Thinking Big:

Determinacy and Large-Scale Solutions in the Sequence Space

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- How do we solve models <u>when both S and N are large?</u>

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  - Using $T(\mathbf{j}^{-1})$ as guess for $\mathbf{J}^{-1}$ gives **rapid iterative solution**, even when N large

Toeplitz and quasi-Toeplitz operators

What is a Toeplitz operator?

$$T(\mathbf{j}) = \begin{pmatrix} j_0 & j_{-1} & j_{-2} & \\ j_1 & j_0 & j_{-1} & \ddots \\ j_2 & j_1 & j_0 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

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- Can define series $j(z) \equiv \sum_{s=-\infty}^{\infty} j_s z^s$, sometimes called **"symbol"** of $T(\mathbf{j})$

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  [First-order impulse responses give full linear rational expectations solution: Boppart-Krusell-Mitman 2018, Auclert-Bardoczy-Rognlie-Straub 2021…]

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$$\begin{pmatrix} -1 & 0 & 0 & \cdots \\ R & -1 & 0 & \ddots \\ 0 & R & -1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

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[= taking one-period-ahead expectations given MIT shock]

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Columns of Toeplitz matrix: costs to reset prices

$$(1-\beta\theta)\times$$

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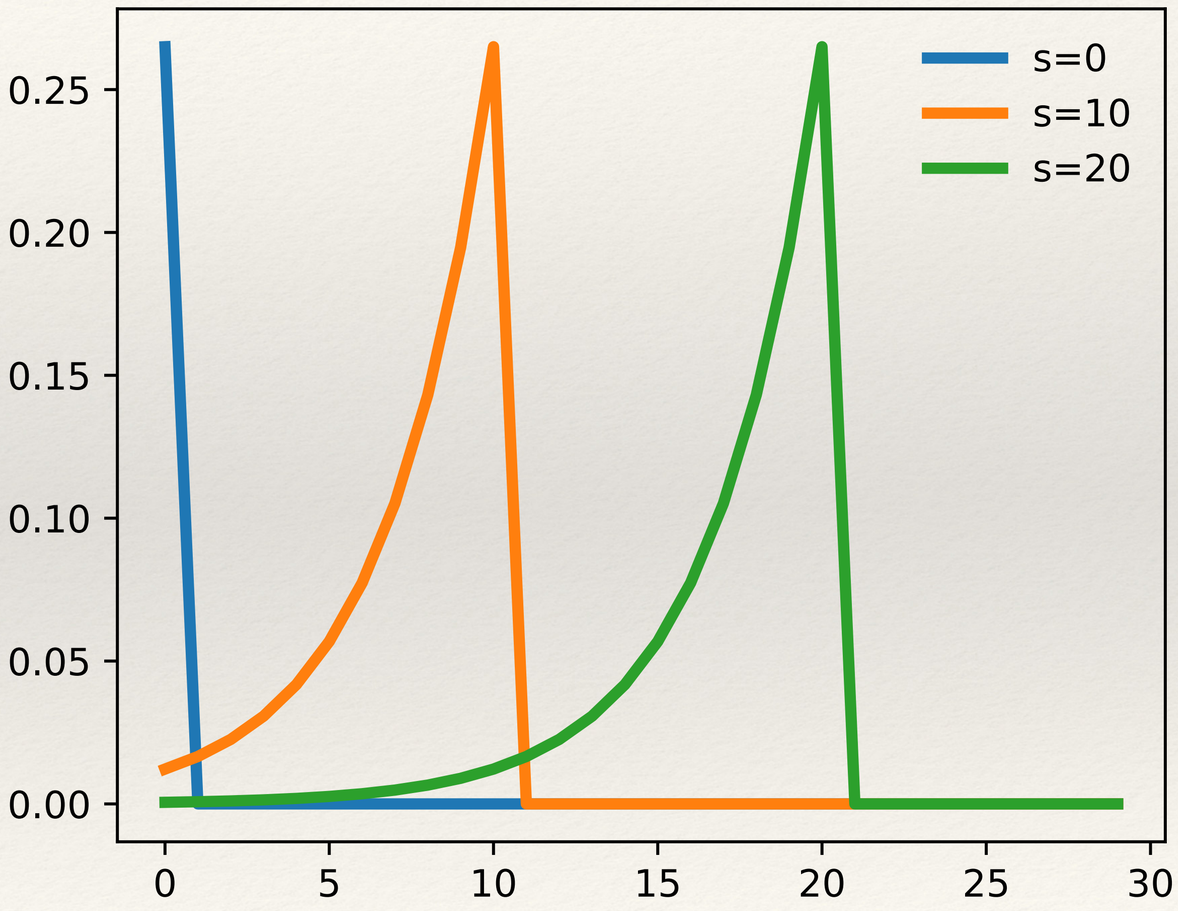
$(\beta = 0.98,\ \ \theta = 0.75)$



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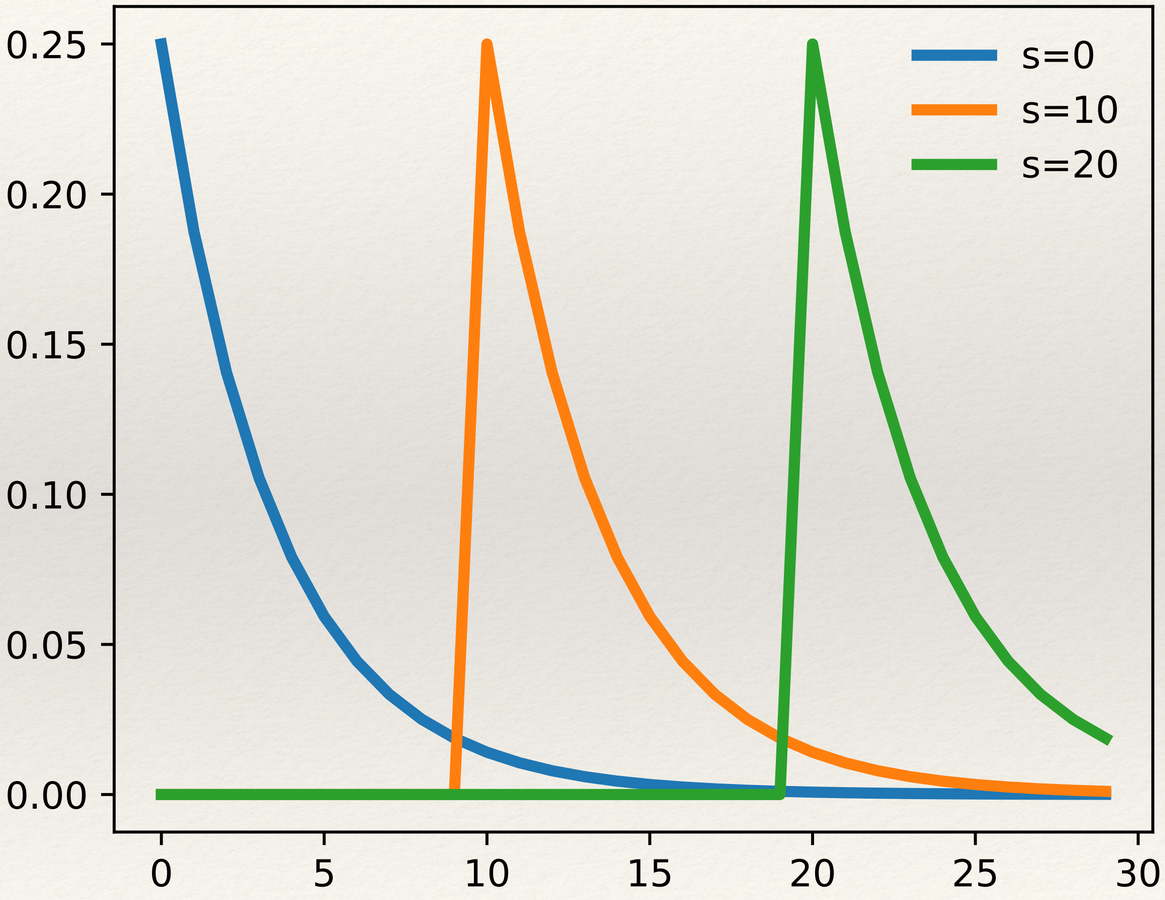

