*Please note, all Griffiths problems come from our class text, the second edition.*

1. **Griffiths 3.27**

**Problem 3.27 Sequential measurements.** An operator $\hat{A}$, representing observable $A$, has two normalized eigenstates $\psi_1$ and $\psi_2$, with eigenvalues $a_1$ and $a_2$, respectively. Operator $\hat{B}$, representing observable $B$, has two normalized eigenstates $\phi_1$ and $\phi_2$, with eigenvalues $b_1$ and $b_2$. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \quad \psi_2 = (4\phi_1 - 3\phi_2)/5.$$

**(a)** Observable $A$ is measured, and the value $a_1$ is obtained. What is the state of the system (immediately) after this measurement?

**(b)** If $B$ is now measured, what are the possible results, and what are their probabilities?

**(c)** Right after the measurement of $B$, $A$ is measured again. What is the probability of getting $a_1$? (Note that the answer would be quite different if I had told you the outcome of the $B$ measurement.)

**Solution**:

(a) $\psi_1$

(b) The outcomes are $b_1$ and $b_2$, with probabilities

$$\Pr(b_1) = |\langle\psi_1|\phi_1\rangle|^2 = \left|\frac{3\langle\phi_1|\phi_1\rangle + 4\langle\phi_2|\phi_1\rangle}{5}\right|^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}, \tag{1}$$

$$\Pr(b_2) = |\langle\psi_1|\phi_2\rangle|^2 = \left|\frac{3\langle\phi_1|\phi_2\rangle + 4\langle\phi_2|\phi_2\rangle}{5}\right|^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}. \tag{2}$$

(c) If we had got outcome $b_1$, the state of the system became $\phi_1$, so

$$\Pr(a_1|b_1) = |\langle\phi_1|\psi_1\rangle|^2 = \frac{9}{25}, \tag{3}$$

while if we had gotten $b_2$, the state of the system was $\phi_2$, so

$$\Pr(a_1|b_2) = |\langle\phi_2|\psi_1\rangle|^2 = \frac{16}{25}. \tag{4}$$

Since we don't know the outcome of the $B$ measurement, we weight them according to their probabilities,

$$\begin{aligned}\Pr(a_1) &= \Pr(b_1, a_1) + \Pr(b_2, a_1) = \Pr(b_1)\Pr(a_1|b_1) + \Pr(b_2)\Pr(a_1|b_2)\\ &= \left(\frac{9}{25}\right)^2 + \left(\frac{16}{25}\right)^2 = \frac{337}{625}.\end{aligned} \tag{5}$$

If we didn't measure $B$ in between, we would have gotten $a_1$ again with certainty. Measuring $B$ changed the state of the system, projecting it into the eigenspace corresponding to the observed value.

2. **Griffiths 4.1**

***Problem 4.1**

**(a)** Work out all of the **canonical commutation relations** for components of the operators **r** and **p**: $[x, y]$, $[x, p_y]$, $[x, p_x]$, $[p_y, p_z]$, and so on. *Answer:*

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0, \qquad [4.10]$$

where the indices stand for $x$, $y$, or $z$, and $r_x = x$, $r_y = y$, and $r_z = z$.

**(b)** Confirm Ehrenfest's theorem for 3-dimensions:

$$\frac{d}{dt}\langle \mathbf{r}\rangle = \frac{1}{m}\langle \mathbf{p}\rangle, \quad \text{and} \quad \frac{d}{dt}\langle \mathbf{p}\rangle = \langle -\nabla V\rangle. \qquad [4.11]$$

(Each of these, of course, stands for *three* equations—one for each component.) *Hint:* First check that Equation 3.71 is valid in three dimensions.

**(c)** Formulate Heisenberg's uncertainty principle in three dimensions. *Answer:*

$$\sigma_x\sigma_{p_x} \geq \hbar/2, \quad \sigma_y\sigma_{p_y} \geq \hbar/2, \quad \sigma_z\sigma_{p_z} \geq \hbar/2, \qquad [4.12]$$

but there is no restriction on, say, $\sigma_x\sigma_{p_y}$.

