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1. **Griffiths 4.28**

***Problem 4.28** For the most general normalized spinor χ (Equation 4.139), compute $\langle S_x\rangle$, $\langle S_y\rangle$, $\langle S_z\rangle$, $\langle S_x^2\rangle$, $\langle S_y^2\rangle$, and $\langle S_z^2\rangle$. Check that $\langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle = \langle S^2\rangle$.

$$|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

$$\langle \chi | S_x | \chi\rangle = (a^* \ b^*) \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \boxed{(a^* b + b^* a)\frac{\hbar}{2} = \langle S_x \rangle} = \hbar \, Re(ab^*)$$

$$\boxed{\langle S_y \rangle = -\hbar \, Im(ab^*)}$$

$$\boxed{\langle S_z \rangle = \frac{\hbar}{2}(|a|^2 - |b|^2)}$$

$$\langle S_x^2 \rangle = \hbar^2/4$$

$$\langle S_y^2 \rangle = \hbar^2/4$$

$$\langle S_z^2 \rangle = \hbar^2/4$$

$$= \langle S^2 \rangle = 3\hbar^2/4 \checkmark$$

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***Problem 4.29**

(a) Find the eigenvalues and eigenspinors of $\mathbf{S}_y$.

(b) If you measured S_y on a particle in the general state χ (Equation 4.139), what values might you get, and what is the probability of each? Check that the probabilities add up to 1. *Note: a and b need not be real!*

(c) If you measured S_y^2, what values might you get, and with what probabilities?

$$S_y \approx \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{e. Vec} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$S_y|\uparrow_y\rangle = \frac{\hbar}{2}|\uparrow_y\rangle \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}} \text{ normalized}$$

$$S_y|\downarrow_y\rangle = -\frac{\hbar}{2}|\downarrow_y\rangle \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}\frac{1}{\sqrt{2}}$$

so
$$\boxed{\begin{aligned} |\downarrow_y\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\rangle - i|\downarrow\rangle\right] \\ |\uparrow_y\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\rangle + i|\downarrow\rangle\right] \end{aligned}}$$

(b) possibilities are $\{\hbar/2, -\hbar/2\}$ with probabilities

$$\left\{\frac{1}{2}|a-ib|^2, \frac{1}{2}|a+ib|^2\right\}$$

$$|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$
$$= \frac{1}{\sqrt{2}}\left[a|\uparrow_y\rangle + a|\downarrow_y\rangle - ib|\uparrow_y\rangle + ib|\downarrow_y\rangle\right]$$
$$= \frac{1}{\sqrt{2}}(a-ib)|\uparrow_y\rangle + \frac{1}{\sqrt{2}}(a+ib)|\downarrow_y\rangle$$

(c) $\frac{\hbar^2}{4}$, probability 1

(a) $S_- |1\ 0\rangle$

$= \left(S_-^{(1)} + S_-^{(2)}\right) \frac{1}{\sqrt{2}}\left(\uparrow\downarrow + \downarrow\uparrow\right)$

$= \frac{\hbar}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right.$

$= \sqrt{2}\hbar\ |\downarrow\downarrow\rangle = \sqrt{2}\hbar\,|1\ {-1}\rangle$

(b)

$S_\pm |0\ 0\rangle = \left[S_\pm^{(1)} + S_\pm^{(2)}\right]\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$\pm\ \hbar/\sqrt{2}\left[|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle\right] = 0$

$= \hbar/\sqrt{2}\left[|\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle\right] = 0$

(c)

$S^2|1\ 1\rangle = \left[(S^1)^2 + (S^2)^2 + 2S^1\cdot S^2\right]|\uparrow\uparrow\rangle$

$= \frac{3}{4}\hbar^2\left(|\uparrow\uparrow\rangle + |\uparrow\uparrow\rangle\right) + 2\left[\frac{\hbar^2}{4}|\downarrow\downarrow\rangle + \frac{(i\hbar)^2}{4}|\downarrow\downarrow\rangle + \frac{\hbar^2}{4}|\uparrow\uparrow\rangle\right]$

