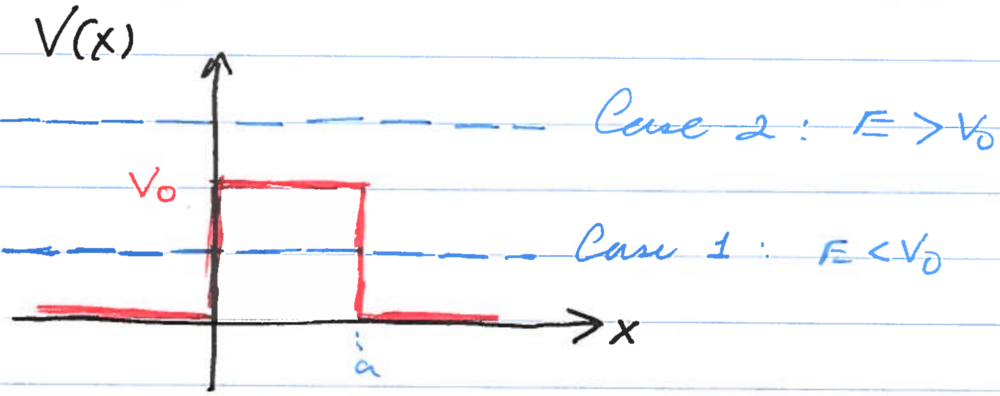
Lecture 11 The Potential Barrier 137A



$$V(x) = \begin{cases} 0 & x<0 \\ V_0 & 0<x<a \\ 0 & x>a \end{cases}$$

- No solution for $E<0$

Case 1:

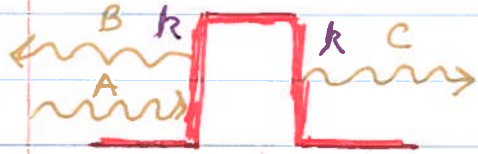
Outside barrier we have a free particle

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x<0 \\ Ce^{ikx} + De^{-ikx}, & x>a \end{cases}$$

$$k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

• Consider the case of a particle incident from the left.

$\rightarrow D = 0$



$$j = \begin{cases} v\left[|A|^2 - |B|^2\right] & x < 0 \\ v\,|C|^2 & x > a \end{cases}$$

$$v = \frac{\hbar k}{m}$$

$$\mathbb{R} = \frac{|B|^2}{|A|^2} \qquad \mathcal{J} = \frac{|C|^2}{|A|^2}$$

Inside the barrier for $E < V_0$

$$\psi(x) = F e^{Kx} + G e^{-Kx} \qquad x \in [0, a]$$

$$K = \left[\frac{2m}{\hbar^2}(V_0 - E)\right]^{1/2}$$

Need to keep both solutions (i.e. F, G) because neither goes to $\pm\infty$ within the region $[0, a]$

- Match ψ and $\frac{d\psi}{dx}$ @ $x = 0, a$

$\underline{x=0}$

ψ: $A + B = F + G$

ψ': $ik(A - B) = K(F - G)$

$\underline{x=0}$

$Ce^{ika} = Fe^{Ka} + Ge^{-Ka}$

$ikCe^{ika} = K(Fe^{Ka} - Ge^{-Ka})$

eliminate F, G, express in terms of $\frac{B}{A}, \frac{C}{A}$

$$\frac{B}{A} = \frac{(k^2 + K^2)(e^{2Ka} - 1)}{e^{2Ka}(k + iK)^2 - (k - iK)^2}$$

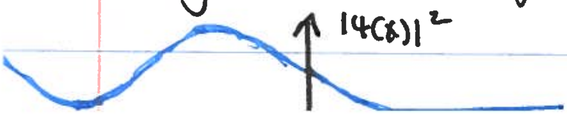
$$\frac{C}{A} = \frac{4ikK\, e^{-ika} e^{Ka}}{e^{2Ka}(k + iK)^2 - (k - iK)^2}$$

$$\mathbb{R} = \frac{|B|^2}{|A|^2} = \left[1 + \frac{4E(V_0 - E)}{V_0^2 \sinh^2(Ka)}\right]^{-1}$$

$$\mathcal{T} = \frac{|C|^2}{|A|^2} = \left[1 + \frac{V_0^2 \sinh(Ka)}{4E(V_0 - E)}\right]^{-1}$$