**Solution**:

(a) Since any operator commutes with itself, $[\hat{r}_i, \hat{r}_i] = [\hat{p}_i, \hat{p}_i] = 0$. Since scalar multiplication is commutative, for any $\psi(\mathbf{r})$,

$$[\hat{r}_i, \hat{r}_j]\psi(\mathbf{r}) = (r_i r_j - r_j r_i)\psi(\mathbf{r}) = 0, \tag{6}$$

and since partial derivatives commute,

$$[\hat{p}_i, \hat{p}_j]\psi(\mathbf{r}) = -\hbar^2\frac{\partial^2\psi}{\partial r_i \partial r_j}(\mathbf{r}) + \hbar^2\frac{\partial^2\psi}{\partial r_j \partial r_i}(\mathbf{r}) = 0, \tag{7}$$

so $[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0$. Finally,

$$[\hat{r}_i, \hat{p}_j]\psi(\mathbf{r}) = -i\hbar r_i\frac{\partial\psi}{\partial r_j}(\mathbf{r}) + i\hbar\frac{\partial(r_i\psi)}{\partial r_j}(\mathbf{r}) = i\hbar\frac{\partial r_i}{\partial r_j}\psi(\mathbf{r}) = i\hbar\delta_{ij}\psi(\mathbf{r}). \tag{8}$$

(b) For any (time-independent) observable $\hat{A}$, Since $\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\langle A\rangle &= \left(\frac{\partial}{\partial t}\langle\psi|\right)\hat{A}\,|\psi\rangle + \langle\psi|\,\hat{A}\left(\frac{\partial}{\partial t}|\psi\rangle\right) = \frac{i}{\hbar}\Big(\langle\psi|\hat{H}^\dagger\hat{A}|\psi\rangle - \langle\psi|\hat{A}\hat{H}|\psi\rangle\Big) \\ &= \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle = \frac{i}{\hbar}\Big\langle[\hat{H},\hat{A}]\Big\rangle.\end{aligned} \tag{9}$$

Thus, since $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ for any operators $\hat{A}, \hat{B}, \hat{C}$,

$$[V(\hat{\mathbf{r}}), \hat{r}_i]\psi(\mathbf{r}) = (V(\mathbf{r})r_i - r_iV(\mathbf{r}))\psi(\mathbf{r}) = 0, \tag{10}$$

$$[V(\hat{\mathbf{r}}), \hat{p}_i]\psi(\mathbf{r}) = -i\hbar\left(V(\mathbf{r})\frac{\partial\psi}{\partial r_i}(\mathbf{r}) - \frac{\partial(V\psi)}{\partial r_i}(\mathbf{r})\right) = i\hbar\frac{\partial V}{\partial r_i}(\mathbf{r})\psi(\mathbf{r}), \tag{11}$$

$$\begin{aligned}[\hat{\mathbf{p}}^2, \hat{r}_i] &= \sum_j [\hat{p}_j^2, \hat{r}_i] = \sum_j \left(\hat{p}_j[\hat{p}_j, \hat{r}_i] + [\hat{p}_j, \hat{r}_i]\hat{p}_j\right) \\ &= \sum_j -2i\hbar\delta_{ij}\hat{p}_j = -2i\hbar\hat{p}_i,\end{aligned} \tag{12}$$

$$[\hat{\mathbf{p}}^2, \hat{p}_i] = \sum_j \left(\hat{p}_j[\hat{p}_j, \hat{p}_i] + [\hat{p}_j, \hat{p}_i]\hat{p}_j\right) = 0, \tag{13}$$

so, since $\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 + V(\hat{\mathbf{r}})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle r_i\rangle = \frac{i}{\hbar}\left\langle[\hat{H}, \hat{r}_i]\right\rangle = \frac{i}{2m\hbar}\left\langle[\hat{\mathbf{p}}^2, \hat{r}_i]\right\rangle = \frac{\langle p_i\rangle}{m}, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle p_i\rangle = \frac{i}{\hbar}\left\langle[\hat{H}, \hat{p}_i]\right\rangle = \frac{i}{\hbar}\langle[V(\hat{\mathbf{r}}), \hat{p}_i]\rangle = -\left\langle\frac{\partial V}{\partial r_i}\right\rangle. \tag{15}$$

(c)

$$\sigma(r_i)\sigma(p_j) \geq \left|\frac{1}{2i}\langle[\hat{r}_i, \hat{p}_j]\rangle\right| = \left|\frac{\hbar}{2}\delta_{ij}\right| = \frac{\hbar}{2}\delta_{ij}. \tag{16}$$

3. **Griffiths 4.18**

∗**Problem 4.18** The raising and lowering operators change the value of $m$ by one unit:

$$L_\pm f_l^m = (A_l^m)f_l^{m\pm1}, \tag{4.120}$$

where $A_l^m$ is some constant. *Question:* What *is* $A_l^m$, if the eigenfunctions are to be *normalized*? *Hint:* First show that $L_\mp$ is the hermitian conjugate of $L_\pm$ (since $L_x$ and $L_y$ are *observables*, you may assume they are hermitian ... but *prove* it if you like); then use Equation 4.112. *Answer:*

$$A_l^m = \hbar\sqrt{l(l+1) - m(m\pm1)} = \hbar\sqrt{(l\mp m)(l\pm m+1)}. \tag{4.121}$$

Note what happens at the top and bottom of the ladder (i.e., when you apply $L_+$ to $f_l^l$ or $L_-$ to $f_l^{-l}$).

