

1)

*Problem 1.7 Calculate $d\langle p \rangle / dt$. Answer:

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle. \quad [1.38]$$

Equations 1.32 (or the first part of 1.33) and 1.38 are instances of **Ehrenfest's theorem**, which tells us that *expectation values obey classical laws*.

$$\langle p \rangle = \int dx \psi^* (-i\hbar \frac{\partial}{\partial x} \psi)$$

$$\frac{\partial}{\partial t} \langle p \rangle = \int dx \left(\frac{\partial}{\partial t} \psi^* \right) (-i\hbar \frac{\partial}{\partial x} \psi) + \int dx \psi^* (-i\hbar \frac{\partial^2}{\partial x \partial t} \psi)$$

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) \psi \implies -i\hbar \frac{\partial}{\partial t} \psi^* = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi^* + V(x) \psi^*$$

$$\frac{\partial}{\partial t} \langle p \rangle = \int dx \left(-\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V(x) \psi^* \right) \left(\frac{\partial}{\partial x} \psi \right) - \int dx \psi^* \frac{\partial}{\partial x} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) \psi \right)$$

$$= \int dx \left(-\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} + V(x) \cancel{\psi^*} \right) \left(\frac{\partial}{\partial x} \psi \right) + \frac{\hbar^2}{2m} \int dx \psi^* \frac{\partial^3}{\partial x^3} \psi - \int dx \psi^* \frac{\partial V}{\partial x} \psi - \int dx \cancel{\psi^* V(x)} \frac{\partial \psi}{\partial x}$$

$$= -\frac{\hbar^2}{2m} \int dx \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \psi}{\partial x} + \frac{\hbar^2}{2m} \int dx \psi^* \frac{\partial^3 \psi}{\partial x^3} - \int dx \psi^* \frac{\partial V}{\partial x} \psi$$

$$= -\frac{\hbar^2}{2m} \int dx \cancel{\psi^*} \frac{\partial^3 \psi}{\partial x^3} + \frac{\hbar^2}{2m} \int dx \cancel{\psi^*} \frac{\partial^3 \psi}{\partial x^3} - \int dx \psi^* \frac{\partial V}{\partial x} \psi$$

\sim integrate by parts twice

$$= - \int dx \psi^* \frac{\partial V}{\partial x} \psi = \left\langle -\frac{\partial V}{\partial x} \right\rangle$$

2)

* **Problem 1.15** Suppose you wanted to describe an **unstable particle**, that spontaneously disintegrates with a “lifetime” τ . In that case the total probability of finding the particle somewhere should *not* be constant, but should decrease at (say) an exponential rate:

$$P(t) \equiv \int_{-\infty}^{+\infty} |\Psi(x, t)|^2 dx = e^{-t/\tau}.$$

A crude way of achieving this result is as follows. In Equation 1.24 we tacitly assumed that V (the potential energy) is *real*. That is certainly reasonable, but it leads to the “conservation of probability” enshrined in Equation 1.27. What if we assign to V an imaginary part:

$$V = V_0 - i\Gamma,$$

where V_0 is the true potential energy and Γ is a positive real constant?

(a) Show that (in place of Equation 1.27) we now get

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P.$$

(b) Solve for $P(t)$, and find the lifetime of the particle in terms of Γ .

a) eg 1.24 now becomes

$$\frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V^* \Psi^*$$

$$V^* = V_0 + i\Gamma$$

$$\frac{\partial}{\partial t} |\Psi|^2 = \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi^*}{\partial x^2} \Psi \right) + \frac{i}{\hbar} |\Psi|^2 (V^* - V)$$

$$\Rightarrow \frac{\partial}{\partial t} \int_{-\infty}^{\infty} |\Psi|^2 dx = \frac{i\hbar}{2m} \left(\Psi^* \frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x} \Psi \right) \Big|_{-\infty}^{\infty} + \frac{i}{\hbar} \int_{-\infty}^{\infty} dx |\Psi|^2 (V^* - V)$$

$$\frac{dP}{dt} = \frac{i}{\hbar} \int_{-\infty}^{\infty} dx |\Psi|^2 (V_0 + i\Gamma - V_0 + i\Gamma) = \frac{i}{\hbar} (2i\Gamma) \underbrace{\int_{-\infty}^{\infty} dx |\Psi|^2}_P$$

