*Problem 2.12 Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and $\langle T \rangle$, for the nth stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty principle is satisfied.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_+ + \hat{a}_-\right) \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}_+ - \hat{a}_-\right)$$

$$\langle x \rangle : \quad \langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a_+ + a_-|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\cancel{\langle n|n+1\rangle}^{0} + \sqrt{n}\,\cancel{\langle n|n-1\rangle}^{0}\right) = 0$$

$$\langle p \rangle : \quad \langle n|\hat{p}|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|a_+ - a_-|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\,\cancel{\langle n|n+1\rangle}^{0} - \sqrt{n}\,\cancel{\langle n|n-1\rangle}^{0}\right) = 0$$

$$\langle x^2 \rangle : \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|(a_+ + a_-)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|a_+a_+ + a_+a_- + a_-a_+ + a_-a_-|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|a_+a_+ + a_+a_- + a_+a_- + 1 + a_-a_-|n\rangle = \frac{\hbar}{2m\omega}\left(\cancel{\langle n|a_+a_+|n\rangle}^{0} + 2\langle n|a_+a_-|n\rangle + \underbrace{\langle n|n\rangle}_{1} + \cancel{\langle n|a_-a_-|n\rangle}^{0}\right)$$

$$= \frac{\hbar}{2m\omega}\left(2\langle n|a_+a_-|n\rangle + 1\right)$$

$$= \frac{\hbar}{2m\omega}\left(2\sqrt{n}\,\langle n|a_+|n-1\rangle + 1\right) = \frac{\hbar}{2m\omega}\left(2\sqrt{n}\,\sqrt{n-1+1}\,\underbrace{\langle n|n\rangle}_{1} + 1\right)$$

$$= \frac{\hbar}{2m\omega}\left(2n+1\right)$$

$$= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

$$\langle p^2 \rangle : \langle n|p^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|(a_+ - a_-)^2|n\rangle = -\frac{\hbar m\omega}{2}\langle n|\cancel{a_+a_+}^{0} - a_+a_- - a_-a_+ + \cancel{a_-a_-}^{0}|n\rangle$$

$$= \frac{\hbar m\omega}{2}\langle n|a_+a_- + a_-a_+|n\rangle = \frac{\hbar m\omega}{2}\left(2n+1\right)$$

$$= \hbar m\omega\left(n+\frac{1}{2}\right)$$

$$\langle T \rangle : \langle n|\hat{T}|n\rangle = \langle n|\frac{\hat{p}^2}{2m}|n\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{2m}\,m\hbar\omega\left(n+\frac{1}{2}\right) = \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

Now we can check if the uncertainty principle is satisfied for the n^{th} eigenstate

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)}$$

$$\sigma_x \sigma_p = \sqrt{\hbar^2\left(n+\frac{1}{2}\right)^2} = \hbar\left(n+\frac{1}{2}\right) \geq \frac{\hbar}{2}$$

Problem 2.13 A particle in the harmonic oscillator potential starts out in the state

$$\Psi(x, 0) = A[3\psi_0(x) + 4\psi_1(x)].$$

(a) Find A.

(b) Construct $\Psi(x, t)$ and $|\Psi(x, t)|^2$.

(c) Find $\langle x \rangle$ and $\langle p \rangle$. Don't get too excited if they oscillate at the classical frequency; what would it have been had I specified $\psi_2(x)$, instead of $\psi_1(x)$? Check that Ehrenfest's theorem (Equation 1.38) holds for this wave function.

(d) If you measured the energy of this particle, what values might you get, and with what probabilities?

a) $\langle \Psi(x,0) | \Psi(x,0) \rangle = 1$

$|A|^2 \big(3\langle 0| + 4\langle 1|\big)\big(3|0\rangle + 4|1\rangle\big) = 1$

$|A|^2 \big(9\langle 0|0\rangle + 12\langle 0|1\rangle + 12\langle 1|0\rangle + 16\langle 1|1\rangle\big) = 1$

$|A|^2 \big(9 + 0 + 0 + 16\big) = 1$

$|A|^2 = \frac{1}{25}$

$A = \frac{1}{5}$

b) $\Psi(x,t) = \frac{1}{5}\left[3\psi_0(x) e^{-i\omega t/2} + 4\psi_1(x) e^{-3i\omega t/2}\right]$

$$|\Psi(x,t)|^2 = \frac{1}{25}\left[3\psi_0 e^{i\omega t/2} + 4\psi_1 e^{i3\omega t/2}\right]\left[3\psi_0 e^{-i\omega t/2} + 4\psi_1 e^{-3i\omega t/2}\right]$$

$$= \frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 12\psi_0\psi_1 e^{i\omega t/2(1-3)} + 12\psi_0\psi_1 e^{i\omega t/2(3-1)}\right]$$

$$= \frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 12\psi_0\psi_1\left(e^{-i\omega t} + e^{i\omega t}\right)\right]$$

$$= \frac{1}{25}\left[9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos(\omega t)\right]$$

c) $\langle x\rangle = \langle \Psi(x,t)|\,\hat{x}\,|\Psi(x,t)\rangle = \frac{1}{25}\left[3e^{i\omega t/2}\langle 0| + 4e^{3i\omega t/2}\langle 1|\right]\hat{x}\left[3e^{-i\omega t/2}|0\rangle + 4e^{-3i\omega t/2}|1\rangle\right]$

