1. **Griffiths 2.18**
Solution: Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, and noting that cos is even and sin is odd,
$$Ae^{ikx} + Be^{-ikx} = (A+B)\cos kx + (A-B)i\sin kx, \tag{1}$$
so
$$\begin{aligned} C &= A+B, & A &= \tfrac{1}{2}(C-Di), \\ D &= (A-B)i, & B &= \tfrac{1}{2}(C+Di). \end{aligned} \tag{2}$$
Note that since $A = \overline{B}$, we still have only two (real) degrees of freedom.

2. **Gaussians** (based on G1.3)
A Gaussian distribution, parametrised by μ, σ, is given by
$$\rho(x) = A\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right). \tag{3}$$

(a) Define
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\,. \tag{4}$$
Show that $I = \sqrt{\pi}$.
(Hint: find I^2 by transforming to polar coordinates and then changing variables)
Solution: Substituting $u = r^2$,
$$I^2 = \int e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^{\infty} re^{-r^2}\,\mathrm{d}r = \pi\int_0^{\infty} e^{-u}\,\mathrm{d}u = \pi. \tag{5}$$

(b) Hence find the normalisation constant A.
Solution: Substituting $y = \frac{x-\mu}{\sqrt{2}\sigma}$,
$$1 = \int \rho(x)\,\mathrm{d}x = A\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,\mathrm{d}x = \sqrt{2}\sigma A\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \sqrt{2\pi}\sigma A, \tag{6}$$
so $A = \frac{1}{\sqrt{2\pi}\sigma}$.

(c) Find $\langle x\rangle$, $\langle x^2\rangle$, $\sigma(x)$ of the given Gaussian.
(Hint: $\int x^2 e^{-ax^2}\,\mathrm{d}x = -\frac{\partial}{\partial a}\int e^{-ax^2}\,\mathrm{d}x$)
Solution:
$$\begin{aligned} \langle x\rangle &= \int x\rho(x)\,\mathrm{d}x = \sqrt{2}\sigma A\int (\sqrt{2}\sigma y + \mu)e^{-y^2}\,\mathrm{d}y \\ &= \sqrt{\frac{2}{\pi}}\sigma\int ye^{-y^2}\,\mathrm{d}y + \frac{\mu}{\sqrt{\pi}}\int e^{-y^2}\,\mathrm{d}y = \mu, \end{aligned} \tag{7}$$
since $\int_{-\infty}^{\infty} ye^{-y^2}\,\mathrm{d}y = 0$ as the integrand is odd. Since
$$\int e^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{a}}\int e^{-y^2}\,\mathrm{d}y = \sqrt{\frac{\pi}{a}}, \tag{8}$$
and
$$\int x^2 e^{-ax^2}\,\mathrm{d}x = \int x^2 e^{-ax^2}\,\mathrm{d}x\bigg|_{a=1} = -\frac{\partial}{\partial a}\sqrt{\frac{\pi}{a}}\bigg|_{a=1} = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}\bigg|_{a=1} = \frac{\sqrt{\pi}}{2}, \tag{9}$$

$$\begin{aligned}\langle x^2\rangle &= \int x^2\rho(x)\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int(\sqrt{2}\sigma y+\mu)^2 e^{-y^2}\,\mathrm{d}y \\ &= \frac{2\sigma^2}{\sqrt{\pi}}\int y^2 e^{-y^2}\,\mathrm{d}y + \frac{\mu^2}{\sqrt{\pi}}\int e^{-y^2}\,\mathrm{d}y = \sigma^2+\mu^2,\end{aligned} \tag{10}$$

$$\sigma(x) = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sigma. \tag{11}$$

(d) Sketch the Gaussian. Indicate μ and σ on your sketch or describe the effect of changing them.

