Please note, all Griffiths problems come from our class text, the second edition.

1. **An Adventure with Hermitian Operators**
Linear operators are linear maps from the Hilbert space to itself, i.e. the functions $\hat{L}$ from wavefunctions to wavefunctions such that $\hat{L}(\alpha\Psi+\beta\Phi) = \alpha\hat{L}\Psi+\beta\hat{L}\Phi$, for all wavefunctions Ψ, Φ and all complex numbers α, β. A Hermitian operator $\hat{A}$ is one whose adjoint is equal to itself, $\hat{A}^\dagger = \hat{A}$, i.e. for all Ψ, Φ,

$$\left\langle \Psi, \hat{A}\Phi \right\rangle = \int \Psi^*(x)[\hat{A}\Phi](x)\,\mathrm{d}x = \int [\hat{A}\Psi]^*(x)\Phi(x)\,\mathrm{d}x = \left\langle \hat{A}\Psi, \Phi \right\rangle. \tag{1}$$

Physical measurables are represented by Hermitian operators. We know two such operators, the position and momentum,

$$\hat{x} : [\hat{x}\Psi](x) = x\Psi(x), \qquad \hat{p} : [\hat{p}\Psi](x) = -i\hbar\frac{\partial\Psi}{\partial x}(x). \tag{2}$$

For the following maps, check whether they are linear operators, and if so, whether they are Hermitian.

(a) $\hat{x}$
Solution: Given any Ψ, Φ, for all x,

$$\begin{aligned}[\hat{x}(\alpha\Psi + \beta\Phi)](x) &= x[\alpha\Psi + \beta\Phi](x) = x(\alpha\Psi(x) + \beta\Phi(x)) = \alpha x\Psi(x) + \beta x\Phi(x)\\ &= \alpha[\hat{x}\Psi](x) + \beta[\hat{x}\Phi](x) = [\alpha\hat{x}\Psi + \beta\hat{x}\Phi](x),\end{aligned} \tag{3}$$

so $\hat{x}(\alpha\Psi + \beta\Phi) = \alpha\hat{x}\Psi + \beta\hat{x}\Phi$ and thus $\hat{x}$ is linear. Given any Ψ, Φ,

$$\langle \Psi, \hat{x}\Phi \rangle = \int \Psi^*(x)[\hat{x}\Phi](x)\,\mathrm{d}x = \int x\Psi^*(x)\Phi(x)\,\mathrm{d}x = \int [\hat{x}\Psi]^*(x)\Phi(x)\,\mathrm{d}x = \langle \hat{x}\Psi, \Phi \rangle, \tag{4}$$

so $\hat{x}$ is Hermitian.

(b) $\hat{A} : \hat{A}\Psi(x) = \Psi^*(x)\Psi(x)$
Solution:

$$\begin{aligned}[\hat{A}(\alpha\Psi + \beta\Phi)](x) &= [\alpha\Psi + \beta\Phi]^*(x)[\alpha\Psi + \beta\Phi](x)\\ &= |\alpha\Psi(x)|^2 + |\beta\Phi(x)|^2 + \alpha^*\beta\Psi^*(x)\Phi(x) + \alpha\beta^*\Psi(x)\Phi^*(x)\\ &\neq \alpha|\Psi(x)|^2 + \beta|\Phi(x)|^2 = \alpha[\hat{A}\Psi](x) + \beta[\hat{A}\Phi](x),\end{aligned} \tag{5}$$

so $\hat{A}$ is not linear.

(c) $\hat{p}$
Solution:

$$[\hat{p}(\alpha\Psi + \beta\Phi)](x) = -i\hbar(\alpha\frac{\partial\Psi}{\partial x}(x) + \beta\frac{\partial\Phi}{\partial x}(x)) = \alpha[\hat{p}\Psi](x) + \beta[\hat{p}\Phi](x), \tag{6}$$

so $\hat{p}$ is linear. Using integration by parts,

$$\begin{aligned}\langle \Psi, \hat{p}\Phi \rangle &= -i\hbar\int \Psi^*(x)\frac{\partial\Phi}{\partial x}(x)\,\mathrm{d}x = -i\hbar\left[\Psi^*(x)\Phi(x)\right]_{-\infty}^{\infty} + i\hbar\int \frac{\partial\Psi^*}{\partial x}(x)\Phi(x)\,\mathrm{d}x\\ &= \int \left(-i\hbar\frac{\partial\Psi}{\partial x}(x)\right)^*\Phi(x)\,\mathrm{d}x = \langle \hat{p}\Psi, \Phi \rangle.\end{aligned} \tag{7}$$

