THE BASIC RBC MODEL

April 2021

OUTLINE

- Outline the model
- Linearize the model
- Solve the model

HOUSEHOLDS

- Households choose consumption and labor to maximize the present discounted value of their utility, subject to the budget constraint and taking prices as given.

$$\max_{C_t, L_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{L_t^{1+\phi}}{1+\psi} \right]$$

$$C_t + K_{t+1} \leq r_t K_t + w_t L_t$$

- You've probably seen this before! It's basically the stochastic neoclassical growth model with labor added in.

HOUSEHOLD OPTIMALITY CONDITIONS

- Optimal labor/consumption tradeoff:

$$\chi C_t^{\sigma} L_t^{\phi} = w_t$$

- This tells you, given wages, about how consumption and labor are allocated. Their exponents give you an idea about their relative importance (Why?).
- Euler equation:

$$C_t{}^{-\sigma} = \beta \mathbb{E}_t [r_{t+1} C_{t+1}{}^{-\sigma}]$$

- This tells you about the optimal path of consumption over time, based on expectations (what happens if expected interest rates and consumption change?)

FIRMS AND MARKET CLEARING

- Firms choose capital and labor to maximize static profits. No intertemporal motive here.

$$\max_{K_t, L_t} A_t K_t^{\alpha} L_t^{1-\alpha} - \left(r_t - (1-\delta)\right) K_t - w_t L_t$$

- We also have some market clearing conditions:
  - w_t and r_t adjust so that the labor and capital supplied by the household are equal to the labor and capital demanded by the firm.
  - The goods market also clears ($Y_t = C_t + I_t$)

FIRM OPTIMALITY CONDITIONS

- The firm optimality conditions help us pin down the prices in terms of marginal products in a perfectly competitive market for inputs.
- Marginal Product of Capital:

$$\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t - (1-\delta)$$

- Marginal Product of Labor:

$$(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} = w_t$$

COLLECTING EQUATIONS

- We have seven equations in 7 unknowns $\{C_t, L_t, K_t, I_t, Y_t, r_t, w_t\}$ plus an exogenous process for productivity A_t:

$$
\begin{aligned}
\chi C_t^{\sigma} L_t^{\phi} &= w_t && (1)\\
C_t^{-\sigma} &= \beta \mathbb{E}_t \left[r_{t+1} C_{t+1}^{-\sigma} \right] && (2)\\
\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} &= r_t - (1-\delta) && (3)\\
(1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} &= w_t && (4)\\
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} && (5)\\
Y_t &= C_t + I_t && (6)\\
K_{t+1} &= (1-\delta) K_t + I_t && (7)
\end{aligned}
$$

HOW DO WE SOLVE THIS?

- This is a non-linear system of equations which depends on expectations.
- Ideally, we would like to find a way to relate current variables to past values, current shocks, and expectations of future shocks.
- We can't really do that in its current form. However, if we use an approximation to this system, we can do that.
- Linear approximations to the system around its steady state are the most common. However, there exist methods to solving higher order approximations (quadratic, cubic, etc.).
- Once the system is approximated, we can get it in a form that allows us to understand how it depends on past values and shocks.

LOG-LINEARIZING

- To log-linearize, you can just apply a couple simple rules and do some algebra.
- This is the cardinal rule:

$$x_t^a \approx x^a(1 + a\hat{x}_t)$$

- That is, the first order approximation of x_t to the power of a around its steady state is x^a (steady state value of x_t to the power of a) times one plus a times the log deviation of x_t from steady state ($\hat{x}_t$).
- The other cardinal rule of log-linearizing is that $(1 + \hat{x}_t)(1 + \hat{y}_t) = 1 + \hat{x}_t + \hat{y}_t$. That means products of log-deviations are assumed to be small enough to be negilible.

