1. Finals Problem :

$$\int_0^{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{d^{k^2}}{dx^{k^2}} (\sin(x)) \right)^k dx$$

The derivatives of $\sin(x)$ based on k^2 modulo 4 are:

$$\frac{d^{k^2}}{dx^{k^2}} \sin(x) = \begin{cases} \sin x & \text{if } k^2 \equiv 0 \pmod 4, \\ \cos x & \text{if } k^2 \equiv 1 \pmod 4, \\ -\sin x & \text{if } k^2 \equiv 2 \pmod 4, \\ -\cos x & \text{if } k^2 \equiv 3 \pmod 4. \end{cases}$$

Using the above patterns, the integral can be expanded into:

$$\int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} \sin^{2n}(x) - \sum_{n=0}^{\infty} \cos^{2n+1}(x) \right) dx$$

This accounts for the alternate signs in the series expansion for odd powers of $\cos(x)$.

The series for even and odd powers of sine and cosine can be written as:

$$\sum_{n=0}^{\infty} \sin^{2n}(x) = \frac{1}{1 - \sin^2(x)} = \sec^2(x)$$

$$\sum_{n=0}^{\infty} \cos^{2n+1}(x) = \cos(x) \sum_{n=0}^{\infty} (\cos^2(x))^n = \frac{\cos(x)}{1 - \cos^2(x)} = \cot(x)\csc(x)$$

Thus, the integral simplifies to:

$$\int_0^{\frac{\pi}{2}} \left(\sec^2(x) - \cot(x)\csc(x)\right) dx = \tan(x) + \csc(x)\Big|_0^{\frac{\pi}{2}}$$

$$= \tan(\frac{\pi}{2}) - \tan(0) + \csc(\frac{\pi}{2}) - \csc(0)$$

$$= \tan(\frac{\pi}{2}) - 0 + 1 - \csc(0)$$

To address the divergences at $x = \frac{\pi}{2}$ and $x = 0$:

$$\lim_{x \to \frac{\pi}{2}} (\tan x - \csc(\frac{\pi}{2} - x))$$

Using the identity $\csc(\frac{\pi}{2} - x) = \sec(x)$, we find:

$$\lim_{x \to \frac{\pi}{2}} (\tan x - \sec x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\cos x}$$

Apply L'Hôpital's Rule due to the indeterminate form $\frac{0}{0}$:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$$

Therefore our final step is: $0 - 0 + 1 = \boxed{1}$

Note: There are many other ways of solving this integral, my favorite in particular is using the Wallis Product to show

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\, dx = \int_0^{\frac{\pi}{2}} \cos^n(x)\, dx$$

to turn our geometric series into either

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin(x)}\, dx \text{ or } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos(x)}\, dx$$

Both of which equal 1.

2.

$$\int_0^{\frac{\pi}{2}} \left(\frac{\cos x}{\sec^2(x) - \tan x \sec x} - \frac{\cos^2(x)}{\sec x + \tan x} \right)^3 dx$$

Convert $\sec x$ and $\tan x$ to their equivalents in terms of $\sin x$ and $\cos x$:

$$\sec x = \frac{1}{\cos x}, \quad \tan x = \frac{\sin x}{\cos x}$$

The first term:

$$\sec^2 x - \tan x \sec x = \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{1 - \sin x}{\cos^2 x}$$

$$\text{Thus, } \frac{\cos x}{\sec^2(x) - \tan x \sec x} = \frac{\cos x}{\frac{1 - \sin x}{\cos^2 x}} = \frac{\cos^3 x}{1 - \sin x}$$

The second term:

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x}$$

$$\text{Thus, } \frac{\cos^2 x}{\sec x + \tan x} = \frac{\cos^2 x}{\frac{1+\sin x}{\cos x}} = \frac{\cos^3 x}{1+\sin x}$$