$\longrightarrow$ Tunneling possible for $E < V_0$!!

eg Solution for $mV_0 a^2/\hbar^2 = 1/4$



Notes:

- $|\psi|^2$ never goes to zero since $|B|^2 < |A|$
- $\lim_{E \to 0} \tilde{T} = 0$
- $$\lim_{E \to V_0} \tilde{T} = \Big(1 + \underbrace{\frac{m V_0 a^2}{2\hbar^2}}\Big)^{-1}$$

opacity of the tunnel barrier

Case 2: $E > V_0$

Solution is now again oscillating plane waves in the barrier region.

$$\psi(x) = F e^{ik'x} + G e^{-ik'x} \quad \text{for } x \in [0, a]$$

$$k' = \left[\frac{2m}{\hbar^2}(E - V_0)\right]^{1/2}$$

* Note "k" is different;
Spatial frequency (wave #) is different.
The available energy E is divided between potential V and KE.

Apply boundary conditions again,
eliminate F, G, express in terms of $\frac{B}{A}, \frac{C}{A}$

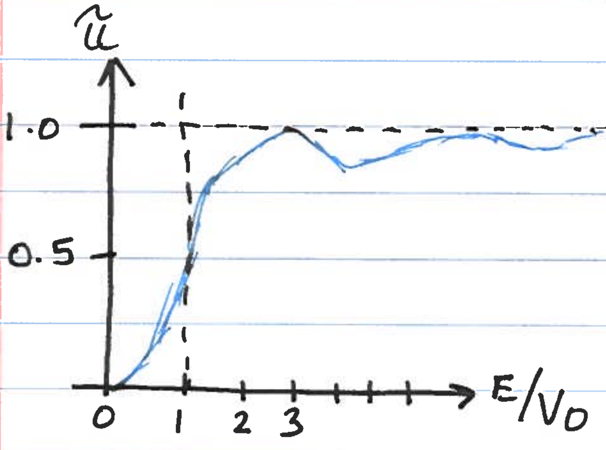
$$\mathbb{R} = \left[1 + \frac{4E(E-V_0)}{V_0^2 \sin^2(k'a)} \right]^{-1}$$

$$\tilde{T} = \left[1 + \frac{V_0^2 \sin^2(k'a)}{4E(E-V_0)} \right]^{-1}$$

- Note T only $\rightarrow 1$ for $k'a = \pi, 2\pi, 3\pi$

Transmission resonances fit half-integer de Broglie wavelengths in the barrier.

$\longrightarrow$ Reflections from $x=0, a$ cancel!

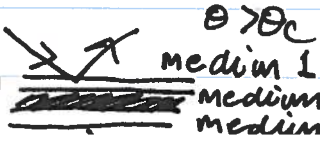


(i) Compare with optics

$\theta > \theta_c$

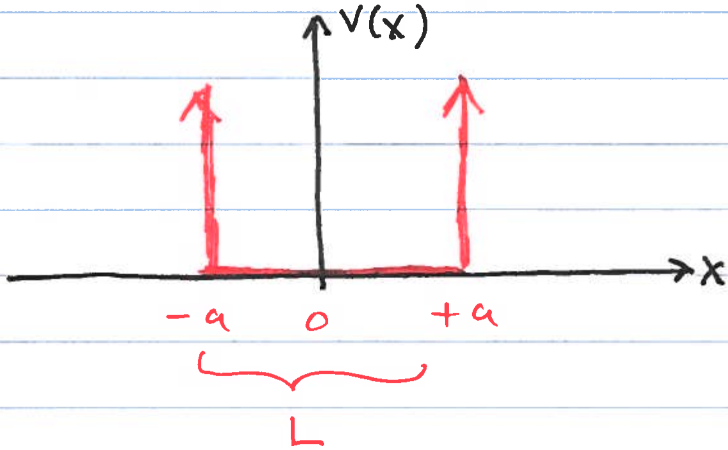
total internal reflection

(ii)



Lecture 12 — Square Potential Wells — 137A

A. Infinite Square Well



$$V(x) = \begin{cases} 0, & x \in [0,a \\ \infty, & |x| > a \end{cases}$$

* When motion is confined in space, E is quantized!

