Problem Set #7 Evan Imata

1. The Accelerating Universe and the Cosmic Event Horizon

OBSERVATIONS HAVE INDICATED THAT not only is our universe expanding, but that this expansion is *accelerating*. This can be explained by invoking a new kind of substance which has negative pressure. One such example is the cosmological constant, which has pressure $P = -\epsilon$ (which implies equation of state parameter $w = P/\epsilon = -1$).

1a) Show that the energy density of a substance with $w = -1$ remains constant even as the scale factor of the universe, $a(t)$, changes.

comment: The constancy of ϵ over cosmic expansion motivates the name "cosmological constant". It is tempting to associate the cosmological constant (called Λ) with the "vacuum energy density" of spacetime itself – that is, to argue that an empty region of space possesses a kind of constant quantum "ground state energy". Unfortunately, simple calculations of this vacuum energy density give estimates that are many many orders of magnitude bigger than the ϵ_Λ inferred from cosmic expansion

$$\varepsilon = \varepsilon_0 a^{-3(1+w)}$$

if $w = -1$: $\varepsilon = \varepsilon_0 a^{-3(1+-1)} = \varepsilon_0$

$\Rightarrow \varepsilon = \varepsilon_0$

so $\frac{d\varepsilon}{da} = 0$

$\Rightarrow$ energy density ε is const. ✓

1b) Show[1] that a substance with $w < -1/3$ leads to cosmic acceleration, $\ddot{a} > 0$.

$\ddot{a}$

[1] For this problem, use the (first) Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc}{R_0^2 a(t)^2}$$

and the fluid equation

$$\frac{d\epsilon}{da} = -\frac{3}{a}\epsilon(1+w) \quad (1)$$

To derive an acceleration equation (sometimes called the Second Friedmann equation)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\epsilon(1+3w) \quad (2)$$

$$\dot{a}^2 = \frac{8\pi G a^2}{3c^2}\varepsilon(t) - \frac{kc}{R_0^2}$$

$$\hookrightarrow\ 2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\cdot \underbrace{\frac{d}{dt}\left(a^2\varepsilon(t)\right)}_{= 2a\varepsilon\frac{da}{dt} + a^2\frac{d\varepsilon}{dt}}$$

$$\frac{2\dot{a}\ddot{a}}{\dot{a}} = \frac{8\pi G}{3c^2}\left(\frac{2a\varepsilon\frac{da}{dt}}{\dot{a}} + \frac{a^2\frac{d\varepsilon}{dt}}{\dot{a}}\right)$$

$$2\ddot{a} = \frac{8\pi G}{3c^2}\left(2a\varepsilon + a^2\cdot \underbrace{\cancel{\frac{d\varepsilon}{dt}\frac{dt}{da}}}_{= d\varepsilon/da}\right)$$

$$\ddot{a} = \frac{4\pi G}{3c^2}\left(2a\varepsilon - 3a\varepsilon(1+w)\right)$$

$$\ddot{a} = \frac{4\pi G}{3c^2} \cdot a\varepsilon \left(2 - 3(1+w)\right)$$

$$2-3-3w = -1-3w$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \varepsilon (1+3w)$$

showing that $\ddot{a}>0$ implies that $w<-\frac{1}{3}$

$$\ddot{a}>0 \implies -\frac{4\pi G}{3c^2}\varepsilon(1+3w) > 0$$

$$\hookrightarrow 1+3w<0$$

$$3w<-1$$

$$\boxed{w<-\frac{1}{3}} \checkmark$$

c) $w=-1$

flat universe $\implies \Omega_0 = 1$

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\underbrace{\frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,r}}{a^4}}_{\text{eliminate b/c we only care about } \Lambda} + \Omega_{0,\Lambda} + \cancel{\frac{(1-\Omega_0)}{a^2}}^{0}\right]$$

$$\implies \frac{\dot{a}^2}{a^2} = H_0^2\Omega_{0,\Lambda} \rightarrow \frac{\dot{a}}{a} = H_0\sqrt{\Omega_{0,\Lambda}} \rightarrow \frac{da}{a} = H_0\Omega_{0,\Lambda}^{1/2}\,dt$$

$$\int_a^{a(t_0)} \frac{da}{a} = H_0\Omega_{0,\Lambda}^{1/2}\int_t^{t_0} dt \implies \cancel{\ln(a(t_0))}^{a(t_0)=1} - \ln(a) = H_0\Omega_{0,\Lambda}^{1/2}(t_0-t)$$

$$\implies \ln(a(t)) = H_0\Omega_{0,\Lambda}^{1/2}(t-t_0)$$

$$\rightarrow \boxed{a(t) = e^{H_0 \overset{\sim 1}{\Omega_{0,\Lambda}^{1/2}}(t-t_0)} = e^{H_0(t-t_0)}}$$

