Project 1: The Number of Five-Cycles in a Triangle-Free Graph

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March 2, 2013

1 Introduction

In [2] Erdös poses the question of how many 5-cycles can be found in a triangle-free graph. He conjectures that the greatest number of 5-cycles is $(\frac{n}{5})^5$. In the following paper, this conjecture is analyzed. First, introductory concepts are explained, then Erdös's conjecture is explored. Lastly, a proof of a near-optimal upper bound and two current results in the attempt to solve the conjecture using a modern mathematical technique, flag algebra, are reviewed.

1.0.1 Triangle-Free Graphs

A triangle-free graph, T_F is a graph which contains no cycles of length 3. Triangle-free graphs are equivalently defined as a graph which contains no cliques (complete induced subgraphs) other than its nodes and edges. Turań's theorem tells us that the maximal triangle-free graph for a graph G with order n contains at most $\frac{n^2}{4}$ edges. Expanding upon this, any maximal triangle-free graph is also a complete bipartite graph.

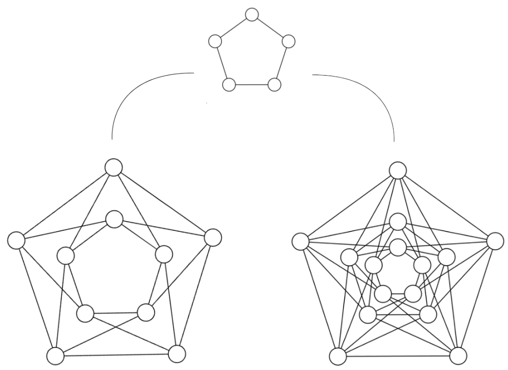
1.1 Erdös's Conjecture

Erdösconjectures that the largest number of 5-cycles found in a triangle-free graph is is $(\frac{n}{5})^5$. He finds that this largest case is achieved by balanced blowups of C_5. That is, a graph where k copies of C_5 are made, and if and edge uv exists in C_5 then u and each corresponding copy of u connects to v and each corresponding copy of v. The blow-ups of C_5 with $k = 1$ and $k = 2$ are shown in Figure 1.1.

1.1.1 Extremal Graph Theory

Each of the approaches here reviewed in looking at Erdös's conjecture are extremal graph theory approaches. Extremal graph theory is a branch of graph

Figure 1: Triangle-free graphs with $(\frac{n}{5})^5$ 5-cycles.



theory which looks at extremal cases of graphs with paricular properties. A common practice in extremal graph theory is to first observe consequences of a particular property on a graph. What structures arise due to this property, and how do these structures limit extremal graphs with that particular property? Extremal graph theory looks at upper and lower bounds - the points at which a graph with a particular property can no longer exist. One classic theorem in extremal graph theory is Turań's theorem which states the following:

Turan's Theorem. *Where G is a $K_r + 1$-free graph of order n, the size of G is at most:*

$$\frac{r-1}{r}\dot{\frac{n^2}{2}} = 1 - \frac{1}{r}\dot{\frac{n^2}{2}}. \tag{1}$$

2 Current Results

2.1 Győri's proof of an upper bound

In [4], Győri proved that a triangle-free graph does not contain more than $c(\frac{n+1}{5})^5$ 5-cycles, where c = $\frac{16875}{16384}$. As stated in [5], this is within 1.03 of the optimal proposed by Erdös. Below, Győri's proof is summarized[1]

Where G is a triangle-free graph with order n and size e, we take $v \in V(G)$ and look at the number of ways we could make a 5-cycle containing v. Because G is triangle-free, the edge opposite to v in a 5-cycle cannot be an element of $N(v)$ (this is shown Figure 2.1, on the left). . So the number of possibilities for an opposite edge is no greater than $\sum_{uv \in E(G)} deg(u)$. If we have v and an opposite edge, xy, then the we want to find the remaining vertices in a 5-cycle with v, $(w, z \in V(G) : w \in N(x), z \in N(y), and w, z \in N(v)$. Because $w, z \in N(v)$, the number of possibilities for w and z is no greater than $\deg(v)^2$. However, because G is triangle-free, w is not adjacent to z and vise-versa (seen in Figure 2.1 on right). Thus the set of vertices which can finish our 5-cycle are characterized as a triangle-free subset of $N(v)$. This subset's size must be less than or equal to the maximum size of a triangle-free graph on $\deg(v)$ vertices - found by Turan's theorem. Altogether then, the greatest number of 5-cycles containing v is shown below.[2]

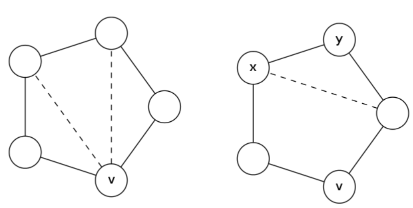
$$\left(e - \sum_{uv \in E(G)} deg(u)\right)\left(\frac{((d(v))^2}{4}\right). \tag{2}$$

Győrinotes that the sum of $\deg(x)^2$ is equivalent to the sum of $\deg(u) : u \in N(x)$ for each $x \in V(G)$, and sets both sums to a fixed value, M. The average value

[1]Because Győri's original proof was written for academics, many details which would be assumptions for those in the field have been left out. I have done my best to add some of those details back into the proof to make it more accessible to the introductory level reader.