- $\psi(x) \to 0$ outside walls
- By continuity $\psi(x) = 0$ for $x = \pm a$
- Cannot match $\frac{d\psi}{dx}$ since $V(x)$ has *infinite* discontinuity

for $|x| < a$:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \qquad k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

or

$$A\cos(kx) + B\sin(ka)$$

<u>BC</u>

$$A\cos(ka) = 0$$
$$B\sin(ka) = 0$$

Two possible classes of solutions.

I. $B = 0$, $\cos(ka) = 0 \longrightarrow k_n = \frac{n\pi}{2a} = \frac{n\pi}{L}$

$n = 1, 3, 5, \ldots$

$$\psi_n(x) = A_n \cos(k_n x)$$

$$\int \psi_n^*(x)\,\psi_n(x)\,dx = 1 \longrightarrow A = \frac{1}{\sqrt{a}}$$

Thus $\psi_n(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right)$

II. $A = 0$, $\sin(ka) = 0$

$$\psi_n(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) \qquad n = 2, 4, 6, \ldots$$

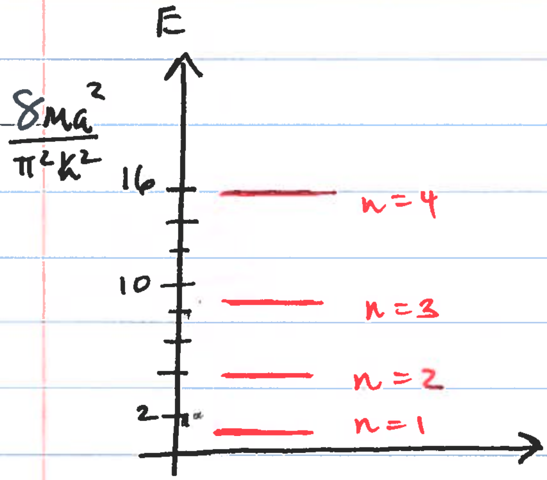
Taking both solutions into account:

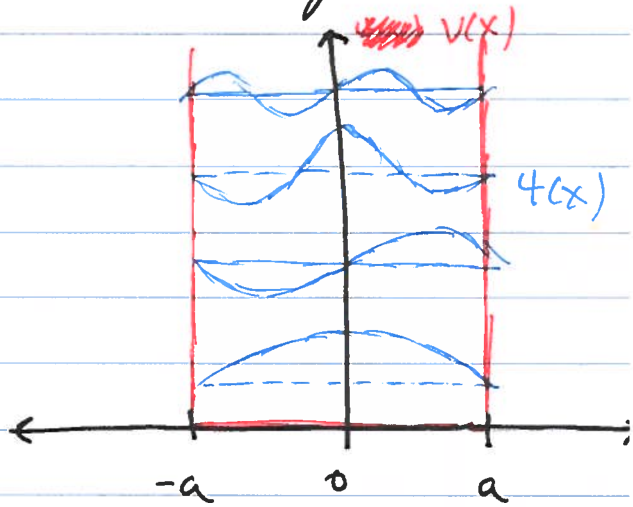
$k_n = \frac{n\pi}{\ }$ with $n = 1, 2, 3, \ldots$

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n} \rightarrow$$ eigenfunctions are only obtained if integer or half-integer # of wavelengths fits in well

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{8ma^2} = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad n = 1,2,3$$

