Physics 112 - Intro to Statistical and Thermal Physics - Spring 2023
Problem Set 10

Due Friday, April 21 at 11:59 PM (PDT)
Last Update: April 18, 2023

- **Highlighted/Most Relevant Reading for the material on this week's problem set:**
  - Schroeder, Sections 7.4
- **Reading for next week:**
  - Schroeder, 7.3, 7.5, 7.6

Problem 10.1 - Photon Gases

In this problem we will explore the statistical mechanics and thermodynamics of our newest system - the ***photon gas***.

First, some place-setting. In dealing with the Bose-Einstein, Fermi-Dirac, and Planck distributions, we often come across integrals of the form

$$\int_0^\infty \frac{x^n dx}{e^x \pm 1}.$$

We encounter integrals of this form with the plus sign in fermionic systems and with the minus sign in bosonic systems like the photon gas we are exploring in this problem.

In lecture we gave the result

$$\int_0^\infty \frac{x^n dx}{e^x - 1} = \Gamma(n+1)\zeta(n+1) = n!\zeta(n+1), \tag{1}$$

where $\Gamma(n)$ is our old friend the gamma function and $\zeta(n)$ is our *new* friend the ***Riemann zeta function***,

$$\zeta(n) \equiv \sum_{k=1}^{\infty} \frac{1}{k^n}. \tag{2}$$

(a) Rewrite the integrand of Eq. 1 by multiplying the numerator and denominator by e^{-x} and then use the geometric series to rewrite the integral as a sum of integrals. Finally, evaluate the integrals to prove Eq. 1.
Hint (highlight to reveal): [Recall that we've used the geometric series a number of times this semester: in the partition function for the Einstein solid/harmonic oscillator; in deriving the Bose-Einstein distribution function; even just last homework when deriving the Planck distribution function.]

(b) **Extra Part** (*Not for Credit*) Sum the first 100 terms in Eq. 2 to determine $\zeta(4)$, $\zeta(3)$, $\zeta(2)$, and $\zeta(3/2)$. Look up or use a program (like Wolfram Alpha or Python) to find these values to the 3rd decimal place.
[*Note: Note that the larger n is the faster the series converges and the fewer terms we need. We need to go out a few thousand terms to get $\zeta(3/2)$ to agree to the third decimal place though!*]

In the previous problem set we found the density of states for a gas of massless particle in a box of volume V,

$$g(E) = \frac{8\pi V E^2}{h^3 c^3}.$$

Let's apply this to the photon gas. The distribution function since $\mu = 0$ is the Planck distribution function,

$$\bar{n}_P = \frac{1}{e^{E/k_B T} - 1}.$$

Recall that we can get the average energy and average particle number using these two functions,

$$\langle U \rangle = \int_0^\infty E g(E) \bar{n}(E) dE, \qquad \langle N \rangle = \int_0^\infty g(E) \bar{n}(E) dE.$$

Also recall that we defined the ***Stefan-Boltzmann constant*** as

$$\sigma \equiv \frac{\pi^2 k_B^4}{60 \hbar^3 c^2}.$$

(c) Show from the above equations that the energy density for the photon gas is $\langle U \rangle / V = 4\sigma T^4 / c$.
Hint (highlight to reveal): [There's a reason I started this problem with a discussion of a certain type of integral...]

(d) Find the average number of photons in a photon gas of volume V and temperature T. You may leave a zeta function in your answer.

(e) **Extra Part** (*Not for Credit*) Using your answer from (c), find the heat capacity at constant volume $C_V(T)$ for the photon gas.
Answer (highlight to reveal): [$C_V = 16\sigma V T^3/c$.]

(f) Find the entropy of the photon gas and then use your answer from (d) to find the entropy-per-photon at temperature T. What is the entropy density (in $\mathrm{J}/(\mathrm{m}^3 \cdot \mathrm{K})$) and energy density (in J/m^3) of the cosmic microwave background $T = 2.73\,\mathrm{K}$.

The radiation pressure/pressure of a photon gas was found to be

$$P = \frac{1}{3}\frac{U}{V} = \frac{4\sigma T^4}{3c}.$$

(g) What is the enthalpy H of a photon gas of temperature T?