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  - $\mathbb{E}_{t-1} x_t = x_t$ for $t > 0$, but $\mathbb{E}_{-1} x_0 = 0 \neq x_0$

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  - Not needed in special case with $\mathbf{j}_1$ forward-looking and $\mathbf{j}_2$ backward-looking
- Quasi-Toeplitz operators closed under multiplication!
  [multiplying Toeplitz parts → quasi-Toeplitz, multiplying any bounded times compact → compact]

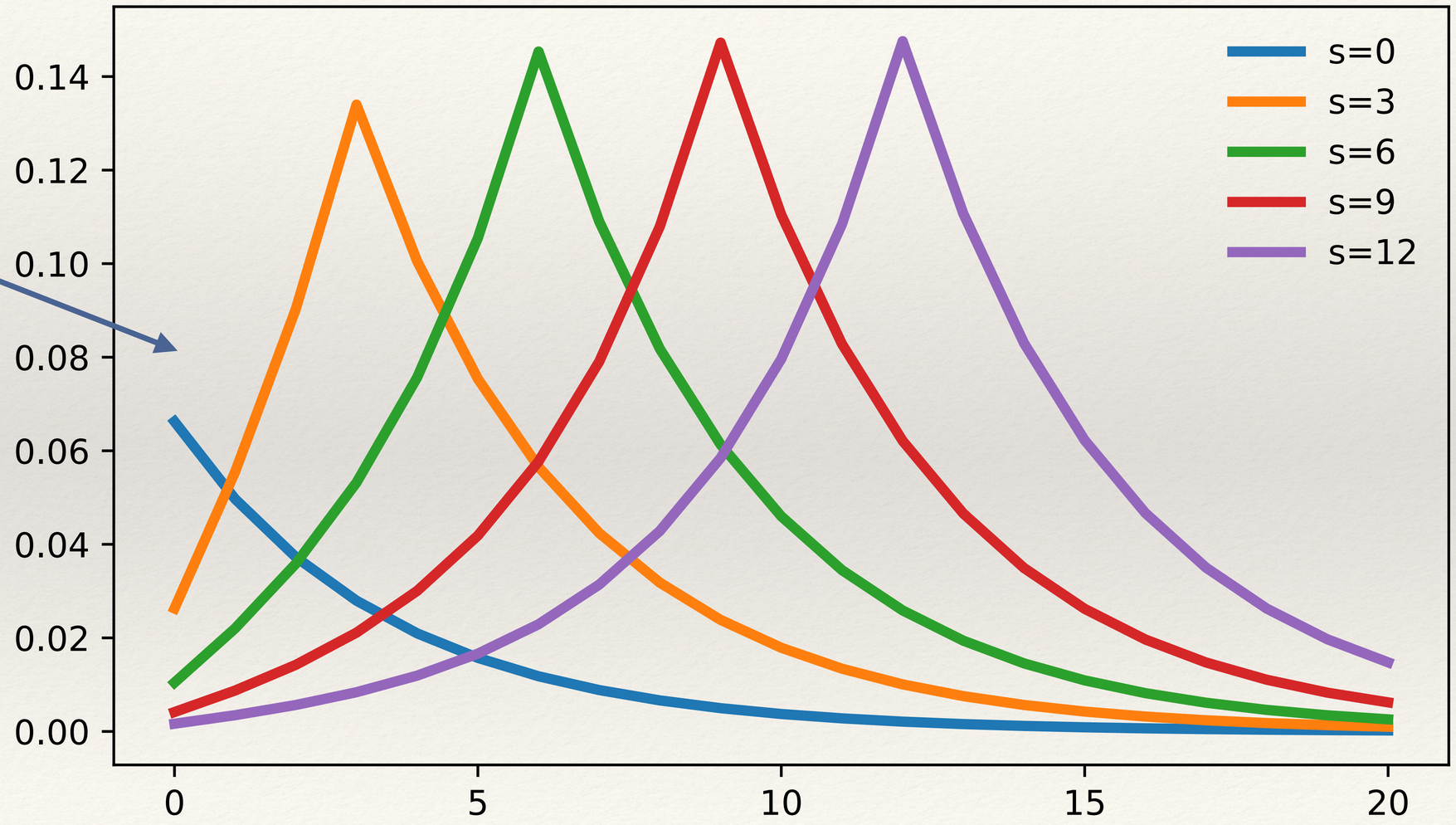
Compose to get map $\mathbf{J}$ from costs to aggregate prices

$$(1-\theta)\times \begin{pmatrix} 1 & 0 & 0 & \\ \theta & 1 & 0 & \ddots \\ \theta^2 & \theta & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \times (1-\beta\theta)\times \begin{pmatrix} 1 & \beta\theta & (\beta\theta)^2 & \\ 0 & 1 & \beta\theta & \ddots \\ 0 & 0 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

(Lag of lead: we should expect a compact "correction" $\mathbf{E}$!)

Visualizing columns of this **J**

Eventually converge to Toeplitz pattern, but "missing anticipation" leads to a limited price response near date 0



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Decays exponentially → compact!

