Midterm 2 review

Fundamentals

1. F, T, T, T, T, F
2. a) H ✓ U ✗
   b) H ✗ (unless $\hat{H}=0$ or $t=0$) U ✓
   c) neither
   d) H ✓ U ✗
   e) neither
   f) H ✓ U ✗

3. a) $[\hat{H},\hat{p}] = \frac{1}{2}m\omega^2[\hat{x}^2,\hat{p}] = \frac{1}{2}m\omega^2\left(\hat{x}[\hat{x},\hat{p}]+[\hat{x},\hat{p}]\hat{x}\right) = i\hbar m\omega^2\hat{x}$
   b) $[\hat{H},\hat{a}_+] = \hbar\omega[\hat{a}_+\hat{a}_-,\hat{a}_+] = \hbar\omega\,\hat{a}_+[\hat{a}_-,\hat{a}_+] = \hbar\omega\,\hat{a}_+$
   c) $[\hat{p}_x,\hat{x}] = -[\hat{x},\hat{p}_x] = -i\hbar$
   d) $[\hat{p}_z,\hat{L}_z] = 0$
   e) $[\hat{L}^2,\hat{L}_\pm] = 0$
   f) $[\hat{L}_z,\hat{L}_\pm] = \pm\hbar\hat{L}_\pm$

$$[\hat{H},\hat{a}_+]^\dagger = [\hat{a}_+^\dagger,\hat{H}^\dagger] = [\hat{a}_-,\hat{H}] = -[\hat{H},\hat{a}_-] = (\hbar\omega\hat{a}_+)^\dagger = \hbar\omega\hat{a}_-$$

$$\therefore [\hat{H},\hat{a}_\pm] = \pm\hbar\omega\hat{a}_\pm$$

4. $|\psi_+\rangle = \dfrac{\hat{A}|\psi\rangle}{\sqrt{\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle}}$ $\qquad \langle\psi_+|\psi_+\rangle = \dfrac{\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle}{\langle\psi|\hat{A}^\dagger\hat{A}|\psi\rangle} = 1$

$$\hat{H}|\psi_+\rangle = \frac{1}{\sqrt{\cdots}}\hat{H}\hat{A}|\psi\rangle = \frac{1}{\sqrt{\cdots}}(\hat{A}\hat{H}+\lambda\hat{A})|\psi\rangle = \frac{\hat{A}}{\sqrt{\cdots}}(\hat{H}+\lambda)|\psi\rangle = \frac{\hat{A}}{\sqrt{\cdots}}(E+\lambda)|\psi\rangle$$

$$= (E+\lambda)\frac{\hat{A}|\psi\rangle}{\sqrt{\cdots}} = (E+\lambda)|\psi_+\rangle$$

$$([\hat{H},\hat{A}])^\dagger = (\hat{H}\hat{A}-\hat{A}\hat{H})^\dagger = \hat{A}^\dagger\hat{H}-\hat{H}\hat{A}^\dagger = -[\hat{H},\hat{A}^\dagger] = (\lambda\hat{A})^\dagger = \lambda^*\hat{A}^\dagger$$

$$\therefore [\hat{H},\hat{A}^\dagger] = -\lambda^*\hat{A}^\dagger$$

but since $\hat{H}$ is Hermitian, it has real eigenvalues, so $E+\lambda$ is real and thus $\lambda=\lambda^*$

$|\psi_-\rangle = \dfrac{\hat{A}^\dagger|\psi\rangle}{\sqrt{\langle\psi|\hat{A}\hat{A}^\dagger|\psi\rangle}}$ $\qquad \langle\psi_-|\psi_-\rangle = 1$

$$\hat{H}|\psi_-\rangle = \cdots\hat{H}\hat{A}^\dagger|\psi\rangle = \cdots(\hat{A}^\dagger\hat{H}-\lambda\hat{A}^\dagger) = \cdots\hat{A}^\dagger(E-\lambda)|\psi\rangle = (E-\lambda)|\psi_-\rangle$$

$\hat{A} = \hat{A}^\dagger \Rightarrow |\psi_-\rangle = |\psi_+\rangle \Rightarrow \lambda = 0$

$\hat{A}^\dagger\hat{A} = \hat{1} \Rightarrow$ maybe?