$= \frac{3}{2}\hbar^2|\uparrow\uparrow\rangle + \frac{\hbar^2}{2}|\uparrow\uparrow\rangle = 2\hbar^2|\uparrow\uparrow\rangle$

$= 2\hbar^2|1\,1\rangle = 1(1+1)\hbar^2\ |1\,1\rangle$

(d) $S^2|1\,{-1}\rangle = \left[(S^1)^2 + (S^2)^2 + 2S^1\cdot S^2\right]|\downarrow\downarrow\rangle$

$= \frac{3}{4}\hbar^2\left(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right) + 2\left[\frac{\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{(i\hbar)^2}{4}|\uparrow\uparrow\rangle + \frac{(-\hbar)^2}{4}|\downarrow\downarrow\rangle\right]$

$= \frac{3}{2}\hbar^2|\downarrow\downarrow\rangle + \frac{\hbar^2}{2}|\downarrow\downarrow\rangle = 2\hbar^2|\downarrow\downarrow\rangle$

$$= \frac{3}{2}\hbar^2 |\downarrow\downarrow\rangle + \frac{\hbar^2}{2} |\downarrow\downarrow\rangle = 2\hbar^2 |\downarrow\downarrow\rangle$$

$$= 2\hbar^2 |1\,{-1}\rangle = 1(1+1)\hbar^2 |1\,{-1}\rangle$$

2 Matrix Representation for $j = 1$

In this problem, we would like to compute probability distribution of measurements of J_x, J_y, and J_z for particles with $j = 1$.

(a) We will do this problem in the $\{|j=1, m_z=1\rangle, |j=1, m_z=0\rangle, |j=1, m_z=-1\rangle\}$ basis. How does the $\hat{J}_z$ operator look in this basis? Note that we will have a 3×3 matrix. (Hint: Think about which eigenvalues the matrix should have.)

Solution: This is its eigenbasis! So

$$\hat{J}_z \to \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Recall that $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$. Show that $\hat{J}_+^\dagger = \hat{J}_-$.

Solution: As $\hat{J}_x$ and $\hat{J}_y$ are observables, they are Hermitian (self-adjoint), so

$$\hat{J}_+^\dagger = (\hat{J}_x + i\hat{J}_y)^\dagger = \hat{J}_x - i\hat{J}_y = \hat{J}_-.$$

(c) Show that $\hat{J}_\mp \hat{J}_\pm = \hat{\mathbf{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z$.

Solution: We have that

$$\begin{aligned}
\hat{J}_\mp \hat{J}_\pm &= (\hat{J}_x \mp i\hat{J}_y)(\hat{J}_x \pm i\hat{J}_y) \\
&= \hat{J}_x^2 + \hat{J}_y^2 \pm i[\hat{J}_x, \hat{J}_y] \\
&= \hat{\mathbf{J}}^2 - \hat{J}_z^2 \pm i(i\hat{J}_z) \\
&= \hat{\mathbf{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z.
\end{aligned}$$

(d) Determine the form of the raising and lowering operators in the $\{|j = 1, m_z\rangle\}$ basis. (Hint: First determine which elements are non-zero by recalling how the raising and lowering operators act on the basis states. Then use the above facts to determine exactly what the non-zero elements are.)

