1. Finals Problem :

$$\int_0^{\frac{\pi}{2}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{d^{k^2}}{dx^{k^2}} \left(\sin(x) \right) \right)^k dx$$

The derivatives of $\sin(x)$ based on k^2 modulo 4 are:

$$\frac{d^{k^2}}{dx^{k^2}} \sin(x) = \begin{cases} \sin x & \text{if } k^2 \equiv 0 \pmod 4, \\ \cos x & \text{if } k^2 \equiv 1 \pmod 4, \\ -\sin x & \text{if } k^2 \equiv 2 \pmod 4, \\ -\cos x & \text{if } k^2 \equiv 3 \pmod 4. \end{cases}$$

Using the above patterns, the integral can be expanded into:

$$\int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} \sin^{2n}(x) - \sum_{n=0}^{\infty} \cos^{2n+1}(x) \right) dx$$

This accounts for the alternate signs in the series expansion for odd powers of $\cos(x)$.

The series for even and odd powers of sine and cosine can be written as:

$$\sum_{n=0}^{\infty} \sin^{2n}(x) = \frac{1}{1 - \sin^2(x)} = \sec^2(x)$$

$$\sum_{n=0}^{\infty} \cos^{2n+1}(x) = \cos(x) \sum_{n=0}^{\infty} (\cos^2(x))^n = \frac{\cos(x)}{1 - \cos^2(x)} = \cot(x)\csc(x)$$

Thus, the integral simplifies to:

$$\int_0^{\frac{\pi}{2}} \left(\sec^2(x) - \cot(x)\csc(x) \right) dx = \tan(x) + \csc(x) \Big|_0^{\frac{\pi}{2}}$$

$$= \tan(\frac{\pi}{2}) - \tan(0) + \csc(\frac{\pi}{2}) - \csc(0)$$

$$= \tan(\frac{\pi}{2}) - 0 + 1 - \csc(0)$$

To address the divergences at $x = \frac{\pi}{2}$ and $x = 0$:

$$\lim_{x \to \frac{\pi}{2}} \left(\tan x - \csc(\frac{\pi}{2} - x) \right)$$

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Using the identity $\csc(\frac{\pi}{2} - x) = \sec(x)$, we find:

$$\lim_{x \to \frac{\pi}{2}} \left(\tan x - \sec x \right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\cos x}$$

Apply L'Hôpital's Rule due to the indeterminate form $\frac{0}{0}$:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin x} = \frac{0}{-1} = 0$$

Therefore our final step is: $0 - 0 + 1 = \boxed{1}$

Note: There are many other ways of solving this integral, my favorite in particular is using the Wallis Product to show

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\, dx = \int_0^{\frac{\pi}{2}} \cos^n(x)\, dx$$

to turn our geometric series into either

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin(x)}\, dx \ \text{ or } \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos(x)}\, dx$$

Both of which equal 1.

2.

$$\int_0^{\frac{\pi}{2}} \left(\frac{\cos x}{\sec^2(x) - \tan x \sec x} - \frac{\cos^2(x)}{\sec x + \tan x} \right)^3 dx$$

Convert $\sec x$ and $\tan x$ to their equivalents in terms of $\sin x$ and $\cos x$:

$$\sec x = \frac{1}{\cos x}, \quad \tan x = \frac{\sin x}{\cos x}$$

The first term:

$$\sec^2 x - \tan x \sec x = \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{1 - \sin x}{\cos^2 x}$$

$$\text{Thus, } \frac{\cos x}{\sec^2(x) - \tan x \sec x} = \frac{\cos x}{\frac{1-\sin x}{\cos^2 x}} = \frac{\cos^3 x}{1 - \sin x}$$

The second term:

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x}$$

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$$\text{Thus, } \frac{\cos^2 x}{\sec x + \tan x} = \frac{\cos^2 x}{\frac{1+\sin x}{\cos x}} = \frac{\cos^3 x}{1 + \sin x}$$