Harmonic oscillator

1. $\psi(x) = A e^{\frac{m\omega}{2\hbar}x^2}$

$\partial_x \psi(x) = A \frac{m\omega}{\hbar} x e^{\frac{m\omega}{2\hbar}x^2}$

$\partial_x^2 \psi(x) = A\left(\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}x\right)^2\right) e^{\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right)\psi(x)$

$\hat{H}\psi(x) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2}m\omega^2 x^2\right)\psi(x)$

$= \left(-\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar} + \frac{m^2\omega^2}{\hbar^2}x^2\right) + \frac{1}{2}m\omega^2 x^2\right)\psi(x)$

$= \left(-\frac{1}{2}\hbar\omega - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}m\omega^2 x^2\right)\psi(x)$

$= -\frac{1}{2}\hbar\omega\,\psi(x)$

$E = -\frac{1}{2}\hbar\omega$

$\psi(x)$ is horribly un-normalisable

2. Any $|\psi(0)\rangle$ can be written as $\sum_{n=0}^{\infty} c_n |n\rangle$ → n^{th} energy eigenstate of H.O. $(c_n = \langle n|\psi(0)\rangle)$

$\therefore |\psi(t)\rangle = \sum_{n=0}^{\infty} e^{-iE_nt/\hbar} c_n|n\rangle = e^{-\frac{1}{2}\omega t}\sum_{n=0}^{\infty} e^{-in\omega t} c_n|n\rangle$

$\therefore |\psi(T)\rangle = e^{-i\pi}\sum_n e^{-2\pi i n} c_n|n\rangle = -\sum_n c_n|n\rangle = -|\psi(0)\rangle$

3. a) $\Psi(x,0) \propto x^3 e^{-\frac{m\omega}{2\hbar}x^2} \propto \xi^3 e^{-\frac{1}{2}\xi^2}$

$\psi_1(x) = B\frac{1}{\sqrt{2}} 2\xi e^{-\frac{1}{2}\xi^2} = B\sqrt{2}\,\xi e^{-\frac{1}{2}\xi^2}$

$\psi_3(x) = B\frac{1}{\sqrt{8\times 6}}(8\xi^3 - 12\xi)e^{-\frac{1}{2}\xi^2} = B\left(\frac{2}{\sqrt{3}}\xi^3 - \sqrt{3}\,\xi\right)e^{-\frac{1}{2}\xi^2}$

$\therefore \Psi(x,0) \propto \sqrt{2}\,\psi_3(x) + \sqrt{3}\,\psi_1(x)$

fix normalisation (~~odd~~ $\Psi = C(\sqrt{2}\psi_3 + \sqrt{3}\psi_1)$ $\quad \langle\Psi|\Psi\rangle = |c|^2(2\langle 3|3\rangle + 3\langle 1|1\rangle)$

$\Psi(x,0) = \sqrt{\frac{2}{5}}\,\psi_3(x) + \sqrt{\frac{3}{5}}\,\psi_1(x)$

b) $\Psi(x,t) = \sqrt{\frac{2}{5}}e^{-i\frac{7}{2}\omega t}\psi_3(x) + \sqrt{\frac{3}{5}}e^{-i\frac{3}{2}\omega t}\psi_1(x) = e^{-i\frac{3}{2}\omega t}\left(\sqrt{\frac{2}{5}}e^{-2i\omega t}\psi_3 + \sqrt{\frac{3}{5}}\psi_1\right)$

c) $E_1 = \frac{3}{2}\hbar\omega \quad p = 3/5$

$E_3 = \frac{7}{2}\hbar\omega \quad p = 2/5$

d) 0 (can see algebraically, since $(\langle 3| + \langle 1|)(\hat{a}_+ \pm \hat{a}_-)(|3\rangle + |1\rangle) = 0$, since $\hat{a}_\pm$ map $|3\rangle, |1\rangle$ to $|0\rangle, |2\rangle, |4\rangle$, which are orthogonal to $|1\rangle, |3\rangle$.
can also see by integral odd/even symmetry)

e) $\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\Psi|(\hat{a}_+ + \hat{a}_-)^2|\Psi\rangle$