Solution: We know that $\hat{J}_+$ raises the value of m in $|j, m\rangle$ unless $m = j$, in which case it maps it to zero. We also know that $\hat{J}_-$ lowers the value of m in $|j, m\rangle$ unless $m = -j$, in which case it maps it to zero. So this gives

$$\hat{J}_+ \to \begin{pmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

As $\hat{J}_+^\dagger = \hat{J}_-$,

$$\hat{J}_- \to \begin{pmatrix} 0 & 0 & 0 \\ c_1^* & 0 & 0 \\ 0 & c_2^* & 0 \end{pmatrix}.$$

Also, as

$$\hat{\mathbf{J}}^2 |1, m\rangle = 2\hbar^2 |j, m\rangle,$$

$$\hat{\mathbf{J}}^2 \to 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So

$$\hat{\mathbf{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z \to 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix},$$

and

$$\hat{J}_-\hat{J}_+ \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & |c_1|^2 & 0 \\ 0 & 0 & |c_2|^2 \end{pmatrix}.$$

So $c_1 = c_2 = \sqrt{2}\hbar$.

(e) Use the raising and lowering operators to construct the representations of $\hat{J}_x$ and $\hat{J}_y$ in the $\{|j=1, m_z\rangle\}$ basis.

Solution: As $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$,

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)$$
$$\hat{J}_y = -\frac{i}{2}\left(\hat{J}_+ - \hat{J}_-\right).$$

So

$$\hat{J}_x \to \frac{1}{2}\left(\sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$\hat{J}_y \to -\frac{i}{2}\left(\sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(f) Use these matrices to find the representations of the eigenstates of both $\hat{J}_x$ and $\hat{J}_y$ in the $\{|j=1, m_z\rangle\}$ basis and their corresponding eigenvalues.

Solution: Here we must find the eigenvalues and eigenvectors of the matrices that we just computed. We find eigenvalues λ of a matrix M by setting

$$\det(M - \lambda I) = 0,$$

where I is the identity matrix. For $\hat{J}_x$, this gives

$$\begin{vmatrix} -\lambda & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\lambda & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - \frac{\hbar^2}{2}) + \lambda\frac{\hbar^2}{2} = -\lambda(\lambda^2 - \hbar^2).$$

The solutions are $\lambda = \pm\hbar, 0$. Now, we get the eigenvectors by solving

$$(M - \lambda I)\mathbf{v} = 0.$$

For $\lambda = \hbar$, we get

$$\begin{pmatrix} -\hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & -\hbar & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & -\hbar \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(To save time, I scaled the first entry to 1, which works unless the entry is supposed to be zero. If that happens, I'll get unsatisfiable equations, which would tell me that the first entry is zero. Then I can scale the second entry to 1 and begin the process anew.) The first line gives the equation

$$-\hbar + a\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = \sqrt{2}$.
The last line, with $a = \sqrt{2}$, gives the equation

$$\hbar - b\hbar = 0,$$

so $b = 1$. Normalizing this eigenvector, we get

$$|1, m_x = 1\rangle \to \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

For $\lambda = 0$, we get

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$a\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = 0$.
The second line, with $a = 0$, gives the equation

$$\frac{\hbar}{\sqrt{2}} + b\frac{\hbar}{\sqrt{2}} = 0,$$

so $b = -1$. Normalizing this eigenvector, we get

$$|1, m_x = 0\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda = -\hbar$, we get

$$\begin{pmatrix} \hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & \hbar & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & \hbar \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$\hbar + a\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = -\sqrt{2}$.

For $\lambda = -\hbar$, we get

$$\begin{pmatrix} \hbar & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & \hbar & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & \hbar \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$\hbar + a\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = -\sqrt{2}$.
The last line, with $a = -\sqrt{2}$, gives the equation

$$-\hbar + b\hbar = 0,$$

so $b = 1$. Normalizing this eigenvector, we get

$$|1, m_x = -1\rangle \to \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

For $\hat{J}_y$, the eigenvalue equation gives

$$\begin{vmatrix} -\lambda & -i\frac{\hbar}{\sqrt{2}} & 0 \\ i\frac{\hbar}{\sqrt{2}} & -\lambda & -i\frac{\hbar}{\sqrt{2}} \\ 0 & i\frac{\hbar}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - \frac{\hbar^2}{2}) + \lambda\frac{\hbar^2}{2} = -\lambda(\lambda^2 - \hbar^2).$$