LINEARIZED SYSTEM

- Once we linearize, this is what we get:

$$
\begin{aligned}
\hat{w}_t &= \sigma \hat{c}_t + \phi \hat{l}_t \\
\mathbb{E}_t[\hat{c}_{t+1}] &= \frac{1}{\sigma}\mathbb{E}_t[\hat{r}_{t+1}] + \hat{c}_t \\
\hat{r}_t &= (1-\beta(1-\delta))\left[\hat{A}_t + (1-\alpha)(\hat{l}_t - \hat{k}_t)\right] \\
\hat{w}_t &= \hat{A}_t + \alpha(\hat{k}_t - \hat{l}_t) \\
\hat{y}_t &= \hat{A}_t + \alpha\hat{k}_t + (1-\alpha)\hat{l}_t \\
\hat{y}_t &= \frac{c}{y}\hat{c}_t + \frac{i}{y}\hat{i}_t \\
\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + \delta\hat{i}_t \\
\hat{A}_t &= \rho\hat{A}_{t-1} + \epsilon_t
\end{aligned}
$$

- We can reduce this system a little by substituting things back into different equations to get something that's easier to work with.

(REDUCED) LINEARIZED SYSTEM

- Once we reduce a little bit, we can get some equations that are simpler.

$$
\begin{aligned}
\frac{\Gamma}{\delta}\hat{k}_{t+1} &= \left(1 - \frac{1-\Gamma}{\sigma}\right)\hat{A}_t + \left(\alpha\left(1 - \frac{1-\Gamma}{\sigma}\right) + \frac{\Gamma(1-\delta)}{\delta}\right)\hat{k}_t \\
&\quad + \left(1 - \alpha + \frac{(1-\Gamma)(\alpha+\phi)}{\sigma}\right)\hat{l}_t \\
\Gamma &= \delta\left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{\sigma-\phi}{\sigma+\phi}}
\end{aligned}
$$

$$
\begin{aligned}
\beta(1-\delta)\mathbb{E}_t[\hat{A}_{t+1}] &+ (\alpha + (1-\alpha)(1-\beta(1-\delta))\hat{k}_{t+1} \\
&- (\alpha + \phi + (1-\alpha)(1-\beta(1-\delta)))\mathbb{E}_t[\hat{l}_{t+1}] = \hat{A}_t + \alpha\hat{k}_t - (\alpha+\phi)\hat{l}_t \\
\hat{A}_t &= \rho\hat{A}_{t-1} + \epsilon_t
\end{aligned}
$$

- Two equations in two variables plus an exogenous process, where $\Gamma = \frac{i}{y}$.

WHAT NOW?

- We want to put this in a form that can tell us more about the evolution of the variables based on past variables, shocks, and expectational errors. That is, it takes the following form (where ϵ is a shock and η is a non-parametric expectational error:

$$\Theta_1 s_{t+1} = \Theta_0 s_t + \Psi \epsilon_{t+1} + \Phi \eta_{t+1}. \tag{8}$$

- This is a canonical form that all linear rational expectations models can be put into. There are several methods of solving this equation to get somethins of the form

$$s_{t+1} = Q s_t + R \epsilon_{t+1} \tag{9}$$

- One such method is Blanchard-Kahn and another is Sims (2002), but there are many others. If you put the larger or the simplified linearized system into Dynare, it can do this for you! These methods rely on using decompositions of the (possibly singular) Θ_1 matrix to get the s_{t+1}s on one side by themselves. If Θ_1 weren't singular, we could simply invert and obtain the equation straightaway. Singularity means we have to be a little more clever.

PUTTING THE SYSTEM IN CANONICAL FORM

- Surpressing the expressions for the coefficients, we have the following three equations:

$$
\begin{aligned}
a_1\hat{k}_{t+1} &= a_2\hat{A}_t + a_3\hat{k}_t + a_4 l_t \\
b_1\mathbb{E}_t[\hat{A}_{t+1}] + b_2\hat{k}_{t+1} + b_3\mathbb{E}_t[\hat{l}_{t+1}] &= \hat{A}_t + b_4\hat{k}_t + b_5\hat{l}_t \\
\hat{A}_t &= \rho A_t + \epsilon_t
\end{aligned}
$$

- Now we can see that s_{t+1} includes $\hat{A}_{t+1}$, $\hat{l}_{t+1}$, and $\hat{k}_{t+1}$ and s_t is the lagged version of that. Notice that we can simplify a little more to get this down to two equations.

