Please note, all Griffiths problems come from our class text, the second edition.

1. **Griffiths 4.9**

∗∗**Problem 4.9** A particle of mass m is placed in a *finite* spherical well:

$$V(r) = \begin{cases} -V_0, & \text{if } r \le a; \\ 0, & \text{if } r > a. \end{cases}$$

Find the ground state, by solving the radial equation with $l = 0$. Show that there is no bound state if $V_0 a^2 < \pi^2\hbar^2/8m$.

Solution: Outside the well, $V = 0$, so plugging this into Eq 4.37 with $\ell = 0$, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \frac{2m(-E)}{\hbar^2} u = \lambda^2 u, \tag{1}$$

where we define λ to be real and positive since $-V_0 \le E < 0$ for a bound state. This has the general solution $u(r) = Ae^{-\lambda r} + Be^{\lambda r}$, but since the second term can't be normalised, $B = 0$. Inside the well,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = -\frac{2m(V_0 + E)}{\hbar^2} u = -k^2 u, \tag{2}$$

where again k is real and non-negative. The general solution is $u(r) = A\cos(kr) + B\sin(kr)$, but since we need $u(0) = 0$ for the second derivative (i.e. the momentum) to not blow up, $A = 0$. Thus, equating $R(a)$ and $R'(a)$ on both sides for continuity and differentiability, which is equivalent to equating $u(a)$ and $u'(a)$, we have

$$u(a) = Ae^{-\lambda a} = B\sin(ka), \qquad u'(a) = -\lambda Ae^{-\lambda a} = Bk\cos(ka) = -\lambda(B\sin(ka)). \tag{3}$$

This determines the allowed energies,

$$\tan(ka) = -\frac{k}{\lambda} = -\sqrt{\frac{V_0 + E}{-E}}, \tag{4}$$

but, as in the case of the 1d finite square well, analytically solving this is difficult. Since the RHS is negative, and $ka \ge 0$, we must have $ka > \frac{\pi}{2}$, so $V_0 > V_0 + E > \frac{\pi^2\hbar^2}{8ma^2}$ for a bound state to exist. And so, defining $C = u(a)$, any $\ell = 0$ bound state must look like

$$u(r) = \begin{cases} C\frac{\sin(kr)}{\sin(ka)} & r \le a, \\ Ce^{-\lambda(r-a)} & r \ge a. \end{cases} \tag{5}$$

Fixing the normalisation,

$$\begin{aligned} 1 &= \int_0^\infty |u(r)|^2 \, \mathrm{d}r = |C|^2 \left(\frac{1}{\sin^2(ka)} \int_0^a \sin^2(kr) \, \mathrm{d}r + \int_a^\infty e^{-2\lambda(r-a)} \, \mathrm{d}r \right) \\ &= |C|^2 \left(\frac{a - \frac{1}{2k}\sin(2ka)}{1 - \cos(2ka)} + \frac{1}{2\lambda} \right), \end{aligned} \tag{6}$$

so $C = \left(\frac{a - \frac{1}{2k}\sin(2ka)}{1-\cos(2ka)} + \frac{1}{2\lambda} \right)^{-\frac{1}{2}}$.

∗ ∗ ∗**Problem 4.39** Because the three-dimensional harmonic oscillator potential (Equation 4.188) is spherically symmetric, the Schrödinger equation can be handled by separation of variables in *spherical* coordinates, as well as cartesian coordinates. Use the power series method to solve the radial equation. Find the recursion formula for the coefficients, and determine the allowed energies. Check your answer against Equation 4.189.

2. **Griffiths 4.39**

(b) The Cartesian and spherical decompositions give us two different energy eigenbases. Check that the degeneracies of each energy level are consistent between them.

(c) For the lowest three energy levels, write the Cartesian eigenstates $|n_x, n_y, n_z\rangle$ as linear combinations of the spherical eigenstates $|n, \ell, m\rangle$ and vice-versa (i.e. find the change-of-basis matrices for the first three energy levels).