Something to think about: We have expressions relating entropy, temperature, pressure, and volume. That's enough to figure out what happens in an isothermal and adiabatic change of a photon gas and thus map out a photon Carnot engine!

Problem 10.2 - The Sun and the Earth

In this problem we will explore the thermodynamics of the earth-sun system and explore some of the consequences of the Stefan-Boltzmann and Planck blackbodyradiation law.

Let's approximate both the sun and the earth asperfect blackbodies (emissivity $\epsilon = 1$) for all wavelengths. We also make the simplifying assumption that the earth maintains a constant temperature on its surface through the day/night cycle. The fact that we are treating earth and the sun as a perfect blackbodies means two things:

- They absorb all radiation incident on them.
- They emit radiation according to the Planck blackbody spectrum and the Stefan-Boltzmann law.

The surface temperature of the sun is $T_\odot = 5770\,\mathrm{K}$. The average distance from the earth to the sun is $1\,\mathrm{AU} \approx 1.5 \times 10^{11}\,\mathrm{m}$. When looking at the sun from earth we find that the sun subtends an angle of 32' - that is "32 minutes of arc", where one minute or arc is 1/60 of a degree.[1]

(a) **Extra Part** (*Not for Credit*) Use the information given above to determine the radius of the sun $R_\odot$.
Hint (highlight to reveal): [This should just be a simple exercise in geometry!]
Answer (highlight to reveal): [$R_\odot \approx 7 \times 10^8\,\mathrm{m}$.]

(b) What is the power emitted by the surface of the sun? Estimate to within an order of magnitude the number of photons emitted by the sun per second.
Hint (highlight to reveal): [We will get better numbers later but you can approximate the photons coming from the sun as mostly coming from the middle of the visible spectrum, $\sim 500\,\mathrm{nm}$.]

The ***solar constant*** G_{SC} is defined as the total power flux density (power-per-unit-area) received from the sun and normal to the rays at a distance of 1 AU (basically, at the Earth). That is, if we had a surface of area A in space next to the earth (or on the surface of the earth if we removed the atmosphere) and the sun was shining directly onto the surface (so the angle between the normal to the surface and the radiation is 0), then the power received would be $G_{SC}A$.

(c) Use your earlier answer from (b) and geometry to determine G_{SC}.

We will use the answer $G_{SC} = 1360\,\mathrm{W/m^2}$ in further parts. Assume 30% of the incident solar radiation gets *reflected* away from earth (e.g. by clouds, snow, ice, etc.) - this is known as the ***albedo*** of the earth.

(d) Neglecting the effects of the atmosphere, calculate the surface temperature of the earth. Does this temperature seem too high, too low, or just right when compared with your experience?
[*Note: Warning! For the incident radiation from the sun we only care about the* cross-sectional *area of the earth! Why?*]
Hint (highlight to reveal): [The earth maintaining a constant temperature means that the net flow of energy absorbed the earth equals the net flow of energy emitted by the earth.]
Hint (highlight to reveal): [While you will need the radius $R_\oplus$ of the earth for individual parts of the problem, it should wind up disappearing from the final answer.]

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[1] Fun fact! Each degree is divided up into 60 minutes of arc. Each minute of arc is then divided again into 60 seconds of arc. That is, one second of arc is a *second* division of a unit into 60. This is why our unit of time is called the second - it is the second subdivision of something into sixtieths.

The ***greenhouse effect*** occurs because gases in the earth's atmosphere - most notably CO_2, but also including water vapor - are ***transparent*** to visible wavelengths of light but ***opaque*** to infrared radiation. That is, infrared radiation gets absorbed by the atmosphere but visible light passes through without being absorbed. If we model the atmosphere as a single thin shell "near" the surface of the earth (the altitude is small compared to Earth's radius so the surface area of the earth is roughly equal to the surface area of the inside and outside surfaces of this atmospheric shell) then we have the following energy transfers:

- About 30% of the incident solar radiation gets reflected back into space and has no effect on the rest of the system.
- Infrared radiation from the sun gets absorbed by the atmospheric shell. We will assume the shell perfectly absorbs all IR radiation and thus none of the IR radiation reaches the surface of the earth.
- Visible (and shorter-wavelength) radiation from the sun passes through the atmosphere and gets absorbed by the earth.
- Infrared radiation from the earth gets absorbed by the atmospheric shell.
- The atmospheric shell emits radiation from its inner surface towards the earth.
- The atmospheric shell emits radiation from its outer surface into space.