**Solution**: Since each component of $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are Hermitian, each component of $\hat{\mathbf{L}}$ must also be, e.g.

$$\hat{L}_x^\dagger = (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)^\dagger = \hat{p}_z^\dagger\hat{y}^\dagger - \hat{p}_y^\dagger\hat{z}^\dagger = \hat{p}_z\hat{y} - \hat{p}_y\hat{z} = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x, \tag{17}$$

and similarly for $\hat{L}_y$ and $\hat{L}_z$. (It follows from this that $\hat{\mathbf{L}}^2$ is also Hermitian.) Hence

$$\hat{L}_\pm^\dagger = (\hat{L}_x \pm i\hat{L}_y)^\dagger = \hat{L}_x^\dagger \mp i\hat{L}_y^\dagger = \hat{L}_x \mp i\hat{L}_y = \hat{L}_\mp. \tag{18}$$

Since $\langle f_\ell^m | f_\ell^m \rangle = 1$ by normalisation,

$$\begin{aligned}\left\langle \hat{L}_\pm f_\ell^m \middle| \hat{L}_\pm f_\ell^m \right\rangle &= \left\langle f_\ell^{m\pm1} \middle| (A_\ell^m)^* A_\ell^m \middle| f_\ell^{m\pm1} \right\rangle = |A_\ell^m|^2 \\ &= \langle f_\ell^m | \hat{L}_\pm^\dagger \hat{L}_\pm | f_\ell^m \rangle = \langle f_\ell^m | \hat{L}_\mp \hat{L}_\pm | f_\ell^m \rangle = \langle f_\ell^m | \hat{\mathbf{L}}^2 - \hat{L}_z^2 \mp \hbar \hat{L}_z | f_\ell^m \rangle \\ &= \langle f_\ell^m | \hbar^2 \ell(\ell+1) - \hbar^2 m^2 \mp \hbar^2 m | f_\ell^m \rangle = \hbar^2 \left(\ell(\ell+1) - m(m\pm1)\right).\end{aligned} \tag{19}$$

This fixes $-\ell \le m \le \ell$ because if we try to go beyond the top or bottom, we get zero,

$$\left\langle \hat{L}_+ f_\ell^\ell \middle| \hat{L}_+ f_\ell^\ell \right\rangle = \hbar^2 \left(\ell(\ell+1) - \ell(\ell+1)\right) = 0, \tag{20}$$

$$\left\langle \hat{L}_- f_\ell^{-\ell} \middle| \hat{L}_- f_\ell^{-\ell} \right\rangle = \hbar^2 \left(\ell(\ell+1) - (-\ell)(-\ell-1)\right) = 0. \tag{21}$$

4. **Griffiths 4.19**

**∗Problem 4.19**

**(a)** Starting with the canonical commutation relations for position and momentum (Equation 4.10), work out the following commutators:

$$\begin{aligned} &[L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0, \\ &[L_z, p_x] = i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0. \end{aligned} \tag{4.122}$$

**(b)** Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from Equation 4.96.

**(c)** Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where, of course, $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

**(d)** Show that the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that $V$ depends only on $r$. (Thus $H$, $L^2$, and $L_z$ are mutually compatible observables.)

**Solution**:

(a)

$$\begin{aligned}[\hat{L}_z, \hat{x}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{x}] = [\hat{x}\hat{p}_y, \hat{x}] - [\hat{y}\hat{p}_x, \hat{x}] \\ &= \hat{x}\cancel{[\hat{p}_y, \hat{x}]} + \cancel{[\hat{x}, \hat{x}]}\hat{p}_y - \hat{y}[\hat{p}_x, \hat{x}] - \cancel{[\hat{y}, \hat{x}]}\hat{p}_y \\ &= -\hat{y}[\hat{p}_x, \hat{x}] = i\hbar\hat{y},\end{aligned} \tag{22}$$