Structure theorem: Jacobians from “stationary” heterogeneous-agent problems are quasi-Toeplitz

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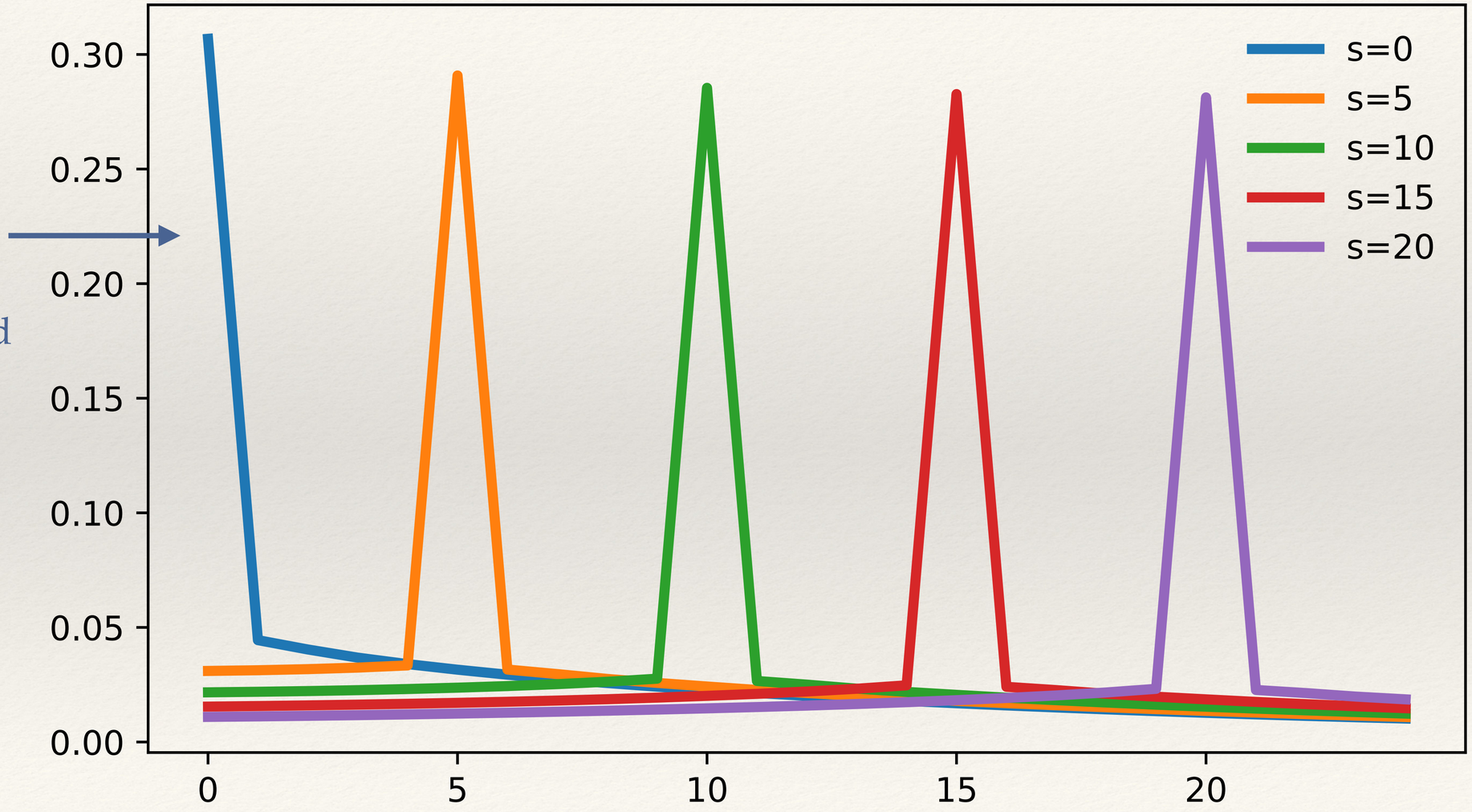
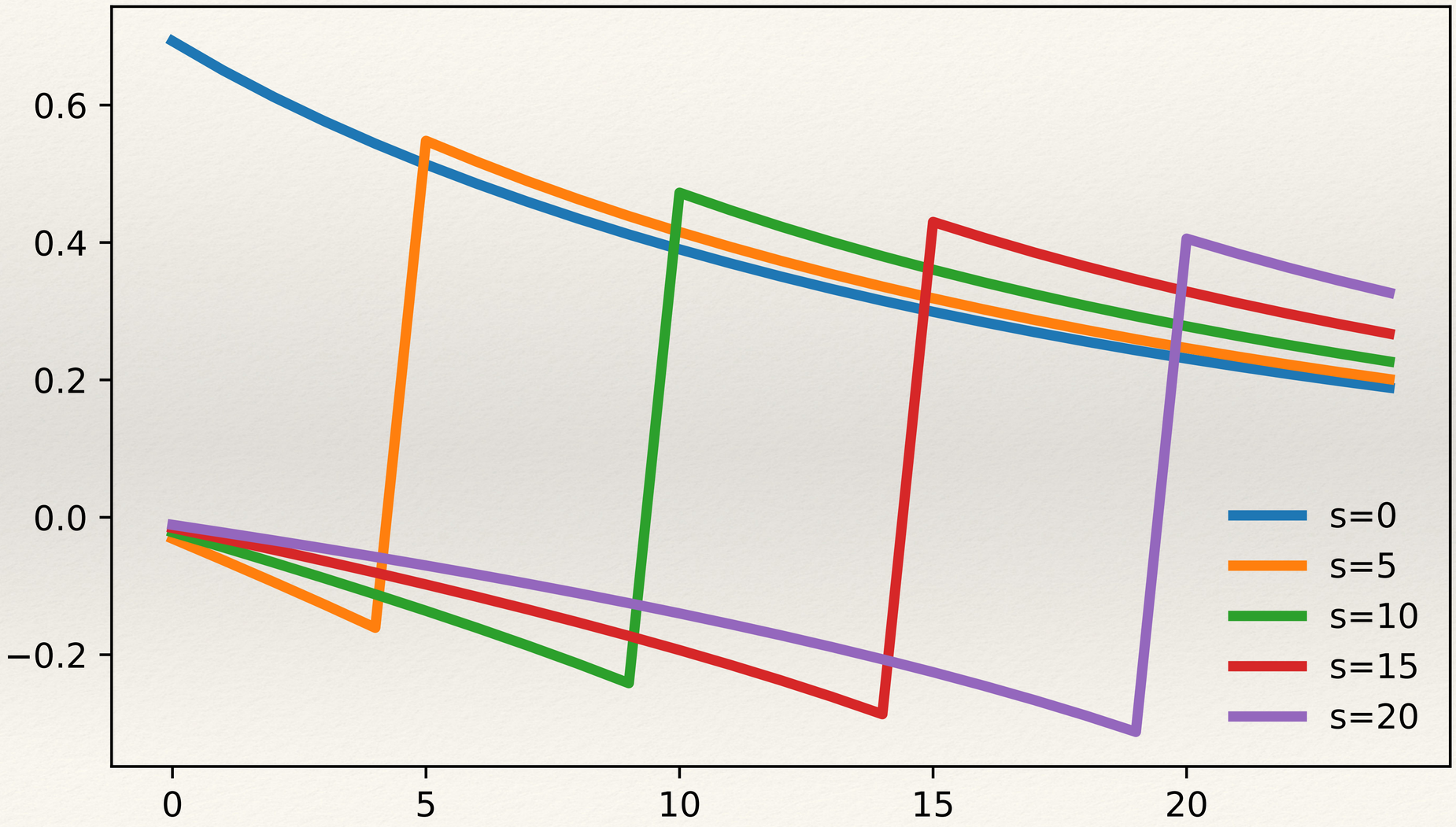
Example: intertemporal MPCs in het-agent model

