Lecture 20 Spin Angular Momentum

- Elementary particles have an <u>internal</u> degree of freedom that behaves as an <u>angular momentum</u> and is termed <u>spin</u>.

$$\hat{S}: \quad [S_x, S_y] = i\hbar S_z$$
$$[S_y, S_x] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y$$

- Can find simultaneous eigenfunctions of $\hat{S}^2, \hat{S}_z$

$$S^2 |s\, m_s\rangle = s(s+1)\hbar^2 |s\, m_s\rangle$$
$$S_z |s\, m_s\rangle = m_s \hbar |s\, m_s\rangle$$

- Can also express as χ_{s, m_s}

- s can be half-integer or integer valued $(0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots)$

integer spin particles are called bosons
half-integer spin particles are called fermions

- m_s has $(2s+1)$ allowed values: $-s, -s+1, \ldots +s$

<u>Eg.</u> $s = 1$

$$S_z = \hbar \begin{array}{c c} & \begin{array}{ccc} |1,1\rangle & |1,0\rangle & |1,-1\rangle \end{array} \\ \begin{array}{c} \langle 1\,1| \\ \langle 1\,0| \\ \langle 1\,{-1}| \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \end{array}$$

(2)

$$S^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

__Note:__ We use $S_\pm = S_x \pm i S_y$

to calculate S_x, S_y

- The eigenvectors are written as column vecs:

eigenvector	$\chi_{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\chi_{1,0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\chi_{1,-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
eigenvalue of S_z	$+\hbar$	0	$-\hbar$
eigenvalue of S^2 $s(s+1)\hbar^2$	$2\hbar^2$	$2\hbar^2$	$2\hbar^2$

- We now have to add a spin degree of freedom to our wavefunctions.

We use σ to denote one of the $2s+1$ values of S_z : $\psi(\vec{r}, t, \sigma)$

(3)

- A general state is then

$$\Psi(\vec{r}, t, \sigma) = \sum_{m_s=-s}^{+s} \Psi_{m_s}(\vec{r}, t)\, \chi_{s,m_s}$$

wavefunction when $S_z = m_s\hbar$

- $|\Psi_{m_s}(\vec{r}, t)|^2 d\vec{r}$ is the probability density of finding the system (particle) in a volume $d\vec{r}$ about $\vec{r}$ at time t with spin $S_z = m_s\hbar$.

- To find the particle independent of spin state:

$$\sum_{m_s=-s}^{s} |\Psi_{m_s}(\vec{r}, t)|^2 d\vec{r}$$

- To find the particle with S_z of $m_s\hbar$ at time t:

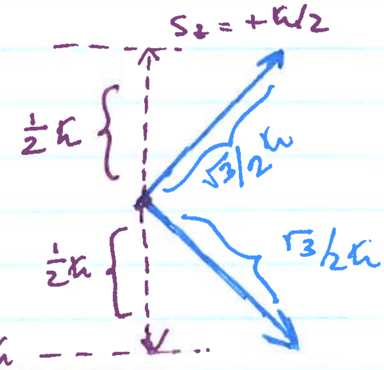
$$\int |\Psi_{m_s}(\vec{r}, t)|^2 d\vec{r}$$

- In certain cases, spin-dependent interactions are negligible, and one can write a separable wavefunction $\Psi(\vec{r}, t, \sigma) = \Psi(\vec{r}, t)\, \chi_s$

Lecture 21 Spin ½ Systems

- Electrons, protons have spin $s = 1/2$

$s = 1/2 \longrightarrow \underbrace{m_s = -1/2,\ 1/2}_{\text{differ by } \hbar}$



Two wavefunctions are:

$$\chi_{\frac{1}{2},\frac{1}{2}} \leftrightarrow |\tfrac{1}{2},\tfrac{1}{2}\rangle \leftrightarrow |\tfrac{1}{2}\rangle \leftrightarrow |\uparrow\rangle \leftrightarrow |+\rangle$$

$$\chi_{\frac{1}{2},-\frac{1}{2}} \leftrightarrow |\tfrac{1}{2},-\tfrac{1}{2}\rangle \leftrightarrow |-\tfrac{1}{2}\rangle \leftrightarrow |\downarrow\rangle \leftrightarrow |-\rangle$$

Several different representations.

- $S_\pm = S_x \pm i S_y$

$S^2 |\tfrac{1}{2}\rangle = (3/4)\hbar^2 |\tfrac{1}{2}\rangle \qquad S_z |\tfrac{1}{2}\rangle = (\hbar/2) |\tfrac{1}{2}\rangle$

$S^2 |-\tfrac{1}{2}\rangle = (3/4)\hbar^2 |-\tfrac{1}{2}\rangle \qquad S_z |-\tfrac{1}{2}\rangle = (-\hbar/2) |-\tfrac{1}{2}\rangle$

$S_+ |\tfrac{1}{2}\rangle = 0$ (top of ladder)

$S_+ |-\tfrac{1}{2}\rangle = \hbar |\tfrac{1}{2}\rangle$ (use formula for raising + lowering operators)

$$S_\pm |s, m_s\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\ |s, m_s \pm 1\rangle$$

$S_- |-\tfrac{1}{2}\rangle = 0$ (bottom of ladder)

$S_- |+\tfrac{1}{2}\rangle = \hbar |-\tfrac{1}{2}\rangle$

- Now, we can calculate S_x, S_y

$$S_x = \frac{S_+ + S_-}{2} \Rightarrow S_x|\tfrac{1}{2}\rangle = \frac{\hbar}{2}|-\tfrac{1}{2}\rangle$$

$$S_y|-\tfrac{1}{2}\rangle = \underbrace{-\frac{i\hbar}{2}|\tfrac{1}{2}\rangle}$$

Note: NOT an eigenequation!