We get the same characteristic equation. The solutions are again $\lambda = \pm\hbar, 0$. Now for the eigenvectors:
For $\lambda = \hbar$, we get

$$\begin{pmatrix} -\hbar & -i\frac{\hbar}{\sqrt{2}} & 0 \\ i\frac{\hbar}{\sqrt{2}} & -\hbar & -i\frac{\hbar}{\sqrt{2}} \\ 0 & i\frac{\hbar}{\sqrt{2}} & -\hbar \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$-\hbar - ai\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = i\sqrt{2}$.
The last line, with $a = i\sqrt{2}$, gives the equation

$$-\hbar - b\hbar = 0,$$

so $b = -1$. Normalizing this eigenvector, we get

$$|1, m_y = 1\rangle \to \frac{1}{2}\begin{pmatrix} 1 \\ i\sqrt{2} \\ -1 \end{pmatrix}.$$

For $\lambda = 0$, we get

$$\begin{pmatrix} 0 & -i\frac{\hbar}{\sqrt{2}} & 0 \\ i\frac{\hbar}{\sqrt{2}} & 0 & -i\frac{\hbar}{\sqrt{2}} \\ 0 & i\frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$-ai\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = 0$.

For $\lambda = 0$, we get

$$\begin{pmatrix} 0 & -i\frac{\hbar}{\sqrt{2}} & 0 \\ i\frac{\hbar}{\sqrt{2}} & 0 & -i\frac{\hbar}{\sqrt{2}} \\ 0 & i\frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$-ai\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = 0$.

The second line, with $a = 0$, gives the equation

$$i\frac{\hbar}{\sqrt{2}} - ib\frac{\hbar}{\sqrt{2}} = 0,$$

so $b = 1$. Normalizing this eigenvector, we get

$$|1, m_y = 0\rangle \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = -\hbar$, we get

$$\begin{pmatrix} \hbar & -i\frac{\hbar}{\sqrt{2}} & 0 \\ i\frac{\hbar}{\sqrt{2}} & \hbar & -i\frac{\hbar}{\sqrt{2}} \\ 0 & i\frac{\hbar}{\sqrt{2}} & \hbar \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first line gives the equation

$$\hbar - ai\frac{\hbar}{\sqrt{2}} = 0,$$

so $a = -i\sqrt{2}$.

The last line, with $a = -i\sqrt{2}$, gives the equation

$$\hbar + b\hbar = 0,$$

so $b = -1$. Normalizing this eigenvector, we get

$$|1, m_y = -1\rangle \to \frac{1}{2} \begin{pmatrix} 1 \\ -i\sqrt{2} \\ -1 \end{pmatrix}.$$

(g) A particle is prepared in the state $|j = 1, m_z = 1\rangle$ and then J_x is measured. What are the possible J_x measurement results, i.e. states, and their respective probabilities? What is the expectation value of the angular momentum in the x-direction of $|j = 1, m_z = 1\rangle$?

Solution: J_x could be $\hbar$, 0, or $-\hbar$, as those are the eigenvalues of $\hat{J}_x$. The probabilities are

$$P(J_x=\hbar) = |\langle j=1, m_x=1|j=1, m_z=1\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4},$$

$$P(J_x=0) = |\langle j=1, m_x=0|j=1, m_z=1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

$$P(J_x=-\hbar) = |\langle j=1, m_x=-1|j=1, m_z=1\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

All of them are observable. We see that the expectation value of angular momentum in the x-direction is

$$\langle J_x\rangle = \frac{1}{4}\times\hbar + \frac{1}{2}\times 0 + \frac{1}{4}\times -\hbar = 0.$$

(h) If we measure $J_x=\hbar$ and then we measure J_y, what is the expectation value of the angular momentum in the y-direction?