Solution:

(a) Plugging in $V(r) = \frac{1}{2}m\omega^2 r^2$, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \left(\frac{m^2\omega^2}{\hbar^2}r^2 + \frac{\ell(\ell+1)}{r^2} - \frac{2mE}{\hbar^2}\right)u. \tag{7}$$

We can construct a length scale $L = \sqrt{\frac{\hbar}{m\omega}}$ from the parameters of the problem so we might expect this to be simpler when we measure distances in the "natural" units, $\xi = r/L$,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} = L^2\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} = \left(\xi^2 + \frac{\ell(\ell+1)}{\xi^2} - \frac{2E}{\hbar\omega}\right)u. \tag{8}$$

For large ξ, $\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} \approx \xi^2 u$ so $u \approx Ae^{-\frac{1}{2}\xi^2}$ and for small ξ, $\frac{\partial^2 u}{\partial \xi^2} \approx \frac{\ell(\ell+1)}{\xi^2}u$ so $u \approx B\xi^{\ell+1}$, since normalisation forbids the $e^{\frac{1}{2}\xi^2}$ and $\xi^{-\ell}$ solutions. Thus we may write $u(\xi) = v(\xi)e^{-\frac{1}{2}\xi^2}$ and expand v in a power series starting at $\ell+1$, $v(\xi) = \sum_{j=\ell+1}^{\infty} c_j\xi^j$. (For $\ell = 0$ the above argument doesn't hold, but we should start at $j = 1$ anyway since we need $u(0) = 0$.) $u' = (v' - \xi v)e^{-\frac{1}{2}\xi^2}$ and $u'' = (v'' - 2\xi v' + (\xi^2 - 1)v)e^{-\frac{1}{2}\xi^2}$, so defining $K = \frac{2E}{\hbar\omega}$,

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - 2\xi\frac{\mathrm{d}v}{\mathrm{d}\xi} + \left(K - \frac{\ell(\ell+1)}{\xi^2} - 1\right)v \\
&= \sum_{j=\ell+1}^{\infty}\left(j(j-1)\xi^{j-2} - 2j\xi^j + (K-1)\xi^j - \ell(\ell+1)\xi^{j-2}\right)c_j \\
&= 2(\ell+1)c_{\ell+2}\xi^{\ell} + \sum_{j=\ell+1}^{\infty}\left([(j+1)(j+2) - \ell(\ell+1)]c_{j+2} - (2j+1-K)c_j\right)\xi^j,
\end{aligned} \tag{9}$$

so $c_{\ell+2} = 0$ and for $j \geq \ell+1$,

$$c_{j+2} = \frac{2j+1-K}{(j+1)(j+2) - \ell(\ell+1)}c_j, \tag{10}$$

so we can find $c_{\ell+1+2n}$ in terms of $c_{\ell+1}$, and all the other coefficients are zero. ($c_{\ell+1}$ is fixed by normalisation.) As before, we need the power series to terminate at some

$j_{\max} = \ell + 1 + 2k$ for integer $k \geq 0$ (otherwise for large j, $c_{j+2} \approx \frac{2}{j} c_j$ so $v \approx Ae^{\xi^2}$). This gives us the energy quantisation,

$$E = \tfrac{1}{2} K\hbar\omega = \tfrac{1}{2}(2j_{\max} + 1)\hbar\omega = \left(2k + \ell + \tfrac{3}{2}\right)\hbar\omega = \left(n + \tfrac{3}{2}\right)\hbar\omega, \tag{11}$$

defining $n = 2k + \ell \geq 0$.

(b) For even n, k can be anything from 0 to $\frac{n}{2}$, while for odd n, k can be anything from 0 to $\frac{n-1}{2}$. So there are $\lfloor \frac{n}{2} \rfloor + 1$ choices for k, each corresponding to different ℓ. For a given n, ℓ, there are $2\ell + 1$ states corresponding to different m, so the total degeneracy is

$$\begin{aligned} g(n) &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2(n - 2k) + 1) = (2n + 1) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 1 - 4 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} k \\ &= (\lfloor \tfrac{n}{2} \rfloor + 1)(2n + 1 - 2\lfloor \tfrac{n}{2} \rfloor) = \begin{cases} (\frac{n}{2} + 1)(n + 1) & n \text{ even}, \\ (\frac{n-1}{2} + 1)(n + 2) & n \text{ odd}, \end{cases} \\ &= \tfrac{1}{2}(n + 1)(n + 2), \end{aligned} \tag{12}$$

which is what we got in the previous HW by analysing it in Cartesian coordinates!

(c) The lowest energy level is $|n, \ell, m\rangle = |0, 0, 0\rangle$ which must be the same as $|n_{x,y,z}\rangle = |0, 0, 0\rangle$ because the ground state degeneracy is 1. We can explicitly check this by constructing the wavefunctions and checking that they are equal,

$$\psi_{0,0,0}(r, \theta, \phi) = \frac{u_{0,0}(r)}{r} Y_0^0(\theta, \phi) \propto e^{-\frac{1}{2}\xi^2} \propto \phi_{0,0,0}(x, y, z), \tag{13}$$

where $\psi_{n,\ell,m}$ are the spherical eigenstates and $\phi_{n_{x,y,z}}$ are the Cartesian ones. Since both are real and normalised, the coefficient must be 1.