E is positive, because not having anticipated income means you didn't spend ahead of time, leaving more to spend now



Same, but Jacobian of assets vs. income



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Determinacy, existence, and inversion

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(Lead of lag: exactly Toeplitz without any correction.)

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Inverting switches order of matrices, so now we're taking lag of a lead → will get compact correction & quasi-Toeplitz!

$$\mathbf{H_K} = \begin{pmatrix} R & -\zeta^{-1} & 0 & \cdots \\ 0 & R & -\zeta^{-1} & \ddots \\ 0 & 0 & R & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} -\zeta & 0 & 0 & \cdots \\ 1 & -\zeta & 0 & \ddots \\ 0 & 1 & -\zeta & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

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  - This agrees with standard criterion (e.g. Blanchard-Kahn)

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- [Example: lag operator has $j(z) = z$, winding number of 1, so not surjective.]

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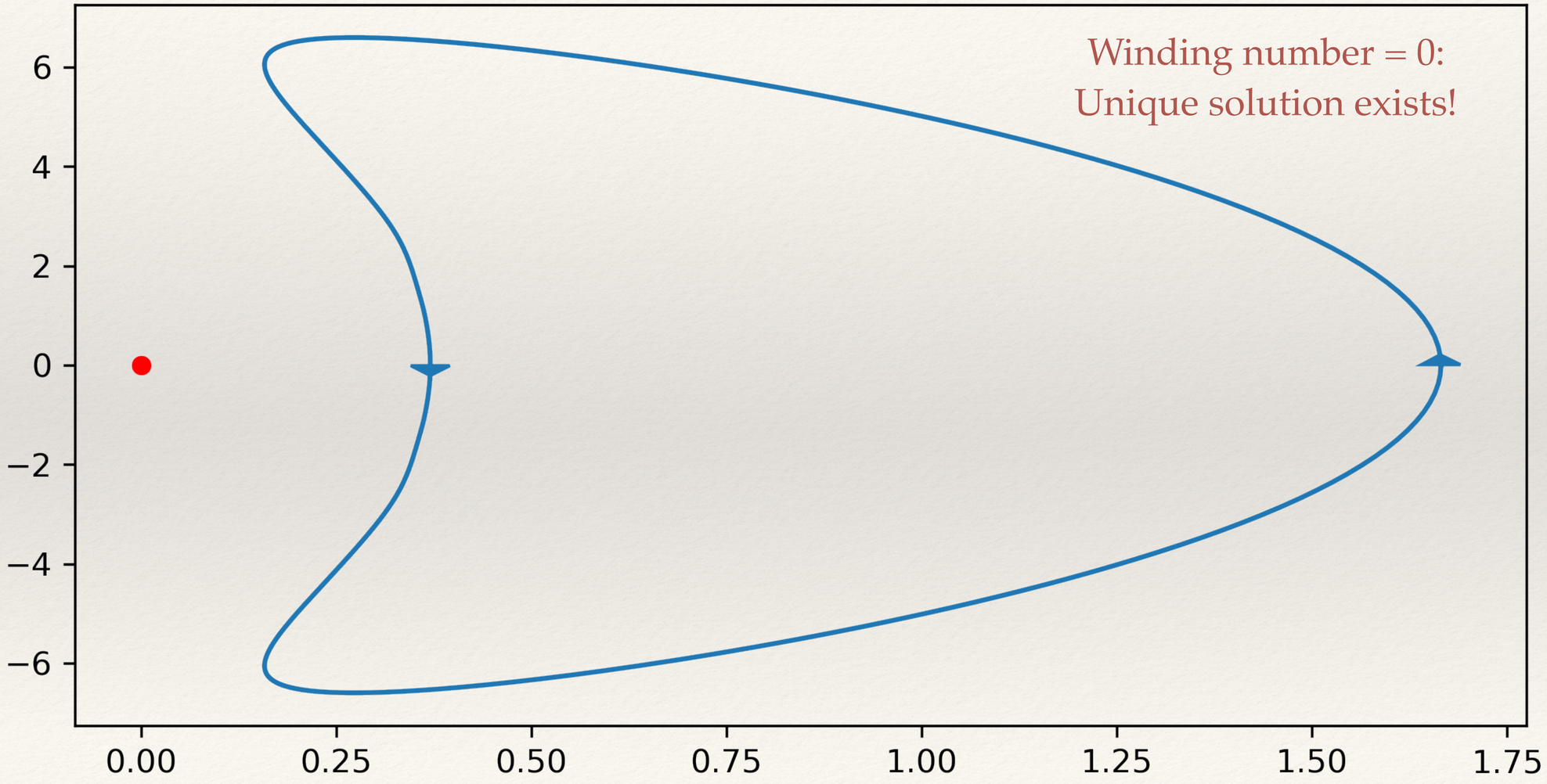
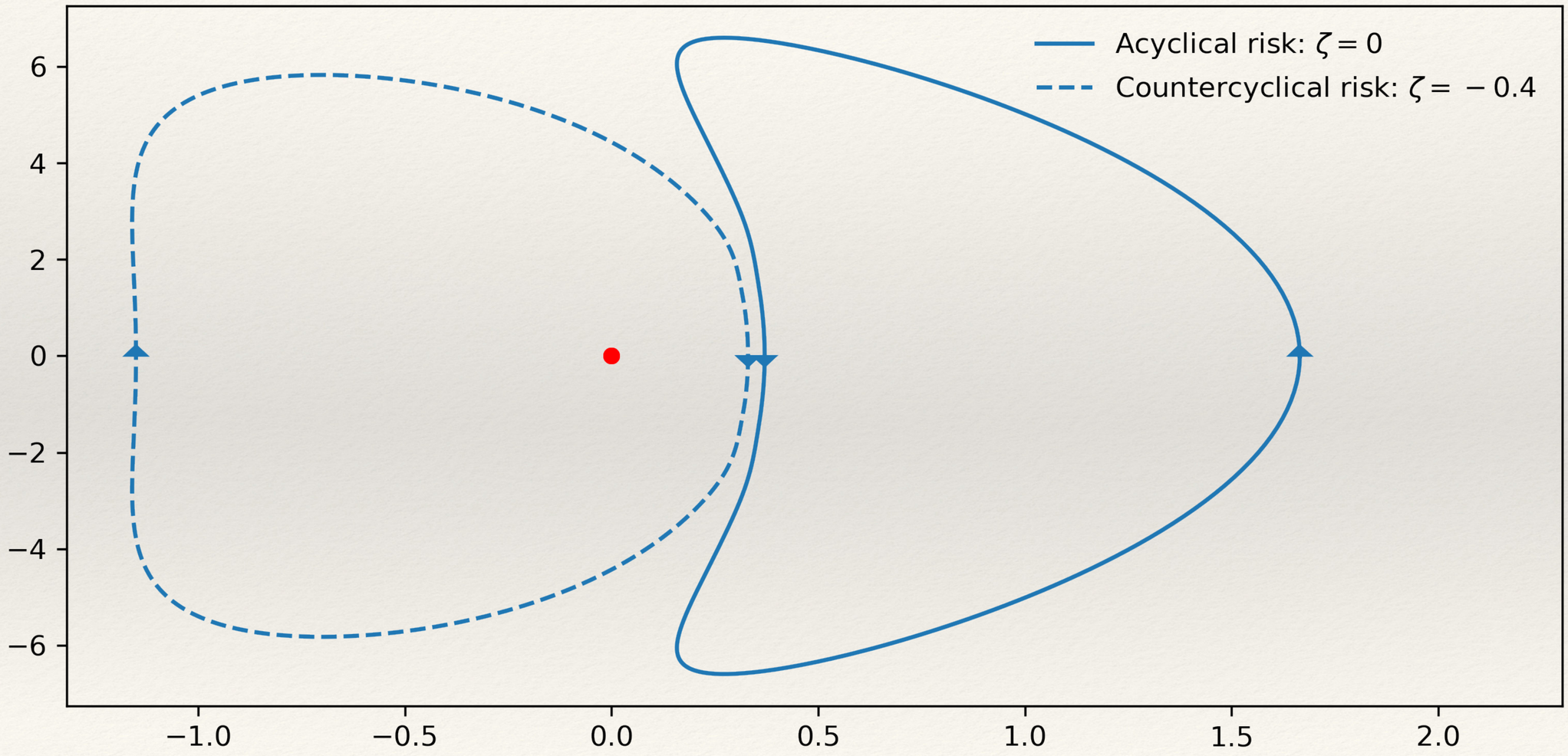
- Multiple equilibria $d\mathbf{Y}$ if winding number is negative!

