WP4:

Problem 1.
We assume $a, b > 0$ since $\pi > 0$, by the definition of π. We see that

$$\pi^2 = \frac{a}{b} \implies \pi = \sqrt{\frac{a}{b}}.$$

This is noteworthy, since π is neither complex nor negative, therefore requiring a positive value of $\frac{a}{b}$. Therefore, both a and b are positive, or both a and b are negative. To account for the loss of generality, we see that choosing either option is equivalent as

$$\frac{a}{b} = \frac{-a}{-b}.$$

This reasoning implies that we are allowed to assume $a, b > 0$ as there is no loss in generality in choosing this.

Problem 2.
We can use the fact that $\mathbb{Q}$ is closed under multiplication to find a contradiction in the statement that $\pi \in \mathbb{Q}$. This implies that $\pi^2 \in \mathbb{Q}$ using the aforementioned closure of $\mathbb{Q}$ under multiplication ($\pi \cdot \pi = \pi^2$). But our proof shows that $\pi^2 \notin \mathbb{Q}$, therefore we have reached our necessary contradiction. Hence π is irrational; showing that the proof of π^2 being irrational also proves that π is irrational.

Problem 3.
By construction,

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) \pm \cdots\right).$$

We can use $\pi^2 = \frac{a}{b}$ by assumption, which implies

$$F(x) = b^n \left(\left(\frac{a}{b}\right)^{2n} f(x) - \left(\frac{a}{b}\right)^{2n-2} f^{(2)}(x) \pm \cdots\right)$$

$$\implies F(x) = \frac{a^{2n}}{b^n} f(x) - \frac{a^{2n-2}}{b^{n-2}} f^{(2)}(x) \pm \cdots,$$

and by part (iii) of the given Lemma, $f^{(k)}(0)$, $f^{(k)}(1) \in \mathbb{Z}$, $\forall k \in \mathbb{Z}_{\geq 0}$ (meaning any linear combination of these terms will be an integer for any k value, after multiplying through by b^n). Since $\pi^2 > 1$, $a > b$, and the power of each a is greater than the powers of b, we must get cancellation.

Since $\mathbb{Z}$ is closed under multiplication and addition, we can use the aforementioned part (iii) of the Lemma to see that $F(1)$ and $F(0)$ are linear combinations of $f^{(k)}(1)$ and $f^{(k)}(0)$ terms, respectively, showing that $F(1)$ and $F(0)$ are integers.

Problem 4.

Using part (ii) of the Lemma, we can see that the upper bound of $f(x)$ is $\frac{1}{n!}$. We can define a slack variable (or slack function of x, more accurately) $s(x) > 0$, by

$$f(x) + s(x) = \frac{1}{n!}$$

$$\implies f(x) = \frac{1}{n!} - s(x).$$

Now we can substitute this into the LHS of our inequality, which we will write as I for simplicity,

$$I = \pi \int_0^1 a^n \left(\frac{1}{n!} - s(x) \right) \sin(\pi x)\, dx.$$

By linearity of the integral,

$$I = \pi a^n \left(\underbrace{\frac{1}{n!} \int_0^1 \sin(\pi x)\, dx}_{I_1} - \underbrace{\int_0^1 s(x) \sin(\pi x)\, dx}_{I_2} \right);$$

for clarity of notation,

$$I = \pi a^n (I_1 - I_2).$$

Since $s(x) > 0$ and $\sin(\pi x) \geq 0$ on $[0, 1]$, I_2 is nonnegative. We can now omit I_2 to create an inequality to construct an upper bound for I, giving

$$I < \pi a^n \cdot I_1 = \pi a^n \frac{1}{n!} \int_0^1 \sin(\pi x)\, dx$$

Because $\sin(\pi x) \leq 1$ for all values of x,

$$\int_0^1 \sin(\pi x)\, dx \leq \int_0^1 1\, dx = 1$$

Therefore,

$$I < \pi a^n (1) = \frac{\pi a^n}{n!},$$

proving our required inequality.