Concavity and Convexity Properties of $\ln\left(\frac{x_1+x_2+\cdots+x_n}{x_0}\right)$ with Integer x_0 and $x_i \geq 1$

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ISEN 320

Abstract

We study the function

$$F\left(x_1, x_2, \ldots, x_n, x_0\right) \;=\; \ln\left(\frac{x_1+x_2+\cdots+x_n}{x_0}\right)$$

under the assumptions

$$x_0 \in \mathbb{Z}_{>0} \quad \text{and} \quad x_i \geq 1 \;\; \text{for each } i = 1, 2, \ldots, n.$$

This document provides a thorough proof that in the full space $(x_1, \ldots, x_n, x_0)$ (treating x_0 as real, however, still contains geometric implications for a mixed-integer valued F), the Hessian of F is indefinite, so F is neither globally convex nor globally concave.

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1 Introduction and Statement of the Problem

Let n be a positive integer. We consider $n+1$ variables:

$$x_1,\ x_2,\ \ldots,\ x_n, \quad \text{and} \quad x_0,$$

subject to the conditions

$$x_0 \in \mathbb{Z}_{>0} \quad \text{(strictly positive integer)}, \quad x_i \geq 1 \text{ for all } i = 1, \ldots, n.$$

Define

$$F(x_1, \ldots, x_n, x_0) \;=\; \ln\!\Big(\tfrac{x_1+x_2+\cdots+x_n}{x_0}\Big).$$

Because $x_1 + \cdots + x_n \geq n \geq 1$ and x_0 is at least 1 (strictly positive), the argument of the logarithm is strictly positive, so F is well-defined.

The purpose of this text is to prove:

- **Concavity in the x_i's (for fixed x_0):** If we hold x_0 fixed, then the map

  $$(x_1, \ldots, x_n) \;\mapsto\; F(x_1, \ldots, x_n, x_0)$$

  is concave in $(x_1, \ldots, x_n)$ over the domain $\{x_i \geq 1\}$.

- **Convexity in x_0 (for fixed $x_1, \ldots, x_n$):** If we hold $(x_1, \ldots, x_n)$ fixed, then the function of x_0 alone

  $$x_0 \;\mapsto\; F(x_1, \ldots, x_n, x_0) \;=\; \ln\!\Big(\tfrac{x_1+\cdots+x_n}{x_0}\Big)$$

  behaves like $-\ln(x_0)$ plus a constant, which is convex in real $x_0 > 0$. Since x_0 is actually integer, we interpret this as F being *monotonically decreasing* in x_0 and having discrete convexity properties for integer steps.

- **Neither concave nor convex in all variables simultaneously:** If we treat all $(x_1, \ldots, x_n, x_0) \in (0, \infty)^{n+1}$ as real variables, the Hessian of F is neither positive semidefinite nor negative semidefinite, i.e. indefinite.

We will prove each of these statements in detail below.

2 Rewrite the Function and Gather Key Observations

Observe that

$$F(x_1, \ldots, x_n, x_0) \;=\; \ln\!\Big(x_1 + \cdots + x_n\Big) \;-\; \ln(x_0).$$

This rewriting splits F into two simpler pieces:

$$F(x_1, \ldots, x_n, x_0) \;=\; \underbrace{\ln\!\big(x_1 + \cdots + x_n\big)}_{=:G(x_1,\ldots,x_n)} \;-\; \underbrace{\ln(x_0)}_{=:H(x_0)}. \tag{1}$$

Hence, when we fix x_0, F differs from $G(x_1, \ldots, x_n)$ by a constant $-\ln(x_0)$. When we fix $(x_1, \ldots, x_n)$, F differs from $-\ln(x_0)$ by a constant $\ln(x_1 + \cdots + x_n)$.