2) $\Omega_{m,0} > \Omega_{\Lambda,0}$

$$1 + z = \frac{1}{a(t_e)}$$

- $\Omega_m = \dfrac{\Omega_{0,m}}{a^3}$
- $\Omega_\Lambda = \Omega_{0,\Lambda}$

$$\Omega_m = \Omega_\Lambda$$

$$\Rightarrow \frac{\Omega_{0,m}}{a^3} = \Omega_{0,\Lambda}$$

$$a^3 = \frac{\Omega_{0,m}}{\Omega_{0,\Lambda}}$$

$$a = \left(\frac{\Omega_{0,m}}{\Omega_{0,\Lambda}}\right)^{1/3}$$

$$z = \frac{1}{a(t_e)} - 1$$

$$= \left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/3} - 1$$

$$\boxed{z_m \gtrsim 0.33}$$

e) 2nd Friedmann Eq: sum of ε_m and ε_Λ

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\sum \varepsilon(1+3w)$$

$$\varepsilon = \varepsilon_0 = \varepsilon_{c,0}\,\Omega_0 = \frac{3c^2 H_0^2}{8\pi G}\,\Omega_0$$

$$= -\frac{4\cancel{\pi G}}{\cancel{3c^2}}\cdot\frac{\cancel{3c^2}H_0^2}{8\cancel{\pi G}}\,\Omega_0\,(1-3)$$

$$= H_0^2\,\Omega_0$$

$$\underset{0}{\frac{\ddot{a}}{a}} = -\frac{4\pi G}{3c^2}\underbrace{\sum\varepsilon(1+3w)}_{(\varepsilon_m + \varepsilon_\Lambda(-2))}$$

- $\varepsilon_m = \dfrac{\varepsilon_{m,0}}{a^3}$
- $\varepsilon_\Lambda = \varepsilon_{\Lambda,0}$

$$\Rightarrow\ 0 = \varepsilon_m - 2\varepsilon_\Lambda$$

$$0 = \frac{\varepsilon_{m,0}}{a^3} - 2\varepsilon_{\Lambda,0}$$

$$2\varepsilon_{\Lambda,0} = \frac{\varepsilon_{m,0}}{a^3}$$

$$2\,\cancel{\varepsilon_{c,0}}\,\Omega_{0,\Lambda} = \cancel{\varepsilon_{c,0}}\,\Omega_{0,m}\,/a^3$$

$$a^3 = \frac{\Omega_{0,m}}{2\Omega_{0,\Lambda}} \Rightarrow a = \left(\frac{\Omega_{0,m}}{2\Omega_{0,\Lambda}}\right)^{1/3}$$

$$1+z = \frac{1}{a} \Rightarrow z = \left(\frac{2\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/3} - 1$$

$$\boxed{z_{accelerate} \gtrsim 0.67}$$

f) $a(t) = e^{H_0(t-t_0)}$

FRW Metric:

$$\underbrace{ds^2}_{=0} = -c^2 dt^2 + a^2(t)\, dr^2$$

$$c^2 dt^2 = a^2 dr^2$$

$$c\,dt = -a\,dr$$

$$c\int_{t_0}^{\infty} \underbrace{e^{H_0(t_0-t)}}_{=e^{H_0 t_0}\, e^{-H_0 t}}\, dt = -\int_{r_H}^{0} dr$$

