Please note, all Griffiths problems come from our class text, the second edition.

1. **Griffiths 4.17**

Problem 4.17 Consider the earth-sun system as a gravitational analog to the hydrogen atom.

(a) What is the potential energy function (replacing Equation 4.52)? (Let m be the mass of the earth, and M the mass of the sun.)

(b) What is the "Bohr radius," a_g, for this system? Work out the actual number.

(c) Write down the gravitational "Bohr formula," and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_o, show that $n = \sqrt{r_o/a_g}$. From this, estimate the quantum number n of the earth.

(d) Suppose the earth made a transition to the next lower level $(n-1)$. How much energy (in Joules) would be released? What would the wavelength of the emitted photon (or, more likely, graviton) be? (Express your answer in light years—is the remarkable answer[20] a coincidence?)

Solution:

(a)

$$V(\mathbf{r}) = -\frac{GMm}{r}. \tag{1}$$

(b) We simply replace $\frac{e^2}{4\pi\epsilon_0} \to GMm$ to get,

$$a_g = \frac{\hbar^2}{GMm^2} \approx 2.35 \times 10^{-138}\,\text{m!} \tag{2}$$

(c)

$$E_n = -\left(\frac{m}{2\hbar^2}(GMm)^2\right)\frac{1}{n^2} = -\frac{\hbar^2}{2ma_g^2}\frac{1}{n^2}. \tag{3}$$

Classically, if a planet is orbiting at radius r_o, it has constant potential energy $-\frac{GMm}{r_o}$, and constant kinetic energy $\frac{1}{2}mv^2 = \frac{GMm}{2r_o}$, since the centripetal force is gravitational, $\frac{mv^2}{r_o} = \frac{GMm}{r_0^2}$. Thus it has total energy $-\frac{GMm}{2r_o}$. If these two energies are equal, since $E_n = \frac{E_1}{n^2}$,

$$n = \sqrt{\frac{E_1}{E}} = \sqrt{\frac{\hbar^2}{2ma_g^2}\frac{2r_o}{GMm}} = \sqrt{\frac{r_o}{a_g}}. \tag{4}$$

For the earth, $r_o \approx 1.50 \times 10^{11}$ m, so $n \approx 2.52 \times 10^{74}$!

(d) The energy of the emitted graviton is

$$E = \frac{E_1}{(n-1)^2} - \frac{E_1}{n^2} = E_1\frac{n^2-(n-1)^2}{n^2(n-1)^2} \approx E_1\frac{2n}{n^4} = \frac{2E_1}{n^3}, \tag{5}$$

and since gravitons travel at the speed of light (as confirmed by gravitational wave astronomy a few years ago), $E = hf = hc/\lambda$

$$\lambda = \frac{hc}{E} \approx \frac{hc}{2E_1} n^3 = \frac{2\pi c m a_g^2}{\hbar} n^3 \approx \frac{n^3}{2} \times 1.18 \times 10^{-208}\,\mathrm{m} \approx 9.46 \times 10^{15}\,\mathrm{m} \approx 1\,\mathrm{ly}. \tag{6}$$

Is this a coincidence? No: the orbital period is

$$T = \frac{2\pi r_o}{v} = 2\pi \sqrt{\frac{r_o^3}{GM}}, \tag{7}$$

and for any system with $r_o \gg a_g$ (i.e. $n \gg 1$), the energy spacing near n is approximately $\frac{2E_1}{n^3}$, so the period of the emitted graviton is

$$T = \frac{1}{f} = \frac{h}{E} = \frac{2\pi m a_g^2}{\hbar} n^3 = \frac{2\pi m}{\hbar} \sqrt{r_o^3 a_g} = 2\pi \sqrt{\frac{r_o^3}{GM}}. \tag{8}$$

2. **Griffiths 4.45**

Problem 4.45 What is the probability that an electron in the ground state of hydrogen will be found *inside the nucleus*?

(a) First calculate the *exact* answer, assuming the wave function (Equation 4.80) is correct all the way down to $r = 0$. Let b be the radius of the nucleus.

(b) Expand your result as a power series in the small number $\epsilon \equiv 2b/a$, and show that the lowest-order term is the cubic: $P \approx (4/3)(b/a)^3$. This should be a suitable approximation, provided that $b \ll a$ (which it *is*).

(c) Alternatively, we might assume that $\psi(r)$ is essentially constant over the (tiny) volume of the nucleus, so that $P \approx (4/3)\pi b^3 |\psi(0)|^2$. Check that you get the same answer this way.