Thus, the study of F reduces to understanding:

$$G(x_1, \ldots, x_n) = \ln(x_1 + \cdots + x_n) \quad \text{and} \quad H(x_0) = \ln(x_0).$$

3 Concavity in $(x_1, \ldots, x_n)$ for Fixed x_0

In this section, *assume x_0 is held fixed* (some positive integer). Then

$$F(x_1, \ldots, x_n, x_0) \;=\; \ln(x_1 + \cdots + x_n) \;-\; \ln(x_0).$$

Since $\ln(x_0)$ is a constant with respect to $(x_1, \ldots, x_n)$, we focus on

$$\ln(x_1 + \cdots + x_n).$$

It is well known (and we repeat the argument for completeness) that

$$\bigl(x_1, \ldots, x_n\bigr) \;\mapsto\; x_1 + \cdots + x_n$$

is an affine (linear) function. Meanwhile, the map

$$t \;\mapsto\; \ln(t) \quad (t > 0)$$

is concave and non-decreasing in the scalar variable t. By a standard composition rule for concave functions:

Theorem 3.1 (Composition with an affine map)**.** *If $\varphi(t)$ is concave and non-decreasing on some interval $(0, \infty)$ and $\phi(\mathbf{x})$ is a linear/affine map into $(0, \infty)$, then $\varphi(\phi(\mathbf{x}))$ is concave in $\mathbf{x}$.*

Here, $\phi(x_1, \ldots, x_n) = x_1 + \cdots + x_n > 0$, and $\varphi(t) = \ln(t)$. Since $\ln(\cdot)$ is indeed concave+non-decreasing on $(0, \infty)$, it follows that

$$G(x_1, \ldots, x_n) \;=\; \ln(x_1 + \cdots + x_n)$$

is concave in $(x_1, \ldots, x_n)$ over the region $x_i \geq 1$ (actually, it holds for $x_i > 0$ in general). Thus G is concave, and subtracting the constant $\ln(x_0)$ does not affect concavity. We conclude:

Proposition 3.2. *For each fixed integer $x_0 > 0$, the function*

$$(x_1, \ldots, x_n) \;\mapsto\; F(x_1, \ldots, x_n, x_0)$$

is concave over the domain $x_i \geq 1$.

4 Convexity in x_0 for Fixed $(x_1, \ldots, x_n)$

Now fix a point $(x_1, \ldots, x_n)$, each $x_i \geq 1$. We look at

$$F(x_0) \;=\; \ln(x_1 + \cdots + x_n) \;-\; \ln(x_0).$$

As a function of the single scalar variable $x_0 > 0$ (if we temporarily allow x_0 to be real), this is a constant $\ln(x_1 + \cdots + x_n)$ minus $\ln(x_0)$. We know from single-variable calculus:

$$\ln(x_0) \quad \text{is concave in } x_0 > 0, \quad \text{so} \quad -\ln(x_0) \quad \text{is convex in } x_0 > 0.$$

Hence

$$x_0 \;\mapsto\; -\ln(x_0)$$

is convex, and adding the constant $\ln(x_1 + \cdots + x_n)$ preserves convexity. Therefore:

Proposition 4.1. *If $(x_1, \ldots, x_n)$ are held fixed, then $x_0 \mapsto F(x_1, \ldots, x_n, x_0)$ is convex in the real variable $x_0 > 0$.*

Discrete Interpretation (Since $x_0 \in \mathbb{Z}_{>0}$)

Because in the actual problem statement x_0 is restricted to be a strictly positive integer, the usual derivative-based definition of convexity (i.e. $f''(x_0) \geq 0$) does not *directly* apply. Nevertheless, if one extends the function to real $x_0 > 0$ and checks convexity, that implies discrete convexity properties when restricting $x_0 \in \{1, 2, 3, \ldots\}$. In particular,

$$\text{since } -\ln(x_0) \text{ is strictly decreasing in } x_0 > 0, \quad F \text{ is also strictly decreasing as an integer function of } x_0.$$

5 Neither Globally Concave nor Globally Convex in All Variables

Finally, consider F as a function of $(x_1, \ldots, x_n, x_0) \in (0, \infty)^{n+1}$. Although the original domain has x_0 integer and $x_i \geq 1$, for the sake of checking global convexity or concavity, one typically examines the Hessian in the continuous domain $(0, \infty)^{n+1}$. We do that here to see whether F could be globally concave/convex. The result is that the Hessian is *indefinite*, hence F cannot be globally one or the other.

