*\psi$ = Prob. particle #1 @ $\vec{r}_1$, 2 @ $\vec{r}_2$, ...

Postulate #2: Superposition principle: If $\psi = c_1\psi_1 + c_2\psi_2$, then it is a solution if ψ_1, ψ_2 are solutions.

Postulate #3 With every dynamical variable A, there is a quantum (linear) operator $\hat{A}$.

eg. $\mathcal{A} = A(\vec{r}_1 \cdots \vec{r}_N, \vec{p}_1 \cdots \vec{p}_N, t)$

$\downarrow$ linear operator in position space

$\hat{A}(\vec{r}_1, \ldots \vec{r}_N, -i\hbar\vec{\nabla}_1, \ldots -i\hbar\vec{\nabla}_N, t)$

Postulate #4: The result of a precise measurement of the dynamical variable A is one of the eigenvalues a_n of the linear operator $\hat{A}$.

$$\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$$

real #

If $\hat{A}$ is Hermitian, eigenvalue a_n is real $\in \mathbb{R}$

Def of Hermitian: $\langle x|A\psi\rangle = \langle Ax|\psi\rangle$

$$\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$$

Take adjoint:

$$\langle\psi_n|\hat{A}^{+} = a_n^{*}(|\psi_n\rangle)^{+} = a_n^{*}\langle\psi_n|$$

Multiply both equations by $|\psi_n\rangle$ and $\langle\psi_n|$

First equation reveals:

$$\langle\psi_n|\hat{A}|\psi_n\rangle = a_n\langle\psi_n|\psi_n\rangle$$
$$\langle\psi_n|\hat{A}^{+}|\psi_n\rangle = a_n^{*}\langle\psi_n|\psi_n\rangle$$

$\rightarrow$ Thus, if $\hat{A} = A^{+}$, $a_n^{*} = a_n \Rightarrow a_n \in \mathbb{R}$

Postulate 5: If a series of measurements are made of the dynamical variable A, the result is called the expectation value $\langle \hat{A} \rangle$.

$$\langle \hat{A} \rangle = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}$$

Note: $\langle \hat{A} \rangle$ is not the average of a classical statistical distribution.

Postulate #6: A wave function representing any dynamical state can be expressed as a linear combination of the eigenfunctions of $\hat{A}$, the linear operator associated with the dynamical variable A.

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle$$

eigenfunctions of $\hat{A}$

Postulate #7

The time evolution of the wave function of a system is given by the time dep. SE: $i\hbar \frac{\partial}{\partial t}\psi(t) = \hat{H}\psi(t)$

Different types of operators & Dirac notation: ④

$|\psi\rangle \leftarrow$ state vector

$\langle\psi| \leftarrow$ adjoint

$$\langle\psi_1|\psi_2\rangle = \int \psi_1^*(\vec{r})\,\psi_2(\vec{r})\,d\vec{r}$$

inner product

Some properties:

- $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*$
- $\langle\psi_1|c\psi_2\rangle = c\langle\psi_1|\psi_2\rangle$

complex #

- $\langle c\psi_1|\psi_2\rangle = c^*\langle\psi_1|\psi_2\rangle$
- $\langle\psi_3|\psi_1+\psi_2\rangle = \langle\psi_3|\psi_1\rangle + \langle\psi_3|\psi_2\rangle$
- $|\psi_1\rangle, |\psi_2\rangle$ are orthogonal if $\langle\psi_1|\psi_2\rangle = 0$
- $|\psi_1\rangle, |\psi_2\rangle$ are orthonormal if $\langle\psi_1|\psi_2\rangle = \delta_{12}$

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- Adjoint Operator:

$\hat{A}^{+}$ is the adjoint or Hermitian conjugate of $\hat{A}$.

$$|\phi\rangle = A|X\rangle$$

$$\langle\phi| = \langle x|\hat{A}^{+}$$

can view as acting to the left.

- Functions of Operators:

$$\left[f(\hat{A})\right]^{+} = f^{*}(\hat{A}^{+})$$

- Identity Operator

$$\hat{I}|\psi\rangle = |\psi\rangle$$

- Inverse: If $\hat{B}\hat{A} = \hat{A}\hat{B} = \hat{I}$, then $\hat{B}$ is said to be the inverse of $\hat{A}$: $\hat{B} = \hat{A}^{-1}$

- Unitary Operators: A linear operator is said to be unitary if $\hat{U}^{-1} = \hat{U}^{+}$. This is the same as $\hat{U}\hat{U}^{+} = \hat{U}^{+}\hat{U} = \hat{I}$

Projection Operator

$\hat{\Lambda}^2 = \hat{\Lambda}$ and Hermitian

For example, $|\psi\rangle = |\phi\rangle + |\chi\rangle$
where $|\phi\rangle = \hat{\Lambda}|\psi\rangle$ and $|\chi\rangle = (\hat{I} - \hat{\Lambda})|\psi\rangle$

Closure Relations

(a) $|\psi\rangle = \sum_n c_n |\psi_n\rangle$

To find coefficients, multiply by each basis vector (eigenfunction) $|\psi_m\rangle$

$$\langle \psi_m | \psi \rangle = \sum_n c_n \langle \psi_m | \psi_n \rangle = c_m$$

For wavefunctions,

$$\psi(\vec{r}, t) = \sum_n \left[\int \psi_n^*(\vec{r}\,') \psi(\vec{r}\,', t)\, d\vec{r}\,' \right] \psi_n(\vec{r})$$

$$= \int \Big[\underbrace{\sum_n \psi_n^*(\vec{r}\,') \psi_n(\vec{r})}_{\delta(\vec{r} - \vec{r}\,')} \Big] \psi(\vec{r}\,', t)\, d\vec{r}\,'$$

$\rightarrow \sum_n \psi_n^*(\vec{r}\,') \psi_n(\vec{r}) = \delta(\vec{r} - \vec{r}\,')$ **

(b) Consider the inner product

$$\langle \chi | \psi \rangle = \int \chi^*(\vec{r},t)\, \psi(\vec{r},t)\, d\vec{r}$$

insert previous relation from (a)

$$= \iint \chi^*(\vec{r},t)\, \underbrace{\delta(\vec{r}-\vec{r}\,')}_{\text{express as } \sum}\, \psi(\vec{r}\,',t)\, d\vec{r}\, d\vec{r}\,'$$

$$= \sum_n \int \chi^*(\vec{r},t)\, \psi_n(\vec{r})\, d\vec{r} \int \psi_n^*(\vec{r}\,') \cdot \psi(\vec{r}\,',t)\, d\vec{r}\,'$$

$$= \sum_n \langle \chi | \psi_n \rangle \langle \psi_n | \psi \rangle$$

$$\Longrightarrow \sum_n |\psi_n\rangle\langle\psi_n| = \underline{I} \; !$$ outer product.