Astro C161 Final Project

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May 5, 2023

1 Methods

For our model of a tidal disruption event, we chose a black hole with mass $M_B = 5M_\odot$ and a star of mass $M_{star} = 1M_\odot$. The star was modeled as a swarm of $N = 10000$ particles randomly distributed around a central point located at a distance equal to the tidal radius away from the black hole. No self-gravity was assumed within the swarm and all swarm particles were confined within a $1R_\odot$ circle around the central point.

In order to run our simulation, the two key parameters we needed to calculate for each particle at each point were $\frac{dr}{d\tau}$ and $\frac{d\phi}{d\tau}$. We extracted these parameters by manipulating the Schwarzchild metric for orbits around a non spinning black hole and substituting in the constant of motions formula. For simplification, we set $\theta = \frac{\pi}{2}$ and $d\theta^2 = 0$ as we only considered motion in the equatorial plane. This in turn required us to determine what the constants of motions $\frac{E}{m}$ and and $\frac{L}{m}$ were, with m being the mass of a particle calculated by $\frac{M_\odot}{N_{particles}}$. This left us with E and L as unknowns for each particle.

$$ds^2 = -c^2(1 - \frac{r_s}{r})dt^2 + (1 - \frac{r_s}{r})^{-1}dr^2 + r^2 d\Omega^2$$

$$\frac{d\phi}{d\tau} = \frac{L}{mr^2}$$

$$\frac{d\phi}{d\tau} = [\frac{E}{m^2c^2} - \frac{L^2}{m^2r^2}(1 - \frac{r_s}{r}) - c^2(1 - \frac{r_s}{r})]^{\frac{1}{2}}$$

To approximate the energy E for each particle, we first chose the central point of the swarm as an example to have $E = 0$ and be in a circular orbit. Knowing E = 0 for this central point, we used Newtonian mechanics to set kinetic energy equal to potential energy at the center of the star and solve for an initial velocity v. Based on our initial positioning of the star, we determined this velocity to be entirely in the angular direction, meaning the calculated v was $\frac{d\phi}{d\tau}$. Figuring the star to be a continuous body at least in the first step, we gave every particle the same v calculated from the center point. Plugging this meant that the kinetic energy contribution was identical for each particle, but the differing initial distances r would result in different potential energies and overall energies. Under this condition, the particles interior to the central point had $E < 0$ and would be in bound systems while particles exterior had $E > 0$ and flew off as unbound particles, which seemed like a reasonable approximation.

To solve for the angular momentum L, we returned to our equation for $\frac{dr}{d\tau}$. Given that we set up the star to have the initial velocity v be solely $\frac{d\phi}{d\tau}$, thus we could set $\frac{dr}{d\tau} = 0$ and rearrange the equation in terms of L.

$$L = (\frac{E^2}{c^2(1 - \frac{r_s}{r})} - c^2 m^2 r^2)^{\frac{1}{2}}$$

From here, we plugged in the previously calculated energy E value and assigned r value for each particle to get L.

As initial value for $\frac{d\phi}{d\tau}$, we chose the v derived from the central point calculation as mentioned above. However, for $\frac{dr}{d\tau}$, we noted that in our equation, the only variable quantity was r, with all other contributing values being constants. However, by initializing the star with $\frac{dr}{d\tau} = 0$ as we did earlier, the simulation would never be able to update r or dr for future steps of $d\tau$. Thus, we decided to give the system an initial kick by introducing a $\frac{dr}{d\tau}$ term of our choosing just for the first interval. As an approximation, we calculated the free-fall acceleration of each particle radially towards the black hole using basic Newtonian mechanics, plugging in the respective r. We then multiplied this by the time step $d\tau$ to get the initial $\frac{dr}{d\tau}$.

$$F = mg = ma$$

$$g = \frac{d^2 r}{d\tau^2}$$

$$\frac{GM}{r^2} = \frac{d^2 r}{d\tau^2}$$

$$\int \frac{GM}{r^2} = \int \frac{d^2 r}{d\tau^2}$$

$$\frac{GM}{r^2} d\tau = \frac{dr}{d\tau}$$

After this first step, we utilized the $\frac{d\phi}{d\tau}$ and $\frac{dr}{d\tau}$ formulas above to recalculate and update the ϕ and r coordinates for each particle for a given number of time steps with size $d\tau$.

Another separate, alternative method to get an equation of motion was by taking different derivatives of the Lagrangian to obtain a second order differential equation. In the Lagrangian

$$\mathscr{L} = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

The velocity v was substituted with the Schwarzchild Metric:

$$v^2 = (\frac{ds}{d\tau})^2 = -(\frac{dr}{d\tau})^2 \frac{1}{\frac{r_s}{r} - 1} + (\frac{r_s}{r}) c^2 + r^2 (\frac{d\Omega}{d\tau})^2$$

Then, the derivatives $\frac{d}{d\tau}(\frac{d\mathscr{L}}{d\dot{r}})$ and $\frac{d\mathscr{L}}{dr}$ are calculated and set equal. Thus, we can solve for the acceleration in the radial direction, $\frac{d^2r}{d\tau^2}$:

$$\frac{d}{d\tau}(\frac{d\mathscr{L}}{d\dot{r}}) = 2[\frac{(\frac{r_s}{r}-1)(\frac{d^2r}{d\tau^2}) + \dot{r}^2\frac{r_s}{r^2}}{(\frac{r_s}{r}-1)^2}]$$

$$\frac{d\mathscr{L}}{dr} = 2\dot{\phi}^2 r + \frac{r_s\dot{r}^2}{(\frac{r_s}{r}-1)^2 r^2} + \frac{c^2 r_s}{r^2} - \frac{GMm}{r^2}$$

$$\rightarrow \frac{d^2r}{d\tau^2} = \dot{\phi}^2 r(\frac{r_s}{r}-1) + \frac{r_s^2}{2r^2(\frac{r_s}{r}-1)} + \frac{c^2 r_s(\frac{r_s}{r}-1)}{2r^2} - \frac{GMm(\frac{r_s}{r}-1)}{2r^2} - \frac{{}^2 r_s}{(\frac{r_s}{r}-1)r}$$