Winding number plot for **A** with standard calibration



Compare to case with countercyclical risk



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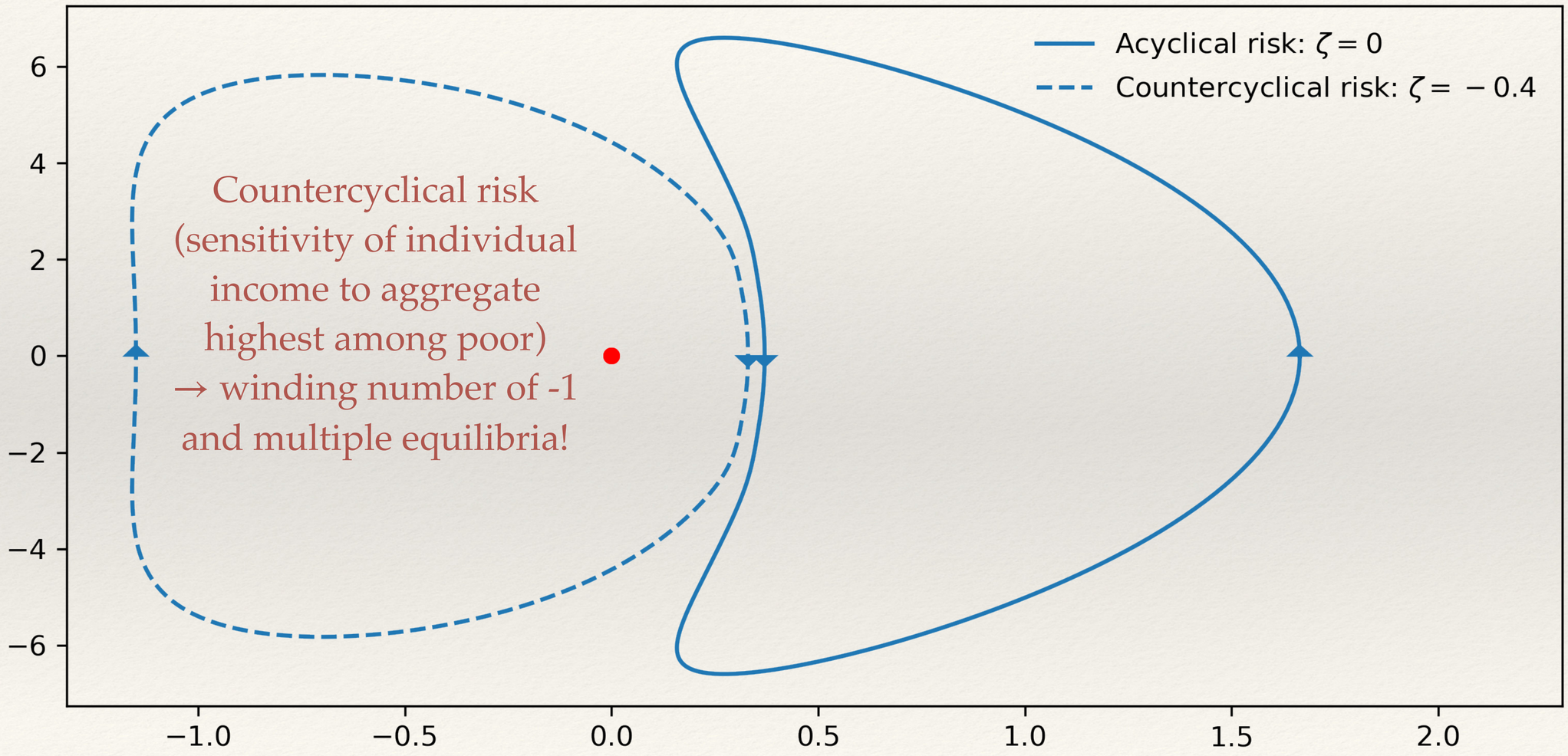