Combining the two terms gives us

$$\frac{\cos^3 x}{1 - \sin x} - \frac{\cos^3 x}{1 + \sin x} = \cos^3 x \left(\frac{(1 + \sin x) - (1 - \sin x)}{1 - \sin^2 x} \right)$$

$$= \frac{2 \sin x \cos^3 x}{\cos^2 x} = 2 \sin x \cos x = \sin(2x)$$

Plugging our combined terms back into the integrand we get

$$\int_0^{\frac{\pi}{2}} (\sin(2x))^3 \, dx$$

Then, we use $u = 2x, du = 2dx$:

$$\frac{1}{2} \int_0^\pi (\sin(u))^3 \, du = \frac{1}{2} \int_0^\pi \sin x (1 - \cos^2 x) \, du$$

$$= \frac{1}{2} \int_0^\pi \sin x \, du - \frac{1}{2} \int_0^\pi \sin x \cos^2 x \, du$$

$$= -\frac{1}{2} \cos(u) \Big|_0^\pi + \frac{1}{2} \cdot \frac{1}{3} \cos^3 u \Big|_0^\pi = -\frac{1}{2}(-1-1) + \frac{1}{6}(-1-1) = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$$

3. Finals Problem :

$$\lim_{n \to \infty} \int_0^1 \underbrace{\sqrt{1 - \sqrt{1 - \sqrt{\cdots}}}}_{n \text{ times}} + \underbrace{\frac{1}{\frac{1}{\frac{1}{\cdots} - 1} - 1}}_{n \text{ times}} \, dx$$

Define α as the limit of nested radicals:

$$\alpha = \sqrt{1 - \alpha}$$

Squaring both sides gives:

$$\alpha^2 = 1 - \alpha \implies \alpha^2 + \alpha - 1 = 0$$

Solving the quadratic equation:

$$\alpha = \frac{-1 \pm \sqrt{5}}{2}$$

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Selecting the non-negative root:

$$\alpha = \frac{-1 + \sqrt{5}}{2}$$

Define β as the limit of nested reciprocals:

$$\beta = \frac{1}{\beta - 1} \implies \beta^2 - \beta - 1 = 0$$

Solving the quadratic equation:

$$\beta = \frac{1 \pm \sqrt{5}}{2}$$

Selecting the non-negative root:

$$\beta = \frac{1 + \sqrt{5}}{2}$$

Combine the stabilized expressions α and β in the integrand:

$$\int_0^1 \left(\frac{-1 + \sqrt{5}}{2} + \frac{1 + \sqrt{5}}{2} \right) dx$$

Simplify the expression inside the integral:

$$\int_0^1 \sqrt{5} \, dx$$

Since the integrand is constant:

$$\sqrt{5} \cdot (1 - 0) = \boxed{\sqrt{5}}$$

4.

$$\int_0^1 x \left(\tanh^{-1}(x^2) \right)^2 dx$$

Letting $u = x^2$, transforms the differential as $du = 2x \, dx$. Substituting into the integral:

$$\frac{1}{2} \int_0^1 \left(\tanh^{-1}(u) \right)^2 du$$

We set $w = \tanh^{-1}(u)$, hence $u = \tanh(w)$ and $du = \text{sech}^2(w) \, dw$. Thus, transforming the integral again:

$$\frac{1}{2} \int_0^\infty w^2 \text{sech}^2(w) \, dw$$

Using the identity $\text{sech}^2(w) = \frac{4}{(e^w + e^{-w})^2}$, simplify further:

$$\frac{1}{2} \int_0^\infty w^2 \frac{4}{(e^w + e^{-w})^2} \, dw = 2 \int_0^\infty w^2 \frac{e^{-2w}}{(1 + e^{-2w})^2} \, dw$$

Consider $\frac{1}{1+e^{-2x}}$ using a geometric series:

$$\frac{1}{1 + e^{-2x}} = \sum_{n=0}^\infty (-1)^n e^{-2nx}$$

Differentiating both sides with respect to x:

$$\frac{2e^{-2x}}{(1 + e^{-2x})^2} = \sum_{n=1}^\infty (-1)^{n+1} 2n e^{-2nx}$$

Incorporating this result into the integral:

$$\int_0^\infty w^2 \left(\sum_{n=1}^\infty (-1)^{n+1} 2n e^{-2nw} \right) dw$$

Each term $\sum_{n=1}^\infty (-1)^n 2n \int_0^\infty w^2 e^{-2nw} \, dw$ involves a substitution $t = 2nw$, where $dw = \frac{dt}{2n}$, thus simplifying our integral:

$$\sum_{n=1}^\infty (-1)^{n+1} 2n \int_0^\infty w^2 e^{-2nw} \, dw = \sum_{n=1}^\infty (-1)^{n+1} 2n \int_0^\infty \frac{t^2}{(2n)^2} e^{-t} \frac{dt}{2n}$$

$$= \frac{1}{4} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} \int_0^\infty t^2 e^{-t} \, dt$$

