Population Growth for Seth

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1 Population Growth and GDP Growth

The importance of population growth for economic growth in raw terms can be seen just by looking at a simple example. Consider the Cobb-Douglas production function:

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}.$$

Then take logs and time-derivatives

$$\begin{aligned}
\log(Y_t) &= \alpha \log(K_t) + (1-\alpha)(\log(N_t) + \log(A_t)) \\
\frac{\dot{Y}_t}{Y_t} &= \alpha \frac{\dot{K}_t}{K_t} + (1-\alpha)\left(\frac{\dot{N}_t}{N_t} + \frac{\dot{A}_t}{A_t}\right) \\
g_t^Y &= \alpha g_t^K + (1-\alpha)(g_t^A + g_t^N).
\end{aligned}$$

In the Solow Model, we have

$$\begin{aligned}
\dot{K}_t &= sK_t^{\alpha}(A_t N_t)^{1-\alpha} - \delta K_t \\
\frac{\dot{K}_t}{K_t} &= s\left(\frac{A_t N_t}{K_t}\right)^{1-\alpha} - \delta.
\end{aligned}$$

Taking logs and time derivatives:

$$\begin{aligned}
\log(g_t^K + \delta) &= \log(s) + (1-\alpha)\left(\log(A_t) + \log(N_t) - \log(K_t)\right) \\
\frac{\dot{g}_t^K}{g_t^K + \delta} &= (1-\alpha)(g_t^A + g_t^N - g_t^K) \\
\dot{g}_t^K &= (1-\alpha)(g_t^K + \delta)(g_t^A + g_t^N - g_t^K).
\end{aligned}$$

Hence, capital growth is dependent on technology growth and population growth. If $g_t^K < g_t^A + g_t^N$, then capital growth declines and vice versa. If g^A and g^N are constant, we eventually converge to a steady growth path where $g^K = g^A + g^N$, which means that $g^Y = g^A + g^N$, so GDP growth is entirely determined by g^A and g^N, productivity growth and population growth. So, permanent declines in population growth lead to permanent declines in GDP growth! That's bad news for a country, so steady growth is important.

2 Population Growth and Interest Rates in a Simple Model

Suppose we have two types of agents: young agents and old agents. Both have the utility function

$$u(c_1, c_2) = \log(c_1) + \beta \log(c_2).$$

However, the two agents have different budget constraints. Young agents only earn income in the second period, while old agents earn income in the first. Hence the young agent's problem is

$$\begin{aligned}\max_{c_1^y, c_2^y, b^y} \ & \log(c_1^y) + \beta \log(c_2^y) \\ \text{s.t. } & c_1^y \leq b^y \\ & c_2^y \leq I^y - Rb^y\end{aligned}$$

and the old agent's problem is

$$\begin{aligned}\max_{c_1^o, c_2^o, b^o} \ & \log(c_1^o) + \beta \log(c_2^o) \\ \text{s.t. } & c_1^o \leq I_1^o - b^o \\ & c_2^o \leq I_2^o + Rb^o.\end{aligned}$$

Since the constraints will always bind, we can write these problems in simplified form (we don't have to use Lagrange multipliers)

$$\begin{aligned}&\max_{b^y} \log(b^y) + \beta \log(I^y - Rb^y) \\ &\max_{b^o} \log(I_1^o - b^o) + \beta \log(I_2^o + Rb^o).\end{aligned}$$

Taking derivatives with respect to b^y and b^o and setting to zero, we obtain

$$\begin{aligned}0 &= \frac{1}{b^y} - \frac{\beta R}{I^y - Rb^y} \\ 0 &= -\frac{1}{I_1^o - b^o} + \frac{\beta R}{I_2^o + Rb^o}.\end{aligned}$$

With some algebra, we obtain

$$\begin{aligned}b^y &= \frac{I^y}{(1+\beta)R} \\ b^o &= \frac{\beta R I_1^o - I_2^o}{(1+\beta)R}.\end{aligned}$$

To pin down R, we need the bond market clearing condition:

$$0 = N^y b^y - N^o b^o.$$

This yields

$$\begin{aligned}0 &= N^y I^y + N^o I_2^o - RN^o \beta I_1^o \\ R &= \frac{N^y I^y + N^o I_2^o}{N^o \beta I_1^o} \\ R &= \frac{1}{\beta}\left(\frac{N^y}{N^o}\frac{I^y}{I_1^o} + \frac{I_2^o}{I_1^o}\right).\end{aligned}$$

Note that $\frac{N^y}{N^o}$ is the population growth rate $(1+g)$ in our simple little economy. We can see then that as g goes down, R goes down too, since $\frac{dR}{dg} = \frac{1}{\beta} \times \frac{I^y}{I_1^o}$ and β, I^y, and I_1^o are all positive.

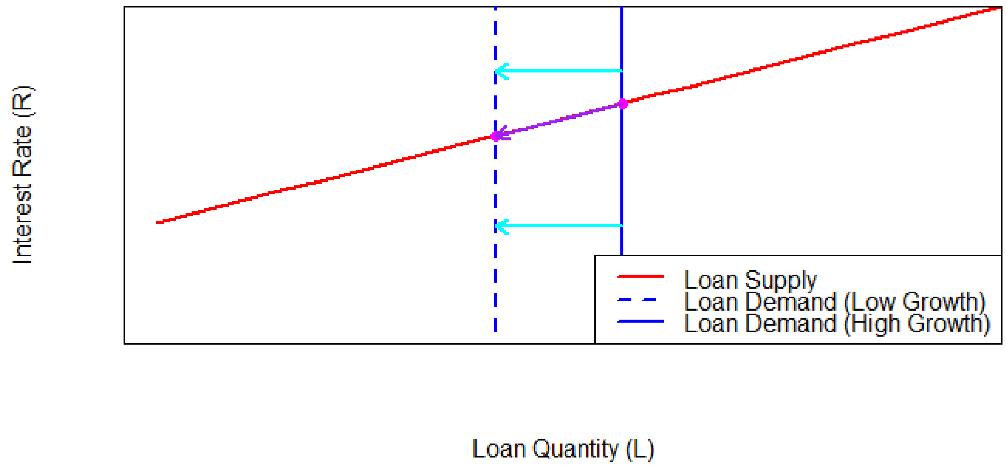
Why is this? It has to do with asset supply and demand. For simplicity, let $I^y = I^o_1 = I^o_2$. Loan demand is (with some normalization)

$$l^d(R) = \frac{1}{\beta}(1+g).$$

Loan supply is (with some normalization)

$$l^s(R) = R - \frac{1}{\beta}.$$

So, if we plot these two equations with R on the y-axis and l on the x-axis, we see that asset demand is completely vertical. This is known as perfectly inelastic demand. Young agents' demand for loans doesn't depend on interest rates. Old agents' supply curves are upward sloping and linear. When population growth decreases, loan demand decreases (shifts to the left). This lowers the quantity of loans made and the interest rate. We can see this in the diagram below.



The cyan arrows show the direction of the shift of the demand curve while the purple arrow shows the change in the equilibrium quantity and price as a result of the shift in demand.