The three first-excited states are $|n, \ell, m\rangle = |1, 1, 1\rangle, |1, 1, 0\rangle, |1, 1, -1\rangle$, and $u_{1,1} = c_2 \xi^2 e^{-\frac{1}{2}\xi^2}$, so

$$\psi_{1,1,0} = \frac{u_{1,1}}{r} Y_1^0 \propto r\cos\theta e^{-\frac{1}{2}\xi^2} = ze^{-\frac{1}{2}\xi^2} \propto \phi_{0,0,1}, \tag{14}$$

$$\psi_{1,1,\pm 1} = \frac{u_{1,1}}{r} Y_1^{\pm 1} \propto \mp r\sin\theta e^{\pm i\phi} e^{-\frac{1}{2}\xi^2} = (\mp x - iy)e^{-\frac{1}{2}\xi^2} \propto \mp\phi_{1,0,0} - i\phi_{0,1,0}, \tag{15}$$

where we can fix the proportionality constant using normalisation (noting that it is always real and positive) to get,

$$\begin{pmatrix} \psi_{1,1,1} \\ \psi_{1,1,0} \\ \psi_{1,1,-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \phi_{1,0,0} \\ \phi_{0,1,0} \\ \phi_{0,0,1} \end{pmatrix}. \tag{16}$$

We can invert the change-of-basis matrix by taking its adjoint since any change of basis between orthonormal bases is unitary, so

$$\begin{pmatrix} \phi_{1,0,0} \\ \phi_{0,1,0} \\ \phi_{0,0,1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{1,1,1} \\ \psi_{1,1,0} \\ \psi_{1,1,-1} \end{pmatrix}. \tag{17}$$

The six second-excited states are $|n, \ell, m\rangle = |2, 0, 0\rangle, |2, 2, \pm 2\rangle, |2, 2, \pm 1\rangle, |2, 2, \pm 0\rangle$ which must correspond to $|n_{x,y,z}\rangle = |2, 0, 0\rangle, |1, 1, 0\rangle, |1, 0, 1\rangle, |0, 2, 0\rangle, |0, 1, 1\rangle, |0, 0, 2\rangle$.

$u_{2,2} = c_3\xi^3 e^{-\frac{1}{2}\xi^2}$, $u_{2,0} = c_1(\xi - \frac{2}{3}\xi^3)e^{-\frac{1}{2}\xi^2}$ (note this has the opposite sign of H_3), and $H_1(\xi) = 2\xi$, $H_2(\xi) = 4\xi^2 - 2$,

$$\begin{aligned}
\psi_{2,0,0} &\propto \left(1 - \tfrac{2}{3}\xi^2\right) e^{-\frac{1}{2}\xi^2} \propto \left[(1 - 2\xi_x^2) + (1 - 2\xi_y^2) + (1 - 2\xi_z^2)\right] e^{-\frac{1}{2}\xi^2} \\
&\propto -(\phi_{2,0,0} + \phi_{0,2,0} + \phi_{0,0,2}), && (18) \\
\psi_{2,2,0} &\propto r^2(3\cos^2\theta - 1)e^{-\frac{1}{2}\xi^2} = (2z^2 - x^2 - y^2)e^{-\frac{1}{2}\xi^2} \\
&\propto \phi_{2,0,0} + \phi_{0,2,0} - 2\phi_{0,0,2}, && (19) \\
\psi_{2,2,\pm1} &\propto \mp r^2 \sin\theta\cos\theta e^{\pm i\phi} e^{-\frac{1}{2}\xi^2} = z(\mp x - iy)e^{-\frac{1}{2}\xi^2} \propto \mp\phi_{1,0,1} - i\phi_{0,1,1}, && (20) \\
\psi_{2,2,\pm2} &\propto (r\sin\theta e^{\pm i\phi})^2 e^{-\frac{1}{2}\xi^2} = (x \pm iy)^2 e^{-\frac{1}{2}\xi^2} = (x^2 - y^2 \pm 2ixy)e^{-\frac{1}{2}\xi^2} \\
&\propto (H_2(\xi_x) - H_2(\xi_y) \pm 2iH_1(\xi_x)H_1(\xi_y))e^{-\frac{1}{2}\xi^2} \propto -\phi_{2,0,0} + \phi_{0,2,0} \pm \sqrt{2}i\phi_{1,1,0}, && (21)
\end{aligned}$$

where in the last line one can use the explicit normalisation of the 1d harmonic oscillator,

$$\phi_{n_{x,y,z}} \propto \frac{1}{\sqrt{2^n n_x! n_y! n_z!}} H_{n_x}(\xi_x)H_{n_y}(\xi_y)H_{n_z}(\xi_z)e^{-\frac{1}{2}\xi^2}, \tag{22}$$