Solution: We are asked for the expectation value of J_y for the state $|1, m_x=1\rangle$. This is

$$\langle J_y\rangle = \langle 1, m_x=1|\,\hat{J}_y\,|1, m_x=1\rangle = \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix} = 0.$$

(i) If we instead measured J_z again after measuring $J_x=\hbar$, what is the probability that we get the original state $|j=1, m_z=1\rangle$? You should find that simply making the measurement of J_x changes the state; you can have the value of J_z change just by measuring J_x!

Solution: The probability of measuring the state $|1, m_x=1\rangle$ in the state $|1, m_z=1\rangle$ (and getting $J_z=\hbar$) is

$$|\langle 1, m_z=1|1, m_x=1\rangle|^2 = \left|\frac{1}{2}\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\ \sqrt{2}\\ 1\end{pmatrix}\right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

We see that this state is not guaranteed to give $J_z=\hbar$, as it would have if it were not measured.

5. Measuring Spin

Imagine you have a beam of spin 1/2 particles moving in the y-direction. We can set up an inhomogeneous magnetic field to interact with the particles, separating them according to their spin component in the direction of the magnetic field, $\mathbf{B}\cdot\hat{\mathbf{S}}$. This is the Stern-Gerlach experiment, depicted in Fig. 1.

(a) You set up a magnetic field in the z-direction. As the beam of particles passes through it, it splits in two equal beams: one goes up, corresponding to the spin-up particles (those whose $\hat{S}_z$ eigenvalue was $+\frac{\hbar}{2}$), and the other goes down, corresponding to the spin-down particles. Now, you take the beam that went up and pass it through another magnetic field in the z-direction. Does the beam split? If so, what fraction of the particles go to each side?

(b) Instead, you pass the beam through a z-field, take the beam that went up, and pass it through a magnetic field in the x-direction. Does the beam split? If so, what fraction of the particles go to each side?

(c) You select one of the beams from part b above, and pass it through another magnetic field in the z-direction. Does the beam split? If so, what fraction of the particles go to each side? Compare with part a and explain.

(d) Suppose we start with N particles. We first pass them through a magnetic field in the z-direction, and block the beam that goes down. After this process, you find that only $\frac{N}{2}$ particles remain. They then go through a magnetic field in the x-z plane, an angle θ from the z-axis, and the beam that goes against the direction of the field is blocked. Then you have a magnetic field in the z-direction again, and block the beam that goes up this time. How many particles come out? Compare with the case without the middle magnetic field.

(a) At first, the particles are split into some spin up and some spin down, and we keep the spin up particles. Now, we pass it through a second magnetic field in the z-direction and keep only the spin down particles. Since the spin up state is orthogonal to the spin down state, particles in the spin up state cannot be measured to be spin down. Therefore we expect to have zero particles at the end of this filtering process.

(b) At first, some undetermined portion of the original beam is split in the z-direction. Let's call this quantity Q. After that, the quantity is split according to spin in the x-direction. Because the z spin up eigenvector has an equal proportion of x spin up and x spin down, we expect to keep 1/2 of the quantity after this split.

(c) Now, the beam is split again in the z-direction. Since the x spin up eigenvector has an equal proportion of z spin up and z spin down, we expect to see 1/2 of what's left going up, and the other half go down. These will each be (1/4)Q. This is more than zero, as was found in (a).

(d) the general eigenspinor for an angle theta and phi = 0 is

$$|\chi_-^{\theta}\rangle = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$$

therefore $\langle \uparrow | \chi_-^\theta \rangle^2 = \sin^2 \frac{\theta}{2}$ ← this is what we keep

and $\langle \uparrow | \chi_+^\theta \rangle^2 = \cos^2 \theta/2$

$$+ \overline{} = 1 \checkmark$$

now we block the beam that goes up, so we only want the overlap with the down state. That will be

$$\langle \chi_-^\theta | \downarrow \rangle^2 = \cos^2 \frac{\theta}{2}$$

So the total amount we keep is

$$\underline{Q \sin^2 \theta/2 \cos^2 \theta/2} \leq Q/4$$