The boundary term is zero, $[\Psi^*(x)\Phi(x)]_{-\infty}^{\infty} = 0$, because Ψ, Φ must both be normalisable, i.e. each satisfies $\int |\Psi(x)|^2\,\mathrm{d}x < \infty$, so must go to zero in the limits $x \to \pm\infty$. Thus $\hat{p}$ is Hermitian (on the space of normalisable wavefunctions).

(d) $\hat{B} : \hat{B}\Psi = -iL^2\frac{\partial^2\Psi}{\partial x^2} + i\frac{x^2}{L^2}\Psi$, for some fixed L
Solution:
$$\hat{B} = \frac{iL^2}{\hbar^2}\hat{p}^2 + \frac{i}{L^2}\hat{x}^2. \tag{8}$$
$\hat{x}$ and $\hat{p}$ are linear operators, as seen above. The composition of linear operators is linear,
$$\hat{L}_1\hat{L}_2(\alpha\Psi + \beta\Phi) = \hat{L}_1(\alpha\hat{L}_2\Psi + \beta\hat{L}_2\Phi) = \alpha\hat{L}_1\hat{L}_2\Psi + \beta\hat{L}_1\hat{L}_2\Phi, \tag{9}$$
and the linear combination of linear operators is also linear,
$$\begin{aligned}(\gamma\hat{L}_1 + \delta\hat{L}_2)(\alpha\Psi + \beta\Phi) &= \gamma\hat{L}_1(\alpha\Psi + \beta\Phi) + \delta\hat{L}_2(\alpha\Psi + \beta\Phi)\\ &= \alpha(\gamma\hat{L}_1 + \delta\hat{L}_2)\Psi + \beta(\gamma\hat{L}_1 + \delta\hat{L}_2)\Phi,\end{aligned} \tag{10}$$
so $\hat{B}$ is linear. The square of a Hermitian operator is Hermitian,
$$\langle\Psi, \hat{A}^2\Phi\rangle = \langle\hat{A}\Psi, \hat{A}\Phi\rangle = \langle\hat{A}^2\Psi, \Phi\rangle, \tag{11}$$
and a linear combination of Hermitian operators with real coefficients is Hermitian,
$$\begin{aligned}\langle\Psi, (a\hat{A}_1 + b\hat{A}_2)\Phi\rangle &= a\langle\Psi, \hat{A}_1\Phi\rangle + b\langle\Psi, \hat{A}_2\Phi\rangle = a\langle\hat{A}_1\Psi, \Phi\rangle + b\langle\hat{A}_2\Psi, \Phi\rangle\\ &= \langle(a\hat{A}_1 + b\hat{A}_2)\Psi, \Phi\rangle.\end{aligned} \tag{12}$$
Thus $\frac{\hat{B}}{i}$ is Hermitian, but i times a Hermitian operator is anti-Hermitian, i.e. its adjoint is minus itself,
$$\langle\Psi, i\hat{A}\Phi\rangle = i\langle\Psi, \hat{A}\Phi\rangle = i\langle\hat{A}\Psi, \Phi\rangle = \langle -i\hat{A}\Psi, \Phi\rangle. \tag{13}$$
$\frac{\hat{B}}{i}$ is proportional to the Hamiltonian for a quadratic potential.