or deduce the coefficients from the unitarity of the change-of-basis matrix. Thus

$$\begin{pmatrix} \psi_{2,0,0} \\ \psi_{2,2,2} \\ \psi_{2,2,1} \\ \psi_{2,2,0} \\ \psi_{2,2,-1} \\ \psi_{2,2,-2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{2} & \frac{i}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & -\sqrt{\frac{2}{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{2,0,0} \\ \phi_{1,1,0} \\ \phi_{1,0,1} \\ \phi_{0,2,0} \\ \phi_{0,1,1} \\ \phi_{0,0,2} \end{pmatrix}, \tag{23}$$

$$\begin{pmatrix} \phi_{2,0,0} \\ \phi_{1,1,0} \\ \phi_{1,0,1} \\ \phi_{0,2,0} \\ \phi_{0,1,1} \\ \phi_{0,0,2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{6}} & 0 & -\frac{1}{2} \\ 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & \frac{i}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{2} & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{2,0,0} \\ \psi_{2,2,2} \\ \psi_{2,2,1} \\ \psi_{2,2,0} \\ \psi_{2,2,-1} \\ \psi_{2,2,-2} \end{pmatrix}. \tag{24}$$

3. **Griffiths 4.43**

Problem 4.43

(a) Construct the spatial wave function (ψ) for hydrogen in the state $n = 3$, $l = 2$, $m = 1$. Express your answer as a function of r, θ, ϕ, and a (the Bohr radius) *only*—no other variables (ρ, z, etc.) or functions (Y, v, etc.), or constants (A, c_0, etc.), or derivatives, allowed (π is okay, and e, and 2, etc.).

(b) Check that this wave function is properly normalized, by carrying out the appropriate integrals over r, θ, and ϕ.

(c) Find the expectation value of r^s in this state. For what range of s (positive and negative) is the result finite?

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(c) Find the expectation value of r^s in this state. For what range of s (positive and negative) is the result finite?

a) $\psi_{321} = R_{32} Y_2^1 = \frac{4}{81\sqrt{30}} \frac{1}{a^{3/2}} \left(\frac{r}{a}\right)^2 e^{-r/3a} \left(-\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \, e^{i\phi}\right)$

$= -\frac{1}{\sqrt{\pi}} \frac{1}{81 a^{7/2}} r^2 e^{-r/3a} \sin\theta \cos\theta \, e^{i\phi}$

b) $\int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, |\psi_{321}|^2 = \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, \frac{1}{\pi (81)^2 a^7} r^4 e^{-2r/3a} \sin^2\theta \cos^2\theta$

$= \frac{2\pi}{\pi (81)^2 a^7} \int_0^\infty dr\, r^6 e^{-2r/3a} \int_0^\pi d\theta \left(1 - \cos^2\theta\right) \cos^2\theta \sin\theta$

$= \frac{2}{(81)^2 a^7} \left(6! \left(\frac{3a}{2}\right)^7\right) \left(-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right)\Big|_0^\pi$

$= \frac{2}{3^8 a^7} \, 6! \, \frac{3^7 a^7}{2^7} \left[\frac{2}{3} - \frac{2}{5}\right] = 1$

c) $\langle r^s \rangle = \int_0^\infty dr\, r^2 |R_{32}|^2 r^s = \left(\frac{4}{81}\right)^2 \frac{1}{30 a^7} \int_0^\infty dr\, r^{s+6} e^{-2r/3a} = (s+6)! \left(\frac{3a}{2}\right)^5 \frac{1}{720}$

This is finite for $s > -7$

Problem 4.46

(a) Use the recursion formula (Equation 4.76) to confirm that when $l = n - 1$ the radial wave function takes the form

$$R_{n(n-1)} = N_n r^{n-1} e^{-r/na},$$

and determine the normalization constant N_n by direct integration.

(b) Calculate $\langle r \rangle$ and $\langle r^2 \rangle$ for states of the form $\psi_{n(n-1)m}$.