$$\Rightarrow c\, e^{H_0 t_0}\left(0 + \frac{1}{H_0} e^{-H_0 t_0}\right) = r_H$$

$$0 + \frac{c}{H_0} = r_H$$

$$\boxed{r_H = \frac{c}{H_0}}$$

g) $r = (1-\delta) r_H \quad , \quad \delta \ll 1$

$a(t_0) = a(t_e) \quad , \quad a(t_{obs}) = a(t)$

$$c\int_{t_e}^{t_{obs}} e^{H_0(t_0-t)}\, dt = -\int_{(1-\delta)r_H}^{0} dr$$

$$c\,e^{H_0 t_0}\int_{t_e}^{t_{obs}} e^{-H_0 t}\,dt = (1-\delta)r_H$$

$$c\,e^{H_0 t_0}\left[-\frac{1}{H_0}e^{-H_0 t}\right]_{\underset{\uparrow}{t_e}\atop =t_0}^{t_{obs}\swarrow =t} = (1-\delta)r_H$$

$$\Rightarrow c\,e^{H_0 t_0}\left[-\frac{1}{H_0}e^{-H_0 t} + \frac{1}{H_0}e^{-H_0 t_0}\right] = (1-\delta)r_H$$

$$\frac{c}{H_0} - \frac{c}{H_0}\cdot\frac{e^{H_0 t_0}}{e^{H_0 t}} = (1-\delta)r_H$$

$$= \cancel{\frac{c}{H_0}}\left(1 - \frac{1}{a(t)}\right) = (1-\delta)r_H = (1-\delta)\cancel{\frac{c}{H_0}}$$

$$1 - \frac{1}{a} = 1-\delta$$

$$1 - \frac{1}{1+z} = 1-\delta$$

$$\delta = \frac{1}{1+z}$$

$$\Rightarrow z+1 = \frac{1}{\delta}$$

$$\boxed{z = \frac{1}{\delta} - 1}$$

$$1+z = \frac{a(t_0)}{a(t_e)}$$

$$\frac{a(t)}{a(t_0)}$$

$$a(t) = a(t_0)(1+z)$$

$$\frac{1}{a(t)} = \frac{1}{1+z}$$

$$1+z = \frac{a(t)}{a(t_0)}$$

$$1+z = a$$

$$\frac{1}{a} = \frac{1}{1+z}$$

2. Phantom Energy and the Big Rip

a) $\frac{d\varepsilon}{da} = -\frac{3}{a}\varepsilon(1+\omega)$

if ε increases, $\frac{d\varepsilon}{da} > 0$

$\hookrightarrow -\frac{3}{a}\varepsilon(1+\omega) > 0$

$1+\omega < 0$

$\Rightarrow \boxed{\omega < -1}$ it is @ $\omega < -1$ in which ε increases as universe expands

b)
- flat $\Rightarrow \Omega_0 = 1$
- $\Omega_{w,0} + \Omega_{m,0}$
- $w = -5/3$

- $\Omega_0 = \sum_i \Omega_{i,0}$
- $\frac{H^2}{H_0^2} = \sum_i \frac{\Omega_{i,0}}{a^{3(1+w_i)}} + \frac{1-\Omega_0}{a^2}$

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\overset{\text{ignore}}{\cancel{\frac{\Omega_{0,m}}{a^3}}} + \underbrace{\frac{\Omega_{0,w}}{a^{3(1+w)}}}_{=a^{3(-2/3)} = a^{-2}} + \overset{0}{\cancel{\frac{(1-\Omega_0)}{a^2}}}\right]$$

$$\frac{\dot{a}^2}{a^2} = H_0^2 a^2 \Omega_{0,w} \rightarrow \frac{\dot{a}^2}{a^4} = H_0^2 \Omega_{0,w}$$

$$\frac{\dot{a}}{a^2} = H_0 \Omega_{0,w}^{1/2} \Rightarrow \int_a^{a(t_0)} \frac{da}{a^2} = H_0 \Omega_{0,w}^{1/2} \int_t^{t_0} dt$$

$$-\frac{1}{a}\Big|_a^{a(t_0)} = H_0 \Omega_{0,w}^{1/2}(t_0 - t)$$

$$-\frac{1}{1} + \frac{1}{a(t)} = H_0 \Omega_{0,w}^{1/2}(t_0 - t)$$

$$\frac{1}{a(t)} = H_0 \Omega_{0,w}^{1/2} (t_0 - t) + 1$$

$$\boxed{a(t) = \left[H_0 \Omega_{0,w}^{1/2} (t_0 - t) + 1 \right]^{-1}}$$

c) Blows up when $\left[H_0 \Omega_{0,w}^{1/2} (t_0 - t) + 1 \right] = 0$

$$H_0 \Omega_{0,w}^{1/2} (t_0 - t_{rip}) = -1$$

$$t_0 - t_{rip} = -\frac{1}{H_0 \Omega_{0,w}^{1/2}}$$

$$\Rightarrow \boxed{t_{rip} = t_0 + \frac{1}{H_0 \Omega_{w,0}^{1/2}}}$$

d) if $\Omega_{m,0} \sim 0.3$, then $\Omega_{w,0} \sim 0.7$ if the rest of the energy density is phantom energy