$= \frac{1}{25}\left[9\cancelto{0}{\langle 0|\hat{x}|0\rangle} + 16\cancelto{0}{\langle 1|\hat{x}|1\rangle} + 12e^{i\omega t}\langle 1|\hat{x}|0\rangle + 12e^{-i\omega t}\langle 0|\hat{x}|1\rangle\right]$

$= \frac{12}{25}\left[e^{i\omega t}\langle 1|\hat{x}|0\rangle + e^{-i\omega t}\langle 1|\hat{x}|0\rangle^*\right]$

$\langle 1|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 1|a_+ + a_-|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 1|a_+|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{1}\langle 1|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}$

$\langle x\rangle = \frac{12}{25}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t} + e^{-i\omega t}\right)$

$= \frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$

$\langle p\rangle = \frac{1}{25}\left[3e^{i\omega t/2}\langle 0| + 4e^{3i\omega t/2}\langle 1|\right]\hat{p}\left[3e^{-i\omega t/2}|0\rangle + 4e^{-3i\omega t/2}|1\rangle\right]$

$= \frac{1}{25}\left[12e^{i\omega t}\langle 1|\hat{p}|0\rangle + 12e^{-i\omega t}\langle 0|\hat{p}|1\rangle\right]$

$= \frac{12}{25}\left[e^{i\omega t}\langle 1|\hat{p}|0\rangle + e^{-i\omega t}\langle 1|\hat{p}|0\rangle^*\right]$

$\langle 1|\hat{p}|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 1|a_+ - a_-|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 1|a_+|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}$

$\langle p\rangle = \frac{12}{25}\sqrt{\frac{\hbar m\omega}{2}}\left[ie^{i\omega t/2} - ie^{-i\omega t/2}\right]$

$\langle p\rangle = -\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin\omega t$

Check Ehrenfest's theorem : $\frac{d\langle p\rangle}{dt} = -\left\langle \frac{\partial V}{\partial x}\right\rangle$

$\frac{d\langle p\rangle}{dt} = -\frac{d}{dt}\frac{24}{25}\sqrt{\frac{\hbar m\omega}{2}}\sin\omega t = -\frac{24}{25}\omega\sqrt{\frac{\hbar m\omega}{2}}\cos\omega t$

$\frac{\partial V}{\partial x} = m\omega^2 x$

$-\left\langle \frac{\partial V}{\partial x}\right\rangle = -m\omega^2\langle x\rangle = -m\omega^2\frac{24}{25}\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t = -\frac{24}{25}\omega\sqrt{\frac{\hbar m\omega}{2}}\cos\omega t$

Ehrenfest's theorem is satisfied

d) We can get $E = \frac{\hbar\omega}{2}$ with a probability of $\frac{9}{25}$ or $E = \frac{3\hbar\omega}{2}$ with a probability of $\frac{16}{25}$.

$* * *$**Problem 3.35 Coherent states of the harmonic oscillator.** Among the stationary states of the harmonic oscillator ($|n\rangle = \psi_n(x)$, Equation 2.67) only $n = 0$ hits the uncertainty limit ($\sigma_x \sigma_p = \hbar/2$); in general, $\sigma_x \sigma_p = (2n + 1)\hbar/2$, as you found in Problem 2.12. But certain *linear combinations* (known as **coherent states**) also minimize the uncertainty product. They are (as it turns out) *eigenfunctions of the lowering operator*:[32]

$$a_-|\alpha\rangle = \alpha|\alpha\rangle$$

(the eigenvalue α can be any complex number).

(a) Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ in the state $|\alpha\rangle$. *Hint:* Use the technique in Example 2.5, and remember that a_+ is the hermitian conjugate of a_-. Do *not* assume α is real.

(b) Find σ_x and σ_p; show that $\sigma_x \sigma_p = \hbar/2$.

(c) Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0.$$

(d) Determine c_0 by normalizing $|\alpha\rangle$. *Answer:* $\exp(-|\alpha|^2/2)$.

