Problem 1

"Fun" Math

In this problem we explore some of the more useful theorems (stated without proof) involving Hermite polynomials.

(a) The **Rodrigues formula** says that

$$H_n(\xi) = (-1)^n e^{\xi^2} \left(\frac{d}{d\xi}\right)^n e^{-\xi^2}. \qquad [2.86]$$

Use it to derive H_3 and H_4.

(b) The following recursion relation gives you H_{n+1} in terms of the two preceding Hermite polynomials:

$$H_{n+1}(\xi) = 2\xi H_n(\xi) - 2n H_{n-1}(\xi). \qquad [2.87]$$

Use it, together with your answer in (a), to obtain H_5 and H_6.

(a)

$$H_3(\xi) = 8\xi^3 - 12\xi$$

$$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$$

(b)

$$H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi$$

$$H_6(\xi) = 64\xi^6 - 480\xi^4 + 720\xi^2 - 120$$

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(a) Write down $\psi_0(x)$, $\psi_1(x)$, and $\psi_2(x)$, and show that they are solutions to the time-independent Schödinger equation (Eq.(1)) with energies $E_0 = \frac{1}{2}\hbar\omega$, $E_1 = \frac{3}{2}\hbar\omega$, and $E_2 = \frac{5}{2}\hbar\omega$, respectively.

(b) A particle begins at time $t = 0$ with the (normalised) wavefunction

$$\Psi(x, t = 0) = \frac{2}{\sqrt{3}\pi^{1/4}}\left(\frac{m\omega}{\hbar}\right)^{5/4} x^2 \exp\left(-\frac{m\omega x^2}{2\hbar}\right).$$

Write this state in terms of the energy eigenfunctions $\psi_n(x)$.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

(c) What is the wavefunction of the particle at a later time t? Keep your answer in terms of the energy eigenfunctions $\psi_n(x)$.

(d) What is the expectation value of energy of the state $\Psi(x, t)$ as a function of time? [Hint: How does the Hamiltonian operator appear in the time-independent Schrödinger equation? There is no need to write out the explicit form of $\psi_n(x)$.]

(a) $$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega x^2}{2\hbar}\right), \qquad n = 0, 1, 2, \ldots$$

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}, \quad \hat{H}\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}\left[-\frac{\hbar^2 m\omega}{2m\hbar}\left(\frac{m\omega}{\hbar}x^2 - 1\right) + \frac{1}{2}m\omega^2 x^2\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{\hbar\omega}{2}\right) e^{-\frac{m\omega x^2}{2\hbar}} = \underbrace{\frac{\hbar\omega}{2}}_{=E}\psi_0$$

$$\psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2}}\left(2\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega x^2}{2\hbar}}, \quad \hat{H}\psi_1 = \underbrace{\frac{3}{2}\hbar\omega}_{=E}\psi_1$$

$$\psi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{8}}\left[4\frac{m\omega}{\hbar}x^2 - 2\right] e^{-\frac{m\omega x^2}{2\hbar}}, \quad \hat{H}\psi_2 = \frac{5}{2}\hbar\omega\psi_2$$

(b) $$\psi_2 = \left(\frac{m\omega}{\hbar}\right)^{5/4}\frac{\sqrt{2}}{(\pi)^{1/4}} x^2 e^{-\frac{m\omega x^2}{2\hbar}} + \frac{1}{\sqrt{2}}\psi_0$$

$$\Rightarrow \Psi = \frac{\sqrt{2}}{\sqrt{3}}\left[\psi_2 - \frac{1}{\sqrt{2}}\psi_0\right]$$

(c) $$\Psi(x,t) = \sqrt{\frac{2}{3}}\left[\psi_2 e^{-\frac{5i\hbar\omega}{2}t} - \frac{1}{\sqrt{2}}\psi_0 e^{-\frac{i\hbar\omega}{2}t}\right]$$

(d) $$\langle\Psi(0)|\hat{H}|\Psi(0)\rangle = \langle E\rangle_{t=0}$$

$$= \langle\Psi(0)|\hat{H}e^{\frac{i\hat{H}t}{\hbar}}e^{-\frac{i\hat{H}t}{\hbar}}|\Psi(0)\rangle$$

$$= \langle\Psi(0)|e^{i\hat{H}t/\hbar}\hat{H}e^{-\frac{i\hat{H}t}{\hbar}}|\Psi(0)\rangle$$

$$= \langle\Psi(t)|\hat{H}|\Psi(t)\rangle = \langle E\rangle_{t=t}$$

$\langle E\rangle$ is constant in time

Problem 3

Half full life

Find the allowed energies of the *half* harmonic oscillator

$$V(x) = \begin{cases} (1/2)m\omega^2 x^2, & \text{for } x > 0, \\ \infty, & \text{for } x < 0. \end{cases}$$