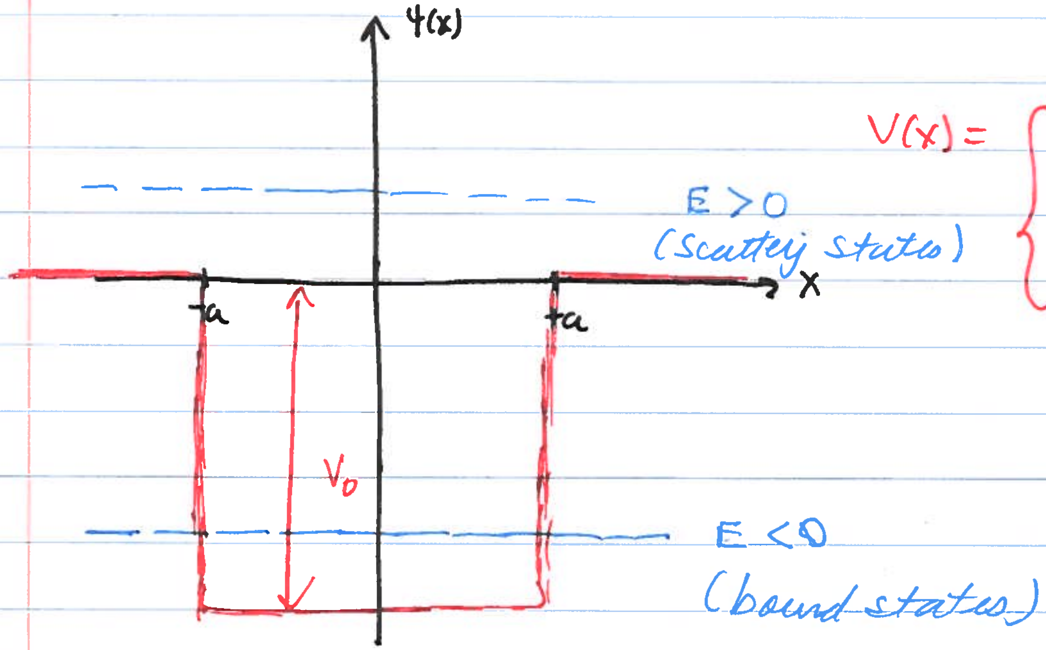
$\Longrightarrow$ Infinite # of bound states (discrete energy levels)





ψ_n has $(n-1)$ nodes

B. Finite Square Well



$$V(x) = \begin{cases} -V_0, & |x| < \\ 0, & |x| > \end{cases}$$

Case I: $E < 0$ (but no $< -V_0$)

$$\frac{d^2\psi(x)}{dx^2} + \alpha^2\psi(x) = 0 \qquad \alpha = \left[\frac{2m}{\hbar^2}(V_0 + E)\right]^{1/2}$$

$$= \left[\frac{2m}{\hbar^2}\left(V_0 - |E|\right)\right]^{1/2}$$

for $|x| < a$ binding energy

Outside the well

$$\frac{d^2\psi(x)}{dx^2} - \beta^2\psi(x) = 0 \qquad \beta = \left(\frac{-2mE}{\hbar^2}\right)^{1/2}$$

Again, since we have a symmetric potential, we can divide into even and odd symmetry solutions.

Even solutions.

$$\psi(x) = A\cos(\alpha x) \quad \text{for} \quad x \in [0,a]$$
$$\psi(x) = Ce^{-\beta x} \quad \text{for} \quad x > a$$

Apply boundary condition at $x=a$

$$A\cos(\alpha a) = Ce^{-\beta a}$$
$$-\alpha A\sin(\alpha a) = -\beta Ce^{-\beta a} \Rightarrow \alpha\tan(\alpha a) = \beta$$

Odd solutions:

$$\psi(x) = B\sin(\alpha x) \quad x \in [0,a]$$
$$\psi(x) = Ce^{-\beta x} \quad x > a$$

with boundary conditions $\alpha\cot(\alpha a) = -\beta$

Graphical method of solution:

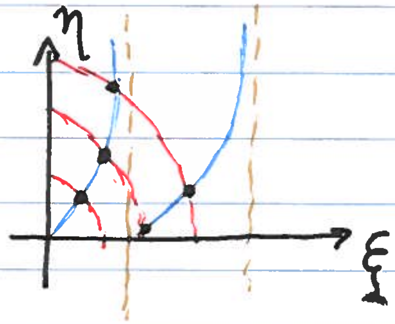
let $\xi = \alpha a$, $\eta = \beta a$

$\xi \tan \xi = \eta$ (even solutions)

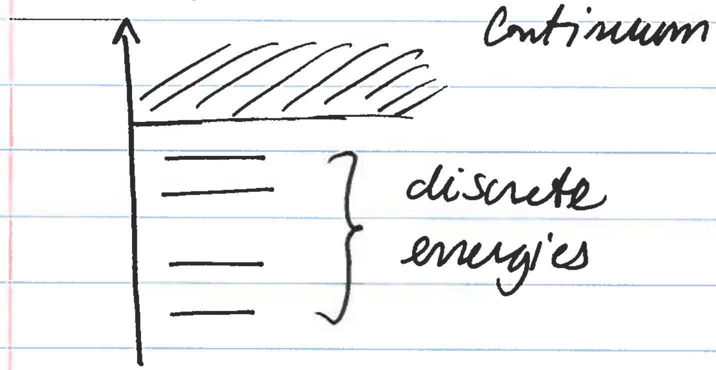
$\xi \cot \xi = -\eta$ (odd solutions)