(e) Now put in the time dependence:

$$|n\rangle \rightarrow e^{-iE_n t/\hbar}|n\rangle,$$

and show that $|\alpha(t)\rangle$ remains an eigenstate of a_-, but the eigen*value* evolves in time:

$$\alpha(t) = e^{-i\omega t}\alpha.$$

So a coherent state *stays* coherent, and continues to minimize the uncertainty product.

(f) Is the ground state ($|n = 0\rangle$) itself a coherent state? If so, what is the eigenvalue?

a) $\langle x \rangle = \langle \alpha | x | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | a_+ + a_- | \alpha \rangle$ $\qquad \langle \alpha | a_+ = \alpha^* \langle \alpha |$

$= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle \alpha | a_+ | \alpha \rangle + \langle \alpha | a_- | \alpha \rangle \right)$

$= \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^* + \alpha \right)$

$a_- a_+ - a_+ a_- = 1$

$\langle x^2 \rangle = \langle \alpha | x^2 | \alpha \rangle = \frac{\hbar}{2m\omega} \langle \alpha | (a_+ + a_-)^2 | \alpha \rangle = \frac{\hbar}{2m\omega} \langle \alpha | a_+ a_+ + a_+ a_- + a_- a_+ + a_- a_- | \alpha \rangle$

$= \frac{\hbar}{2m\omega} \langle \alpha | a_+ a_+ + 2 a_+ a_- + 1 + a_- a_- | \alpha \rangle = \frac{\hbar}{2m\omega} \langle \alpha | \alpha^{*2} + 2\alpha^* \alpha + 1 + \alpha^2 | \alpha \rangle$

$= \frac{\hbar}{2m\omega} \left[1 + (\alpha + \alpha^*)^2 \right]$

$\langle p \rangle = i \sqrt{\frac{\hbar m\omega}{2}} \langle \alpha | a_+ - a_- | \alpha \rangle = i \sqrt{\frac{\hbar m\omega}{2}} \langle \alpha | \alpha^* - \alpha | \alpha \rangle = i \sqrt{\frac{\hbar m\omega}{2}} \left(\alpha^* - \alpha \right)$

$\langle p^2 \rangle = -\frac{\hbar m\omega}{2} \langle \alpha | (a_+ - a_-)^2 | \alpha \rangle = -\frac{\hbar m\omega}{2} \langle \alpha | a_+ a_+ - a_+ a_- - a_- a_+ + a_- a_- | \alpha \rangle$

$= -\frac{\hbar m\omega}{2} \langle \alpha | a_+ a_+ - 2 a_+ a_- - 1 + a_- a_- | \alpha \rangle$

$= -\frac{\hbar m\omega}{2} \langle \alpha | \alpha^{*2} - 2\alpha^* \alpha - 1 + \alpha^2 | \alpha \rangle = \frac{\hbar m\omega}{2} \left[1 - (\alpha^{*2} - 2\alpha^* \alpha + \alpha^2) \right]$

$= \frac{\hbar m\omega}{2} \left[1 - (\alpha - \alpha^*)^2 \right]$

b) $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left(\frac{\hbar}{2m\omega} \left[1 + (\alpha + \alpha^*)^2 \right] - \frac{\hbar}{2m\omega} (\alpha + \alpha^*)^2 \right)^{1/2} = \sqrt{\frac{\hbar}{2m\omega}}$

$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \left(\frac{\hbar m\omega}{2} \left[1 - (\alpha - \alpha^*)^2 \right] - i^2 \frac{\hbar m\omega}{2} (\alpha - \alpha^*)^2 \right)^{1/2} = \sqrt{\frac{\hbar m\omega}{2}}$

$\sigma_x \sigma_p = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m\omega}{2}} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$

c) $c_n = \langle n | \alpha \rangle = \left(\langle 0 | \frac{(a_+)^n}{\sqrt{n!}} \right) | \alpha \rangle = \frac{1}{\sqrt{n!}} \langle 0 | (a_-)^n | \alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \langle 0 | \alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} c_0$

d) $1 = \sum_{n=0}^{\infty} |c_n|^2 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$

$c_0 = e^{-|\alpha|^2/2}$

e) $|\alpha(t)\rangle = \sum_{n=0}^{\infty} C_n e^{-i(n+\frac{1}{2})\omega t} |n\rangle = e^{-i\frac{\omega t}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} C_0 e^{-in\omega t} |n\rangle = e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle$

This still has the form of a coherent state but with α shifted to $\alpha e^{-i\omega t}$

$\alpha(t) = \alpha e^{-i\omega t}$

f) $a_- |0\rangle = 0 |0\rangle$

The ground state is a coherent state with an eigenvalue of 0

**Problem 3.39

(a) For a function $f(x)$ that can be expanded in a Taylor series, show that

$$f(x + x_0) = e^{i\hat{p}x_0/\hbar} f(x)$$

(where x_0 is any constant distance). For this reason, $\hat{p}/\hbar$ is called the **generator of translations in space.** *Note:* The exponential of an *operator* is defined by the power series expansion: $e^{\hat{Q}} \equiv 1 + \hat{Q} + (1/2)\hat{Q}^2 + (1/3!)\hat{Q}^3 + \ldots$.