$$= \frac{\hbar}{2m\omega}\left(\frac{\sqrt{6}}{5}\langle 1|\hat{a}_-^2|3\rangle e^{-2i\omega t} + \frac{2}{5}\underbrace{\langle 3|(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)|3\rangle}_{} + \frac{3}{5}\underbrace{\langle 1|(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)|1\rangle}_{=\langle 1|(\hat{a}_+|0\rangle + \hat{a}_-\sqrt{2}|2\rangle) = \langle 1|(|1\rangle + 2|1\rangle) = 3}\right.$$

$$\left. + \frac{\sqrt{6}}{5}\underbrace{\langle 3|\hat{a}_+^2|1\rangle}_{=\langle 3|\hat{a}_+\sqrt{2}|2\rangle = \sqrt{6}\langle 3|3\rangle = \sqrt{6}} e^{2i\omega t}\right)$$

$$\langle 3|(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)|3\rangle = \langle 3|(\hat{a}_+\sqrt{3}|2\rangle + \hat{a}_-\sqrt{4}|4\rangle) = \langle 3|(3|3\rangle + 4|3\rangle) = 7$$

$$= \frac{\hbar}{2m\omega}\left(\frac{6}{5}e^{-2i\omega t} + \frac{14}{5} + \frac{9}{5} + \frac{6}{5}e^{2i\omega t}\right) = \frac{\hbar}{10m\omega}\left(23 + 12\cos(2\omega t)\right)$$

$$\therefore \sigma(x) = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{10m\omega}(23 + 12\cos 2\omega t)}$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\langle\Psi|(\hat{a}_+ - \hat{a}_-)^2|\Psi\rangle$$

$$= -\frac{\hbar m\omega}{2}\left(\frac{\sqrt{6}}{5}\langle 1|\hat{a}_-^2|3\rangle e^{-2i\omega t} - \frac{2}{5}\langle 3|\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+|3\rangle - \frac{3}{5}\langle 1|\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+|3\rangle\right.$$

$$\left. + \frac{\sqrt{6}}{5}\langle 3|\hat{a}_+^2|1\rangle e^{2i\omega t}\right)$$

$$= -\frac{\hbar m\omega}{2}\left(\frac{6}{5}e^{-2i\omega t} - \frac{14}{5} - \frac{9}{5} + \frac{6}{5}e^{2i\omega t}\right) = \frac{\hbar m\omega}{10}\left(23 - 12\cos(2\omega t)\right)$$

$$\therefore \sigma(p) = \sqrt{\frac{\hbar m\omega}{10}(23 - 12\cos 2\omega t)}$$

Note $\sigma(x)\sigma(p) = \frac{\hbar}{10}\sqrt{23^2 - 12^2\cos^2 2\omega t} = \frac{\hbar}{10}\sqrt{457 - 72\cos 4\omega t}$

f) $\Psi(x,t)$ is odd, $\Psi(-x,t) = \cdots\psi_3(-x) + \cdots\psi_1(-x) = -\cdots\psi_3(x) - \cdots\psi_1(x)$
$= -\Psi(x,t)$

$\therefore Pr(x>0) = \int_0^\infty |\Psi(x,t)|^2\,dx = \frac{1}{2}\int_{-\infty}^\infty |\Psi(x,t)|^2\,dx = \frac{1}{2}$

4.a) $\psi = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$ $\quad E = \frac{1}{2}\hbar\omega$

New gs: $V \to 4V$, so $k \to 4k$ so $\omega = \sqrt{\frac{k}{m}} \to 2\omega$

$\therefore \psi_{new} = \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{\hbar}x^2}$, $E = \hbar\omega$

b) originally, $\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle 0|(\hat{a}_+ + \hat{a}_-)^2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|\hat{a}_-\hat{a}_+|0\rangle = \frac{\hbar}{2m\omega}$

$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\langle 0|(\hat{a}_+ - \hat{a}_-)^2|0\rangle = \frac{\hbar m\omega}{2}\langle 0|\hat{a}_+\hat{a}_-|0\rangle = \frac{\hbar m\omega}{2}$

$\therefore \langle K\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{4}\hbar\omega$, $\quad \langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{4}\hbar\omega$