$$
\begin{aligned}
\left(b_1 - \frac{b_3 a_2}{a_4}\right)\mathbb{E}_t[\hat{A}_{t+1}] + \left(b_2 - \frac{b_5 a_1}{a_4}\right)\hat{k}_{t+1} + \frac{b_3 a_1}{a_4}\mathbb{E}_t[\hat{k}_{t+2}]& \\
= \left(1 - \frac{b_5 a_2}{a_4}\right)\hat{A}_t + \left(b_4 - \frac{b_5 a_3}{a_1}\right)\hat{k}_t& \\
\hat{A}_t = \rho A_t + \epsilon_t&
\end{aligned}
$$

PUTTING THE SYSTEM IN CANONICAL FORM

- Again surpressing the expressions for the coefficients, we have the simplified system:

$$
\begin{aligned}
c_1 \mathbb{E}_t[\hat{A}_{t+1}] + c_2 \hat{k}_{t+1} + c_3 \mathbb{E}_t[\hat{k}_{t+2}] &= c_4 \hat{A}_t + c_5 \hat{k}_t \\
\hat{A}_t &= \rho A_t + \epsilon_t
\end{aligned}
$$

- This is interesting because it implies that the state variables $\hat{k}$ and $\hat{A}$ completely determine the values for the entire system. If you know the states, then you can get everything else. That means policy functions (i.e. consumption function and labor supply) only depend on the state variables.

THE CANONICAL EQUATION

- Now we write the 3 equation system in matrix form.

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ b_2 & b_1 & b_3 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{A}_{t+1} \\ \hat{l}_{t+1} \end{bmatrix} = \begin{bmatrix} a_3 & a_2 & a_4 \\ 0 & \rho & 0 \\ b_4 & 1 & b_5 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{A}_t \\ \hat{l}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \epsilon_{t+1} + \begin{bmatrix} 0 \\ 0 \\ -b_3 \end{bmatrix} \eta_{t+1}$$

- We can do the same for the system with only capital.

$$\begin{bmatrix} c_3 & c_2 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+2} \\ \hat{k}_{t+1} \\ \hat{A}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & c_5 & c_4 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ \hat{k}_t \\ \hat{A}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \epsilon_{t+1} + \begin{bmatrix} -c_3 \\ 0 \\ 0 \end{bmatrix} \eta_{t+1}$$

- Now these can be solved using your favorite method! In fact, both of these have matrices that can be inverted, so it's relatively easy to do yourself. Once that's done, you can look at the dynamics of the system, assess its stability, and even estimate the parameters of the model.

FURTHER AFIELD

- Take a look at Sims' 2002 paper and the `gensys` programs provided to solve linear rational expectatations models. There are also plenty of resources on understanding the Blanchard-Kahn method, which is a somewhat simpler, though more limited method.
- Take a look at Kydland and Prescott's 1982 paper ("Time to Build"). It's a slightly different version of the model detailed here, but they're close in spirit. This is the genesis of much of the current DSGE literature and it's an interesting look at the past.

DETAILS ON HOUSEHOLD EQ. CONDITIONS

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{L_t^{1+\phi}}{1+\psi} \right] + \sum_{t=0}^{\infty} \lambda_t (r_t K_t + w_t L_t - C_t - K_{t+1})$$

$$\begin{aligned}
\partial C &: \beta^t C_t^{-\sigma} - \lambda_t = 0 \\
\partial L &: -\beta^t \chi L_t^{\phi} + w_t \lambda_t = 0 \\
\partial K_{t+1} &: -\lambda_t + \mathbb{E}_t[r_{t+1} \lambda_{t+1}] = 0 \\
(\partial C + \partial L) &: w_t \beta^t C_t^{-\sigma} = \beta^t \chi L_t^{\phi} \Rightarrow w_t = \chi C_t^{\Sigma} L_t^{\phi} \\
(\partial C + \partial K_{t+1}) &: \beta^t C_t^{-\sigma} = \mathbb{E}_t[r_{t+1} \beta^{t+1} C_{t+1}^{-\sigma}] \Rightarrow C_t^{-\sigma} = \beta \mathbb{E}_t[r_{t+1} C_{t+1}^{-\sigma}]
\end{aligned}$$