(c) Show that the "uncertainty" in r (σ_r) is $\langle r \rangle / \sqrt{2n+1}$ for such states. Note that the fractional spread in r decreases, with increasing n (in this sense the system "begins to look classical," with identifiable circular "orbits," for large n). Sketch the radial wave functions for several values of n, to illustrate this point.

a) $c_j = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} c_0$

$c_1 = \frac{2(n-n)}{(1)(2n)} c_0 = 0$ $\quad l = n-1$

$v(\rho) = c_0$

$R_{n(n-1)} = N_n r^{n-1} e^{-r/na}$

$1 = \int_0^\infty dr\, r^2 |R|^2 = (N_n)^2 \int_0^\infty dr\, r^{2n} e^{-2r/na} = (N_n)^2 (2n)! \left(\frac{na}{2}\right)^{2n+1}$

$N_n = \left(\frac{2}{na}\right)^n \sqrt{\frac{2}{na\,(2n)!}}$

b) $\langle r^l \rangle = \int_0^\infty dr\, |R|^2 r^{l+2} = N_n^2 \int_0^\infty dr\, r^{2n+l} e^{-2r/na}$

$\langle r \rangle = \left(\frac{2}{na}\right)^{2n+1} \frac{1}{(2n)!} (2n+1)! \left(\frac{na}{2}\right)^{2n+2} = \left(n+\frac{1}{2}\right)na$

$\langle r^2 \rangle = \left(\frac{2}{na}\right)^{2n+1} \frac{1}{(2n)!} (2n+2)! \left(\frac{na}{2}\right)^{2n+3} = \left(n+\frac{1}{2}\right)(n+1)(na)^2$

c) $\sigma_r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$

$$= \left[\left(n + \tfrac{1}{2}\right)(n+1)(na)^2 - \left(n + \tfrac{1}{2}\right)^2 (na)^2\right]^{1/2}$$

$$= \left[\tfrac{1}{2}\left(n+\tfrac{1}{2}\right)(na)^2\right]^{1/2} = \left[\frac{1}{2\left(n+\frac{1}{2}\right)} \langle r \rangle^2\right]^{1/2}$$

$$\sigma_r = \frac{\langle r \rangle}{\sqrt{2n+1}}$$

7.2 Consider an anisotropic harmonic oscillator described by the Hamiltonian

$$H = \frac{1}{2\mu}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}k_1(x^2 + y^2) + \frac{1}{2}k_2 z^2.$$

(a) Find the energy levels and the corresponding energy eigenfunctions using Cartesian coordinates. What are the degeneracies of the levels, assuming that $\omega_1 = (k_1/\mu)^{1/2}$ and $\omega_2 = (k_2/\mu)^{1/2}$ are incommensurable?

(b) Can the stationary states be eigenstates of $\mathbf{L}^2$? of L_z?

a) The energy levels are

$$E_{n_x,n_y,n_z} = (n_x + \tfrac{1}{2})\hbar\omega_1 + (n_y + \tfrac{1}{2})\hbar\omega_1 + (n_z + \tfrac{1}{2})\hbar\omega_2$$

$$= (n_x + n_y + 1)\hbar\omega_1 + (n_z + \tfrac{1}{2})\hbar\omega_2$$

The ground state is nondegenerate

$$E_{000} = \hbar\omega_1 + \tfrac{1}{2}\hbar\omega_2$$

Excited state degeneracies: since ω_1 and ω_2 are incommensurable so we only need to consider degeneracies from n_x and n_y.

$E_{10n_z} = E_{01n_z}$ degeneracy = 2

$E_{20n_z} = E_{02n_z} = E_{11n_z}$ degeneracy = 3

$E_{30n_z} = E_{03n_z} = E_{21n_z} = E_{12n_z}$ degeneracy = 4

This has the same degeneracy of a 2D Harmonic Oscillator which is $n+1$ where $n = n_x + n_y$.

b) Rewrite the Hamiltonian as

$$H = \frac{1}{2m}p^2 + \frac{1}{2}K_1(x^2+y^2) + \frac{1}{2}K_2 z^2 + \frac{1}{2}K_1 z^2 - \frac{1}{2}K_1 z^2$$

$$= \frac{p^2}{2m} + \frac{K_1}{2}r^2 + \frac{1}{2}(K_1 - K_2)z^2$$

Now it's simple to check the commutation relations.

$$[H, L^2] = \frac{1}{2m}[p^2, L^2] + \frac{k_1}{2}[r^2, L^2] + \frac{1}{2}(k_1 - k_2)[z^2, L^2]$$

$$= 0 + 0 + \frac{1}{2}(k_1 - k_2)(-4\hbar^2 z^2)$$

$$= 2(k_2 - k_1)\hbar^2 z^2 \neq 0$$

$$[H, L_z] = \frac{1}{2m}[p^2, L_z] + \frac{k_1}{2}[r^2, L_z] + \frac{1}{2}(k_1 - k_2)[z^2, L_z]$$

$$= 0 + 0 + 0 = 0$$

Since $[H, L^2] \neq 0$ stationary states are not eigenstates of L^2, but since H and L_z commute, stationary states can be eigenstates of L_z.