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P$$

$$b) \frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P$$

$$\int_{P(t_0)}^P \frac{dP}{P} = \int_{t_0}^t -\frac{2\Gamma}{\hbar} dt$$

$$\ln P \Big|_{P(t_0)}^P = -\frac{2\Gamma}{\hbar} t$$

$$\ln \frac{P}{P(0)} = -\frac{2\Gamma}{\hbar} t$$

$$\frac{P}{P(0)} = e^{-\frac{2\Gamma}{\hbar} t}$$

$$P(t) = P(0) e^{-\frac{2\Gamma}{\hbar} t} = P(0) e^{-t/\tau}$$

The lifetime of the particle is $\tau = \frac{\hbar}{2\Gamma}$

3)

a) $[\hat{x}, \hat{x}] = \hat{x}^2 - \hat{x}^2 = 0$

b) act with $[\hat{x}, \hat{p}]$ on an arbitrary wavefunction $\Psi(x)$

$$\begin{aligned} [\hat{x}, \hat{p}] \Psi(x) &= \hat{x} \hat{p} \Psi(x) - \hat{p} \hat{x} \Psi(x) = x \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi(x) + i\hbar \frac{\partial}{\partial x} (x \Psi(x)) \\ &= -i\hbar x \frac{\partial \Psi}{\partial x} + i\hbar \Psi(x) + i\hbar x \frac{\partial \Psi}{\partial x} = i\hbar \Psi(x) \end{aligned}$$

Since $\Psi(x)$ was arbitrary we can see $[\hat{x}, \hat{p}] = i\hbar$

c) $[\hat{x}^2, \hat{p}]$

Here we can use the identity $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

$$[\hat{x}^2, \hat{p}] = [\hat{x}\hat{x}, \hat{p}] = \hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x} = i\hbar \hat{x} + i\hbar \hat{x} = 2i\hbar \hat{x}$$

d) $[\hat{x}, f(\hat{x})] = \hat{x}f(\hat{x}) - f(\hat{x})\hat{x} = x f(x) - f(x)x = x f(x) - x f(x) = 0$

e) We can use the same trick as in part b.

$$\begin{aligned} [\hat{p}, f(\hat{x})] \Psi(x) &= \hat{p} f(\hat{x}) \Psi(x) - f(\hat{x}) \hat{p} \Psi(x) = -i\hbar \frac{\partial}{\partial x} (f(x) \Psi(x)) + i\hbar f(x) \frac{\partial \Psi}{\partial x} \\ &= -i\hbar \frac{\partial f}{\partial x} \Psi(x) - i\hbar f(x) \frac{\partial \Psi}{\partial x} + i\hbar f(x) \frac{\partial \Psi}{\partial x} \end{aligned}$$

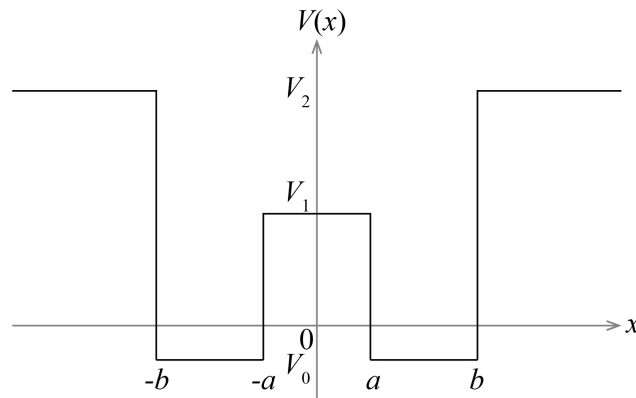
$$\Rightarrow [\hat{p}, f(\hat{x})] = -i\hbar \frac{\partial f}{\partial x}$$