Combining the two terms gives us

$$\frac{\cos^3 x}{1-\sin x} - \frac{\cos^3 x}{1+\sin x} = \cos^3 x \left(\frac{(1+\sin x) - (1-\sin x)}{1-\sin^2 x} \right)$$

$$= \frac{2\sin x \cos^3 x}{\cos^2 x} = 2 \sin x \cos x = \sin(2x)$$

Plugging our combined terms back into the integrand we get

$$\int_0^{\frac{\pi}{2}} (\sin(2x))^3 \, dx$$

Then, we use $u = 2x, du = 2dx$:

$$\frac{1}{2}\int_0^{\pi} (\sin(u))^3 \, du = \frac{1}{2}\int_0^{\pi} \sin(u)(1-\cos^2(u))\, du$$

$$= \frac{1}{2}\int_0^{\pi} \sin(u)\, du - \frac{1}{2}\int_0^{\pi} \sin(u)\cos^2(u)\, du$$

$$= -\frac{1}{2}\cos(u)\Big|_0^{\pi} + \frac{1}{2}\cdot\frac{1}{3}\cos^3 u\Big|_0^{\pi} = -\frac{1}{2}(-1-1) + \frac{1}{6}(-1-1) = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

3. Finals Problem :

$$\lim_{n\to\infty} \int_0^1 \underbrace{\sqrt{1-\sqrt{1-\sqrt{\cdots}}}}_{n \text{ times}} + \underbrace{\frac{1}{\frac{1}{\frac{1}{\cdots}-1} - 1}}_{n \text{ times}} \, dx$$

Define α as the limit of nested radicals:

$$\alpha = \sqrt{1-\alpha}$$

Squaring both sides gives:

$$\alpha^2 = 1 - \alpha \implies \alpha^2 + \alpha - 1 = 0$$

Solving the quadratic equation:

$$\alpha = \frac{-1 \pm \sqrt{5}}{2}$$

Selecting the non-negative root:

$$\alpha = \frac{-1+\sqrt{5}}{2}$$

Define β as the limit of nested reciprocals:

$$\beta = \frac{1}{\beta - 1} \implies \beta^2 - \beta - 1 = 0$$

Solving the quadratic equation:

$$\beta = \frac{1 \pm \sqrt{5}}{2}$$

Selecting the non-negative root:

$$\beta = \frac{1+\sqrt{5}}{2}$$

Combine the stabilized expressions α and β in the integrand:

$$\int_0^1 \left(\frac{-1+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} \right) dx$$

Simplify the expression inside the integral:

$$\int_0^1 \sqrt{5}\, dx$$

Since the integrand is constant:

$$\sqrt{5} \cdot (1-0) = \boxed{\sqrt{5}}$$

4.

$$\int_0^1 x \left(\tanh^{-1}(x^2) \right)^2 dx$$

Letting $u = x^2$, transforms the differential as $du = 2x\,dx$. Substituting into the integral:

$$\frac{1}{2} \int_0^1 \left(\tanh^{-1}(u) \right)^2 du$$

We set $w = \tanh^{-1}(u)$, hence $u = \tanh(w)$ and $du = \operatorname{sech}^2(w)\,dw$. Thus, transforming the integral again:

$$\frac{1}{2} \int_0^\infty w^2 \operatorname{sech}^2(w)\, dw$$

Using the identity $\operatorname{sech}^2(w) = \frac{4}{(e^w+e^{-w})^2}$, simplify further:

$$\frac{1}{2}\int_0^\infty w^2 \frac{4}{(e^w+e^{-w})^2}\,dw = 2\int_0^\infty w^2 \frac{e^{-2w}}{(1+e^{-2w})^2}\,dw$$