DETAILS ON FIRM EQ. CONDITIONS

$$A_t K_t^{\alpha} L_t^{1-\alpha} - \left(r_t - (1-\delta)\right) K_t - w_t L_t$$

$$\partial K : \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} - \left(r_t - (1-\delta)\right) = 0$$

$$\partial L : (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha} - w_t = 0$$

DETAILS ON LINEARIZING CONDITIONS (1)

$$\begin{aligned}
\chi C_t^\sigma L_t^\phi &= w_t \\
\chi C^\sigma L^\phi (1+\sigma\hat{c}_t)(1+\phi\hat{l}_t) &= w(1+\hat{w}_t) \\
\chi C^\sigma L^\phi (1+\sigma\hat{c}_t+\phi\hat{l}_t) &= w(1+\hat{w}_t) \\
(1+\sigma\hat{c}_t+\phi\hat{l}_t) &= (1+\hat{w}_t) \\
\sigma\hat{c}_t+\phi\hat{l}_t &= \hat{w}_t
\end{aligned}$$

DETAILS ON LINEARIZING CONDITIONS (2)

$$
\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t \left[r_{t+1} C_{t+1}^{-\sigma} \right] \\
C^{-\sigma}(1 - \sigma \hat{c}_t) &= \beta r C^{-\sigma} \mathbb{E}_t [(1 + \hat{r}_{t+1})(1 - \sigma \hat{c}_{t+1})] \\
C^{-\sigma}(1 - \sigma \hat{c}_t) &= \beta r C^{-\sigma} \mathbb{E}_t [(1 + \hat{r}_{t+1} - \sigma \hat{c}_{t+1})] \\
1 &= \beta r \\
(1 - \sigma \hat{c}_t) &= \mathbb{E}_t [(1 + \hat{r}_{t+1} - \sigma \hat{c}_{t+1})] \\
-\sigma \hat{c}_t &= \mathbb{E}_t [\hat{r}_{t+1} - \sigma \hat{c}_{t+1}]
\end{aligned}
$$

DETAILS ON LINEARIZING CONDITIONS (3)

$$
\begin{aligned}
\alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} &= r_t - (1-\delta) \\
\alpha K^{\alpha-1} L^{1-\alpha}(1+\hat{A}_t)(1+(1-\alpha)\hat{l}_t)(1-(1-\alpha)\hat{k}_t) &= r(1+\hat{r}_t) - (1-\delta) \\
\alpha K^{\alpha-1} L^{1-\alpha}(1+\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= r(1+\hat{r}_t) - (1-\delta) \\
\alpha K^{\alpha-1} L^{1-\alpha} &= r - (1-\delta) \\
(r-(1-\delta))(1+\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= r(1+\hat{r}_t) - (1-\delta) \\
(r-(1-\delta)) + (r-(1-\delta))(\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= r + r\hat{r}_t - (1-\delta) \\
(r-(1-\delta))(\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= r\hat{r}_t \\
(1/\beta-(1-\delta))(\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= \frac{\hat{r}_t}{\beta} \\
(1-\beta(1-\delta))(\hat{A}_t+(1-\alpha)(\hat{l}_t-\hat{k}_t)) &= \hat{r}_t
\end{aligned}
$$

DETAILS ON LINEARIZING CONDITIONS (4)

$$
\begin{aligned}
(1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha} &= w_t \\
(1-\alpha)K^{\alpha}L^{-\alpha}(1+\hat{A}_t)(1+\alpha\hat{k}_t)(1-\alpha\hat{l}_t) &= w(1+\hat{w}_t) \\
(1-\alpha)K^{\alpha}L^{-\alpha}(1+\hat{A}_t+\alpha(\hat{k}_t-\hat{l}_t)) &= w(1+\hat{w}_t) \\
(1+\hat{A}_t+\alpha(\hat{k}_t-\hat{l}_t)) &= (1+\hat{w}_t) \\
\hat{A}_t+\alpha(\hat{k}_t-\hat{l}_t) &= \hat{w}_t
\end{aligned}
$$