Note: $\xi^2 + \eta^2 = \gamma^2$ where $\gamma = \left(\frac{2 m V_0 a^2}{\hbar^2} \right)$

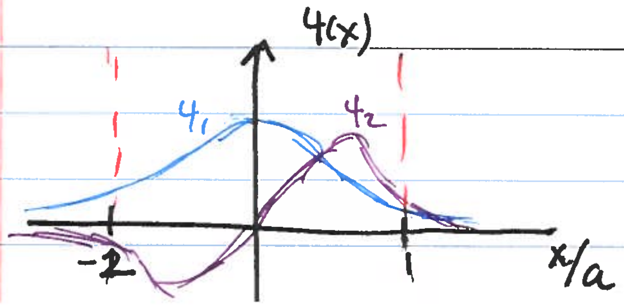
Can plot



Energy spectrum



* • Note: The deeper the well the more the # of bound states.



* Wavefunctions leak outside of finite barrier!

Case II $\quad E > 0$

- In the external region $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < \\ Ce^{ikx}, & x > a \end{cases}$

for a left incident particle

$$k = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$$

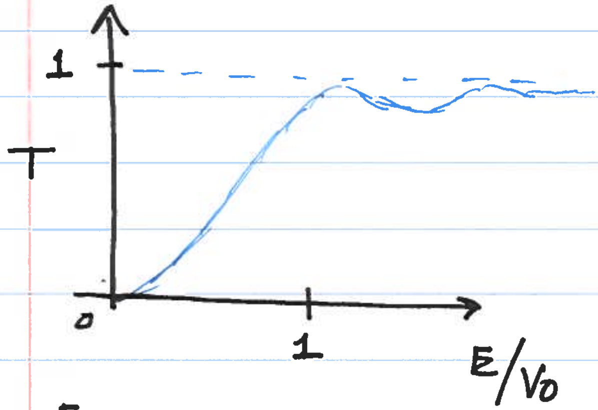
- In the internal region $\psi(x) = Fe^{i\alpha x} + Ge^{-i\alpha x}$

$$\alpha = \left[\frac{2m}{\hbar^2}(V_0 + E)\right]^{1/2}$$

Very similar situation to the potential barrier.

Difference here is that you have more energy available for k in the notch than the barrier.

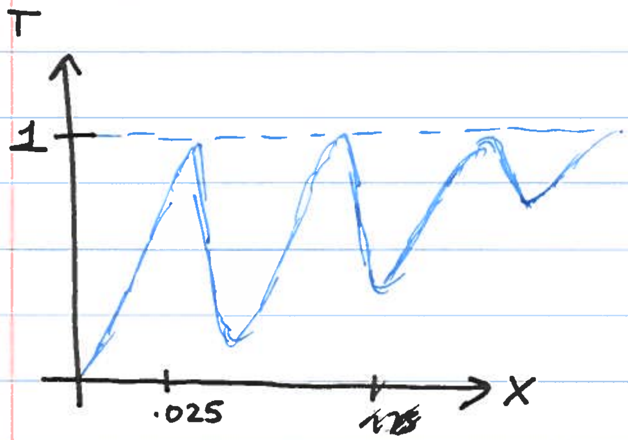
$$\mathbb{R} = \left[1 + \frac{4E(V_0+E)}{V_0^2 \sin^2(\alpha L)}\right]^{-1} \qquad \mathbb{T} = \left[1 + \frac{V_0^2 \sin^2(\alpha L)}{4E(V_0+E)}\right]^{-}$$



Shallow well

$E=0,\ T=0$

$\gamma = 10$ $\qquad \lambda_{dB} = \frac{2\pi}{\alpha}$ satisfies resonance condition



$\gamma = 100$ deep well with resonances.

max $\Rightarrow$ $\alpha L = n\pi$

min $\Rightarrow$ $\alpha L = (2n+1)\pi/2$ $\qquad n = 1, 2, 3$