Consider $\frac{1}{1+e^{-2x}}$ using a geometric series:

$$\frac{1}{1+e^{-2x}} = \sum_{n=0}^{\infty}(-1)^n e^{-2nx}$$

Differentiating both sides with respect to x:

$$\frac{2e^{-2x}}{(1+e^{-2x})^2} = \sum_{n=1}^{\infty}(-1)^{n+1}2ne^{-2nx}$$

Incorporating this result into the integral:

$$\int_0^\infty w^2\left(\sum_{n=1}^{\infty}(-1)^{n+1}2ne^{-2nw}\right)dw$$

Each term $\sum_{n=1}^{\infty}(-1)^n 2n\int_0^\infty w^2 e^{-2nw}\,dw$ involves a substitution $t = 2nw$, where $dw = \frac{dt}{2n}$, thus simplifying our integral:

$$\sum_{n=1}^{\infty}(-1)^{n+1}2n\int_0^\infty w^2 e^{-2nw}\,dw = \sum_{n=1}^{\infty}(-1)^{n+1}2n\int_0^\infty \frac{t^2}{(2n)^2}e^{-t}\,\frac{dt}{2n}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\int_0^\infty t^2e^{-t}\,dt$$

Since $\int_0^\infty t^2 e^{-t}\,dt = \Gamma(3) = 2$, this part evaluates to:

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ evaluates to $\frac{\pi^2}{12}$, thus leading to the integral value:

$$I = \frac{1}{2}\left(\frac{\pi^2}{12}\right) = \boxed{\frac{\pi^2}{24}}$$

5.
$$\int \left(\sec^{-1}(x) + \cos^{-1}(x) + \csc^{-1}(x) + \sin^{-1}(x)\right)\, dx$$

Using the properties:

$$\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2} \text{ and } \sec^{-1}(x) + \csc^{-1}(x) = \frac{\pi}{2}$$

We can see our integrand is π:

$$\int \pi \, dx = \boxed{\pi x}$$

6.
$$\int \frac{2x^2 \sec^{-1}(x) - \sqrt{x^2-1}}{(x^2\sqrt{x} - \sqrt{x})^2}\, dx$$

After some simplification, our integral turns into

$$\int \frac{2x \sec^{-1}(x) - \frac{1}{x} \cdot \frac{(x^2-1)}{\sqrt{x^2-1}}}{(x^2-1)^2}\, dx = \int \frac{\frac{(1-x^2)}{x\sqrt{x^2-1}} - (-2x)\sec^{-1}(x)}{(1-x^2)^2}\, dx$$

From here, we can clearly see that this is the reverse quotient rule of our antiderivative in question:

$$\int \frac{2x^2 \sec^{-1}(x) - \sqrt{x^2-1}}{(x^2\sqrt{x} - \sqrt{x})^2}\, dx = \boxed{\frac{\sec^{-1}(x)}{1-x^2}}$$

7.
$$\int_0^1 \ln^3\left(\prod_{n=1}^{\infty}\left(x^{n^{-2}}\right)\right) dx$$

Turning our logarithm product into a sum:

$$\int_0^1 \left(\sum_{n=1}^{\infty}\left(\ln^3\left(x^{n^{-2}}\right)\right)\right)^3 dx = \int_0^1 \left(\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\ln(x)\right)\right)^3 dx$$

Using $\sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$ we can see:

$$\frac{\pi^6}{216}\int_0^1 \ln^3(x)\, dx = \frac{\pi^6}{216}\left((-1)^3(3!)\right) = \boxed{\frac{-\pi^6}{36}}$$

8.

$$\int \frac{dx}{x^3+6x^2+11x+6}$$

$$= \int \frac{dx}{(x+1)(x+2)(x+3)} = \int \left(\frac{1/2}{x+1} + \frac{-1}{x+2} + \frac{1/2}{x+3}\right) dx$$

$$= \boxed{\ln\left(\frac{\sqrt{(x+1)(x+3)}}{x+2}\right)}$$

9.

$$\int_0^1 \max\left(\cos^{-1}(x), \sin^{-1}(x)\right) dx$$

We need to find where our bounds split which we can do pretty simply by noting these three values:

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\sin(0) = 0 < 1 = \cos(0)$$

$$\therefore \int_0^1 \max\left(\cos^{-1}(x), \sin^{-1}(x)\right) dx = \int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x)\, dx + \int_{\frac{1}{\sqrt{2}}}^1 \sin^{-1}(x)\, dx$$

These two integrals both contain similar solution developments through IBP.

$$= x\cos^{-1}(x)\Big]_0^{\frac{1}{\sqrt{2}}} + \int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^2}}\, dx + x\sin^{-1}(x)\Big]_{\frac{1}{\sqrt{2}}}^1 - \int_{\frac{1}{\sqrt{2}}}^1 \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{\pi}{4\sqrt{2}} - \frac{\pi}{2} - \sqrt{1-x^2}\Big]_0^{\frac{1}{\sqrt{2}}} + \frac{\pi}{2} - \frac{\pi}{4\sqrt{2}} + \sqrt{1-x^2}\Big]_{\frac{1}{\sqrt{2}}}^1$$

After numerical evaluation, we get

$$\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x)\, dx + \int_{\frac{1}{\sqrt{2}}}^1 \sin^{-1}(x)\, dx = \boxed{1 - \sqrt{2} + \frac{\pi}{2}}$$

10.

$$\int_0^{\frac{\pi}{6}} \frac{1}{1+\tan(3x)}\,dx$$

This integral can be easily tackled with a u-sub of $u = 3x$, then after multiplying both numerator and denominator by $\cos(x)$ we get

$$\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\cos(u)+\sin(u)}\,du$$

Then after applying Queen's Rule ($t = \frac{\pi}{2} - u$). This gives us two forms of the same integral that we can add together.

$$2I = \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\cos(u)+\sin(u)}\,du + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\sin(t)}{\sin(t)+\cos(t)}\,dt$$

After changing out variable of integration in our first integral from u to t, we can add our integrals and simplify.

$$2I = \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{\sin(t)+\cos(t)}{\sin(t)+\cos(t)}\,dt = \frac{1}{3}\int_0^{\frac{\pi}{2}} 1\,dt = \frac{\pi}{6}$$

$$\therefore\ I = \boxed{\frac{\pi}{12}}$$

11.

$$\int_0^1 \left(\left(\frac{1-x^2}{1+x^2}\right)^2 \frac{1}{1+x^2}\right) dx$$

$$\tan(u) = x \implies \int_0^{\frac{\pi}{4}} \left(\frac{1-\tan^2(u)}{\sec^2(u)}\right)^2 du$$

$$= \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x)\right)^2\,du = \int_0^{\frac{\pi}{4}} \left(\cos(2x)\right)^2\,du$$

$$t = 2u \implies \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \boxed{\frac{\pi}{8}} \text{ by Wallis Product.}$$

12.

$$\lim_{a,b\to 0}\int_0^{\infty} \frac{e^{(a-e)x} - e^{(b-e)x}}{a-b}\,dx$$

$$\lim_{a,b\to 0}\frac{1}{a-b}\int_0^{\infty}\left(e^{(a-e)x} - e^{(b-e)x}\right)dx = \lim_{a,b\to 0}\frac{1}{a-b}\left(\frac{1}{b-e} - \frac{1}{a-e}\right)$$

$$= \lim_{a,b\to 0} \frac{1}{a-b}\left(\frac{(a-e)-(b-e)}{(b-e)(a-e)}\right) = \lim_{a,b\to 0} \frac{1}{a-b}\left(\frac{a-b}{(b-e)(a-e)}\right)$$

$$= \lim_{a,b\to 0}\left(\frac{1}{(b-e)(a-e)}\right) = \frac{1}{(0-e)(0-e)} = \boxed{\frac{1}{e^2}}$$

13.

$$\int_0^{\frac{\pi}{4}} \sin(2x)\prod_{n=0}^{\infty}\left(e^{(-1)^n(\tan x)^{2n}}\right)\,dx$$