(e) $\hat{P}_\Phi : \hat{P}_\Phi\Psi(x) = \Phi(x)\int\Phi^*(y)\Psi(y)\,\mathrm{d}y$, for some fixed $\Phi(x)$
Solution: Written in terms of the inner product, $\hat{P}_\Phi\Psi = \langle\Phi, \Psi\rangle\,\Phi$. Such an operator is known as a projection operator. $\hat{P}_\Phi$ is linear because the inner product is linear on the right, $\langle\Phi, (\alpha\Psi_1 + \beta\Psi_2)\rangle = \alpha\,\langle\Phi, \Psi_1\rangle + \beta\,\langle\Phi, \Psi_2\rangle$. It is Hermitian because the inner product is conjugate-symmetric, $\langle\Psi, \Phi\rangle^* = \langle\Phi, \Psi\rangle$:
$$\begin{aligned}\langle\Psi_1, \hat{P}_\Phi\Psi_2\rangle &= \langle\Psi_1, \langle\Phi, \Psi_2\rangle\,\Phi\rangle = \langle\Psi_1, \Phi\rangle\,\langle\Phi, \Psi_2\rangle = \langle\Phi, \Psi_1\rangle^*\,\langle\Phi, \Psi_2\rangle\\ &= \langle\langle\Phi, \Psi_1\rangle\,\Phi, \Psi_2\rangle = \langle\hat{P}_\Phi\Psi_1, \Psi_2\rangle.\end{aligned} \tag{14}$$

(f) $\hat{Q}_\Phi : \hat{Q}_\Phi\Psi(x) = \Phi(x)\int\Psi^*(y)\Phi(y)\,\mathrm{d}y$, for some fixed $\Phi(x)$
Solution: $\hat{Q}_\Phi\Psi = \langle\Psi, \Phi\rangle\,\Phi$, and since the inner product is anti-linear on the left, $\langle\alpha\Phi_1 + \beta\Phi_2, \Psi\rangle = \alpha^*\,\langle\Phi_1, \Psi\rangle + \beta^*\,\langle\Phi_2, \Psi\rangle$, $\hat{Q}_\Phi$ is also anti-linear.

(g) $\hat{T}_a : \hat{T}_a\Psi(x) = \Psi(x + a)$, for some fixed a
Solution: $\hat{T}_a$ is linear,
$$[\hat{T}_a(\alpha\Psi + \beta\Phi)](x) = \alpha\Psi(x + a) + \beta\Phi(x + a) = \alpha[\hat{T}_a\Psi](x) + \beta[\hat{T}_a\Phi](x). \tag{15}$$
But it is not Hermitian, its adjoint is $\hat{T}_{-a}$, which is not $\hat{T}_a$ (unless $a = 0$):
$$\langle\Phi, \hat{T}_a\Psi\rangle = \int\Phi(x)\Psi(x + a)\,\mathrm{d}x = \int\Phi(x - a)\Psi(x)\,\mathrm{d}x = \langle\hat{T}_{-a}\Phi, \Psi\rangle. \tag{16}$$

(h) $\hat{x}\hat{p}$
Solution: As we saw earlier, the composition/product of two linear operators is linear. However this is not in general true for Hermitian operators. For Hermitian $\hat{A}_1, \hat{A}_2$, the adjoint of $\hat{A}_1\hat{A}_2$ is $\hat{A}_2\hat{A}_1$:

$$\langle \Phi, \hat{A}_1\hat{A}_2\Psi\rangle = \langle \hat{A}_1\Phi, \hat{A}_2\Psi\rangle = \langle \hat{A}_2\hat{A}_1\Phi, \Psi\rangle, \tag{17}$$

so the product is Hermitian iff they commute. In our case, $\hat{x}$ and $\hat{p}$ do not commute,

$$[\hat{p}\hat{x}\Psi](x) = -i\hbar\frac{\partial(x\Psi)}{\partial x}(x) = -i\hbar x\frac{\partial\Psi}{\partial x}(x) - i\hbar\Psi(x) \neq -i\hbar x\frac{\partial\Psi}{\partial x}(x) = [\hat{x}\hat{p}\Psi](x), \tag{18}$$

so $\hat{x}\hat{p}$ is not Hermitian.