Since $\int_0^\infty t^2 e^{-t} \, dt = \Gamma(3) = 2$, this part evaluates to:

$$= \frac{1}{2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}$$

The series $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}$ evaluates to $\frac{\pi^2}{12}$, thus leading to the integral value:

$$\frac{1}{2} \left(\frac{\pi^2}{12} \right) = \boxed{\frac{\pi^2}{24}}$$

5.

$$\int \left(\sec^{-1}(x) + \cos^{-1}(x) + \csc^{-1}(x) + \sin^{-1}(x) \right) dx$$

Using the properties:

$$\cos^{-1}(x) + \sin^{-1}(x) = \frac{\pi}{2} \text{ and } \sec^{-1}(x) + \csc^{-1}(x) = \frac{\pi}{2}$$

We can see our integrand is π:

$$\int \pi \, dx = \boxed{\pi x}$$

6.

$$\int \frac{2x^2 \sec^{-1}(x) - \sqrt{x^2 - 1}}{(x^2\sqrt{x} - \sqrt{x})^2} \, dx$$

After some simplification, our integral turns into

$$\int \frac{2x \sec^{-1}(x) - \frac{1}{x} \cdot \frac{(x^2-1)}{\sqrt{x^2-1}}}{(x^2 - 1)^2} \, dx = \int \frac{\frac{(1-x^2)}{x\sqrt{x^2-1}} - (-2x) \sec^{-1}(x)}{(1 - x^2)^2} \, dx$$

From here, we can clearly see that this is the reverse quotient rule of our antiderivative in question:

$$\int \frac{2x^2 \sec^{-1}(x) - \sqrt{x^2 - 1}}{(x^2\sqrt{x} - \sqrt{x})^2} \, dx = \boxed{\frac{\sec^{-1}(x)}{1-x^2}}$$

7.

$$\int_0^1 \ln^3 \left(\prod_{n=1}^{\infty} \left(x^{n^{-2}} \right) \right) dx$$

Turning our logarithm product into a sum:

$$\int_0^1 \left(\sum_{n=1}^{\infty} \left(\ln^3 \left(x^{n^{-2}} \right) \right) \right)^3 dx = \int_0^1 \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^2} \ln(x) \right) \right)^3 dx$$

Using $\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$ we can see:

$$\frac{\pi^6}{216} \int_0^1 \ln^3(x) \, dx = \frac{\pi^6}{216} \left((-1)^3(3!) \right) = \boxed{\frac{-\pi^6}{36}}$$

8.

$$\int \frac{dx}{x^3 + 6x^2 + 11x + 6}$$

$$= \int \frac{dx}{(x+1)(x+2)(x+3)} = \int \left(\frac{1/2}{x+1} + \frac{-1}{x+2} + \frac{1/2}{x+3} \right) dx$$

$$= \boxed{\ln \left(\frac{\sqrt{(x+1)(x+3)}}{x+2} \right)}$$

9.

$$\int_0^1 \max \left(\cos^{-1}(x), \sin^{-1}(x) \right) dx$$

We need to find where our bounds split which we can do pretty simply by noting these three values:

$$\cos \left(\frac{\pi}{4} \right) = \sin \left(\frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}$$

$$\sin(0) = 0 < 1 = \cos(0)$$

$$\therefore \int_0^1 \max \left(\cos^{-1}(x), \sin^{-1}(x) \right) dx = \int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x) \, dx + \int_{\frac{1}{\sqrt{2}}}^1 \sin^{-1}(x) \, dx$$

These two integrals both contain similar solution developments through IBP. After numerical evaluation, we get

$$\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1}(x) \, dx + \int_{\frac{1}{\sqrt{2}}}^1 \sin^{-1}(x) \, dx = \boxed{1 - \sqrt{2} + \frac{\pi}{2}}$$

10.

$$\int_0^{\frac{\pi}{6}} \frac{1}{1 + \tan(3x)} \, dx$$

This integral can be easily tackled with a u-sub of $u = 3x$, then after multiplying both numerator and denominator by $\cos(x)$ we get

$$\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\cos(u) + \sin(u)} \, du$$

Then after applying Queen's Rule ($t = \frac{\pi}{2} - u$). This gives us two forms of the same integral that we can add together.