$$[\hat{L}_z, \hat{y}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{y}] = \hat{x}[\hat{p}_y, \hat{y}] = -i\hbar\hat{x}, \tag{23}$$

$$[\hat{L}_z, \hat{z}] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{z}] = 0, \tag{24}$$

$$[\hat{L}_z, \hat{p}_x] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_x] = [\hat{x}, \hat{p}_x]\hat{p}_y = i\hbar\hat{p}_y, \tag{25}$$

$$[\hat{L}_z, \hat{p}_y] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_y] = -[\hat{y}, \hat{p}_y]\hat{p}_x = -i\hbar\hat{p}_x, \tag{26}$$

$$[\hat{L}_z, \hat{p}_z] = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{p}_z] = 0. \tag{27}$$

(b)

$$\begin{aligned}[\hat{L}_z, \hat{L}_x] &= [\hat{L}_z, \hat{y}\hat{p}_z - \hat{z}\hat{p}_y] = [\hat{L}_z, \hat{y}\hat{p}_z] - [\hat{L}_z, \hat{z}\hat{p}_y] \\ &= \hat{y}\cancel{[\hat{L}_z, \hat{p}_z]} + [\hat{L}_z, \hat{y}]\hat{p}_z - \hat{z}[\hat{L}_z, \hat{p}_y] - \cancel{[\hat{L}_z, \hat{z}]}\hat{p}_y \\ &= -i\hbar\hat{x}\hat{p}_z + i\hbar\hat{z}\hat{p}_x = i\hbar(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z) = i\hbar\hat{L}_y. \end{aligned} \tag{28}$$

(c)

$$\begin{aligned}[\hat{L}_z, \hat{\mathbf{r}}^2] &= [\hat{L}_z, \hat{x}^2] + [\hat{L}_z, \hat{y}^2] + \cancel{[\hat{L}_z, \hat{z}^2]} = \hat{x}[\hat{L}_z, \hat{x}] + [\hat{L}_z, \hat{x}]\hat{x} + \hat{y}[\hat{L}_z, \hat{y}] + [\hat{L}_z, \hat{y}]\hat{y} \\ &= i\hbar(\hat{x}\hat{y} + \hat{y}\hat{x} - \hat{y}\hat{x} - \hat{x}\hat{y}) = 0, \end{aligned} \tag{29}$$

$$\begin{aligned}[\hat{L}_z, \hat{\mathbf{p}}^2] &= \hat{p}_x[\hat{L}_z, \hat{p}_x] + [\hat{L}_z, \hat{p}_x]\hat{p}_x + \hat{p}_y[\hat{L}_z, \hat{p}_y] + [\hat{L}_z, \hat{p}_y]\hat{p}_y \\ &= i\hbar(\hat{p}_x\hat{p}_y + \hat{p}_y\hat{p}_x - \hat{p}_y\hat{p}_x - \hat{p}_x\hat{p}_y) = 0. \end{aligned} \tag{30}$$

(d) Since $\hat{L}_z$ commutes with $\hat{\mathbf{r}}^2$, it commutes with any polynomial of it, and hence "any" (analytic) function $V(\hat{r}) = V(\sqrt{\hat{\mathbf{r}}^2}) = f(\hat{\mathbf{r}}^2)$ of it, and since it also commutes with $\hat{\mathbf{p}}^2$, it commutes with the Hamiltonian. And since the Hamiltonian is rotationally invariant, if it commutes with $\hat{L}_z$, it commutes with any component of $\hat{\mathbf{L}}$.

To see this explicitly, note that all the definitions and canonical commutation relations are unchanged by replacing $(x, y, z) \to (y, z, x)$, e.g. $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \to \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$, so all the commutation relations that derive from it must also be the same after this substitution, e.g. $[\hat{L}_z, \hat{x}] = i\hbar\hat{y} \to [\hat{L}_x, \hat{y}] = i\hbar\hat{z}$. Since $\hat{\mathbf{r}}^2$ and $\hat{\mathbf{p}}^2$ are unchanged, so is $\hat{H}$, and $[\hat{L}_z, \hat{H}] = 0 \to [\hat{L}_x, \hat{H}] = 0 \to [\hat{L}_y, \hat{H}] = 0$.