~~new~~ new $\langle \cancel{E} \rangle = \langle T \rangle + 4\langle V \rangle = \frac{5}{4}\hbar\omega$

$\therefore$ ⟨energy cost⟩ $= \langle \cancel{E} \; E_{new} - E_{old} \rangle = \frac{5}{4}\hbar\omega - \frac{1}{2}\hbar\omega = \frac{3}{4}\hbar\omega$

c) $Pr(\text{new gs}) = |\langle \psi_{new} | \psi \rangle|^2$

$\langle \psi_{new} | \psi \rangle = \int \psi_{new}^* \psi \, dx = \sqrt{\frac{m\omega}{\pi\hbar}} \, 2^{1/4} \int e^{-\frac{3m\omega}{2\hbar}x^2} dx$ $\quad y = \sqrt{\frac{3m\omega}{2\hbar}}\, x$

$= \sqrt{\frac{m\omega}{\pi\hbar}} \, 2^{1/4} \sqrt{\frac{2\hbar}{3m\omega}} \underbrace{\int e^{-y^2} dy}_{\sqrt{\pi}} = \frac{2^{3/4}}{\sqrt{3}}$

$\therefore Pr(\text{new gs}) = \frac{\sqrt{8}}{3}$

If we see this, then the system is in new gs (ψ_{new}) after measurement

Aug. momentum

1. a) $H = L^2/2I$, $I = \Sigma m r^2 = 2m\left(\frac{a}{2}\right)^2 = \frac{1}{2}ma^2$

$\therefore H = L^2/ma^2$

eigenvals of $\hat{L}^2$ are $\hbar^2 l(l+1)$

$\therefore$ eigenvals of $\hat{H}$ are $\frac{\hbar^2 l(l+1)}{ma^2}$

b) The angular momentum eigenstates (i.e. spherical harmonics)

At n^{th} energy level, there are $2n+1$ eigenstates ($m = -n$ to $m = n$)

2. a) $\hbar^2 l(l+1) = 2\hbar^2$

b) $-\hbar, 0, \hbar$

c) $|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $|1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\hat{L}_z|1,1\rangle = \hbar|1,1\rangle$ $\quad \hat{L}_z|1,0\rangle = 0$ $\quad \hat{L}_z|1,-1\rangle = -\hbar|1,-1\rangle$

$\therefore \hat{L}_z = \begin{pmatrix}\hbar & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -\hbar\end{pmatrix}$

d) $\psi = f(r)\left(\sqrt{\frac{3}{8\pi}}\cos\theta + \frac{1}{2i}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} - \frac{1}{2i}\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$, $Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$

$\psi = f(r)\left(\frac{1}{\sqrt{2}}Y_1^0 - \frac{1}{2i}Y_1^1 - \frac{1}{2i}Y_1^{-1}\right) = f(r)\left(\frac{1}{\sqrt{2}}|1,0\rangle + \frac{i}{2}|1,1\rangle + \frac{i}{2}|1,-1\rangle\right)$

$= f(r)\begin{pmatrix}i/2\\ 1/\sqrt{2}\\ i/2\end{pmatrix}$

e) $\langle L_z \rangle = \langle \psi | \hat{L}_z | \psi \rangle$

$\langle \psi | = (|\psi\rangle)^\dagger = \begin{pmatrix} i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix}^\dagger = (-i/2, 1/\sqrt{2}, -i/2)$

$\langle L_z \rangle = (-i/2, 1/\sqrt{2}, -i/2) \begin{pmatrix} \hbar & & \\ & 0 & \\ & & -\hbar \end{pmatrix} \begin{pmatrix} +i/2 \\ 1/\sqrt{2} \\ +i/2 \end{pmatrix} = \hbar (-i/2, 1/\sqrt{2}, -i/2) \begin{pmatrix} +i/2 \\ 0 \\ -i/2 \end{pmatrix}$

$= \hbar \left(\frac{1}{4} + 0 - \frac{1}{4} \right) = 0$

$\langle L_z^2 \rangle = (-i/2, 1/\sqrt{2}, -i/2) \begin{pmatrix} \hbar & & \\ & 0 & \\ & & -\hbar \end{pmatrix}^2 \begin{pmatrix} i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix} = \hbar^2 (-i/2, 1/\sqrt{2}, -i/2) \begin{pmatrix} +i/2 \\ 0 \\ +i/2 \end{pmatrix}$

