Problem Set 6 Evan Imata

1. Redshift and Cosmological Time Dilation

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)}, \qquad a(t_0)=1$$

- emitted @ t_e and $t_e + \Delta t_e$
- received @ t_0 and $t_0 + \Delta t_0$
- $\Delta t_0 > \Delta t_e$

a) FRW Metric:

$$ds^2 = -c^2 dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2)$$

- lightlike $\Rightarrow ds = 0$
- i guess $d\Omega^2 = 0$? b/c radial

$$\Rightarrow 0 = -c^2 dt^2 + a(t)^2 dr^2$$

$$c^2 dt^2 = a(t)^2 dr^2 \Rightarrow c\,dt = \pm a(t)\, dr$$

first pulse:

$$-\int_{t_e}^{t_0} \frac{c}{a(t)} dt = \int_r^0 dr \rightarrow \int_{t_e}^{t_0} \frac{c}{a(t)} dt = r$$ ✓

second pulse:

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c}{a(t)}\,dt = r$$ ✓

b) $$\int_{t_e}^{t_0} \frac{c}{a(t)}\,dt = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{c}{a(t)}\,dt$$

$$= \int_{t_e}^{t_e+\Delta t_e} \frac{\cancel{c}}{a(t)}\,dt + \cancel{\int_{t_e+\Delta t_e}^{t_0} \frac{c}{a(t)}\,dt} = \cancel{\int_{t_e+\Delta t_e}^{t_0} \frac{c}{a(t)}\,dt} + \int_{t_0}^{t_0+\Delta t_0} \frac{\cancel{c}}{a(t)}\,dt$$

$$\Rightarrow \boxed{\int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)}}$$ ✓

c) $a(t)$ essentially constant

$$\Rightarrow \frac{1}{a(t_e)}(\cancel{t_e} + \Delta t_e - \cancel{t_e}) = \underbrace{\frac{1}{a(t_0)}}_{=1}(\cancel{t_0} + \Delta t_0 - \cancel{t_0})$$

$$\boxed{\frac{1}{a(t_e)}\Delta t_e = \Delta t_0}$$ ✓

d) • Redshift of $\frac{\lambda_0}{\lambda_e} = \frac{1}{a(t_e)}$

• $\lambda = cT$

$$z = (\lambda_0 - \lambda_e)/\lambda_e$$

$$1 + z = \cancel{1} + \frac{\lambda_0}{\lambda_e} - \cancel{\frac{\lambda_e}{\lambda_e}} = \frac{1}{a(t_e)}$$ ✓

e) • $ds^2 = \underbrace{-c^2}_{g_{tt}} dt^2 + \underbrace{a^2(t)}_{g_{rr}} dr^2$ $\qquad \frac{P_r}{m} = a^2 \frac{dr}{d\tau}$

• $ds^2 = -c^2 d\tau^2$

• $\frac{E}{m} = -c^2 \frac{dt}{d\tau} \Rightarrow \frac{dt^2}{d\tau^2} = \frac{E^2}{m^2c^4}$

• let $\frac{P_r}{m} = a^2 \frac{dr}{d\tau} \Rightarrow \frac{dr^2}{d\tau^2} = \frac{P_r^2}{m^2a^4}$

$$\rightarrow d\tau^2 = dt^2 - \frac{a^2(t)}{c^2} dr^2$$

$$1 = \frac{dt^2}{d\tau^2} - \frac{a^2(t)}{c^2} \frac{dr^2}{d\tau^2}$$

$$1 = \frac{E^2}{m^2c^4} - \frac{a^2}{c^2} \cdot \frac{P_r^2}{m^2a^4} = -\frac{E^2}{m^2c^4} - \frac{P_r^2}{a^2m^2c^2}$$

$$\frac{E^2}{m^2c^4} = 1 + \frac{P_r^2}{a^2m^2c^2}$$

$$\hookrightarrow \boxed{E^2 = (mc^2)^2 + \frac{c^2}{a^2} \cdot P_r^2}$$ ✓

2. A Matter Filled Universe

- Friedmann Equation:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{kc^2}{R_0^2 a^2}$$

- Fluid Equation:

$$\frac{d\epsilon}{dt} = -3\frac{\dot{a}}{a}(\epsilon + P)$$

- Equation of State:

$$P = w\epsilon$$

a)

$$\frac{d\epsilon}{dt} = -3\frac{\dot{a}}{a}(\epsilon + w\epsilon) = -3\frac{\dot{a}}{a}\epsilon(1+w)$$

$$\int_{\epsilon_0}^{\epsilon}\frac{d\epsilon}{\epsilon} = -3(1+w)\int_{a(t_0)}^{a(t)}\frac{da}{a}$$

$a(t_0) = 1$

$$\ln\left(\frac{\epsilon}{\epsilon_0}\right) = -3(1+w)\ln\left(a(t)\right)$$

$$\Rightarrow \boxed{\epsilon = \epsilon_0 a^{-3(1+w)}}$$

b)