Moving the product into the exponential turns it into the sum, which then we can turn into a geometric series based on the convergence of $\tan(x)$:

$$\int_0^{\frac{\pi}{4}} \sin(2x)e^{\left(\sum_{n=0}^{\infty}(-1)^n(\tan x)^{2n}\right)}\,dx = \int_0^{\frac{\pi}{4}} \sin(2x)e^{\left(\frac{1}{1+\tan^2(x)}\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sin(2x)e^{\cos^2(x)}\,dx \;\; u=\cos^2(x) \implies \int_{\frac{1}{2}}^{1} e^u\,du$$

$$= \boxed{e-\sqrt{e}}$$

14.

$$\int_0^{\infty} \frac{\cos^{-1}(x)}{(1+x^2)(\sec^{-1}(x)+\cos^{-1}(x))}\,dx$$

Given that $\cos^{-1}(1/x) = \sec^{-1}(x)$, we can make the u-sub $u = 1/x$:

$$\int_0^{\infty} \frac{\sec^{-1}(u)}{(1+u^2)(\cos^{-1}(u)+\sec^{-1}(u))}\,du$$

$$\therefore\; 2I = \int_0^{\infty} \frac{\sec^{-1}(x)+\cos^{-1}(x)}{(1+x^2)(\sec^{-1}(x)+\cos^{-1}(x))}\,dx = \int_0^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$I = \boxed{\frac{\pi}{4}}$$

15.

$$\int_{-1}^{1} \underbrace{\sin\left(\cos^{-1}\left(\sin\left(\cos^{-1}\left(\cdots(x)\right)\right)\right)\right)}_{2024\ (\sin(\cos^{-1}(\cdots)))\text{'s}}\, dx$$

See that any even number of compositions of $\sin(\cos^{-1}(\cdots)) = |x|$ with an example below of 2 compositions:

$$\sin(\cos^{-1}(\sin(\cos^{-1}(x)))) = \sqrt{1 - \left(\sqrt{1-x^2}\right)^2} = \sqrt{x^2} = |x|$$

This holds for 2024 compositions as well, making our integral

$$\int_{-1}^{1} |x|\, dx = 2\int_{0}^{1} x\, dx = 2 \cdot \frac{1}{2} = \boxed{1}$$

16.

$$\sum_{n=0}^{\infty} \int \sin^{2n+1}(x)\, dx$$

$\sin(x)$ is absolutely convergent, meaning we can make our sum into a geometric series:

$$\int \sum_{n=0}^{\infty} \sin^{2n+1}(x)\, dx = \int \frac{\sin(x)}{1-\sin^2(x)}\, dx = \int \frac{\sin(x)}{\cos^2(x)}\, dx$$

$$= \int \tan(x)\sec(x)\, dx = \boxed{\sec(x)}$$

17.

$$\int_{0}^{\frac{\pi}{2}} \left(\log_{10}(\sin^2(x)) + \log_{10}(\cos^2(x))\right)\, dx$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{2}{\log(10)} \left(\log(\sin(x)) + \log(\cos(x))\right)\, dx$$

$$= \frac{2}{\log(10)} \int_{0}^{\frac{\pi}{2}} \log\left(\frac{\sin(2x)}{2}\right)\, dx$$

$$u = 2x \implies \frac{2}{\log(10)}\left(-\frac{\pi}{2}\log(2) + \frac{1}{2}\int_{0}^{\pi} \log\left(\sin(u)\right)\, du\right)$$

$$t = \frac{\pi}{2} - u \implies \frac{2}{\log(10)}\left(-\frac{\pi}{2}\log(2) + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\cos(t)\right)\, dt\right)$$

$$= \frac{2}{\log(10)} \left(-\frac{\pi}{2} \log(2) + \int_0^{\frac{\pi}{2}} \log\left(\cos(t)\right) \, dt \right) = \frac{-\pi \log(2)}{\log(10)} + \frac{1}{2} I$$

$$\therefore \frac{1}{2} I = \frac{-\pi \log(2)}{\log(10)} \implies I = \frac{-2\pi \log(2)}{\log(10)} = \boxed{-\pi \log_{10}(4)}$$