(i) $\hat{x}\hat{p} + \hat{p}\hat{x}$
Solution: This is a linear combination of linear operators, so is linear, as we saw earlier. And since the adjoint of $\hat{x}\hat{p}$ is $\hat{p}\hat{x}$ (and vice-versa, since the adjoint of the adjoint of any operator is itself), $\hat{x}\hat{p} + \hat{p}\hat{x}$ is its own adjoint and thus Hermitian.

2. **Becoming Friends with Gaussian wave-packets** *inspired by Griffiths 2.22*

***Problem 2.22 The gaussian wave packet.** A free particle has the initial wave function

$$\Psi(x,0) = Ae^{-ax^2},$$

where A and a are constants (a is real and positive).

(a) Normalize $\Psi(x,0)$.

(b) Find $\Psi(x,t)$. *Hint:* Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}\,dx$$

can be handled by "completing the square": Let $y \equiv \sqrt{a}\,[x+(b/2a)]$, and note that $(ax^2+bx) = y^2 - (b^2/4a)$. *Answer:*

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)}}.$$

(c) Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1+(2\hbar at/m)^2}}.$$

Sketch $|\Psi|^2$ (as a function of x) at $t=0$, and again for some very large t. Qualitatively, what happens to $|\Psi|^2$, as time goes on?

(d) Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, σ_x, and σ_p. *Partial answer:* $\langle p^2\rangle = a\hbar^2$, but it may take some algebra to reduce it to this simple form.

(e) Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?

(f) Consider a microscopic particle with the mass of an electron localised in a space of 10^{-10} m, about the size of an atom. How long does it take for σ_x to double its initial value? Compare with a macroscopic particle of mass 1 g localised in a space of 10^{-6} m

Solution:

(a)

$$\int |\Psi(x,0)|^2\,\mathrm{d}x = |A|^2 \int e^{-2ax^2}\,\mathrm{d}x = |A|^2\sqrt{\frac{\pi}{2a}} = 1, \tag{19}$$

using the result for Gaussians from the previous HW with $\mu = 0$ and $\sigma = \frac{1}{2\sqrt{a}}$, so $A = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}$.

(b) In momentum space,

$$\begin{aligned}\Phi(k,0) &= \frac{1}{\sqrt{2\pi}}\int \Psi(x,0)e^{-ikx}\,\mathrm{d}x = \frac{A}{\sqrt{2\pi}}\int e^{-ax^2-ikx}\,\mathrm{d}x \\ &= \frac{Ae^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x+\frac{ik}{2a})^2}\,\mathrm{d}x = \frac{Ae^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty+\frac{ik}{2a}}^{\infty+\frac{ik}{2a}} e^{-ax^2}\,\mathrm{d}x \\ &= \frac{Ae^{-\frac{k^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}\,\mathrm{d}x = \frac{Ae^{-\frac{k^2}{4a}}}{\sqrt{2a}} = \frac{e^{-\frac{k^2}{4a}}}{(2\pi a)^{\frac{1}{4}}},\end{aligned} \tag{20}$$

where we can displace the integral any finite distance above or below the real line without changing it because the integrand is analytic and goes to zero for large $\operatorname{Re} x$. Since the stationary states for the free particle are the momentum eigenstates, each $\Phi(k)$ oscillates independently at frequency $\omega(k) = \frac{E(k)}{\hbar} = \frac{\hbar k^2}{2m}$, so, defining $b = \frac{1}{4a} + \frac{i\hbar t}{2m} = \frac{1}{4a}(1+\frac{2i\hbar at}{m})$,

$$\Phi(k,t) = e^{-i\omega(k)t}\Phi(k,0) = \frac{e^{-bk^2}}{(2\pi a)^{\frac{1}{4}}}, \tag{21}$$

and Fourier transforming back to position space gives us

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int \Phi(k,t)e^{ikx}\,\mathrm{d}k = \frac{1}{(2\pi)^{\frac{3}{4}}a^{\frac{1}{4}}}\int e^{-bk^2+ikx}\,\mathrm{d}k \\ &= \frac{e^{-\frac{x^2}{4b}}}{(2\pi)^{\frac{3}{4}}a^{\frac{1}{4}}}\int e^{-b(k-\frac{ix}{2b})^2}\,\mathrm{d}k = \frac{e^{-\frac{x^2}{4b}}}{(2\pi)^{\frac{3}{4}}a^{\frac{1}{4}}}\sqrt{\frac{\pi}{b}} \\ &= \frac{1}{(2\pi)^{\frac{3}{4}}a^{\frac{1}{4}}}\sqrt{\frac{4\pi a}{1+\frac{2i\hbar at}{m}}}\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right) \\ &= \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\frac{\exp\left(-\frac{ax^2}{1+\frac{2i\hbar at}{m}}\right)}{\sqrt{1+\frac{2i\hbar at}{m}}}.\end{aligned} \tag{22}$$

