Please note, all Griffiths problems come from our class text, the second edition.

1. **An Adventure with Hermitian Operators**
Linear operators are linear maps from the Hilbert space to itself, i.e. the functions $\hat{L}$ from wavefunctions to wavefunctions such that $\hat{L}(\alpha\Psi+\beta\Phi) = \alpha\hat{L}\Psi+\beta\hat{L}\Phi$, for all wavefunctions Ψ, Φ and all complex numbers α, β. A Hermitian operator $\hat{A}$ is one whose adjoint is equal to itself, $\hat{A}^\dagger = \hat{A}$, i.e. for all Ψ, Φ,

$$\left\langle \Psi, \hat{A}\Phi \right\rangle = \int \Psi^*(x)[\hat{A}\Phi](x)\,\mathrm{d}x = \int [\hat{A}\Psi]^*(x)\Phi(x)\,\mathrm{d}x = \left\langle \hat{A}\Psi, \Phi \right\rangle. \tag{1}$$

Physical measurables are represented by Hermitian operators. We know two such operators, the position and momentum,

$$\hat{x} : [\hat{x}\Psi](x) = x\Psi(x), \qquad \hat{p} : [\hat{p}\Psi](x) = -i\hbar\frac{\partial\Psi}{\partial x}(x). \tag{2}$$

For the following maps, check whether they are linear operators, and if so, whether they are Hermitian.

(a) $\hat{x}$

(b) $\hat{A} : \hat{A}\Psi(x) = \Psi^*(x)\Psi(x)$

(c) $\hat{p}$

(d) $\hat{B} : \hat{B}\Psi = -iL^2\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi$, for some fixed L

(e) $\hat{P}_\Phi : \hat{P}_\Phi\Psi(x) = \Phi(x)\int \Phi^*(y)\Psi(y)\,\mathrm{d}y$, for some fixed $\Phi(x)$

(f) $\hat{Q}_\Phi : \hat{Q}_\Phi\Psi(x) = \Phi(x)\int \Psi^*(y)\Phi(y)\,\mathrm{d}y$, for some fixed $\Phi(x)$

(g) $\hat{T}_a : \hat{T}_a\Psi(x) = \Psi(x+a)$, for some fixed a

(h) $\hat{x}\hat{p}$

(i) $\hat{x}\hat{p} + \hat{p}\hat{x}$

2. **Becoming Friends with Gaussian wave-packets** *inspired by Griffiths 2.22*

∗Problem 2.22 The gaussian wave packet. A free particle has the initial wave function

$$\Psi(x, 0) = Ae^{-ax^2},$$

where A and a are constants (a is real and positive).

(a) Normalize $\Psi(x, 0)$.

(b) Find $\Psi(x, t)$. *Hint:* Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\,dx$$

can be handled by "completing the square": Let $y \equiv \sqrt{a}\,[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$. *Answer:*

$$\Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1 + (2i\hbar at/m)}}.$$

(c) Find $|\Psi(x, t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1 + (2\hbar at/m)^2}}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t = 0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$, as time goes on?

(d) Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, σ_x, and σ_p. *Partial answer:* $\langle p^2\rangle = a\hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

(f) Consider a microscopic particle with the mass of an electron localised in a space of 10^{-10} m, about the size of an atom. How long does it take for σ_x to double its initial value? Compare with a macroscopic particle of mass 1 g localised in a space of 10^{-6} m

3. **Some useful results using the Schrödinger equation** *Griffiths 2.1*

$*$**Problem 2.1** Prove the following three theorems:

(a) For normalizable solutions, the separation constant E must be *real*. *Hint:* Write E (in Equation 2.7) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if Equation 1.20 is to hold for all t, Γ must be zero.

(b) The time-independent wave function $\psi(x)$ can always be taken to be *real* (unlike $\Psi(x, t)$, which is necessarily complex). This doesn't mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to ψ's that are real. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does its complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.

(c) If $V(x)$ is an **even function** (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.

4. **Alas, it's a Question about Probability Currents** *inspired by Griffiths 1.14*

Problem 1.14 Let $P_{ab}(t)$ be the probability of finding a particle in the range $(a < x < b)$, at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t),$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

What are the units of $J(x,t)$? *Comment:* J is called the **probability current,** because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other.

(b) Show that if at any time t, $\Psi(x,t)$ is real or has spatially constant phase, i.e. $\Psi(x,t) = e^{i\theta(t)}f(x,t)$ for real functions θ, f, then $J(x,t) = 0$. What does this imply for energy eigenstates?

(c) Calculate $J(x,0)$ for a Gaussian wavepacket $\Psi(x,t)$.