(d) Use $b \approx 10^{-15}$ m and $a \approx 0.5 \times 10^{-10}$ m to get a numerical estimate for P. Roughly speaking, this represents the "fraction of its time that the electron spends inside the nucleus."

Solution:

(a)

$$\Pr(r \le b) = \int_0^b r^2 |R_{1,0}|^2 \,\mathrm{d}r = 4a^{-3} \int_0^b r^2 e^{-2r/a} \,\mathrm{d}r = \tfrac{1}{2} \int_0^{2b/a} x^2 e^{-x} \,\mathrm{d}x\,, \tag{9}$$

$$I_n(y) = \int_0^y x^n e^{-x} \,\mathrm{d}x = -\int_0^y \left((x^n e^{-x})' - n x^{n-1} e^{-x} \right) \mathrm{d}x = n I_{n-1}(y) - y^n e^{-y}, \tag{10}$$

$$I_0(y) = \int_0^y e^{-x} \,\mathrm{d}x = 1 - e^{-y}, \tag{11}$$

$$\Pr(r \le b) = \tfrac{1}{2} I_2(\epsilon) = I_1(\epsilon) - \tfrac{1}{2}\epsilon^2 e^{-\epsilon} = I_0(\epsilon) - (\epsilon + \tfrac{1}{2}\epsilon^2) e^{-\epsilon} = 1 - (1 + \epsilon + \tfrac{1}{2}\epsilon^2) e^{-\epsilon}. \tag{12}$$

You can show by induction that $I_n(y) = n!(1 - e^{-y} \sum_{k=0}^{n} \frac{y^k}{k!})$.

(b) Noting that $\sum_{k=0}^{n} \frac{y^k}{k!} = e^y - \sum_{k=n+1}^{\infty} \frac{y^k}{k!}$,

$$\begin{aligned}\Pr(r \leq b) &= 1 - (e^\epsilon - \tfrac{1}{6}\epsilon^3 - \tfrac{1}{24}\epsilon^4 - \ldots)e^{-\epsilon} = (\tfrac{1}{6}\epsilon^3 + \tfrac{1}{24}\epsilon^4 + \ldots)e^{-\epsilon} \\ &= \tfrac{1}{6}\epsilon^3 + \mathcal{O}\big(\epsilon^4\big) \approx \frac{1}{6}\left(\frac{2b}{a}\right)^3 = \frac{4b^3}{3a^3}.\end{aligned} \tag{13}$$

(c)

$$\Pr(r \leq b) \approx \frac{4\pi b^3}{3}|\psi_{1,0,0}(0)|^2 = \frac{4\pi b^3}{3}\frac{1}{\pi a^3} = \frac{4b^3}{3a^3}. \tag{14}$$

(d) $\epsilon = \frac{2b}{a} = 4 \times 10^{-5}$, so

$$\Pr(r \leq b) \approx \tfrac{1}{6}\epsilon^3 = \frac{64}{6} \times 10^{-15} \approx 1.07 \times 10^{-14}. \tag{15}$$

3. **Hydrogen wavefunctions**

1 Hydrogen Wavefunctions

You may find the following integral useful in this problem:

$$\int_0^\infty x^n e^{-x/\alpha}\, dx = n!\alpha^{n+1}.$$

(a) States of a hydrogen atom are typically expressed in terms of the basis states $|n, l, m\rangle$. Write down three operators and their corresponding eigenvalue equations that define the quantum numbers n, l, and m. What values can n take? For each n, what values can l take? For each l, when values can m take? [Hint: By eigenvalue equation, we mean something like $\hat{O}\,|nlm\rangle = \lambda\,|nlm\rangle$, where λ is a number that depends on n, l, and m.]

(b) For the state $|2,0,0\rangle$, write down its wavefunction $\psi_{2,0,0}(r,\theta,\phi) \equiv \langle r,\theta,\phi|2,0,0\rangle$. Compute the expectation values $\langle x\rangle$, $\langle r^2\rangle$, and $\langle x^2\rangle$ for this state. [Hint: Use symmetry arguments wherever possible. For $\langle x^2\rangle$, note that $r^2 = x^2+y^2+z^2$, and then use symmetry.]