$= \hbar^2 \left(\frac{1}{4} + 0 + \frac{1}{4} \right) = \hbar^2/2$

$\therefore \sigma(L_z) = \hbar/\sqrt{2}$

f) $\hat{L}_+ |1,1\rangle = 0$, $\hat{L}_+ |1,0\rangle = \hbar\sqrt{1(1+1) - 0(0+1)}\, |1,1\rangle = \hbar\sqrt{2}\, |1,1\rangle$

$\hat{L}_+ |1,-1\rangle = \hbar\sqrt{1(1+1) - (-1)(-1+1)}\, |1,0\rangle = \hbar\sqrt{2}\, |1,0\rangle$

$\therefore \hat{L}_+ = \begin{pmatrix} 0 & \hbar\sqrt{2} & 0 \\ 0 & 0 & \hbar\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$

$\hat{L}_- = \hat{L}_+^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ \hbar\sqrt{2} & 0 & 0 \\ 0 & \hbar\sqrt{2} & 0 \end{pmatrix}$

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) = \frac{i}{2}(\hat{L}_- - \hat{L}_+) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

3.a) $\langle \psi | \psi \rangle = \int |\psi(\vec{r})|^2 d^3r = \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \; r^2 \sin\theta \, |\psi|^2 = 1$

if $\psi = f(r)\, A(\theta,\phi)$,

$\langle \psi | \psi \rangle = \underbrace{\left(\int_0^\infty dr\, r^2 |f(r)|^2 \right)}_{=1} \left(\int d\theta\, d\phi\, |A(\theta,\phi)|^2 \right) = 1$

the point here is that we have the r^2 from the Jacobian

b) $\langle \psi | \psi \rangle = |A|^2 \underbrace{\left(\int dr \cdots \right)}_{1} \underbrace{\left(\int_0^\pi d\theta \sin\theta \right)}_{2} \underbrace{\left(\int_0^{2\pi} d\phi \cos^2\phi \right)}_{\pi} = 2\pi |A|^2 = 1$

$\therefore A = 1/\sqrt{2\pi}$

c) $\cos\phi = \dfrac{e^{i\phi} + e^{-i\phi}}{2}$

$\langle l, m | \psi \rangle = 0$ for $m \neq 1$ since $Y_l^m \propto e^{im\phi}$, and $\int_0^{2\pi} e^{im\phi} e^{\pm i\phi} d\phi = 2\pi \delta_{m,\pm 1}$

and, $\langle l, 1 | \psi \rangle = \pm \langle l, -1 | \psi \rangle$ since coefficients are the same

$\therefore$ we can write $|\psi\rangle = \sum_l c_l \left(|l, 1\rangle \pm |l, -1\rangle \right)$ for some c_l

$\therefore$ possible outcomes are $\pm\hbar$, $p = 1/2$

state after measurement $= \cancel{B} f(r) e^{\pm i\phi} = \frac{1}{\sqrt{4\pi}} f(r) e^{\pm i\phi}$

d) Since $m = \pm 1$, $l \geq 1$. So $2\hbar^2$ is lowest outcome.

$$p = |\langle 1,1|\psi\rangle|^2 + |\langle 1,0|\psi\rangle|^2 + |\langle 1,-1|\psi\rangle|^2$$

$$\langle 1,1|\psi\rangle = -\sqrt{\frac{3}{8\pi}} \frac{1}{\sqrt{2\pi}} \int d\theta d\phi \sin\theta \left(\sin\theta e^{i\phi}\right)^* \cos\phi = -\frac{\sqrt{3}}{4\pi} \left(\int_0^\pi \sin^2\theta \, d\theta \right) \left(\int_0^{2\pi} e^{-i\phi} \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) d\phi \right)$$

$$= -\frac{\sqrt{3}}{8\pi} \left(\frac{\pi}{2}\right)(\pi) = -\frac{\sqrt{3}\pi}{16}$$