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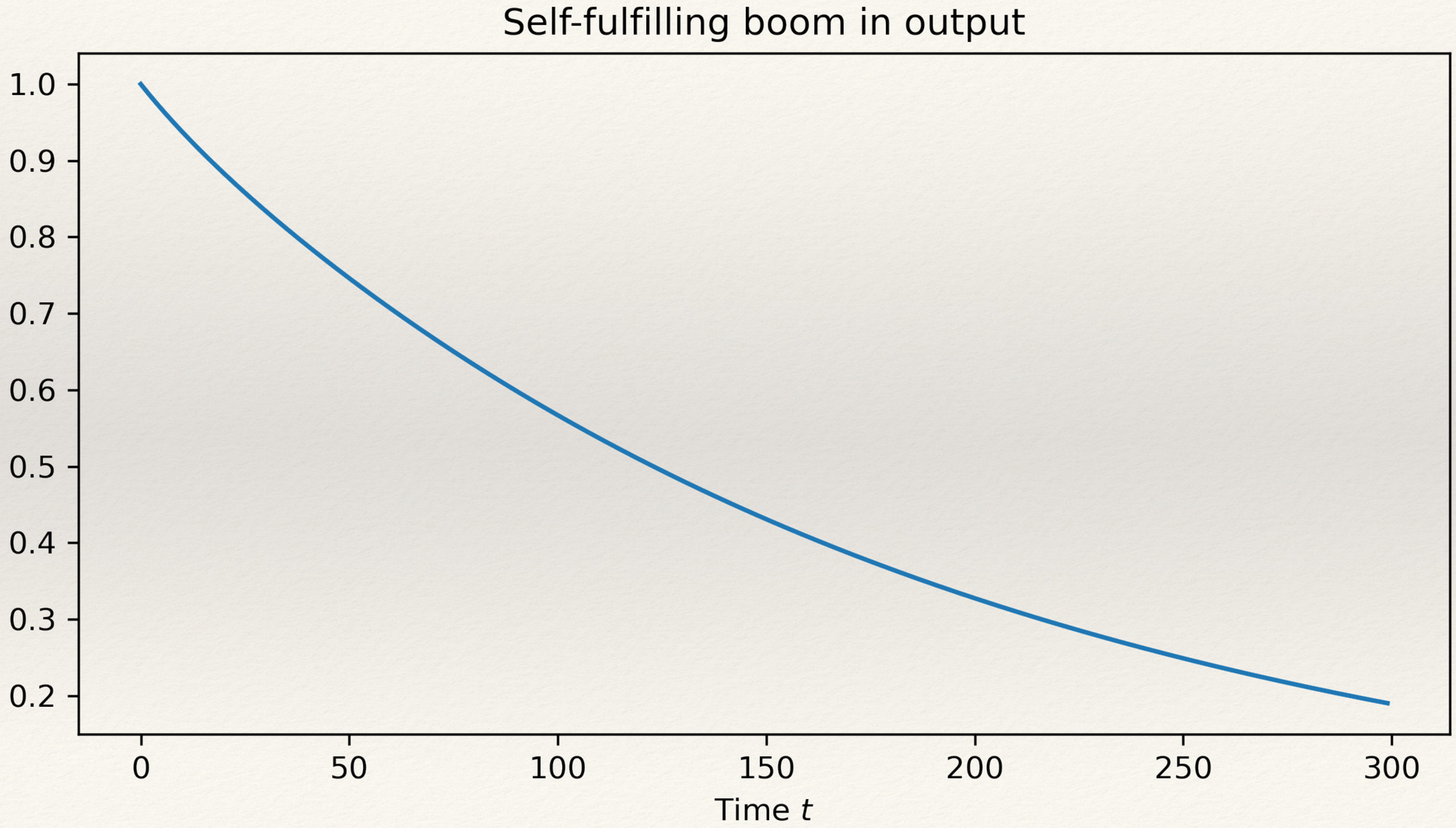

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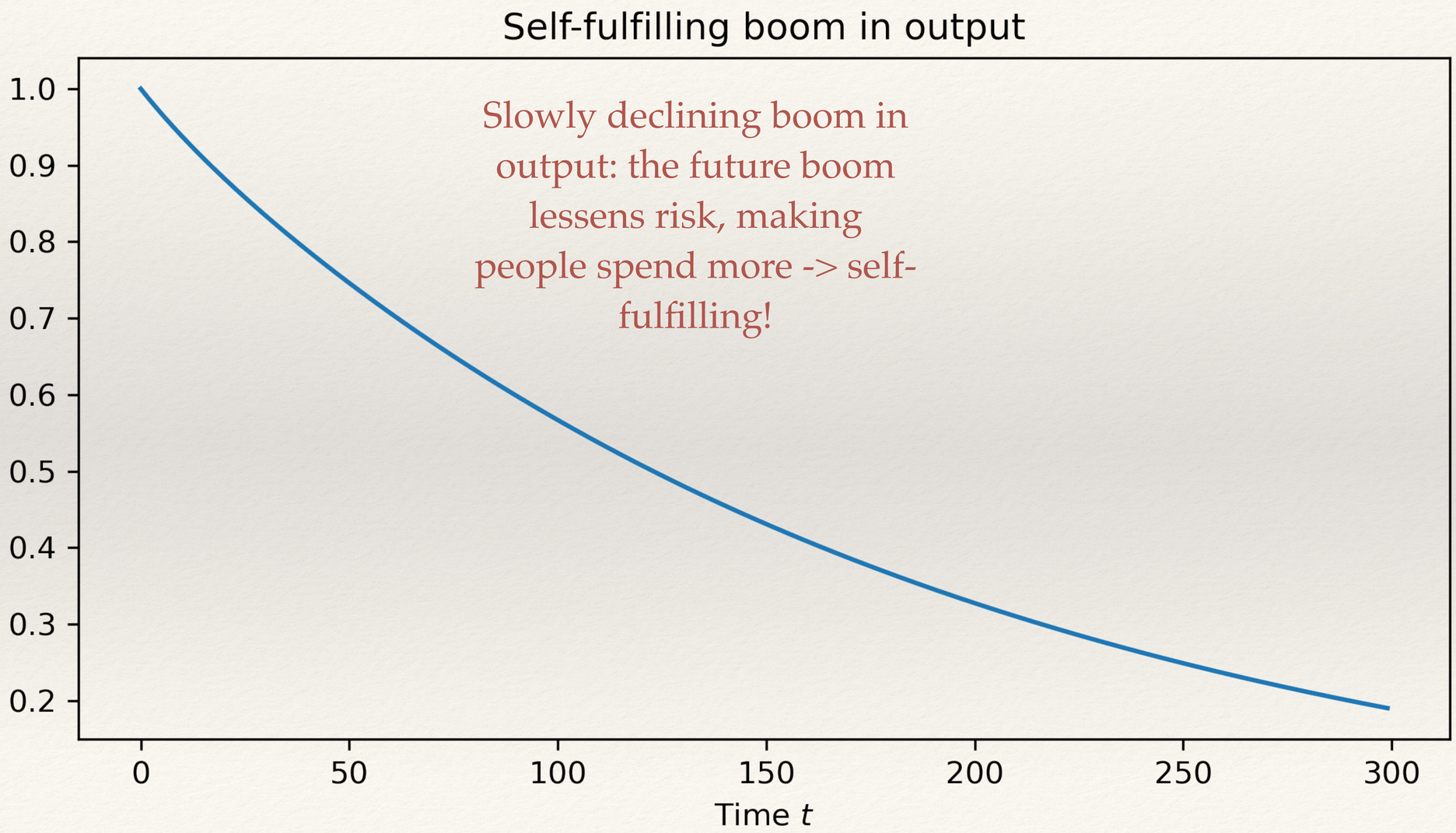