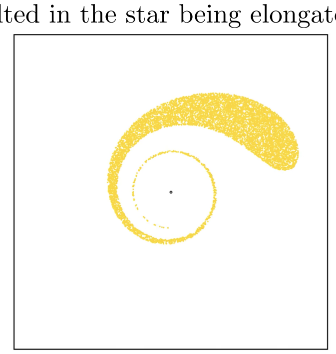
An equation of motion for the angular component can also be calculated in a similar way to obtain:

$$2r\dot{r}\dot{\phi} + r^2\frac{d^2\phi}{d\tau^2} = 0$$

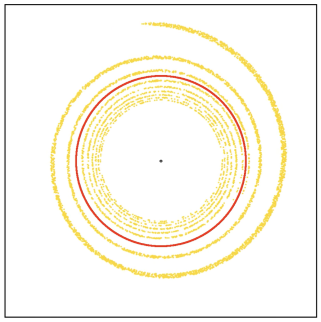
This second equation of motion for the angular component is used in one of the simulation attempts to calculate the change in angular velocity, given the radii of each of the solar swarm particles and radial and angular velocities.

2 Results

In one of the first simulations using the Lagrangian approach to deriving equations of motion, only the angular velocity component was simulated. This resulted in the star being elongated.

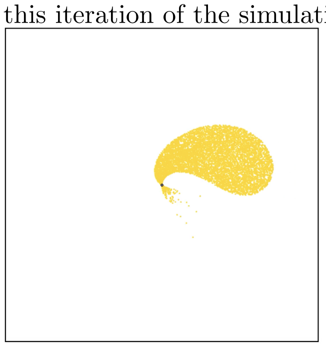


After the simulation was run for a higher number of time step iterations, a disk shape started to emerge.

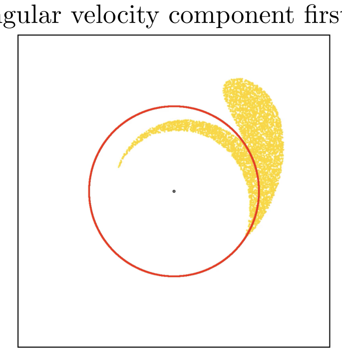


Where the red circle indicates the impact parameter.

Then, we attempted to add the energy component and the radial velocity to the simulation. This resulted in particles of the star swarm being shot outward. It is important to note that the following figure is not to scale; even though it appears that the swarm particles are going through the black hole, the scaling for this iteration of the simulation was incorrect.

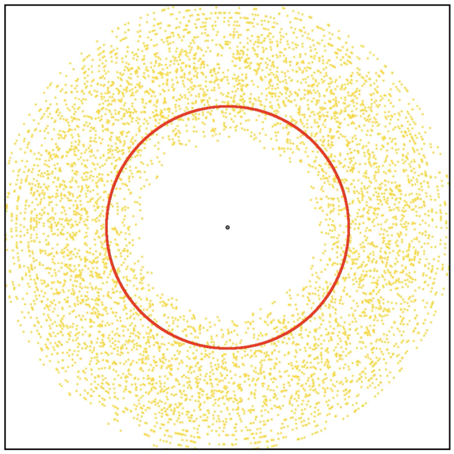


Using the other method involving Newtonian mechanics in conjunction with the Schwarzchild Metric, we attempted to create a simulation with just the angular velocity component first.

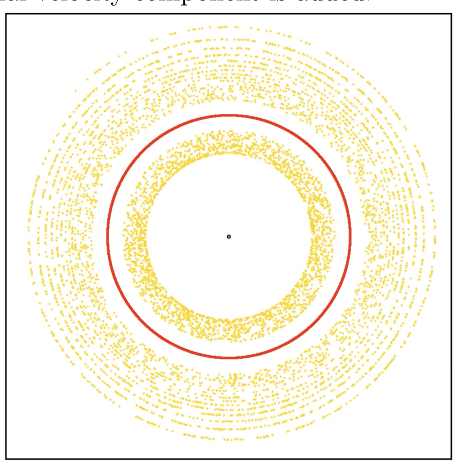


This version of the simulation had a few promising features. For instance, the part of the star outside of the tidal radius became more elongated whereas the part of the star within the tidal radius began to spiral faster.

Once this simulation, which lacks radial velocity was extended to longer time frames, the particles of the star began to disperse into what appeared as a large disk.



A final test of the simulation was performed in which the energy and the radial velocity component is added.



The segment of the star outside of the tidal radius is ejected outward; however, it does not move outward at the rate that was expected. It appears to form its own separate disk, which should be incorrect. The section of the stellar swarm inside of the tidal radius begins to form a disk. The physical origin of the inner disk limit is unknown, as it is not the tidal radius outlined in red or the Schwarzchild radius, which is orders of magnitude smaller.

Given enough simulation run time, the outer particles eventually leaves the frame, which is the correct behavior given they began with $E > 0$. The formation of a disk may also be explainable as the radius of the swarm is small, meaning the difference in tidal forces between the inner and outer edge are not significantly large. Coupled with a relatively low initial entry velocity, this may explain why the inner and outer particles display similar paths as the forces are similar, although ultimately matching our expectations for different fates.