[2](size of $G-$ sum of the degrees of $N(v)$) $*$ the size of a maximal triangle-free graph of size $\deg(v)$

Figure 2: Graphs containing a 5-cycle which are not triangle-free.



of a squared degree, and the average sum of a vertex's neighors' degree for $x \in V(G)$, then, is just $\frac{M}{n}$. Győrithen proves the general case that for a graph H with order n and size e, there exists $m \in V(H)$ such that the following is true:

$$\left(e - \sum_{im \in E(H)} \deg(i)\right) \deg(m)^2 \leq \left(e - \frac{M}{n}\right) \frac{M}{n}. \tag{3}$$

Where m produces a minimum value for the left side of 3, and the right side forms an upper-bound the the minimum. To get from 3 to something which looks like 2, Győritakes from 3 that $\deg(m) \leq \sqrt{\frac{M}{n}}$. In order to find an upper bound to the greatest number of 5-cycle subgraphs a single node can be a member of, Győrisubstitutes all occurences of $d(x)$ with $\sqrt{\frac{M}{n}}$ in 2, leaving us with:

$$\left(e - \sum_{uv \in E(G)} deg(u)\right) \left(\frac{((d(v))^2}{4}\right) \leq \left(e - \frac{M}{n}\right) \frac{M}{4n} \tag{4}$$

however, from the handshaking lemma (see [8] for reference) we know that $e = \frac{1}{2} \sum_{v} \deg(v)$. Substituting with $\sqrt{\frac{m}{n}}$, we get $e \leq \frac{\sqrt{Mn}}{2}$ thus we finally get:

$$\left(e - \sum_{uv \in E(G)} deg(u)\right) \left(\frac{((d(v))^2}{4}\right) \leq \left(\frac{\sqrt{Mn}}{2} - \frac{M}{n}\right) \frac{M}{4n}. \tag{5}$$

This equation gives us an upper bound to the greatest number of 5-cycle subsets of a triangle-free graph a single vertex can be a member of. Győrifinds that the right side of 5 is greatest when $M = \frac{9}{64}n^3$. From this he uses induction to prove that a triangle-free graph G contains no more than:

$$\frac{3^8}{5\dot{2}^1 4} n^5 + \frac{3^8}{2^1 4} n^4 \leq \frac{3^8}{5\dot{2}^1 4} (n+1)^5. \tag{6}$$

5-cycles.

2.2 Flag Algebras

Flag algebras are a method of extremal graph theory and combinatorics used in general to investigate the number (density) of smaller graphs in larger graphs, among other related subjects. They are syntactic tools for use in extremal problems, intended to make calculations simpler and results more universal. A flag algebra is a calculus, developed and appended to a problem's specific needs, which is based on simple algebraic objects. In [6] Razborobv claims that this dynamic structure of flag algebras allows them to draw upon a broad and deep foundations, found elsewhere in mathematics.

The fundamental objects of flag algebras are the following:

- models - structures which define an interpretation of a formal language
- theories - groups of sentences, which can be satisfied by a model (for example, a model might satisfy the undirected or directed case of graph theory)
- types - a model M of a theory T with vertices, $V(M) = [k]$, where $[k]$ is a combinatorial collection of k elements.
- σ-flags - pairs $F = (M, \theta) : M$ is a finite model, and θ is a model-embedding (such as a subgraph embedded in a graph) from σ to θ.

2.3 Proofs using Flag Algebras

Two proported proofs of Erdös's conjecture, [5] and [3] were recently developed using flag algebras. Both proofs are complex, and thus will not be summarized here.

3 Conclusion: Why Flag Algebras?

In [5], the authors remark that that they were unaware that Grzesik had concurrently come up with a completely different proof of Erdös's conjecture using flag algebra in [3]. What is it that makes flag algebra a good foundation for solving Erdös's conjecture? I answer this with speculations:

- flag algebras were built to work with problems like those in extremal graph theory
- as described in [6], flag algebras do not skip over challenging-to-prove minimal or trivial cases, as they are required to be proven to satisfy the appropriate theory. As such, flag algebras might provide a more accurate "model" of the structure of a graph from the bottom-up - allowing more clear an thorough inferences to be made.
- flag-algebras are designed to be dynamically altered to fit the problem. As a result, proofs of theorems in the papers seem to have results that are more direct.
- flag-algebras structural and logical aspects allow an interface to problems which allows the parameters to be easily manipulated on a computer. Both proofs seem to take advantage of this.

All and all, the flag algebras method seems to be one which is fundamentally different than the methodology used Győri.Győri's proof reflected a more intuitive approach, where results, or the next step in the proof were not immediately apparent. Flag algebras, designed to be computable and thorough - I would imagine rely much less on this sort of intuition. Is there anything to be said about these different methodologies?

References

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