Schrödinger Equation for a 2-body System — Lecture 23

- Consider two particles with mass m_1 and m_2
- These bodies interact with a time indep. potential $\hat{V}(\vec{r}_1 - \vec{r}_2)$ which *only* depends on $\vec{r}_1 - \vec{r}_2$.
- Schrödinger Equation reads:

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}_1, \vec{r}_1, t) = \left[\frac{-\hbar^2}{2m_1} \nabla^2_{\vec{r}_1} - \frac{\hbar^2}{2m_2} \vec{\nabla}^2_{r_2} + \hat{V}(\vec{r}_1 - \vec{r}_2) \right] \psi$$

→ seven dimensional partial differential equation

* Introduce relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$ and center of mass coordinate $\vec{R} = \frac{m_1 \vec{r}_1 + m \vec{r}_2}{m_1 + m_2}$

- We can readily show:

$$\frac{-\hbar^2}{2m_1} \nabla_{r_1}^2 - \frac{\hbar^2}{2m_2} \nabla^2_{r_2} = \frac{-\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2$$

where $M = m_1 + m_2$ total mass

$\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

Schrödinger Equation now reads:

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{R}, \vec{r}, t) = \left[-\frac{\hbar^2}{2M} \nabla_R^2 - \frac{\hbar^2}{2\mu} \nabla_r^2 + \hat{V}(\vec{r}) \right] \psi$$

Introduce separations: i. time indep solns since $V(\vec{r})$ is time indep

ii. $\psi(\vec{R}, \vec{r})$ can be separated

Seek solutions:

$$\Psi(\vec{R}, \vec{r}, t) = \Phi(\vec{R}) \psi(\vec{r}) e^{-i(E_{CM} + E)t/\hbar}$$

where $\frac{-\hbar^2}{2M} \nabla_R^2 \Phi(R) = E_{CM} \Phi(\vec{R})$

and

$$\left[\frac{-\hbar^2}{2\mu} \nabla_r^2 + \hat{V}(\vec{r}) \right] \psi(\vec{r}) = E \psi(\vec{r})$$

<u>Thus</u> * We have <u>decoupled</u> the original two-body problem into two <u>independent</u> one-body problems:

- Free particle of mass M
- Particle of mass μ in potential $\hat{V}(\vec{r})$