$$2I = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\cos(u) + \sin(u)} \, du + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sin(t)}{\sin(t) + \cos(t)} \, dt$$

After changing out variable of integration in our first integral from u to t, we can add our integrals and simplify.

$$2I = \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sin(t) + \cos(t)}{\sin(t) + \cos(t)} \, dt = \frac{1}{3} \int_0^{\frac{\pi}{2}} 1 \, dt = \frac{\pi}{6}$$

$$\therefore \ I = \boxed{\frac{\pi}{12}}$$

11.

$$\int_0^1 \left(\left(\frac{1 - x^2}{1 + x^2} \right)^2 \frac{1}{1 + x^2} \right) dx$$

$$\tan(u) = x \implies \int_0^{\frac{\pi}{4}} \left(\frac{1 - \tan^2(u)}{\sec^2(u)} \right)^2 du$$

$$= \int_0^{\frac{\pi}{4}} \left(\cos^2(x) - \sin^2(x) \right)^2 du = \int_0^{\frac{\pi}{4}} \left(\cos(2x) \right)^2 du$$

$$t = 2u \implies \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2(t) \, dt = \boxed{\frac{\pi}{8}} \text{ by Wallis Product.}$$

12.

$$\lim_{a,b \to 0} \int_0^\infty \frac{e^{(a-e)x} - e^{(b-e)x}}{a - b} \, dx$$

$$\lim_{a,b \to 0} \frac{1}{a - b} \int_0^\infty \left(e^{(a-e)x} - e^{(b-e)x} \right) dx = \lim_{a,b \to 0} \frac{1}{a - b} \left(\frac{1}{b - e} - \frac{1}{a - e} \right)$$

$$= \lim_{a,b \to 0} \frac{1}{a - b} \left(\frac{(a - e) - (b - e)}{(b - e)(a - e)} \right) = \lim_{a,b \to 0} \frac{1}{a - b} \left(\frac{a - b}{(b - e)(a - e)} \right)$$

$$= \lim_{a,b \to 0} \left(\frac{1}{(b - e)(a - e)} \right) = \frac{1}{(0 - e)(0 - e)} = \boxed{\frac{1}{e^2}}$$

13.

$$\int_0^{\frac{\pi}{4}} \sin(2x) \prod_{n=0}^{\infty} \left(e^{(-1)^n (\tan x)^{2n}} \right) dx$$

Moving the product into the exponential turns it into the sum, which then we can turn into a geometric series based on the convergence of $\tan(x)$:

$$\int_0^{\frac{\pi}{4}} \sin(2x) e^{\left(\sum_{n=0}^{\infty} (-1)^n (\tan x)^{2n} \right)} dx = \int_0^{\frac{\pi}{4}} \sin(2x) e^{\left(\frac{1}{1+\tan^2(x)} \right)} dx$$

$$= \int_0^{\frac{\pi}{4}} \sin(2x) e^{\cos^2(x)} dx \quad u = \cos^2(x) \implies \int_{\frac{1}{2}}^1 e^u \, du$$

$$= \boxed{e - \sqrt{e}}$$

14.

$$\int_0^{\infty} \frac{\cos^{-1}(x)}{(1+x^2)(\sec^{-1}(x) + \cos^{-1}(x))} dx$$

Given that $\cos^{-1}(1/x) = \sec^{-1}(x)$, we can make the u-sub $u = 1/x$:

$$\int_0^{\infty} \frac{\sec^{-1}(u)}{(1+u^2)(\cos^{-1}(u) + \sec^{-1}(u))} du$$

$$\therefore 2I = \int_0^{\infty} \frac{\sec^{-1}(x) + \cos^{-1}(x)}{(1+x^2)(\sec^{-1}(x) + \cos^{-1}(x))} dx = \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

$$I = \boxed{\frac{\pi}{4}}$$

15.

$$\int_{-1}^1 \underbrace{\sin\left(\cos^{-1}\left(\sin\left(\cos^{-1}\left(\cdots(x)\right)\right)\right)\right)}_{2024\ (\sin\,(\cos^{-1}(\cdots)))\text{'s}} dx$$

See that any even number of compositions of $\sin(\cos^{-1}(\cdots)) = |x|$ with an example below of 2 compositions:

$$\sin(\cos^{-1}(\sin(\cos^{-1}(x)))) = \sqrt{1 - \left(\sqrt{1-x^2}\right)^2} = \sqrt{x^2} = |x|$$