(c)

$$\begin{aligned}|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t) &= \sqrt{\frac{2a}{\pi}}\frac{\exp\left(-ax^2\left(\frac{1}{1+\frac{2i\hbar at}{m}}+\frac{1}{1-\frac{2i\hbar at}{m}}\right)\right)}{\sqrt{\left(1+\frac{2i\hbar at}{m}\right)\left(1-\frac{2i\hbar at}{m}\right)}} \\ &= \sqrt{\frac{2a}{\pi\left(1+\left(\frac{2\hbar at}{m}\right)^2\right)}}\exp\left(-\frac{2a}{1+\left(\frac{2\hbar at}{m}\right)^2}x^2\right) = \sqrt{\frac{2}{\pi}}we^{-2w^2x^2},\end{aligned} \tag{23}$$

which is a Gaussian (centred around 0) with $\sigma = \frac{1}{2w}$. Since w decreases with time, σ increases with time, i.e. the particle disperses (gets more spread out).

(d) Comparing with the previous HW, we see that $|\Psi(x,t)|^2$ is a Guassian with zero mean, $\langle x\rangle = 0$, and standard deviation $\sigma_x = \sqrt{\langle x^2\rangle} = \frac{1}{2w}$. And, since

$$|\Phi(k,t)|^2 = |\Phi(k,0)|^2 = \frac{e^{-\frac{x^2}{2a}}}{\sqrt{2\pi a}}, \tag{24}$$

k is also distributed as a Gaussian with zero mean and variance a. Thus, since $p = \hbar k$, $\langle p\rangle = \hbar\langle k\rangle = 0$ and $\sigma_p = \sqrt{\langle p^2\rangle} = \hbar\sigma_k = \hbar\sqrt{a}$. (Or you can do the integrals again.)

(e)

$$\sigma_x \sigma_p = \frac{\hbar}{2}\frac{\sqrt{a}}{w} = \frac{\hbar}{2}\sqrt{1+\left(\frac{2\hbar a t}{m}\right)^2} \geq \frac{\hbar}{2}, \tag{25}$$

with saturation at $t = 0$.

(f) Since $\sigma_x = \frac{1}{2w}$, we need $\frac{\sqrt{a}}{w} = 2$, so

$$t = \frac{\sqrt{3}m}{2\hbar a} = \frac{2\sqrt{3}m\sigma^2}{\hbar}. \tag{26}$$

Plugging in the numbers, $t \approx \frac{(10^{-30}\,\mathrm{kg})(10^{-10}\,\mathrm{m})^2}{10^{-34}\,\mathrm{kg\,m^2/s}} = 10^{-16}\,\mathrm{s}$ for the electron and $t \approx \frac{(10^{-3}\,\mathrm{kg})(10^{-6}\,\mathrm{m})^2}{10^{-34}\,\mathrm{kg\,m^2/s}} = 10^{19}\,\mathrm{s} \approx 10^{12}\,\mathrm{yr}$ for the macroscopic particle.

3. **Some useful results using the Schrödinger equation** *Griffiths 2.1*

∗**Problem 2.1** Prove the following three theorems:

(a) For normalizable solutions, the separation constant E must be *real*. *Hint:* Write E (in Equation 2.7) as $E_0 + i\Gamma$ (with E_0 and Γ real), and show that if Equation 1.20 is to hold for all t, Γ must be zero.

(b) The time-independent wave function $\psi(x)$ can always be taken to be *real* (unlike $\Psi(x,t)$, which is necessarily complex). This doesn't mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to ψ's that are real. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does its complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i(\psi - \psi^*)$.