(b) If $\Psi(x, t)$ satisfies the (time-dependent) Schrödinger equation, show that

$$\Psi(x, t + t_0) = e^{-i\hat{H}t_0/\hbar}\Psi(x, t)$$

(where t_0 is any constant time); $-\hat{H}/\hbar$ is called the **generator of translations in time.**

(c) Show that the expectation value of a dynamical variable $Q(x, p, t)$, at time $t + t_0$, can be written[34]

$$\langle Q\rangle_{t+t_0} = \langle\Psi(x, t)|e^{i\hat{H}t_0/\hbar}\hat{Q}(\hat{x}, \hat{p}, t + t_0)e^{-i\hat{H}t_0/\hbar}|\Psi(x, t)\rangle.$$

Use this to recover Equation 3.71. *Hint:* Let $t_0 = dt$, and expand to first order in dt.

a) Taylor expand $f(x+x_0)$

$$f(x+x_0) = \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{d}{dx}\right)^n f(x) \qquad \frac{d}{dx} = \frac{ip}{\hbar}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x_0^n \left(\frac{ip}{\hbar}\right)^n f(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{ipx_0}{\hbar}\right)^n}{n!} f(x)$$

$$= e^{\frac{ipx_0}{\hbar}} f(x)$$

b) $i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$

$$\Psi(x, t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(\frac{\partial}{\partial t}\right)^n \Psi(x,t)$$

$$\Psi(x,t+t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} t_0^n \left(-\frac{i}{\hbar} H\right)^n \Psi(x,t) = e^{-iHt_0/\hbar}\Psi(x,t)$$

c) $\langle Q \rangle_{t+t_0} = \langle \Psi(x, t+t_0) | Q(x,p,t+t_0) | \Psi(x,t+t_0) \rangle$

$$|\Psi(x,t+t_0)\rangle = e^{-iHt_0/\hbar} |\Psi(x,t)\rangle \implies \langle \Psi(x,t+t_0)| = \langle \Psi(x,t)| e^{iHt_0/\hbar}$$

$$\langle Q \rangle_{t+t_0} = \langle \Psi(x,t)| e^{iHt_0/\hbar} Q(x,p,t+t_0) e^{-iHt_0/\hbar} |\Psi(x,t)\rangle$$

Now consider an infinitesimal time $t_0 = dt$

$$\langle Q \rangle_{t+dt} = \langle Q \rangle_t + \frac{d\langle Q \rangle}{dt} dt + O(dt^2)$$

$$\langle Q \rangle_{t+dt} = \langle \Psi(x,t)| e^{iHdt/\hbar} Q(x,p,t+t_0) e^{-iHdt/\hbar} |\Psi(x,t)\rangle$$

$$e^{iHdt/\hbar} Q(x,p,t+t_0) e^{-iHdt/\hbar} = \left(1 + \frac{iH}{\hbar} dt + \dots\right)\left(Q(x,p,t) + \frac{\partial Q}{\partial t} dt + \dots\right)\left(1 - \frac{iH}{\hbar} dt + \dots\right)$$

$$= Q(x,p,t) + \frac{iH}{\hbar} dt\, Q - Q \frac{iH}{\hbar} dt + \frac{\partial Q}{\partial t} dt + O(dt^2)$$

$$= Q(x,p,t) + \frac{i}{\hbar}[H, Q] dt + \frac{\partial Q}{\partial t} dt + O(dt^2)$$

$$\langle Q \rangle_{t+dt} = \langle \Psi(x,t)| e^{iHdt/\hbar} Q(x,p,t+t_0) e^{-iHdt/\hbar} |\Psi(x,t)\rangle = \langle Q \rangle_t + \frac{i}{\hbar}\langle [H,Q] \rangle dt + \left\langle \frac{\partial Q}{\partial t} \right\rangle dt + O(dt^2)$$

$$\implies \langle Q \rangle + \frac{d\langle Q \rangle}{dt} dt = \langle Q \rangle_t + \frac{i}{\hbar}\langle [H,Q] \rangle dt + \left\langle \frac{\partial Q}{\partial t} \right\rangle dt$$

$$\frac{d\langle Q \rangle}{dt} = \frac{i}{\hbar}\langle [H,Q] \rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$