5. **B&J 6.12**

**6.12** Let $\hat{\mathbf{n}}$ be a unit vector in a direction specified by the polar angles $(\theta, \phi)$. Show that the component of the angular momentum in the direction $\hat{\mathbf{n}}$ is

$$\begin{aligned} L_n &= \sin\theta\cos\phi L_x + \sin\theta\sin\phi L_y + \cos\theta L_z \\ &= \tfrac{1}{2}\sin\theta(\mathrm{e}^{-\mathrm{i}\phi}L_+ + \mathrm{e}^{\mathrm{i}\phi}L_-) + \cos\theta L_z. \end{aligned}$$

If the system is in simultaneous eigenstates of $\mathbf{L}^2$ and $L_z$ belonging to the eigenvalues $l(l+1)\hbar^2$ and $m\hbar$,

(a) what are the possible results of a measurement of $L_n$?

(b) what are the expectation values of $L_n$ and $L_n^2$?

**Solution**: Since $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in Cartesian coordinates, and $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + L_-)$ and $\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) = \frac{i}{2}(L_- - L_+)$,

$$\begin{aligned} \hat{L}_n = \mathbf{n}\cdot\hat{\mathbf{L}} &= \sin\theta\cos\phi\hat{L}_x + \sin\theta\sin\phi\hat{L}_y + \cos\theta\hat{L}_z \\ &= \tfrac{1}{2}\sin\theta\left(\cos\phi(\hat{L}_+ + \hat{L}_-) + i\sin\phi(\hat{L}_- - \hat{L}_+)\right) + \cos\theta\hat{L}_z \\ &= \tfrac{1}{2}\sin\theta\left(e^{-i\phi}\hat{L}_+ + e^{i\phi}\hat{L}_-\right) + \cos\theta\hat{L}_z. \end{aligned} \tag{31}$$

(a) By rotational symmetry, $\hat{L}_n$ is just like any other component of $\hat{\mathbf{L}}$, so the eigenvalues (which are the possible outcomes of the measurement) are $m\hbar$ for integer $-\ell \le m \le \ell$. (We can deduce the eigenvalues of $\hat{L}_n$ exactly the same way as we did those of $\hat{L}_z$.)

(b) Supposing the system is in state $|\ell, m\rangle$, where $\hat{\mathbf{L}}^2|\ell, m\rangle = \ell(\ell+1)\hbar^2|\ell, m\rangle$ and $\hat{L}_z|\ell, m\rangle = \hbar m|\ell, m\rangle$, (note that in 3D each $|\ell, m\rangle$ isn't a unique wavefunction since we haven't specified the radial part, but here the radial part remains unchanged throughout, so is irrelevant)

$$\begin{aligned} \langle L_n\rangle &= \langle \ell, m|\hat{L}_n|\ell, m\rangle = \langle \ell, m|\tfrac{1}{2}\sin\theta\left(e^{-i\phi}\hat{L}_+ + e^{i\phi}\hat{L}_-\right) + \cos\theta\hat{L}_z|\ell, m\rangle \\ &= \langle \ell, m|\cos\theta\hat{L}_z|\ell, m\rangle = \hbar m\cos\theta, \end{aligned} \tag{32}$$

$$\begin{aligned} \left\langle L_n^2\right\rangle &= \langle \ell, m|\hat{L}_n^2|\ell, m\rangle = \langle \ell, m|\tfrac{1}{4}\sin^2\theta\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) + \cos^2\theta\hat{L}_z^2|\ell, m\rangle \\ &= \langle \ell, m|\tfrac{1}{2}\sin^2\theta\left(\hat{\mathbf{L}}^2 - \hat{L}_z^2\right) + \cos^2\theta\hat{L}_z^2|\ell, m\rangle \\ &= \tfrac{1}{2}\hbar^2(\ell(\ell+1) - m^2)\sin^2\theta + \hbar^2 m^2\cos^2\theta, \end{aligned} \tag{33}$$

$$\sigma(L_n) = \sqrt{\langle L_n^2\rangle - \langle L_n\rangle^2} = \tfrac{1}{\sqrt{2}}\hbar\sin\theta\sqrt{\ell(\ell+1) - m^2}. \tag{34}$$

We can understand/check this by trying $\mathbf{n} = \mathbf{z}$, i.e. $\theta = 0$, and we get $\langle L_n\rangle = \hbar m$ and $\sigma(L_n) = 0$ as expected. We can also try $\mathbf{n} = \mathbf{x}$ or $\mathbf{y}$ (we expect them to be the same by symmetry), i.e. $\theta = \pi/2$, and we get $\langle L_n\rangle = 0$, and $\langle L_n^2\rangle = \frac{1}{2}\langle \mathbf{L}^2 - L_z^2\rangle = \frac{1}{2}\hbar^2(\ell(\ell+1) - m^2)$, as expected.