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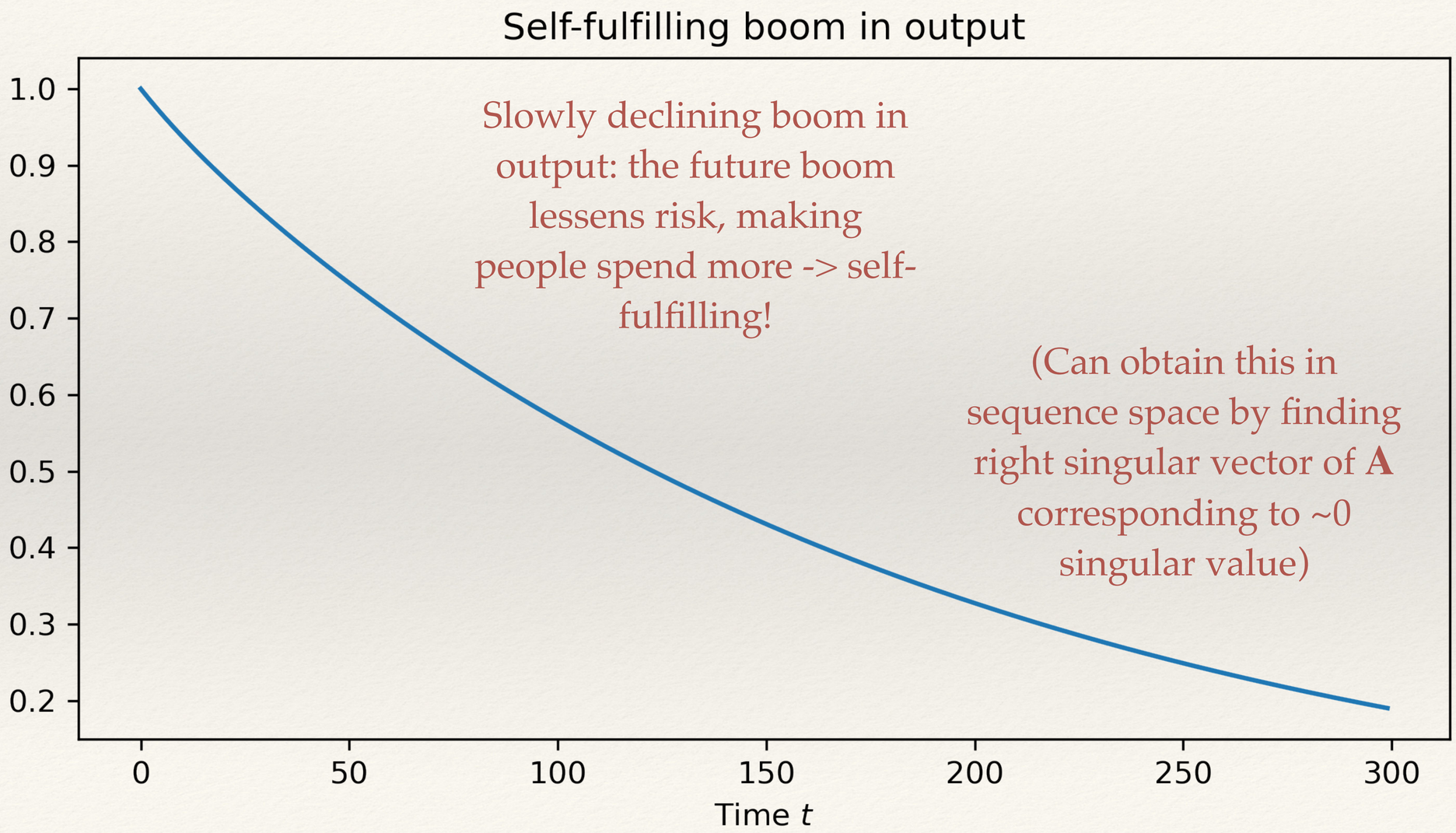