(c) Consider the state

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|2,1,1\rangle + |2,1,-1\rangle).$$

First write down its wavefunction $\psi_\alpha(r,\theta,\phi) = \langle r,\theta,\phi|\alpha\rangle$ using $R_{nl}(r)$ and $Y_{lm}(\theta,\phi)$. Then, write down the full wavefunction. [For example, if the state is $|1,0,0\rangle$, we are looking for $R_{10}(r)Y_{00}(\theta,\phi)$ for the first part and $\frac{1}{\sqrt{\pi a^3}}e^{-r/a}$ for the second part.]

(d) Find $\langle y\rangle$ for the state $|\alpha\rangle$. [Hint: Recall $y = r\sin\theta\sin\phi$.]

Solution:

(a)

$$\hat{H}\,|n,\ell,m\rangle = \frac{E_1}{n^2}\,|n,\ell,m\rangle\,, \tag{16}$$

$$\hat{\mathbf{L}}^2\,|n,\ell,m\rangle = \hbar^2\ell(\ell+1)\,|n,\ell,m\rangle\,, \tag{17}$$

$$\hat{L}_z\,|n,\ell,m\rangle = \hbar m\,|n,\ell,m\rangle\,. \tag{18}$$

n can be any positive integer, and ℓ, m are integers bounded by $0 \le \ell \le n-1$, $-\ell \le m \le \ell$.

(b)

$$\psi_{2,0,0}(\mathbf{r}) = R_{20}(r)Y_0^0(\theta,\phi) = \sqrt{\frac{1}{8\pi a^3}}\left(1-\frac{r}{2a}\right)e^{-r/2a}. \tag{19}$$

$\langle x\rangle = 0$ by symmetry since the particle is equally likely to be found at x as $-x$ (since Y_0^0 is rotationally invariant).

$$\begin{aligned}\langle r^2\rangle &= \int_0^\infty r^2\,\mathrm{d}r \int \mathrm{d}\Omega\, r^2|\psi_{2,0,0}|^2 = \int_0^\infty |R_{2,0}|^2 r^4\,\mathrm{d}r = \frac{1}{2a^3}\int_0^\infty \left(1-\frac{r}{2a}\right)^2 e^{-r/a} r^4\,\mathrm{d}r \\ &= \frac{a^2}{2}\int_0^\infty (1 - x + \tfrac{1}{4}x^2)x^4 e^{-x}\,\mathrm{d}x = \frac{4!}{2}\left(1 - 5 + \tfrac{30}{4}\right)a^2 = 42a^2,\end{aligned} \tag{20}$$

and since by symmetry $\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle$, $\langle r^2\rangle = \langle x^2+y^2+z^2\rangle = 3\langle x^2\rangle$, so $\langle x^2\rangle = 14a^2$.

(c) Note that Griffiths defines the Y_ℓ^ms with an additional minus sign for $m > 0$ (but I think this problem is intended to use the convention that doesn't have this additional sign), so

$$\begin{aligned}
\psi_\alpha(\mathbf{r}) &= \frac{R_{2,1}(r)}{\sqrt{2}}(Y_1^1(\theta,\phi) + Y_1^{-1}(\theta,\phi)) \\
&= \frac{1}{\sqrt{48a^3}}\frac{r}{a}e^{-r/2a}\left(\pm\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} + \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right) \\
&= \begin{cases} \frac{1}{\sqrt{32\pi a^3}}\frac{r}{a}e^{-r/2a}\sin\theta\cos\phi & \text{Prof's convention,} \\ \frac{-i}{\sqrt{32\pi a^3}}\frac{r}{a}e^{-r/2a}\sin\theta\sin\phi & \text{Griffiths' convention.} \end{cases}
\end{aligned} \tag{21}$$

(d) Under either convention, $\langle y\rangle = 0$ by symmetry, since $\psi_\alpha \propto xe^{-r/2a}$ or $\psi_\alpha \propto ye^{-r/2a}$, so $\Pr(x,-y,z) = \Pr(x,y,z)$. Or you can see it explicitly in the integral:

$$\begin{aligned}
\langle y\rangle &= \int \mathrm{d}^3\mathbf{r}\, y|\psi_\alpha|^2 \\
&= \begin{cases} \frac{1}{32\pi a^5}\left(\int_0^\infty r^5 e^{-r/a}\,\mathrm{d}r\right)\left(\int_0^\pi \sin^4\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\cos^2\phi\sin\phi\,\mathrm{d}\phi\right) & \text{Prof,} \\ \frac{1}{32\pi a^5}\left(\int_0^\infty r^5 e^{-r/a}\,\mathrm{d}r\right)\left(\int_0^\pi \sin^4\theta\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\sin^3\phi\,\mathrm{d}\phi\right) & \text{Griffiths,} \end{cases}
\end{aligned} \tag{22}$$

and in either case the integral over ϕ is zero.