$$\langle 1,0|\psi\rangle = 0$$

$$\langle 1,-1|\psi\rangle = +\sqrt{\frac{3}{8\pi}} \frac{1}{\sqrt{2\pi}} \left(\int_0^\pi \sin^2\theta \, d\theta \right) \left(\int_0^{2\pi} e^{i\phi} \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) d\phi \right)$$

$$= \frac{\sqrt{3}}{8\pi} \left(\frac{\pi}{2}\right)(\pi) = \frac{\sqrt{3}\pi}{16}$$

$$\therefore p = 2 \left(\frac{\sqrt{3}\pi}{16} \right)^2 = \frac{3\pi^2}{128}$$

state after measurement $\propto \langle 1,1|\psi\rangle |1,1\rangle + \langle 1,-1|\psi\rangle |1,-1\rangle$

$$\propto f(r) \sin\theta \cos\phi$$

normalise $\Rightarrow$ ~~$\frac{\sqrt{2}}{\pi} f(r) \sin\theta\cos\phi$~~

$$|B|^2 \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi = |B|^2 \left(\frac{4}{3}\right)(\pi) = 1$$

$$\int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi \sin\theta (1 - \cos^2\theta) d\theta = -\int_1^{-1} (1 - u^2) du = \left[u - \frac{u^3}{3}\right]_{-1}^{1} = 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

$u = \cos\theta \quad du = -\sin\theta d\theta$

$\therefore$ state after meas. $= \sqrt{\frac{3}{4\pi}} f(r) \sin\theta \cos\phi$

e) Yes, because they commute

<u>Note</u>: our fn is ~~not~~ ill-defined / discontinuous on z-axis $(\theta = 0, \pi)$
that's why we need a $g(\theta)$ that goes to zero for $\theta = 0, \pi$
in the sph. harmonics, e.g. $g(\theta) = \sin\theta$

4 a) $\langle\psi|\psi\rangle = \frac{2}{3}\langle 2,1|2,1\rangle + \frac{1}{6}\langle 3,0|3,0\rangle + \frac{1+1}{12}\langle 3,1|3,1\rangle = \frac{2}{3}+\frac{1}{6}+\frac{1}{6} = 1$ ✓

b) $0 \quad p = 1/6$

$\hbar \quad p = 2/3 + 1/6 = 5/6$

c) $\hbar^2 2(2+1) = 6\hbar^2 \quad p = 2/3$

$\hbar^2 3(3+1) = 12\hbar^2 \quad p = 1/6 + 1/6 = 1/3$

d) after 1st meas. $|\psi\rangle \rightarrow |\psi_2\rangle \propto \sqrt{\frac{2}{3}}|2,1\rangle - \frac{1+i}{\sqrt{12}}|3,1\rangle$

normalise $\Longrightarrow |\psi_2\rangle = A(\sqrt{8}|2,1\rangle - (1+i)|3,1\rangle)$

$\langle\psi_2|\psi_2\rangle = |A|^2(8\langle 2,1|2,1\rangle + 2\langle 3,1|3,1\rangle) = 10|A|^2 = 1$

$\therefore |\psi_2\rangle = \sqrt{4/5}\,|2,1\rangle - \frac{(1+i)}{\sqrt{10}}|3,1\rangle$

$\therefore$ when meas. L^2, get $\begin{cases} 6\hbar^2 & p = 4/5 \\ 12\hbar^2 & p = 1/5 \end{cases}$

e) $\langle\psi|\hat{L}_x|\psi\rangle = \frac{1}{2}\langle\psi|\hat{L}_+ + \hat{L}_-|\psi\rangle = Re(\langle\psi|\hat{L}_+|\psi\rangle)$

$\langle\psi|\hat{L}_+|\psi\rangle = \left(\frac{1-i}{\sqrt{12}}\right)\left(\frac{-i}{\sqrt{6}}\right)\langle 3,1|\hat{L}_+|3,0\rangle = \frac{-i-1}{\sqrt{72}}\langle 3,1|\hbar\sqrt{12}|3,1\rangle = -\frac{(1+i)}{\sqrt{6}}\hbar$

$\therefore \langle L_x\rangle = Re\left(-\frac{1+i}{\sqrt{6}}\hbar\right) = -\frac{\hbar}{\sqrt{6}}$