f) $[\hat{x}, \hat{H}] = [\hat{x}, \frac{\hat{p}^2}{2m}] + [\hat{x}, V(\hat{x})]$

$$= \frac{1}{2m} [\hat{x}, \hat{p}^2] = \frac{1}{2m} [\hat{x}, \hat{p}] \hat{p} + \frac{1}{2m} \hat{p} [\hat{x}, \hat{p}] = \frac{1}{2m} (i\hbar \hat{p} + i\hbar \hat{p}) = \frac{i\hbar}{m} \hat{p}$$

4) Qualitative Sketches of Wavefunctions

Consider the following potential $V(x)$:



We will be looking at solutions to the time-independent Schrödinger equation (Eq.(1)).

(a) For what range of values E

(i) are we certain that there does not exist a solution?

Solution: $E < V_0$.

(ii) do we have solutions which are bound states?

Solution: $V_0 < E < V_2$.

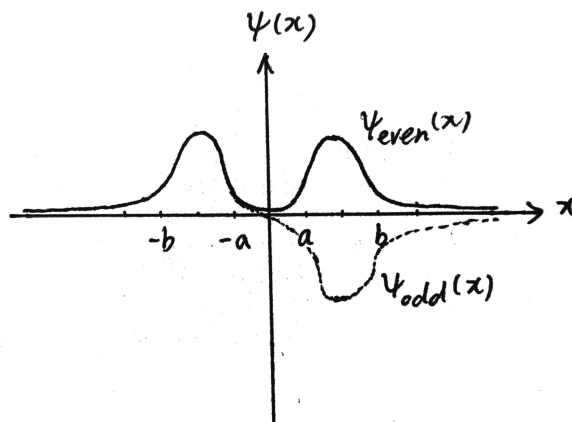
(iii) do we have solutions which are unbound states?

Solution: $V_2 < E$.

(b) Consider the case $V_0 < E < V_1$. Where are the classically allowed and disallowed regions? In other words, if a classical particle has total energy E , where is it possible or impossible for the particle to be?

Solution: Classically allowed: $-b < x < -a$ and $a < x < b$. Classically forbidden: $x < -b$, $-a < x < a$, and $b < x$.

(c) The two lowest-energy solutions (*i.e.* the ground state and first excited state) are sketched below:



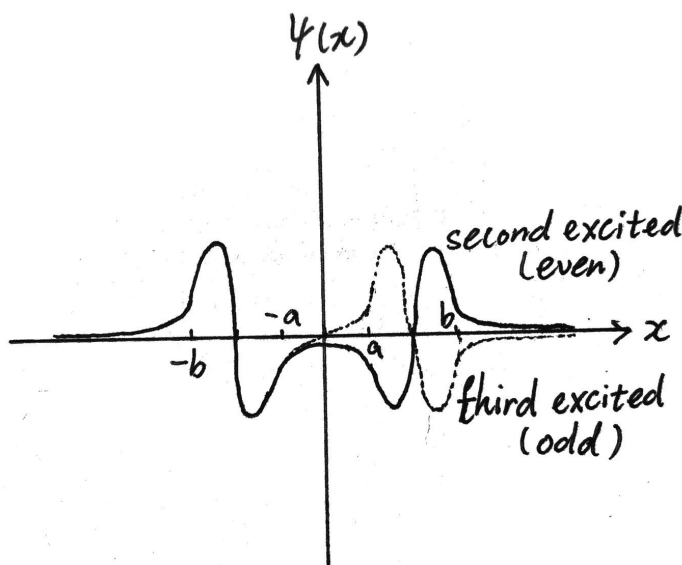
Which wavefunction, ψ_{even} or ψ_{odd} , has a lower energy? Explain why qualitatively.