Suppose a fraction f_V of the incident solar radiation is visible (or shorter wavelength) and is able to reach the earth's surface while the rest gets absorbed by the atmosphere.

(e) Show that if the earth and atmosphere are maintaining constant temperatures, then the temperature of the *atmosphere* is the answer you found in (d) but the temperature of the *earth* is now increased by a factor of $(1+f_V)^{1/4}$. What temperature does this predict in the limit $f_V = 1$? Does this temperature seem too high, too low, or just right when compared with your experience?

(f) **Extra Part** (*Not for Credit*) Given the known temperature of the surfact of the sun and the Planck blaockbody radiation law, determine numerically the fraction f_V of energy received from the sun that is in the visible-and-shorter range of the spectrum. Use this new f_V to perform a final calculation of the earth's temperature.

Problem 10.3 - DataHub - Color and Temperature

The Planck blackbody radiation law gives the spectral energy density - the energy per unit volume per unit frequency or per unit wavelength. The formulas relating these are

$$u_\omega(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_B T}-1}, \qquad u_\lambda(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_B T}-1}.$$

(a) Find the frequency λ_{peak} that maximizes $u_\lambda(\lambda)$.
[*Note: It may help to define the dimensionless parameter* $y = hc/(\lambda k_{BT})$ *and first find* y_{peak}. *When you are solving you will encounter a transcendental equation. A space has been left in the DataHub page for you to perform the numerical calculation to solve the equation.*]

(b) **Extra Part** (*Not for Credit*) Similarly, find the frequency ω_{peak} that maximizes $u_\omega(\omega)$.
[*Note: It will help to define the dimensionless parameter* $x = \hbar\omega/k_B T$ *and first find* x_{peak}. *When you are solving you will encounter a transcendental equation. Again, a space has been left in the DataHub page for you to perform the numerical calculation to solve the equation.*]
Answer (highlight to reveal): [$\omega_{\text{peak}} \approx 2.821 k_B T/\hbar$]

(c) Find the values of $E_{\text{peak}} = \hbar\omega$ (in eV) and λ_{peak} (in nm) for radiation from the surface of the sun ($T = 5770\,\text{K}$). [*Note: You should find that, even though $E = hc/\lambda$, we do* not *have $E_{peak} = hc/\lambda_{peak}$.*]

The cell below the heading 10.3(d) in the DataHub page contains an interactive plot of the blackbody spectrum as a function of temperature. Feel free to play around with it!

(d) Compare the axes to a chart of the EM spectrum, which can be easily found online. At what frequency range do everyday objects radiate the most? At what temperature do objects radiate visible light the most? *Assuming* our eyes have evolved to match the spectrum of the sun (ignoring the scattering properties of the atmosphere and environment), roughly (order-of-magnitude) estimate the temperature of the sun. Does this match the temperature we found in (c)?

You can find the rest of this problem on the DataHub:
http://datahub.berkeley.edu/user-redirect/interact?account=ajh38&repo=phy112-001_spring_2023&branch=main&path=Sp23-112-Hw10-Python

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Not for Credit - DataHub - Thermal Radiation from Quark Gluon Plasma

I have added a second problem to the data hub, titled "*Thermal Radiation from Quark Gluon Plasma*". This is a great problem written by Prof. Barbara Jacak, who is teaching the other section of 112 this semester. This is directly based on her research!

Not a Problem - Interesting Windows into the Early Universe!

Discussions and analysis of the Planck distribution gives us fascinating windows into the early universe. We already introduced the cosmic microwave background. Look at Schroeder problems 7.47, 7.48, 7.49, and 7.50 for a great, accessible discussion.

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