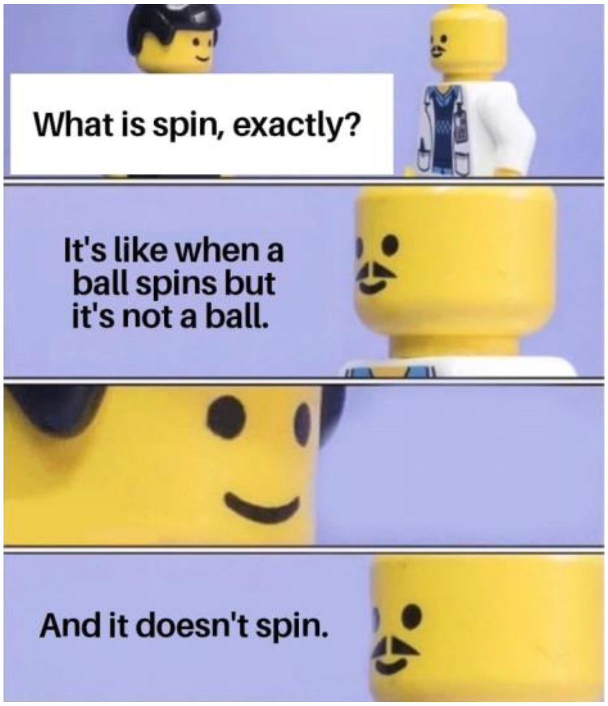
(This represents, for example, a spring that can be stretched, but not compressed.) *Hint:* This requires some careful thought, but very little actual computation.

Everything about the harmonic oscillator applies to the right half. Now we have b.c. $\psi(0) = 0$

$\Rightarrow$ Only Odd Solutions are good.

$$E_n = \frac{3}{2}\hbar w, \frac{7}{2}\hbar w, \frac{11}{2}\hbar w \ldots$$

$$= \left(2n - 1 + \frac{\hbar\omega}{2}\right) \qquad n \in \mathbb{Z}^+$$


What is spin, exactly?
It's like when a ball spins but it's not a ball.
And it doesn't spin.

Problem 4
"Fun" Math 2

Show that two noncommuting operators cannot have a complete set of common eigenfunctions. *Hint:* Show that if $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions, then $[\hat{P}, \hat{Q}]f = 0$ for any function in Hilbert space.

Suppose $\hat{P}, \hat{Q}$, have a complete set of common eigenfunctions ψ_i with eigenvalues q_i, p_i respectively

Then, for any f, $f = \sum_n c_n \psi_n$

Then, $[P,Q]f = PQf - QPf$

$$= \sum_n \left[c_n PQ \psi_n - c_n QP \psi_n \right]$$

$$= \sum_n c_n \left[(P\psi_n) q_n - (Q\psi_n) p_n \right]$$

$$= \sum_n c_n \left[\psi_n p_n q_n - \psi_n q_n p_n \right]$$

$$\underline{= 0}$$

So they commute.

$\therefore$ if two operators have a common, complete set of eigenfunctions, they commute.

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Problem 5

do some quantum

Consider a three-dimensional vector space spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, $|3\rangle$. Kets $|\alpha\rangle$ and $|\beta\rangle$ are given by

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle, \quad |\beta\rangle = i|1\rangle + 2|3\rangle.$$

(a) Construct $\langle\alpha|$ and $\langle\beta|$ (in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$).

(b) Find $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$, and confirm that $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$.

(c) Find all nine matrix elements of the operator $\hat{A} \equiv |\alpha\rangle\langle\beta|$, in this basis, and construct the matrix **A**. Is it hermitian?

(a) $\langle\alpha| = -\langle 1|i + \langle 2|2 + \langle 3|i$

$\langle\beta| = -\langle 1|i + \langle 3|2$

(b) $\langle\alpha|\beta\rangle = (-\langle 1|i + \langle 2|2 + \langle 3|i)(i|1\rangle + 2|3\rangle)$

$= \langle 1|1\rangle + 2i\langle 3|3\rangle$ + stuff like $\langle 1|2\rangle$ $\langle 2|3\rangle$...

$= 1 + 2i$

$\langle\beta|\alpha\rangle = \langle 1|1\rangle - 2i\langle 3|3\rangle$

$= 1 - 2i = \langle\alpha|\beta\rangle^*$

(c) $\hat{A} = |\alpha\rangle\langle\beta|$

$= (i|1\rangle - 2|2\rangle - i|3\rangle)(-i\langle 1| + 2\langle 3|)$

$$\begin{array}{l|lll} & 1 & 2 & 3 \\ \hline 1 & = |1\rangle\langle 1| & + 0 & + 2i|1\rangle\langle 3| \\ 2 & + 2i|2\rangle\langle 1| & + 0 & - 4|2\rangle\langle 3| \\ 3 & + |3\rangle\langle 1| & + 0 & - 2i|3\rangle\langle 3| \end{array}$$

$\rightarrow A = \big(1 \quad 0 \quad 2i$

$$\Rightarrow \underline{A} = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ 1 & 0 & -2i \end{pmatrix}$$

$$\neq A^{\dagger}$$