$$\Delta t = t_{rip} - t_0 = \frac{1}{H_0 \Omega_{w,0}^{1/2}}$$

$$H_0 = 70 \frac{\cancel{km}}{s\, Mpc} \times \frac{1000 \cancel{m}}{1 \cancel{km}} \times \frac{1 Mpc}{3.086 \times 10^{22} \cancel{m}} = 2.268 \times 10^{-18} s^{-1}$$

$$\Rightarrow \boxed{\Delta t = 5.26 \times 10^{17} s \sim 1.67 \times 10^{10} yrs.}$$

time until big rip

e) $v = \sqrt{GM/R}$, $T = \frac{2\pi R}{v} \Rightarrow \frac{T}{2\pi} = \frac{R}{v}$

$$m \frac{\ddot{a}}{a} R = \frac{GMm}{R^2} \rightarrow \frac{\ddot{a}}{a} = \frac{GM}{R^3} = \frac{v^2}{R^2} = \frac{4\pi^2}{T^2}$$

$$\frac{GM}{R} = v^2$$

2nd Friedmann Eq:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \varepsilon (1 + 3\underbrace{w}_{w=-5/3})$$

$$= -\frac{4\pi G}{3c^2} \varepsilon(-4) = \frac{16\pi G}{3c^2} \varepsilon = \frac{16\pi G}{3c^2} \varepsilon_0 a^2$$

$$\varepsilon_0 = \varepsilon_{c,0} \Omega_0 = \frac{3c^2 H_0^2}{8\pi G} \Omega_0$$

$$\Rightarrow \frac{16\cancel{\pi G}}{\cancel{3c^2}} \cdot \frac{\cancel{3c^2} H_0^2}{8\cancel{\pi G}} \Omega_0 a^2 = 2H_0^2 \underbrace{\Omega_0}_{=1} a^2$$

$$= 2H_0^2 a^2 = \frac{4\pi^2}{T^2}$$

$$a H_0 \sqrt{2} = \frac{2\pi}{T} \rightarrow a = \frac{1}{H_0} \cdot \frac{2\pi}{T\sqrt{2}}$$

$$\Rightarrow \frac{1}{a} = H_0 \cdot \frac{T\sqrt{2}}{2\pi}$$

relating $\frac{1}{a}$ term to part b:

$$\frac{1}{a} = H_0 \Omega_{0,w}^{1/2} (t_0 - t) + 1$$

$$\Rightarrow \Omega^{1/2} t_0 - \Omega^{1/2} t + \frac{1}{H_0} = \frac{\sqrt{2}}{2\pi} T$$

abbreviate $\Omega_{w,0}$ as Ω for my sake

$$\rightarrow t_0 = t - \frac{1}{\Omega^{1/2} H_0} + \frac{1}{\Omega^{1/2}} \frac{\sqrt{2}}{2\pi} T$$

Substitute into eq. from part c:

$$t_{rip} = t - \cancel{\frac{1}{\Omega^{1/2} H_0}} + \frac{1}{\Omega^{1/2}} \frac{\sqrt{2}}{2\pi} T + \cancel{\frac{1}{\Omega^{1/2} H_0}}$$

$$\Rightarrow t_{rip} = t + \frac{1}{\Omega^{1/2}} \frac{\sqrt{2}}{2\pi} T$$

$$t = t_{rip} - \frac{1}{\underbrace{\Omega^{1/2}}} \frac{\sqrt{2}}{2\pi} T$$

Near the big rip, phantom energy dominates and $\Omega_{w,0} \sim 1$

$$\Rightarrow \boxed{t = t_{rip} - \frac{\sqrt{2}}{2\pi} T} \checkmark$$