(c) If $V(x)$ is an **even function** (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given E, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.

Solution:

(a) Following the hint, for a stationary state

$$\int |\Psi(x,t)|^2 \, \mathrm{d}x = \int |\psi(x)|^2 e^{i(E^*-E)t/\hbar} \, \mathrm{d}x = e^{2\Gamma t} \int |\psi(x)|^2 \, \mathrm{d}x = 1, \tag{27}$$

so $\Gamma = 0$.

(b) Writing the real imaginary parts of ψ as ψ_R and ψ_I, we have

$$\begin{aligned} 0 &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\psi_R + i\psi_I) + V(\psi_R + i\psi_I) - E(\psi_R + i\psi_I) \\ &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_R}{\mathrm{d}x^2} + V\psi_R - E\psi_R + i\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_I}{\mathrm{d}x^2} + V\psi_I - E\psi_I\right). \end{aligned} \tag{28}$$

Taking the real and imaginary parts of the above equation, since ψ_R and ψ_I are real, they are each independently (real) stationary states with the same energy as ψ. One of them may be zero, but at least one of them must be normalisable if ψ is.

(c) $\psi_-(x) = \psi(-x)$ also satisfies the Schrödinger eqn (with the same energy) since $\frac{\mathrm{d}^2\psi_-}{\mathrm{d}x^2}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big(-\frac{\mathrm{d}\psi}{\mathrm{d}x}(-x)\big) = \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(-x)$,

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi_-}{\mathrm{d}x^2}(x) + V(x)\psi_-(x) &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(-x) + V(x)\psi(-x) \\ &= -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}(-x) + V(-x)\psi(-x) \\ &= E\psi(-x) = E\psi_-(x). \end{aligned} \tag{29}$$

Then $\psi \pm \psi_-$ gives us odd and even stationary states with the same energy. Again, one of them may be zero, but at least one of them must be normalisable if ψ is.

4. **Alas, it's a Question about Probability Currents** *inspired by Griffiths 1.14*

Problem 1.14 Let $P_{ab}(t)$ be the probability of finding a particle in the range $(a < x < b)$, at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t),$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

What are the units of $J(x,t)$? *Comment:* J is called the **probability current,** because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other.

(b) Show that if at some time t, $\Psi(x,t)$ is real or has spatially constant phase, i.e. $\Psi(x,t) = e^{i\theta}f(x)$ for real θ, f, then $J(x,t) = 0$ at that t. What does this imply for energy eigenstates?

(c) Calculate $J(x,0)$ for a Gaussian wavepacket $\Psi(x,t)$.

Solution:

(a)

$$\begin{aligned}
\frac{\mathrm{d}P_{ab}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b |\Psi|^2\,\mathrm{d}x = \int_a^b \left(\frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t}\right)\mathrm{d}x \\
&= \int_a^b \left(\frac{i}{\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^*\right)\Psi - \frac{i}{\hbar}\Psi^*\left(\frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right)\right)\mathrm{d}x \\
&= \frac{i\hbar}{2m}\int_a^b\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)\mathrm{d}x = \frac{i\hbar}{2m}\int_a^b\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\mathrm{d}x \\
&= -[J(x,t)]_a^b.
\end{aligned} \tag{30}$$

J has units $\frac{1}{\text{time}}$. (Since $|\Psi|^2$ integrates to 1, Ψ has units length$^{-\frac{1}{2}}$, so J has units $\frac{ML^2T^{-1}}{M}L^{-2} = T^{-1}$.)

(b) $\Psi\frac{\partial\Psi^*}{\partial x} = f\frac{\partial f}{\partial x} = \Psi^*\frac{\partial\Psi}{\partial x}$ so $J = 0$. Since we can always choose energy eigenstates to be real, they are indeed stationary (for all times, since they oscillate at constant frequency). Also note that this question was confusingly worded: we don't need the phase to be spatially constant for all times for $J = 0$ at some specific time.

(c) Since $\Psi(x,0)$ is real, $J(x,0) = 0$.