18.

$$\int_0^5 \left(\frac{1}{\sqrt{5x}} \left(\frac{1+\sqrt{5}}{2} \right)^{\lceil x \rceil} - \frac{1}{\sqrt{5x}} \left(\frac{1-\sqrt{5}}{2} \right)^{\lceil x \rceil} \right) dx$$

After factoring out $1/\sqrt{x}$, we can see Binet's Formula for the Fibonacci sequence. We can convert our integral into a sum to show this:

$$\sum_{n=1}^{5} \left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n} \right) \int_{n-1}^{n} \frac{dx}{\sqrt{x}}$$

$$= \sum_{n=1}^{5} F_n \cdot 2 \left(\sqrt{n} - \sqrt{n-1} \right) = \boxed{10\sqrt{5} - 2\sqrt{3} - 2\sqrt{2} - 8}$$

19.

$$\int_0^\infty e^{-x^{1/3}} \, dx$$

$$u^3 = x \implies 3 \int_0^\infty u^2 e^{-u} \, du = 3 \cdot \Gamma(3) = \boxed{6}$$

20.

$$\int_1^\infty \frac{1}{\left\lceil \sum_{n=1}^{\lfloor x \rfloor} \frac{\lfloor x \rfloor (-1)^{\lfloor x+1 \rfloor}}{n^{\lfloor x \rfloor}} \right\rceil} dx$$

We can turn this integral into a summation based on the integer value returned by the floor function:

$$\sum_{k=1}^{\infty} \frac{1}{\left\lceil \sum_{n=1}^{k} \frac{k(-1)^{k+1}}{n^k} \right\rceil}$$

$$= \frac{1}{\left\lfloor \sum_{n=1}^{1} \frac{(1)(-1)^2}{n^1} \right\rfloor} + \frac{1}{\left\lfloor \sum_{n=1}^{2} \frac{(2)(-1)^3}{n^2} \right\rfloor} + \ldots = 1 + \frac{1}{\left\lfloor 2\left(-1-\frac{1}{2^2}\right) \right\rfloor} \cdots$$

Using $\lfloor -x \rfloor = -\lceil x \rceil$ we can see all of our terms after 1 cancel.

$$= 1 - \frac{1}{\left\lceil 2\left(1+\frac{1}{2^2}\right) \right\rceil} + \frac{1}{\left\lfloor 3\left(1+\frac{1}{2^3}+\frac{1}{3^3}\right) \right\rfloor} \cdots = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \cdots = \boxed{1}$$

21.

$$\int \frac{3x^4 - 4x^2 + 5}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= \int \frac{3x^4 - 4x^2 + 5 + (5x^4 - 5x^4) + (5x^2 - 5x^2)}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= \int \frac{5\sqrt{x^4 - x^2 + 1}}{x^6} + \frac{-2x^4 + x^2}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

Then we can split our integral by the linearity of the integration operator.

$$I = I_0 + I_1$$

$$\text{where} \quad I_0 = \int \frac{5\sqrt{x^4 - x^2 + 1}}{x^6}\, dx \quad \text{and} \quad I_1 = \int \frac{-2x^4 + x^2}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

Then we can use IBP on I_0.

$$I_0 = -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} + \int \frac{1}{x^5} \cdot \frac{2x^3 - x}{\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} - I_1$$

$$\therefore I = -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} - I_1 + I_1 \implies I = \boxed{-\frac{\sqrt{x^4 - x^2 + 1}}{x^5}}$$

22.

$$\int \frac{\cos^2(3x)}{1 + \cos(3x)}\, dx$$

$$u = 3x \implies \frac{1}{3} \int \frac{\cos^2(u)}{1 + \cos(u)}\, du$$

Using polynomial division we can simplify our integral.