This holds for 2024 compositions as well, making our integral

$$\int_{-1}^{1} |x| \, dx = 2 \int_0^1 x \, dx = 2 \cdot \frac{1}{2} = \boxed{1}$$

16.
$$\sum_{n=0}^{\infty} \int \sin^{2n+1}(x) \, dx$$

$\sin(x)$ is absolutely convergent, meaning we can make our sum into a geometric series:

$$\int \sum_{n=0}^{\infty} \sin^{2n+1}(x) \, dx = \int \frac{\sin(x)}{1 - \sin^2(x)} \, dx = \int \frac{\sin(x)}{\cos^2(x)} \, dx$$

$$= \int \tan(x) \sec(x) \, dx = \boxed{\sec(x)}$$

17.
$$\int_0^{\frac{\pi}{2}} \left(\log_{10}(\sin^2(x)) + \log_{10}(\cos^2(x)) \right) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2}{\log(10)} \left(\log(\sin(x)) + \log(\cos(x)) \right) \, dx$$

We can quickly see using Queen's Rule that

$$\int_0^{\frac{\pi}{2}} \log(\sin(x)) \, dx = \int_0^{\frac{\pi}{2}} \log(\cos(x)) \, dx = -\frac{\pi}{2} \log(2)$$

We know this integral as Euler's Log-Sine Integral.

$$\frac{-2\pi \log(2)}{\log(10)} = \boxed{-\pi \log_{10}(4)}$$

18.
$$\int_0^5 \left(\frac{1}{\sqrt{5}x} \left(\frac{1 + \sqrt{5}}{2} \right)^{\lceil x \rceil} - \frac{1}{\sqrt{5}x} \left(\frac{1 - \sqrt{5}}{2} \right)^{\lceil x \rceil} \right) \, dx$$

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After factoring out $1/\sqrt{x}$, we can see Binet's Formula for the Fibonacci sequence. We can convert our integral into a sum to show this:

$$\sum_{n=1}^{5} \left(\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \int_{n-1}^{n} \frac{dx}{\sqrt{x}}$$

$$= \sum_{n=1}^{5} F_n \cdot 2\left(\sqrt{n} - \sqrt{n-1} \right) = \boxed{10\sqrt{5} - 2\sqrt{3} - 2\sqrt{2} - 8}$$

19.

$$\int_{0}^{\infty} e^{-x^{1/3}} \, dx$$

$$u^3 = x \implies 3\int_{0}^{\infty} u^2 e^{-u} \, du = 3 \cdot \Gamma(3) = \boxed{6}$$

20.

$$\int_{1}^{\infty} \frac{1}{\left\lceil \displaystyle\sum_{n=1}^{\lfloor x \rfloor} \frac{\lfloor x \rfloor \, (-1)^{\lfloor x+1 \rfloor}}{n^{\lfloor x \rfloor}} \right\rceil} \, dx$$

We can turn this integral into a summation based on the integer value returned by the floor function:

$$\sum_{k=1}^{\infty} \frac{1}{\left\lceil \displaystyle\sum_{n=1}^{k} \frac{k(-1)^{k+1}}{n^k} \right\rceil}$$

$$= \frac{1}{\left\lceil \displaystyle\sum_{n=1}^{1} \frac{(1)(-1)^2}{n^1} \right\rceil} + \frac{1}{\left\lceil \displaystyle\sum_{n=1}^{2} \frac{(2)(-1)^3}{n^2} \right\rceil} + \ldots = 1 + \frac{1}{\left\lceil 2\left(-1 - \frac{1}{2^2} \right) \right\rceil} \ldots$$

Using $\lfloor -x \rfloor = -\lceil x \rceil$ we can see all of our terms after 1 cancel.

$$= 1 - \frac{1}{\left\lceil 2\left(1 + \frac{1}{2^2} \right) \right\rceil} + \frac{1}{\left\lceil 3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} \right) \right\rceil} \ldots = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} \ldots = \boxed{1}$$

21.

$$\int \frac{3x^4 - 4x^2 + 5}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= \int \frac{3x^4 - 4x^2 + 5 + (5x^4 - 5x^4) + (5x^2 - 5x^2)}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= \int \frac{5\sqrt{x^4 - x^2 + 1}}{x^6} + \frac{-2x^4 + x^2}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$$