DETAILS ON LINEARIZING CONDITIONS (5)

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} L_t^{1-\alpha} \\
Y(1+\hat{y}_t) &= K_t^{\alpha} L_t^{1-\alpha}(1+\hat{A}_t)(1+\alpha\hat{k}_t)(1+(1-\alpha)\hat{l}_t) \\
Y(1+\hat{y}_t) &= K_t^{\alpha} L_t^{1-\alpha}(1+\hat{A}_t+\alpha\hat{k}_t+(1-\alpha)\hat{l}_t) \\
1+\hat{y}_t &= 1+\hat{A}_t+\alpha\hat{k}_t+(1-\alpha)\hat{l}_t \\
\hat{y}_t &= \hat{A}_t+\alpha\hat{k}_t+(1-\alpha)\hat{l}_t
\end{aligned}
$$

DETAILS ON LINEARIZING CONDITIONS (6)

$$\begin{aligned}
Y_t &= C_t + I_t \\
Y(1+\hat{y}_t) &= C(1+\hat{c}_t) + I(1+\hat{i}_t) \\
Y + Y\hat{y}_t &= C + C\hat{c}_t + I + I\hat{i}_t \\
Y\hat{y}_t &= C\hat{c}_t + I\hat{i}_t \\
\hat{y}_t &= \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t
\end{aligned}$$

DETAILS ON LINEARIZING CONDITIONS (7)

$$
\begin{aligned}
K_{t+1} &= (1-\delta)K_t + I_t \\
K(1+\hat{k}_{t+1}) &= (1-\delta)K(1+\hat{k}_t) + I(1+\hat{i}_t) \\
K + K\hat{k}_{t+1} &= (1-\delta)K + (1-\delta)K\hat{k}_t + I + I\hat{i}_t \\
K\hat{k}_{t+1} &= (1-\delta)K\hat{k}_t + I\hat{i}_t \\
K &= (1-\delta)K + I \Rightarrow I = \delta K \\
K\hat{k}_{t+1} &= (1-\delta)K\hat{k}_t + \delta K\hat{i}_t \\
\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + \delta\hat{i}_t
\end{aligned}
$$

STEADY STATE EQUATIONS

- We have seven equations in 7 unknowns $\{C, L, K, I, Y, r, w\}$:

$$
\begin{aligned}
\chi C^{\sigma} L^{\phi} &= w \\
1 &= \beta r \\
\alpha K^{\alpha-1} L^{1-\alpha} &= r - (1-\delta) \\
(1-\alpha) K^{\alpha} L^{-\alpha} &= w \\
Y &= K^{\alpha} L^{1-\alpha} \\
Y &= C + I \\
\delta K &= I
\end{aligned}
$$

- With some substitution and algebra, we can recover the steady state expressions in terms of the parameters.

IMPORTANT STEADY STATES

$$\frac{K}{L} = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{1}{1-\alpha}}$$

$$\Psi = \left(\frac{1-\alpha}{\chi} \right)^{\frac{1}{\sigma}} \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{\alpha}{\sigma(1-\alpha)}}$$

$$L = \left(\frac{1}{\Psi} \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha} - \delta \right) \right)^{-\frac{\sigma}{\phi+\sigma}} \left(\frac{K}{L} \right)^{-\frac{\sigma}{\sigma+\phi}}$$

$$K = \left(\frac{1}{\Psi} \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha} - \delta \right) \right)^{-\frac{\sigma}{\phi+\sigma}} \left(\frac{K}{L} \right)^{-\frac{\phi}{\sigma+\phi}}$$

$$Y = \left(\frac{1}{\Psi} \left(\frac{\frac{1}{\beta} - (1-\delta)}{\alpha} - \delta \right) \right)^{-\frac{\sigma}{\phi+\sigma}} \left(\frac{K}{L} \right)^{-\frac{\alpha\phi+(1-\alpha)\sigma}{\sigma+\phi}}$$

$$\Gamma = \delta \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{\sigma-\phi}{\sigma+\phi}}$$