Solution: The even function has a lower energy than the odd function, because it has one fewer node (zero). In fact, the even wavefunction is the ground state.

- (d) Sketch the third and fourth energy levels (*i.e.* the second and third excited states), assuming their energies still satisfy $V_0 < E < V_1$. Clearly label the two wavefunction, as well as the locations of $\pm a, \pm b$.

Solution: Let's first see how the ground state and first excited state are drawn, as above. First, consider only the left well. The lowest state in that well is a single bump in the classically allowed region, and it tapers off outside of the region. By problem 1(b), we know that the overall wavefunction for the double well (which is symmetric) has to be either odd or even. In the right well, we can either mirror the bump in the left well (ψ_{even}) or rotate the curve for the left well about the origin by 180 degrees (ψ_{odd}).

To construct the second and third excited states, we go through the same procedure. The next lowest state in the left well has two bumps, one above and one below the x -axis; it tapers off outside the classically allowed region. For the right well, we can either mirror the curve in the left well across the vertical axis (even) or rotate it about the origin by 180 degrees (odd). Notice that the even function has two nodes, each in a well, while the odd function has three nodes, including the one at the origin. This means that the even function is the second excited state, while the odd function is the third excited state.



Remember that the wavefunction is always concave towards the x -axis inside the classically allowed regions and is concave away from the x -axis in the classically forbidden regions.

5)

3.12 Let E_n denote the bound-state energy eigenvalues of a one-dimensional system and let $\psi_n(x)$ denote the corresponding energy eigenfunctions. Let $\Psi(x, t)$ be the wave function of the system, normalised to unity, and suppose that at $t = 0$ it is given by

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}} e^{i\alpha_1} \psi_1(x) + \frac{1}{\sqrt{3}} e^{i\alpha_2} \psi_2(x) + \frac{1}{\sqrt{6}} e^{i\alpha_3} \psi_3(x)$$

where the α_i are real constants.

- Write down the wave function $\Psi(x, t)$ at time t .
- Find the probability that at time t a measurement of the energy of the system gives the value E_2 .
- Does $\langle x \rangle$ vary with time? Does $\langle p_x \rangle$ vary with time? Does $E = \langle H \rangle$ vary with time?

$$a) \quad \Psi(x, t) = \frac{1}{\sqrt{2}} e^{i\alpha_1} \psi_1(x) e^{-\frac{iE_1 t}{\hbar}} + \frac{1}{\sqrt{3}} e^{i\alpha_2} \psi_2(x) e^{-\frac{iE_2 t}{\hbar}} + \frac{1}{\sqrt{6}} e^{i\alpha_3} \psi_3(x) e^{-\frac{iE_3 t}{\hbar}}$$

$$b) \quad P = |C_{E_2}(t)|^2 = \left| \frac{1}{\sqrt{3}} e^{i\alpha_2} e^{-\frac{iE_2 t}{\hbar}} \right|^2 = \frac{1}{3}$$

c) $\langle x \rangle$ and $\langle p \rangle$ will vary in time, but $\langle E \rangle$ will be constant.

The time dependence comes from cross terms like $\frac{1}{\sqrt{6}} e^{i(\alpha_2 - \alpha_1)} e^{-\frac{i}{\hbar}(E_2 - E_1)t} \langle \psi_1 | \hat{x} | \psi_2 \rangle$. For \hat{x} and

\hat{p} these cross terms will be nonzero and we have time dependence of the form $e^{-\frac{i}{\hbar}(E_2 - E_1)t}$.

For $\langle H \rangle$ these cross terms are zero. This is because ψ_i are eigenstates of the Hamiltonian and these

states are orthogonal to each other: $\langle \psi_1 | \hat{H} | \psi_2 \rangle = E_2 \langle \psi_1 | \psi_2 \rangle = 0$