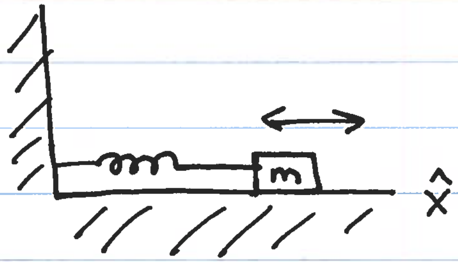
Lecture 13 — The Simple Harmonic Oscillator

137A

Recall a classical oscillator:

Hooke's law: $F = -kx$ — spring constant

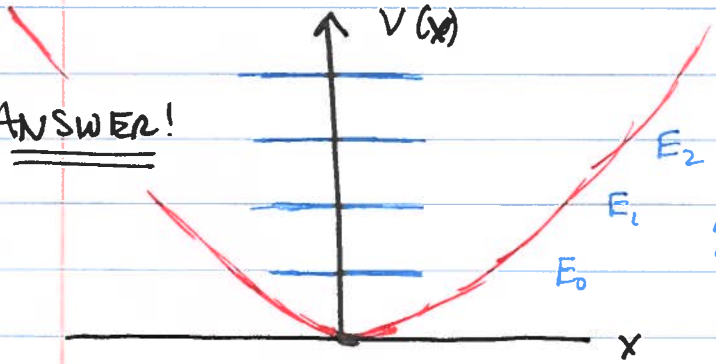


$$V(x) = \frac{1}{2} k x^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

** Useful approximation to many physical systems.

ANSWER!



levels equally spaced by $\hbar\omega$ $(E > 0)$

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} k x^2$$

Schrödinger Eigenvalue Equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2} k x^2 \psi(x) = E\,\psi(x)$$

let's go to dimensionless variables

$$\lambda = \frac{2E}{\hbar\omega}, \quad \xi = \alpha x \qquad \alpha = \left(\frac{mk}{\hbar^2}\right)^{1/4}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{1/2}$$

$$\longrightarrow \frac{d^2\psi(\xi)}{d\xi^2} + (\lambda - \xi^2)\,\psi(\xi) = 0$$

- Consider first asymptotic behavior

As $|\xi| \to \infty$, λ is negligible for finite E

The equation then becomes

$$\left(\frac{d^2}{d\xi^2} - \xi^2\right)\psi(\xi) = 0$$

Thus, for large ξ, $\psi(\xi) = \xi^p e^{\pm \xi^2/2}$

This satisfies the equation for any finite value of p.

* You correct behavior as

We can drop $e^{+\xi^2/2}$ since this blows up at $\xi \to \infty$. (3)

More generally:

$$\psi(\xi) = e^{-\xi^2/2} H(\xi)$$

any function that does not change asymptotic behavior.

Substitute back:

$$\frac{d^2H}{d\xi^2} - 2\xi \frac{dH}{d\xi} + (\lambda - 1) H = 0$$

Hermite Equation!

Solve by power series expansion

- Even states $\psi(-\xi) = \psi(\xi)$, thus $H(-\xi) = H(\xi)$

$$H(\xi) = \sum_{l=0}^{\infty} c_l \xi^{2l} \qquad c_0 \neq 0$$

Substitute:

$$\sum_{l=0}^{\infty} \left[2l(l-1) c_l \xi^{2(l-1)} + (\lambda - 1 - 4l) c_l \xi^{2l} \right] =$$

Think of the two terms as an array or group that must all add to zero in the end.

* This works if the coefficients for all the powers of ξ all go to zero.

Let's rewrite sum: [for term w/ ξ^{2l}]

$$\sum_{l=0}^{\infty} \underbrace{\left[2(l+1)(2l+1)\,c_{l+1} + (\lambda-1-4l)\,c_l\right]}_{=0} \xi^{2l} =$$

$= 0 \rightarrow$ gives us a recursion relation

$$c_{l+1} = \frac{4l+1-\lambda}{2(l+1)(2l+1)}\, c_l$$

* Now... does the series terminate or is it infinite?

Let's consider an infinite series,

for large l $\quad \dfrac{c_{l+1}}{c_l} \sim \dfrac{1}{l}$

Let's assume for a minute that

$$H(\xi) \sim e^{\xi^2} = \sum_\ell \frac{(\xi^2)^\ell}{\ell!} \rightarrow$$ This series will have coefficients $\sim \frac{1}{\ell}$ for large ℓ

In this case, $\psi(\xi) \sim e^{\xi^2} \cdot \xi^p e^{-\xi^2/2} \sim \xi^p e^{+\xi^2/2}$ which blows up as $|\xi| \to \infty$!

Thus, the series for H must terminate and $H(\xi)$ must be a polynomial in the variable ξ^2.