$$\frac{1}{3}\int \cos(u) - 1 + \frac{1}{1+\cos(u)}\,du = \frac{1}{3}\left(\sin(u) - u + \int \frac{1-\cos(u)}{\sin^2(u)}\,du\right)$$

$$= \frac{1}{3}\left(\sin(u) - u + \int \csc^2(u) - \csc(u)\cot(u)\,du\right)$$

$$= \frac{1}{3}\left(\sin(u) - u - \cot(u) + \csc(u)\right) = \boxed{\tfrac{1}{3}\left(\sin(3x) - 3x - \cot(3x) + \csc(3x)\right)}$$

23.

$$\int \frac{\sin x}{\cot x}\cdot\frac{\tan x}{\cos x}\,dx$$

$$= \int \tan^3(x)\,dx = \int \tan(x)\left(\sec^2(x) - 1\right)\,dx$$

$$= \log\left(\cos(x)\right) + \int \tan(x)\sec^2(x)\,dx = \boxed{\log\left(\cos(x)\right) + \tfrac{1}{2}\sec^2(x)}$$

Note: This solution used the substitution $u = \sec(x)$ to solve the integral, however, $u = \tan(x)$ can also be used for the result of $\boxed{\log\left(\cos(x)\right) + \tfrac{1}{2}\tan^2(x)}$

24.

$$\int \frac{\tan\left(\frac{x}{4}\right)}{\sec\left(\frac{x}{2}\right)}\,dx$$

$$u = \frac{x}{4} \implies 4\int \cos(2u)\tan(u)\,du = 4\int \left(2\cos^2(u) - 1\right)\tan(u)\,du$$

$$= 4\int 2\cos(u)\sin(u) - \tan(u)\,du = 4\left(-\cos^2(u) - \log(\sec(x))\right)$$

$$= \boxed{4\log\left(\cos\left(\tfrac{x}{4}\right)\right) - 4\cos^2\left(\tfrac{x}{4}\right)}$$

25.

$$\int \tan^{-1}\left(\tan(x)\right)\,dx$$

$$u = \tan^{-1}\left(\tan(x)\right) \implies \int u\,du = \frac{u^2}{2} = \boxed{\tfrac{1}{2}\left(\tan^{-1}\left(\tan(x)\right)\right)^2}$$

26.

$$\int_1^\infty \frac{x-\lfloor x \rfloor}{x^3}\,dx$$

$$= \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^2} - \frac{n}{x^3}\,dx = \sum_{n=1}^{\infty} \frac{n}{2}\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) - \left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{(n+1)^2} - \frac{1}{n} = 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n+(1-1)}{(n+1)^2} - \frac{1}{n}$$

$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{(n+1)^2} - \frac{1}{n} = 1 + \frac{1}{2}\left(-1 - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{2} - \frac{1}{2}\left(\zeta(2) - 1\right) = \boxed{1 - \frac{\pi^2}{12}}$$

27.

$$\int \frac{x^8}{x^6+1}\,dx$$

$$= \int \frac{x^8 + (x^2 - x^2)}{x^6+1}\,dx = \int x^2 + \frac{x^2}{x^6+1}\,dx = \frac{x^3}{3} + \int \frac{x^2}{x^6+1}\,dx$$

$$u = x^3 \implies \frac{x^3}{3} + \frac{1}{3}\int \frac{du}{u^2+1} = \boxed{\frac{1}{3}\left(x^3 - \tan^{-1}\left(x^3\right)\right)}$$

28.

$$\int \left(\frac{1}{\cot(x)+\csc(x)} + \frac{1}{\cot(x)-\csc(x)}\right)dx$$

$$= \int \frac{(\cot(x)-\csc(x)) + (\cot(x)+\csc(x))}{\cot^2(x)-\csc^2(x)}\,dx = \int \frac{2\cot(x)}{(-1)}\,dx$$

$$u = \sin(x) \implies -2\int \frac{du}{u} = -2\log(\sin(x)) = \boxed{2\ln\left(\csc(x)\right)}$$