5.1 Hessian Computation

Write

$$F(\mathbf{x}, x_0) \;=\; \ln\bigl(x_1 + \cdots + x_n\bigr) \;-\; \ln(x_0).$$

Let us denote $S = x_1 + \cdots + x_n$ for shorthand. Then

$$\nabla F \;=\; \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial x_0}\right).$$

First derivatives:

$$\frac{\partial}{\partial x_i} \ln(S) = \frac{1}{S}, \quad \text{for each } i = 1, \ldots, n,$$

and

$$\frac{\partial}{\partial x_0}\bigl[-\ln(x_0)\bigr] = -\frac{1}{x_0}.$$

Hence

$$\nabla F(\mathbf{x}, x_0) = \Bigl(\tfrac{1}{S},\ \tfrac{1}{S},\ \ldots,\ \tfrac{1}{S},\ -\tfrac{1}{x_0}\Bigr).$$

Second derivatives:

$$\frac{\partial^2}{\partial x_i \partial x_j} \ln(S) = -\frac{1}{S^2}, \quad \text{for all } i, j = 1, \dots, n,$$

because $\frac{\partial}{\partial x_j}\left(\frac{1}{S}\right) = -\frac{1}{S^2}$. Also,

$$\frac{\partial^2}{\partial x_i \partial x_0} \ln(S) = 0, \qquad \frac{\partial^2}{\partial x_0^2}\big[-\ln(x_0)\big] = \frac{1}{x_0^2}.$$

Hence the Hessian matrix $\nabla^2 F$ takes the form:

$$\nabla^2 F = \begin{pmatrix} -\frac{1}{S^2} & -\frac{1}{S^2} & \cdots & -\frac{1}{S^2} & 0 \\ -\frac{1}{S^2} & -\frac{1}{S^2} & \cdots & -\frac{1}{S^2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{1}{S^2} & -\frac{1}{S^2} & \cdots & -\frac{1}{S^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{x_0^2} \end{pmatrix}_{(n+1)\times(n+1)}.$$

The top-left $n \times n$ block is

$$-\tfrac{1}{S^2} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

which is *negative semidefinite* (it has one strictly negative eigenvalue and $n-1$ zeros). Meanwhile, the bottom-right element $\frac{1}{x_0^2}$ is strictly positive. Therefore the full Hessian has both negative and positive eigenvalues, i.e. it is *indefinite*.

5.2 Implication

An indefinite Hessian means the function cannot be globally concave or globally convex in $(x_1, \dots, x_n, x_0)$ if we treat all variables as real. In the stricter setting that $x_0 \in \mathbb{Z}$, one typically uses alternative definitions of concavity/convexity for "mixed" discrete-continuous variables, but the key point remains that F does not admit a unifying global concavity or convexity property across all variables simultaneously.

6 Conclusion

We have thoroughly demonstrated:

1. **Concavity in the x_i's** (for fixed integer x_0):

   $$(x_1, \dots, x_n) \;\mapsto\; \ln\!\left(\tfrac{x_1 + \cdots + x_n}{x_0}\right)$$

   is concave in $(x_1, \dots, x_n)$, thanks to the concavity of $\ln(x_1 + \cdots + x_n)$.

2. **Convexity in** x_0 (for fixed $x_1, \ldots, x_n$):

$$x_0 \mapsto \ln\Bigl(\tfrac{x_1+\cdots+x_n}{x_0}\Bigr) = \ln(x_1 + \cdots + x_n) - \ln(x_0)$$

   is "$-\ln(x_0)$ plus a constant," which is convex in real $x_0 > 0$. Restricted to $x_0 \in \mathbb{Z}_{>0}$, it is monotonically decreasing and satisfies a discrete version of convexity.

3. **Neither globally convex nor concave in all variables**: In the continuous extension $(0, \infty)^{n+1}$, F has an indefinite Hessian, so it is neither globally convex nor globally concave.

Hence we have completely characterized the basic convexity/concavity properties of

$$F(x_1, \ldots, x_n, x_0) = \ln\Bigl(\tfrac{x_1+\cdots+x_n}{x_0}\Bigr),$$

under the given domain assumptions $x_0 \in \mathbb{Z}_{>0}$, $x_i \geq 1$.

Remark. Should one need more advanced concepts like "integer-valued convexity" or "mixed-integer convexity," the negativity/positivity in the Hessian blocks still underlies the fundamental geometry: the function is concave in $(x_1, \ldots, x_n)$ for each fixed integer x_0 and has a convex shape in x_0 for each fixed $(x_1, \ldots, x_n)$. These remain consistent no matter the details of the domain's integrality constraints.