MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

A3 **MATH 6201** DUE: MARCH 13, 2017

Hand in your solutions at the beginning of class on Monday March 13, 2017 (send all codes used to `abihlo@mun.ca`).

1. Consider a system of differential equations of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) := Q(x) + Ax + b,$$

where $x \in \mathbb{R}^n$, Q is a quadratic form, $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. The discretization

$$\frac{x^{n+1} - x^n}{\Delta t} = Q(x^n, x^{n+1}) + \frac{1}{2}A(x^n + x^{n+1}) + b,$$

where

$$Q(x_n, x_{n+1}) := \frac{1}{2}(Q(x^n + x^{n+1}) - Q(x^n) - Q(x^{n+1}))$$

is called the *Kahan method*. Show that the Kahan method coincides with the Runge–Kutta method

$$\frac{x^{n+1} - x^n}{\Delta t} = -\frac{1}{2}f(x) + 2f\left(\frac{x^n + x^{n+1}}{2}\right) - \frac{1}{2}f(x^{n+1})$$

restricted to quadratic right-hand sides.

2. Plotting the stability domains for linear multistep methods is considerably more difficult than for the one step methods we have encountered in class. For linear multistep methods, the following *boundary locus method* for plotting the stability domains can be used. If $z \in \mathbb{C}$ is on the boundary of the stability domain of the method, then the stability polynomial $\pi(\zeta, z)$ must have at least on root ζ_j of the form $\zeta_j = \exp(i\theta)$, $\theta \in [0, 2\pi]$, i.e. a root of magnitude 1. We thus have

$$\pi(e^{i\theta}, z) = 0$$

for this root and thus $\rho(e^{i\theta}) - z\sigma(e^{i\theta}) = 0$ and therefore

$$z = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}.$$

Thus, plotting $z = z(\theta)$ we obtain the locus of all points which are potentially on the boundary of the stability domain. It then remains to find whether the stability domain is on the inside or the outside of this parameterized curve, which can be easily accomplished by evaluating one point on the inside to see if the associated polynomial obeys the root condition.

   (a) Use the boundary locus method to plot the stability domains of the 2-step, 3-step and 4-step Adams–Bashforth and Adams–Moulton methods.

(b) Plot the stability domains of the 2-step, 3-step and 4-step BDF methods.

3. Consider the famous two-species Lotka–Volterra system

$$u_1' = u_1(\alpha - \beta u_2), \quad u_2' = u_2(\beta u_1 - \alpha)$$

which describes the interaction of one predator (variable u_2) and one prey (variable u_1) species. Here α and β are constants of the model.

(a) Show analytically that $H = \alpha \ln(u_1 u_2) - \beta(u_1 + u_2)$ is a conserved quantity of the above system.

(b) Integrate the Lotka–Volterra model with parameters $\alpha = \beta = 1$ and initial conditions $u_1(0) = u_2(0) = 0.5$ using the leapfrog scheme. Use a time step of $\Delta t = 0.01$ and integrate up to (i) $t = 100$, (ii) $t = 150$ and (iii) $t = 200$. Plot u_2 against u_1 and interpret the results. **Note:** Since the leapfrog scheme needs both u^n and u^{n-1} to compute the solution u^{n+1}, use an Euler forward scheme to compute the value u^1 given the initial data $u^0 = (u_1(0), u_2(0))$. After this first step, the leapfrog scheme can be used.

(c) Repeat the integration using again the leapfrog scheme but this time replace the leapfrog scheme every n steps with a two-step scheme consisting of an Euler forward method with time step $\Delta t/2$ and a leapfrog scheme with $\Delta t/2$. Experiment with the value of n, e.g. $n = 10, n = 50, n = 100$ and compare the resulting scheme with the pure leapfrog method. Interpret your results!

(d) Plot the time evolution of H up to $t = 150$ using (i) the leapfrog method, (ii) the leapfrog method using every $n = 100$ steps an Euler forward/leapfrog step. Interpret the results.

(e) One can show that every solution of the Lotka–Volerra system with initial conditions $u_1 > 0$ and $u_2 > 0$ is a closed curve. Is this also the case for the numerical solution?