Then we can split our integral by the linearity of the integration operator.

$$I = I_0 + I_1$$

where $\quad I_0 = \int \frac{5\sqrt{x^4 - x^2 + 1}}{x^6}\, dx \quad$ and $\quad I_1 = \int \frac{-2x^4 + x^2}{x^6\sqrt{x^4 - x^2 + 1}}\, dx$

Then we can use IBP on I_0.

$$I_0 = -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} + \int \frac{1}{x^5} \cdot \frac{2x^3 - x}{\sqrt{x^4 - x^2 + 1}}\, dx$$

$$= -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} - I_1$$

$$\therefore I = -\frac{\sqrt{x^4 - x^2 + 1}}{x^5} - I_1 + I_1 \implies I = \boxed{-\frac{\sqrt{x^4 - x^2 + 1}}{x^5}}$$

22.

$$\int \frac{\cos^2(3x)}{1 + \cos(3x)}\, dx$$

$$u = 3x \implies \frac{1}{3}\int \frac{\cos^2(u)}{1 + \cos(u)}\, du$$

Using polynomial division we can simplify our integral.

$$\frac{1}{3}\int \cos(u) - 1 + \frac{1}{1 + \cos(u)}\, du = \frac{1}{3}\left(\sin(u) - u + \int \frac{1 - \cos(u)}{\sin^2(u)}\, du\right)$$

$$= \frac{1}{3}\left(\sin(u) - u + \int \csc^2(u) - \csc(u)\cot(u)\, du\right)$$

$$= \frac{1}{3}\left(\sin(u) - u - \cot(u) + \csc(u)\right) = \boxed{\frac{1}{3}\left(\sin(3x) - 3x - \cot(3x) + \csc(3x)\right)}$$

23.

$$\int \frac{\sin x}{\cot x} \cdot \frac{\tan x}{\cos x}\, dx$$

$$= \int \tan^3(x)\, dx = \int \tan(x)\left(\sec^2(x) - 1\right)\, dx$$

$$= \log\left(\cos(x)\right) + \int \tan(x)\sec^2(x)\, dx = \boxed{\log\left(\cos(x)\right) + \tfrac{1}{2}\sec^2(x)}$$

Note: This solution used the substitution $u = \sec(x)$ to solve the integral, however, $u = \tan(x)$ can also be used for the result of $\boxed{\log\left(\cos(x)\right) + \tfrac{1}{2}\tan^2(x)}$

24.

$$\int \frac{\tan\left(\frac{x}{4}\right)}{\sec\left(\frac{x}{2}\right)}\, dx$$

$$u = \frac{x}{4} \implies 4\int \cos(2u)\tan(u)\, du = 4\int \left(2\cos^2(u) - 1\right)\tan(u)\, du$$

$$= 4\int 2\cos(u)\sin(u) - \tan(u)\, du = 4\left(-\cos^2(u) - \log(\sec(x))\right)$$

$$= \boxed{4\log\left(\cos\left(\frac{x}{4}\right)\right) - 4\cos^2\left(\frac{x}{4}\right)}$$

25.

$$\int \tan^{-1}\left(\tan(x)\right)\, dx$$

$$u = \tan^{-1}\left(\tan(x)\right) \implies \int u\, du = \frac{u^2}{2} = \boxed{\tfrac{1}{2}\left(\tan^{-1}\left(\tan(x)\right)\right)^2}$$

26.

$$\int_1^\infty \frac{x - \lfloor x \rfloor}{x^3}\, dx$$

$$= \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{x^2} - \frac{n}{x^3}\, dx = \sum_{n=1}^\infty \frac{n}{2}\left(\frac{1}{(n+1)^2} - \frac{1}{n^2}\right) - \left(\frac{1}{n+1} - \frac{1}{n}\right)$$

$$= 1 + \frac{1}{2}\sum_{n=1}^\infty \frac{n}{(n+1)^2} - \frac{1}{n} = 1 + \frac{1}{2}\sum_{n=1}^\infty \frac{n + (1-1)}{(n+1)^2} - \frac{1}{n}$$

$$= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{(n+1)^2} - \frac{1}{n} = 1 + \frac{1}{2} \left(-1 - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \right)$$

$$= \frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{2} - \frac{1}{2} \left(\zeta(2) - 1 \right) = \boxed{1 - \frac{\pi^2}{12}}$$