- $\Omega_0 = \frac{\epsilon_0}{\epsilon_{c,0}}$, $\epsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G}$, $H_0 = \frac{\dot{a}(t_0)}{a(t_0)}$

$$\frac{kc^2}{R_0^2 a^2} = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\dot{a}^2}{a^2}$$

$$\frac{k}{R_0^2} = \frac{8\pi G a^2}{3c^4}\epsilon(t) - \frac{\dot{a}^2}{c^2}$$

$$= \frac{8\pi G a^2}{3c^4}\epsilon_0 a^{-3(1+w)} - \frac{\dot{a}^2}{c^2}$$

$$\epsilon_0 = \epsilon_{c,0}\Omega_0 = \frac{3c^2H_0^2}{8\pi G}\Omega_0$$

$$\Rightarrow \frac{K}{R_0^2} = \frac{H_0^2 a^2}{c^2}\Omega_0 a^{-3(1+w)} - \frac{\dot{a}^2}{c^2} \quad \dot{a}^2 = H_0^2$$

$$= \frac{H_0^2}{c^2}\left(\Omega_0 a^{2-3(1+w)} - 1\right)$$

$a = a(t_0) = 1$

$$\Rightarrow \boxed{\frac{K}{R_0^2} = \frac{H_0^2}{c^2}\left(\Omega_0 - 1\right) = -\frac{H_0^2}{c^2}\left(1-\Omega_0\right)} \checkmark$$

c)

- $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{Kc^2}{R_0^2a^2}$
- $\epsilon = \epsilon_0 a^{-3(1+w)}$
- $\epsilon_0 = \epsilon_{c,0}\Omega_0$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\epsilon_{c,0}\Omega_0 a^{-3(1+w)} - \frac{Kc^2}{R_0^2a^2}$$

$$= \frac{\cancel{8\pi G}}{\cancel{3c^2}}\cdot\frac{\cancel{3c^2}H_0^2}{\cancel{8\pi G}}\Omega_0 a^{-3(1+w)} - \frac{Kc^2}{R_0^2a^2}$$

$$= H_0^2\Omega_0 a^{-3(1+w)} - \frac{Kc^2}{R_0^2a^2}$$

$$\frac{K}{R_0^2} = -\frac{H_0^2}{c^2}(1-\Omega_0)$$

$$= H_0^2\Omega_0 a^{-3(1+w)} + \frac{H_0^2}{\cancel{c^2}}\cdot(1-\Omega_0)\cdot\frac{\cancel{c^2}}{a^2}$$

$$\Rightarrow \boxed{\frac{\dot{a}^2}{a^2} = H_0^2\left[\frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2}\right]}$$

d) • $\Omega_0 = \sum_i \Omega_{i,0}$

• $\frac{H^2}{H_0^2} = \sum_i \frac{\Omega_{i,0}}{a^{3(1+w_i)}} + \frac{1-\Omega_0}{a^2}$

• consider flat universe w/ $\Omega_0 = 1$, w/ pressureless matter $w=0$

• only matter $\Rightarrow$ no dark energy or radiation

solve for $a(t)$

$\Omega_0 = 1 = \sum_i \Omega_{0,i}$

$$\frac{\dot{a}^2}{a^2} = H_0^2\left[\frac{1}{a^3}\right] \Rightarrow a^{1/2}da = H_0 dt$$

$$\rightarrow \int_0^a a^{1/2}da = H_0\int_0^t dt \rightarrow \frac{2}{3}a^{3/2} = H_0 t$$

$$\boxed{a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}}$$

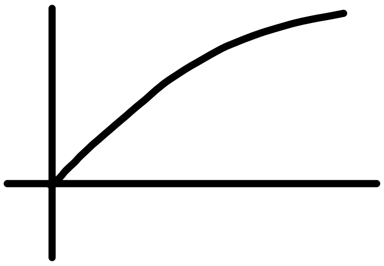
e) combining eqs. from parts c and d:

since $a(t_0) = 1$, $1 = \left(\frac{3}{2}H_0 t_0\right)^{2/3}$

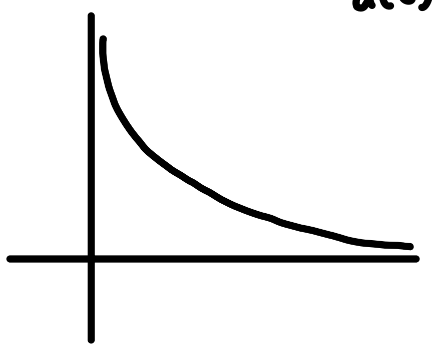
$1 = \left(\frac{3}{2}H_0 t_0\right)^{2/3}$

$$\boxed{t_0 = \frac{2}{3}\frac{1}{H_0}}$$

f) $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$ => $a(t)$ is always increasing