Let the highest power be ξ^{2N} where $N = 0, 1, 2, 3, \ldots$

$$H(\xi) = \sum_{\ell=0}^{\cancel{\infty}} c_\ell \xi^{2\ell}$$ with $c_N \neq 0$ and $c_{N+1} = 0$ to terminate series

Using the recursion relation,

$$\lambda = 4N + 1 \quad , \quad N = 0, 1, 2, \ldots$$
$$= 1, 5, 9, \ldots$$

and

$$\psi(\xi) = e^{-\xi^2/2} H(\xi)$$ ← polynomial of order $2N$ in ξ and even function.

Odd states

When $\psi(-\xi) = -\psi(\xi)$,
$H(-\xi) = -H(\xi)$

Follow same procedure:

$$H(\xi) = \sum_{l=0}^{\infty} d_l \, \xi^{2l+1} \quad , \; d_0 \neq 0$$

The resulting recursion relation is

$$d_{l+1} = \frac{4l+3-\lambda}{2(l+1)(2l+3)} d_l$$

Must terminate series, highest power ξ^{2N+1}, $N = 0, 1, 2 \ldots$

$$\longrightarrow \lambda = 4N+3 \quad , \; N = 0, 1, 2,$$
$$= 3, 7, 11, \ldots$$

Combining both solutions: (convert λ back to E)

$$E_n = (n+\tfrac{1}{2})\hbar\omega \quad , \; n = 0, 1, 2, \ldots$$

* equally spaced by $\hbar\omega$

next

The Wavefunctions are Thus

$$\psi_n(\xi) = e^{-\xi^2/2} H_n(\xi)$$

Hermite polynomials

We can use a generating function

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n}$$

or

$$= e^{\xi^2/2} \left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}$$

eg. $H_0(\xi) = 1$ $\quad H_2(\xi) = 4\xi^2 - 2$

$H_1(\xi) = 2\xi$ $\quad H_3(\xi) = 8\xi^3 - 12\xi$

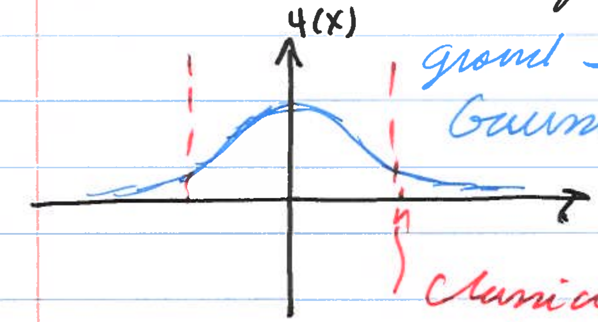
$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$

Normalize wavefunctions, convert back to x

$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{1/2} e^{-\alpha^2 x^2/2} H_n(\alpha x)$$

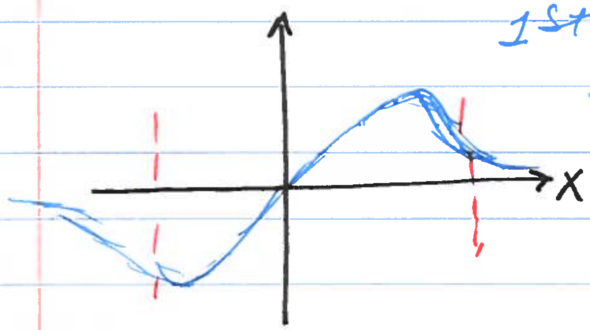
$$\alpha = \left(\frac{mk}{\hbar^2}\right)^{1/4} = \left(\frac{m\omega}{\hbar}\right)^{1/2}.$$

Two lowest wavefunctions



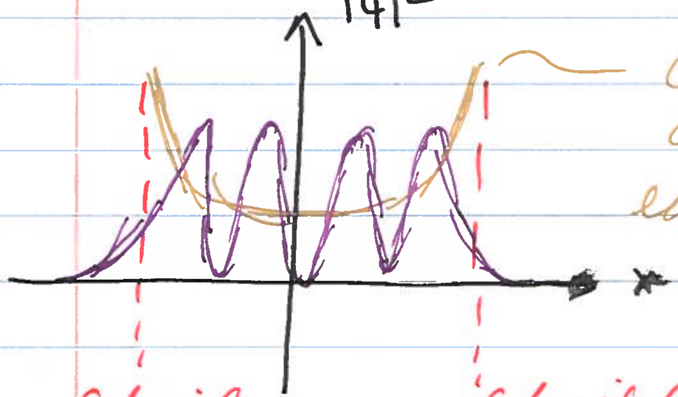
ground state
Gaussian function

classical turning
points → have leakage.



1st excited
state

Classical probability density comparison.



classical probability
density peaks near
edges where oscillator
is the slowest.

classical
turning
pt

classical
turning
point.

• see similarities
for large n between
quantum + classical

• Very differing for n=0