3. **G2.20**
Solution:

(a) We are essentially applying the analysis of G2.18 to each term in the sum:

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty}(a_n\sin(n\pi x/a)+b_n\cos(n\pi x/a)) \\ &= \sum_{n=0}^{\infty}\tfrac{1}{2}\left((b_n - ia_n)e^{in\pi x/a} + (b_n+ia_n)e^{-in\pi x/a}\right),\end{aligned} \tag{12}$$

$$c_n = \begin{cases}\frac{1}{2}(b_n - ia_n) & n\ge 0,\\ \frac{1}{2}(b_n+ia_n) & n\le 0.\end{cases} \tag{13}$$

Note that $c_n = \overline{c_{-n}}$, for real input and $c_0 = b_0$ is the constant part of the function (we can always choose $a_0 = 0$ as convention).

(b)

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,\mathrm{d}x &= \frac{1}{2a}\sum_{m=-\infty}^{\infty} c_m\int_{-a}^{a} e^{i(m-n)\pi x/a}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\sum_m c_m\int_{-\pi}^{\pi} e^{i(m-n)\theta}\,\mathrm{d}\theta \\ &= \sum_m c_m\delta_{m,n} = c_n,\end{aligned} \tag{14}$$

since for any integer $n\neq 0$, $\int e^{in\theta}\,\mathrm{d}\theta$ over a range of 2π cancels out, and for $n=0$, is simply 2π. Also

$$\delta_{a,b} = \begin{cases}1 & a=b,\\ 0 & \text{else},\end{cases} \tag{15}$$

is the Kronecker delta function.

(c)

$$\begin{aligned}f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/a} = \frac{1}{a}\sqrt{\frac{\pi}{2}}\sum_{n=-\infty}^{\infty} F(k)e^{ikx} = \frac{1}{\sqrt{2\pi}}\sum_n \frac{\pi}{a}F(k)e^{ikx} \\ &= \frac{1}{\sqrt{2\pi}}\sum_n \Delta k\, F(k)e^{ikx}, \\ F(k) &= \sqrt{\frac{2}{\pi}}ac_n = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-in\pi x/a}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} f(x)e^{-ikx}\,\mathrm{d}x\,.\end{aligned} \tag{16}$$

(d) As $a \to \infty$, k becomes continuous rather than discrete (it can take on any value), so we replace $\sum_n \Delta k \to \int \mathrm{d}k$, to get

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} \, \mathrm{d}k \,, \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x \,. \end{aligned} \tag{17}$$

4. **G2.21**
Solution:

(a)

$$\int_{-\infty}^{\infty} \Psi^*(x,0)\Psi(x,0)\,\mathrm{d}x = A^2 \int_{-\infty}^{\infty} e^{-2a|x|}\,\mathrm{d}x = 2A^2 \int_0^{\infty} e^{-2ax}\,\mathrm{d}x = \frac{A^2}{a} = 1, \tag{18}$$

so $A = \sqrt{a}$.

(b)

$$\begin{aligned} \phi(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\,\mathrm{d}x = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|-ikx}\,\mathrm{d}x \\ &= \sqrt{\frac{a}{2\pi}} \int_0^{\infty} \left(e^{-(a+ik)x} + e^{-(a-ik)x}\right)\mathrm{d}x = \sqrt{\frac{a}{2\pi}} \left(\frac{1}{a+ik} + \frac{1}{a-ik}\right) \\ &= \sqrt{\frac{a}{2\pi}} \frac{2a}{a^2+k^2} = \sqrt{\frac{2}{\pi a}} \left(1 + \left(\frac{k}{a}\right)^2\right)^{-1}. \end{aligned} \tag{19}$$

(c) Defining $E_k = \hbar^2 k^2/2m$,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} e^{-iE_k t/\hbar}\,\mathrm{d}k = \frac{1}{\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{e^{ikx} e^{-iE_k t/\hbar}}{1 + \left(\frac{k}{a}\right)^2}\,\mathrm{d}k \,. \tag{20}$$

(d) Since a sets the scale for k and $\frac{1}{x}$, for small a, the wavefunction is localised in momentum and spread out in position, while for large a it is localised in position and spread out in momentum.