$\dot{a}(t) = \frac{2}{3} t^{-1/3} t_0^{-2/3}$ => $\dot{a}(t)$ is always decreasing, $\dot{a}(t) \propto t^{-1/3}$



since $a(t)$ is always increasing and $\dot{a}(t)$ is always decreasing, the flat, matter-dominated universe is always expanding and decelerating

3. Radiation Universe

- for radiation:

  $p = \epsilon/3, \ \omega = 1/3$

a)
- $\Omega_0 = 1$
- $\omega = 1/3$
- Solve for $a(t)$

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\frac{\Omega_0}{a^{3(1+w)}} + \cancelto{0}{\frac{1-\Omega_0}{a^2}} \right]$$

$$= H_0^2 \cdot \frac{1}{a^4}$$

$$\Rightarrow \frac{\dot{a}}{a} = \frac{H_0}{a^2}$$

$$a \cdot \frac{da}{dt} = H_0$$

$$a\,da = H_0\,dt \quad \rightarrow \quad \int_0^a a\,da = H_0 \int_0^t dt$$

$$\Rightarrow \frac{1}{2} a^2 = H_0 t$$

$$a^2 = 2 H_0 t$$

$$\boxed{a = (2 H_0 t)^{1/2}}$$

b) $a(t_0) = 1$

$$\Rightarrow 1 = (2H_0 t_0)^{1/2}$$

$$1 = 2H_0 t_0$$

$$\boxed{t_0 = \frac{1}{2H_0}}$$ smaller than matter-dominated universe

c) $$\frac{\dot{a}^2}{a^2} = H_0^2\left[\frac{\Omega_0}{a^{3(1+w)}} + \frac{1-\Omega_0}{a^2}\right], \quad (\Omega_0 > 1)$$

$\dot{a} = 0$ @ extrema

$$0 = \frac{\Omega_0}{a^4} + \frac{1-\Omega_0}{a^2}$$

$$0 = \frac{\Omega_0}{a^2} + 1 - \Omega_0$$

$$\Omega_0 - 1 = \frac{\Omega_0}{a^2}$$

$$a^2(\Omega_0 - 1) = \Omega_0$$

$$a^2 = \frac{\Omega_0}{\Omega_0 - 1}$$

$$\Rightarrow \boxed{a(t) = \left(\frac{\Omega_0}{\Omega_0 - 1}\right)^{1/2}}$$

@ the turnaround point

4. The Observable Universe

- $d_{max} = ct_0$

a) $\Omega_0 = 1$, $w = 0$

FRW Metric:

$$\cancelto{0}{ds^2} = -c^2 dt^2 + a(t)^2 (dr^2 + \cancel{r^2 d\Omega^2})$$

$$\hookrightarrow \quad c^2 dt^2 = a(t)^2 dr^2$$

$$c\,dt = a(t)\,dr$$

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\frac{\Omega_0}{a^{3(1+w)}} + \cancelto{0}{\frac{1-\Omega_0}{a^2}} \right]$$

$$\hookrightarrow \quad \frac{\dot{a}^2}{a^2} = H_0^2 \cdot \frac{1}{a^3}$$

$$\Rightarrow \quad a^2\, \dot{a}^2 = H_0^2$$

$$a^{1/2} da = H_0 dt \rightarrow \int_0^a a^{1/2} da = H_0 \int_0^t dt$$

$$\frac{2}{3} \underbrace{a^{3/2}}_{a(t_0)=1} = H_0 t_0$$

$$a(t) = \left(\frac{3}{2} H_0 t \right)^{2/3}$$

$$c\,dt = \pm a(t)\,dr \quad \rightarrow \quad c\,dt = -a(t)\,dr$$

$$c\,dt = -\left(\tfrac{3}{2} H_0 t\right)^{2/3} dr$$

$$c\int_0^{t_0} t^{-2/3}\,dt = \left(\tfrac{3}{2} H_0\right)^{2/3} \int_0^{r_{max}} dr$$

$$c \cdot 3 t_0^{1/3} = \Big(\tfrac{3}{2} \underset{\uparrow\ \frac{1}{t_0}}{H_0}\Big)^{2/3} r_{max}$$

$$c \cdot 3 t_0^{1/3} = \left(\tfrac{3}{2} \cdot \tfrac{1}{t_0}\right)^{2/3} r_{max}$$

$$3ct_0 = \left(\tfrac{3}{2}\right)^{2/3} r_{max}$$

$$\Rightarrow r_{max} = \underbrace{\left(\tfrac{2}{3}\right)^{2/3} \cdot 3}_{=12^{1/3}} \cdot ct_0$$

$$\boxed{r_{max} = ct_0 \sqrt[3]{12} = ct_0 (12)^{1/3}}$$