$|\pm\frac{1}{2}\rangle$ are not eigenstates of S_z.

$$S_y = \frac{S_+ - S_-}{2i}$$

$$\Downarrow$$

$$S_y|\tfrac{1}{2}\rangle = -\frac{\hbar}{2i}|-\tfrac{1}{2}\rangle = \frac{i\hbar}{2}|-\tfrac{1}{2}\rangle$$

$$S_y|-\tfrac{1}{2}\rangle = -\frac{i\hbar}{2}|\tfrac{1}{2}\rangle$$

- Matricies

$$[S_z] = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad [S_x] = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[S_y] = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- A general spin state can be written as

$$|\chi\rangle = a|\tfrac{1}{2}\rangle + b|-\tfrac{1}{2}\rangle, \quad |a|^2 + |b|^2 = 1$$

A measurement of S_z yields spin $+\frac{1}{2}$ with prob. $|a|^2$

spin $-\frac{1}{2}$, $|b|^2$

- Derive some useful relations for spin 1/2. ③

$S^2 = \frac{3}{4}\hbar^2 \hat{I}$

$S_x^{\,2} = S_y^{\,2} = S_z^{\,2} = \frac{\hbar^2}{4} I$ (spin² distributed along axes)

$S_\pm^{\,2} = 0$ for spin 1/2

$0 = (S_x \pm i S_y)^2 = \underbrace{S_x^{\,2} - S_y^{\,2}}_{\text{equal}} \pm i (S_x S_y + S_y S_x)$

$\longrightarrow \quad S_x S_y + S_y S_x = 0$ } anti-commutator

$[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}$

$[S_x, S_y]_+ = 0$
$[S_y, S_z]_+ = 0$
$[S_z, S_x]_+ = 0$

when combined with usual commutator,

$\Rightarrow \quad S_i S_j = \frac{i\hbar}{2} S_k$ ✱

<u>Pauli Spin Matrices</u>

$\hat{S} = (\hbar/2)\,\hat{\sigma}$ $\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathrm{Tr}\, \sigma_i = 0$
$\det \sigma_i = 1$

$\longrightarrow$ ✱ $\{\hat{\sigma}, \hat{I}\}$ form a basis for all 2×2 matrices

Lecture 22 Addition of Angular Momentum

- Consider a particle with spin $\vec{S}$ and orbital angular momentum $\vec{L}$.

$$\vec{J} = \vec{L} + \vec{S}$$

$\vec{L}$: only depends on (θ, φ)

$\vec{S}$: only operates on spin variables

Thus $[\vec{L}, \vec{S}] = 0$

- To execute a rotation about $\hat{n}$ by angle α

$$\hat{U}_n(\alpha) = \exp\left(-\frac{i}{\hbar}\alpha\, \hat{n}\cdot\vec{J}\right)$$

- For an <u>isolated system</u>, total angular momentum is conserved.

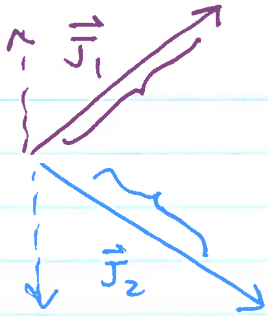
$[\vec{J}, \hat{H}] = 0$!

$\longrightarrow$ can Find common eigenfunctions of $\hat{H}, \hat{J}^2, \hat{J}_z$

$\longrightarrow$ Energy must only depend on j (orientation doesn't matter)

- Eg. Addition of $\vec{J}$ for two particles

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$



Can describe with $\underbrace{J_1^2}_{j_1}, \underbrace{J_{1z}}_{m_1}$

Describe as a product $|j_1 j_2 m_1 m_2\rangle$

Can describe with $\underbrace{J_2^2}_{j_2}, \underbrace{J_{2z}}_{m_2}$

- Since $[\vec{J}_1, \vec{J}_2] = 0$

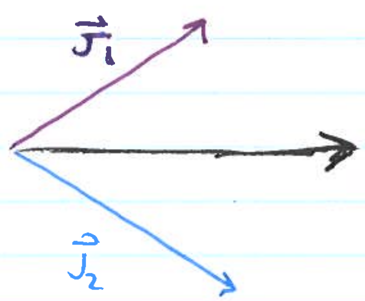
$$|j_1 j_2 m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle$$

- **Note:** Need 4 #'s to describe 2 particles

- For a given value of j_1, j_2, we have $(2j_1+1)(2j_2+2)$ direct product states.

