WP3: A Proof of the Irrationality of e

Introduction

We will follow Ivan Niven's proof that e is irrational from *Proofs by the Book* by Aigner and Ziegler. We will also see that Fourier used the same ideas in his proof. This proof only entails basic calculus facts that we will list in our Facts from Calculus below.

Facts from Calculus

Before presenting the main argument, we recall some fundamental facts from calculus:

1. $$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

2. $\forall n \in \mathbb{Z}_{\geq 0}$
$$n! = 1 \cdot 2 \cdot 3 \cdots n,$$

3. $\forall x \in \mathbb{R}$ s.t. $|x| < 1$,
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Proof of Irrationality

Assume that e is rational (proceeding by contradiction). Then by definiton, $\exists a, b \in \mathbb{Z}_{>0}$ s.t.
$$e = \frac{a}{b}.$$

Multiplying both sides of this equation by $n!$ ($n \in \mathbb{Z}_{\geq 0}$) yields
$$n!\, e = n! \cdot \frac{a}{b} \iff bn!e = n!a$$

by $\boxed{2}$ from our Facts from Calculus.

Now, express $b\,n!\,e$ using the series definition of e from $\boxed{1}$:
$$bn!\, e = bn! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) + bn! \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right)$$

For simplicity, we can denote this as:
$$L = bn! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$$

and
$$R = bn! \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots\right).$$

Observe that $L \in \mathbb{Z}, \forall n \geq b$ because $n!/m! \in \mathbb{Z} \quad \forall m, n$ s.t. $0 \leq m \leq n$, and integers are closed under additon.

Now we turn to R,

$$R = b\left(\frac{n!}{(n+1)n!} + \frac{n!}{n!(n+1)(n+2)} + \frac{n!}{n!(n+1)(n+2)(n+3)} + \cdots\right)$$
$$= b\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right)$$

Each term in this series is $\geq \frac{b}{n+1}$, which means $R > \frac{b}{n+1}$.

Now we will utilize $\boxed{3}$, by bounding R with a geometric series (which we can do as $(n+1)(n+2) > (n+1)^2$ in this case):

$$R < \frac{b}{n+1} + \frac{b}{(n+1)^2} + \frac{b}{(n+1)^3} + \cdots$$
$$= \frac{b}{n+1}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right)$$

We can see that the RHS is $\boxed{3}$ where $x = \frac{1}{n+1}$:

$$\frac{1}{n+1}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)^n = \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n}.$$

From this, we can see that this term has to be greater than R, therefore

$$\frac{b}{n+1} < R < \frac{b}{n} \iff \frac{b}{n+1} < b\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots\right) < \frac{b}{n}.$$

Dividing out b, as it is a natural number (> 0) by definition:

$$\implies \frac{1}{n+1} < \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots < \frac{1}{n}$$

These bounds on R allow for us to see that there is no $n \in \mathbb{Z}$ such that we can have an integer solution to the system of inequalities bounding R.

In conclusion, our original assumption for e to be rational was built upon $L + R \in \mathbb{Z} \wedge L \in \mathbb{Z} \implies R \in \mathbb{Z}$ since integers are closed under addition. However as seen above, $R \in (\frac{b}{n+1}, \frac{b}{n})$, and for adequately large n (like any $n \geq b$), no integer exists in this interval. This contradiction implies that our assumption that e is rational must be false.