4. **Hydrogen atom eigenstates**

2 Hydrogen Atom Eigenstates

The state of an electron in a hydrogen atom at time $t = 0$ is given by

$$|\psi(0)\rangle = \frac{1}{\sqrt{6}} \left(|2,0,0\rangle + C\,|2,1,-1\rangle - |3,1,0\rangle \right),$$

where the states $|n,l,m\rangle$ are the simultaneous energy, angular momentum and z component of angular momentum eigenstates for the electron in the hydrogen atom.

(a) Find the real and positive C that normalizes the state.

(b) What are $\langle E\rangle$, $\langle L^2\rangle$, and $\langle L_z\rangle$ for this state?

(c) Which of these change as a function of time?

(d) At time $t = 0$, the magnitude of the angular momentum $|\mathbf{L}| = \sqrt{L^2}$ is measured. What results are possible, with what probability do you get each result, and what are the states immediately after the measurement is made?

(e) If 2000 hydrogen atoms are in the state $|\psi(0)\rangle$, and the z component of angular momentum is measured, how many do you expect to have each possible value?

(f) If the energy is measured, which energy value(s) can the experimenter measure that would tell them what they would get if they measured L_z afterwards, and which energy value(s) would leave them uncertain of the result if they were to measure L_z afterwards?

Solution:

(a)

$$\langle\psi(0)|\psi(0)\rangle = \frac{1 + |C|^2 + 1}{6} = 1, \tag{23}$$

so $C = 2$.

(b)

$$\begin{aligned}
\langle E\rangle &= \langle\psi(0)|\hat{H}|\psi(0)\rangle = \frac{1}{\sqrt{6}} \langle\psi(0)|\left(E_2\,|2,0,0\rangle + 2E_2\,|2,1,-1\rangle - E_3\,|3,1,0\rangle\right) \\
&= \frac{E_2 + 4E_2 + E_3}{6} = \frac{E_1}{6}\left(\frac{5}{2^2} + \frac{1}{3^2}\right) = \frac{49}{216}E_1,
\end{aligned} \tag{24}$$

$$\begin{aligned}
\langle \mathbf{L}^2\rangle &= \langle\psi(0)|\hat{\mathbf{L}}^2|\psi(0)\rangle = \frac{1}{\sqrt{6}} \langle\psi(0)|\left(2\times 2\hbar^2\,|2,1,-1\rangle - 2\hbar^2\,|3,1,0\rangle\right) \\
&= \frac{8\hbar^2 + 2\hbar^2}{6} = \tfrac{5}{3}\hbar^2,
\end{aligned} \tag{25}$$

$$\langle L_z\rangle = \langle\psi(0)|\hat{L}_z|\psi(0)\rangle = \frac{1}{\sqrt{6}} \langle\psi(0)|\left(-2\hbar\,|2,1,-1\rangle\right) = -\tfrac{2}{3}\hbar. \tag{26}$$

(c) None of them, since they all commute with the Hamiltonian.

(d) Since the possible values of ℓ are 0 and 1, the possible results of the measurement are 0 and $\sqrt{2}\hbar$. The probability of measuring 0 is $\frac{1}{6}$ and the state afterward is $|2,0,0\rangle$. The probability of measuring $\sqrt{2}\hbar$ is $\frac{5}{6}$ and the state afterward is $\frac{2}{\sqrt{5}}\,|2,1,-1\rangle - \frac{1}{\sqrt{5}}\,|3,1,0\rangle$.

(e) For a single atom, when measuring L_z we get 0 with probability $\frac{1}{3}$ and $-\hbar$ with probability $\frac{2}{3}$. Thus, for 2000 atoms, we expect to see $\frac{2000}{3}$ of them with $L_z = 0$ and $\frac{4000}{3}$ of them with $L_z = -\hbar$ on average.

(f) If we see E_3, then the state afterward is $|3, 1, 0\rangle$, which has definite $L_z = 0$. The probability of this is $\frac{1}{6}$. However, if we see E_2, the the state afterward is $\frac{1}{\sqrt{5}} |2, 0, 0\rangle + \frac{2}{\sqrt{5}} |2, 1, -1\rangle$ which does not have definite L_z. The probability of this is $\frac{5}{6}$. No other outcomes are possible.