* Let's now consider total J^2 and total J_z.

For many cases, we may not have access to $J_1^2, J_2^2, J_{1z}, J_{2z}$ for all particles.



$\vec{J}_1 + \vec{J}_2 = \vec{J}$ → can be described of J^2 and J_z

$$j_{max} = j_1 + j_2$$
$$j_{min} = |j_1 - j_2|$$

$$\hat{J_z}\,|j_1 j_2 m_1 m_2\rangle = (J_{1z}+J_{2z})\,|j_1 m_1\rangle|j_2 m_2\rangle \quad ③$$

$$= (m_1+m_2)\hbar\,|j_1 j_2 m_1 m_2\rangle$$

$$J^2 = (J_1+J_2)^2 = J_1^2 + J_2^2 + 2\underbrace{J_1\cdot J_2}_{J_{1x}J_{2x} + J_{1y}J_{2y} + J_{1z}J_{2z}}$$

J^2 commutes with J_1^2, J_2^2

~~Thus~~ J^2 does not commute with J_{1z} or J_{2z} since ($J_1\cdot J_2$ has J_x, J_y terms)

$\longrightarrow$ Thus eigenfunctions of J^2 and J_z

* are eigenfunctions of J_1^2 and J_2^2 in general, but not of J_{1z} and J_{2z}

$\Longrightarrow$ Thus a complete description of the system can include

J^2, J_z*, J_1^2, J_2^2 but not $J_1^2 J_z$, $\cancel{J_{1z}}$, J_{2z}

Here we have used the total J and J_z <u>plus</u> two additional descriptors from $\vec{J_1}$ and $\vec{J_2}$.

Thus, we have two possible bases: ④

$|j_1 j_2 m_1 m_2\rangle$ and $|j m j_1 j_2\rangle$

They are connected by the Clebsch-Gordan coefficients:

$$|j m j_1 j_2\rangle = \sum_{m_1, m_2} \underbrace{\langle j_1 j_2 m_1 m_2 | j m\rangle}_{\text{Clebsch-Gordon coefficients}} |j_1 j_2 m_1 m_2\rangle$$

Eg. Addition of two particles with $S = \frac{1}{2}$.

For particle 1, we have $\chi_{\frac{1}{2},\frac{1}{2}}(1)$, $\chi_{\frac{1}{2},-\frac{1}{2}}(1)$

For particle 2, we have $\chi_{\frac{1}{2},\frac{1}{2}}(2)$, $\chi_{\frac{1}{2},-\frac{1}{2}}(2)$

$\lvert S_1\, S_2\;\; M_{S_1}\;\; M_{S_2}\rangle$	Total M_S $(M_{S_1} + M_{S_2})$
$\chi_{\frac{1}{2},\overset{\uparrow}{\frac{1}{2}}}(1)\,\chi_{\frac{1}{2},\overset{\uparrow}{\frac{1}{2}}}(2)$	1
$\chi_{\frac{1}{2},\overset{\uparrow}{\frac{1}{2}}}(1)\,\chi_{\frac{1}{2},\overset{\downarrow}{-\frac{1}{2}}}(2)$	0
$\chi_{\frac{1}{2},\overset{\downarrow}{-\frac{1}{2}}}(1)\,\chi_{\frac{1}{2},\overset{\uparrow}{\frac{1}{2}}}(2)$	0
$\chi_{\frac{1}{2},\overset{\downarrow}{-\frac{1}{2}}}(1)\,\chi_{\frac{1}{2},\overset{\downarrow}{-\frac{1}{2}}}(2)$	-1

The allowed values of the total spin S are 0, 1

For $S=0$

$$\chi_{0,0} = \frac{1}{\sqrt{2}}\left[\chi_{\frac{1}{2},\frac{1}{2}}(1)\,\chi_{\frac{1}{2},-\frac{1}{2}}(2) - \chi_{\frac{1}{2},-\frac{1}{2}}(1)\,\chi_{\frac{1}{2},\frac{1}{2}}(2)\right]$$

(Note: each term has $S=1$)

$\rightarrow$ Anti-symmetric spin singlet

For $S=1$,

$$\chi_{1,1} = \chi_{\frac{1}{2},\frac{1}{2}}(1)\,\chi_{\frac{1}{2},\frac{1}{2}}(2)$$

$$\chi_{1,0} = \frac{1}{\sqrt{2}}\left[\chi_{\frac{1}{2},\frac{1}{2}}(1)\,\chi_{\frac{1}{2},-\frac{1}{2}}(2) + \chi_{\frac{1}{2},-\frac{1}{2}}(1)\,\chi_{\frac{1}{2},\frac{1}{2}}(2)\right]$$

$$\chi_{1,-1} = \chi_{\frac{1}{2},-\frac{1}{2}}(1)\,\chi_{\frac{1}{2},-\frac{1}{2}}(2)$$

$\rightarrow$ Symmetric spin triplet.