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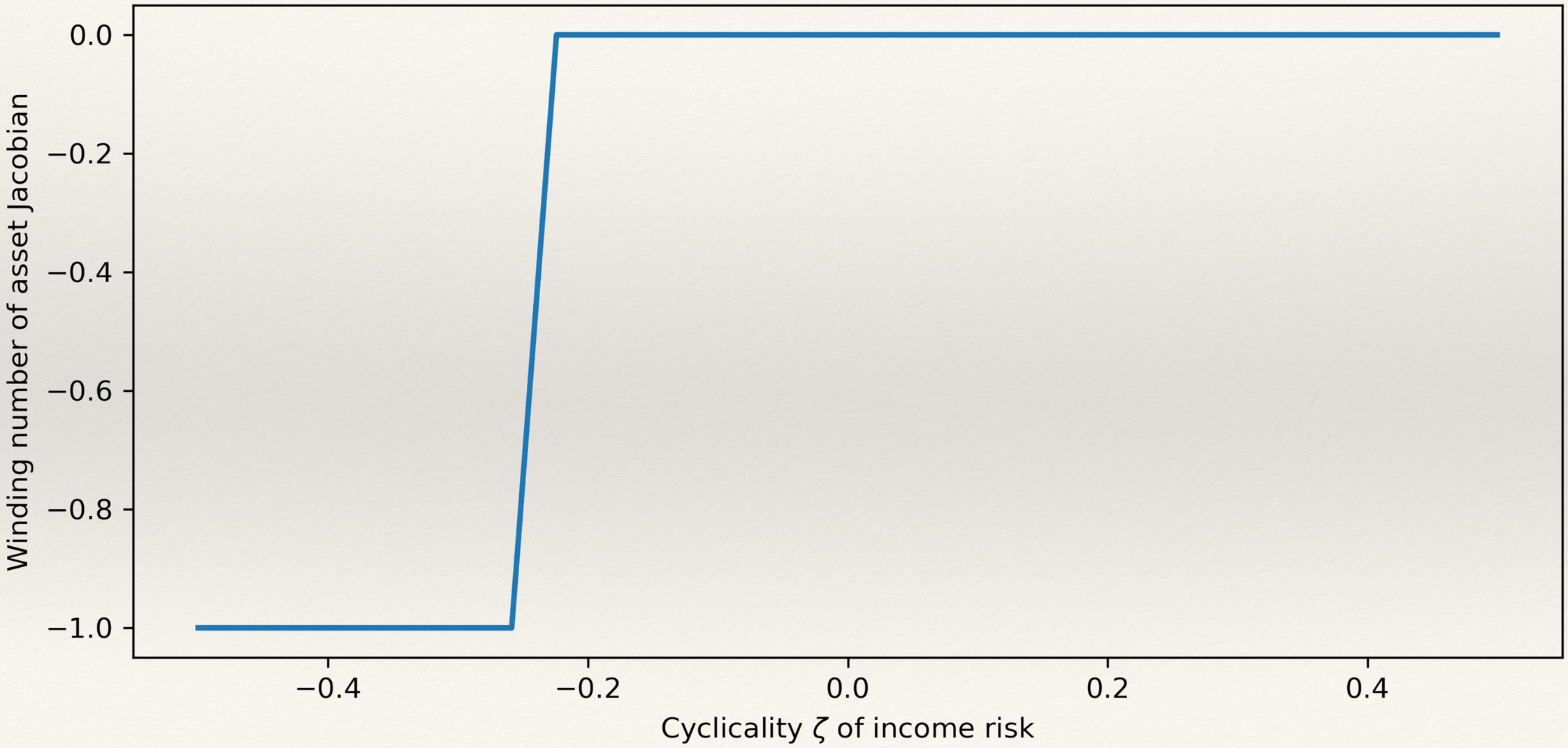

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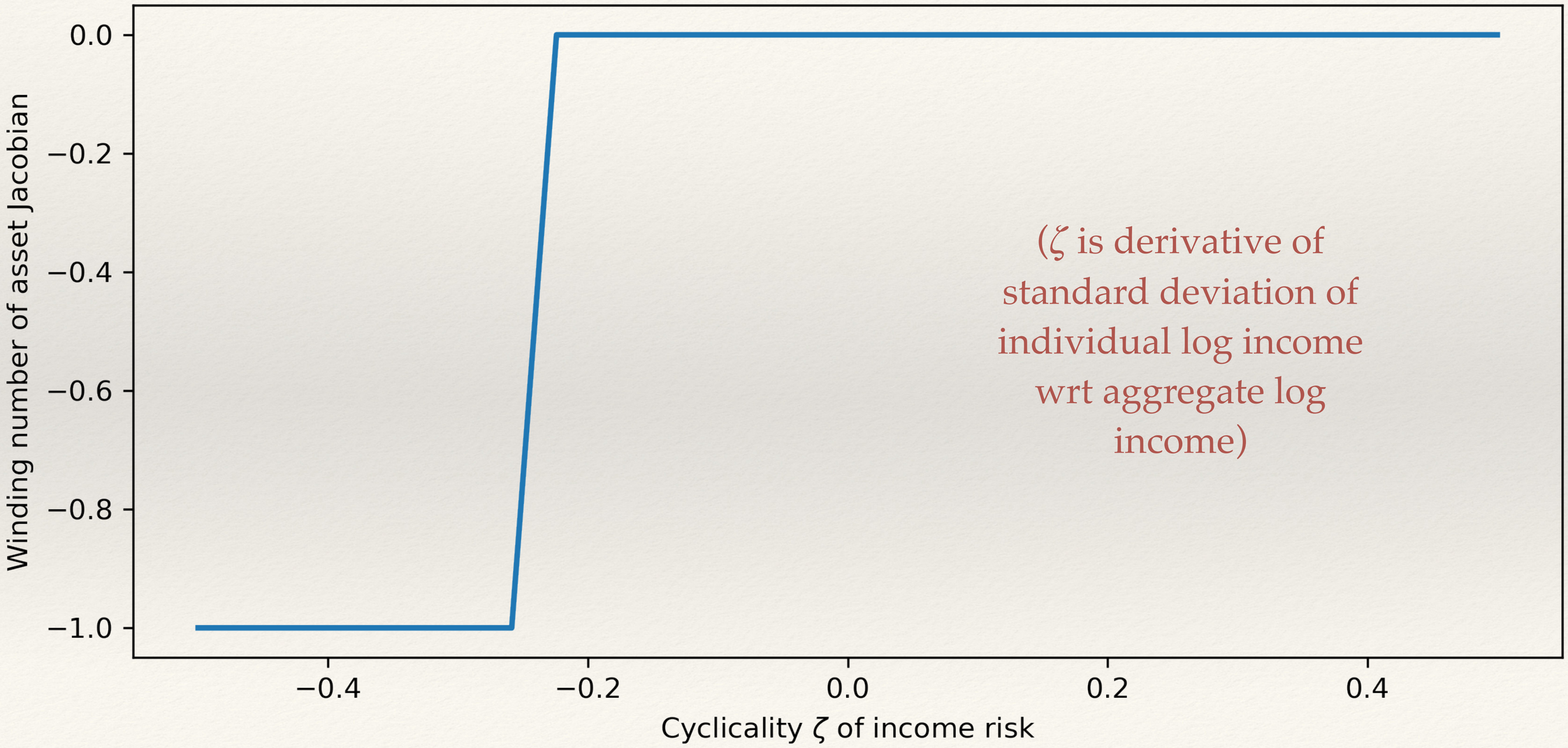

(ζ is derivative of standard deviation of individual log income wrt aggregate log income)

Operations with quasi-Toeplitz operators: No more truncation!

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  - $\mathbf{E}$ often well-approximated by low-rank $\mathbf{E} \approx \mathbf{U}\mathbf{V}'$, with $\mathbf{U}, \mathbf{V}$ dimension $T \times k$ for some low k

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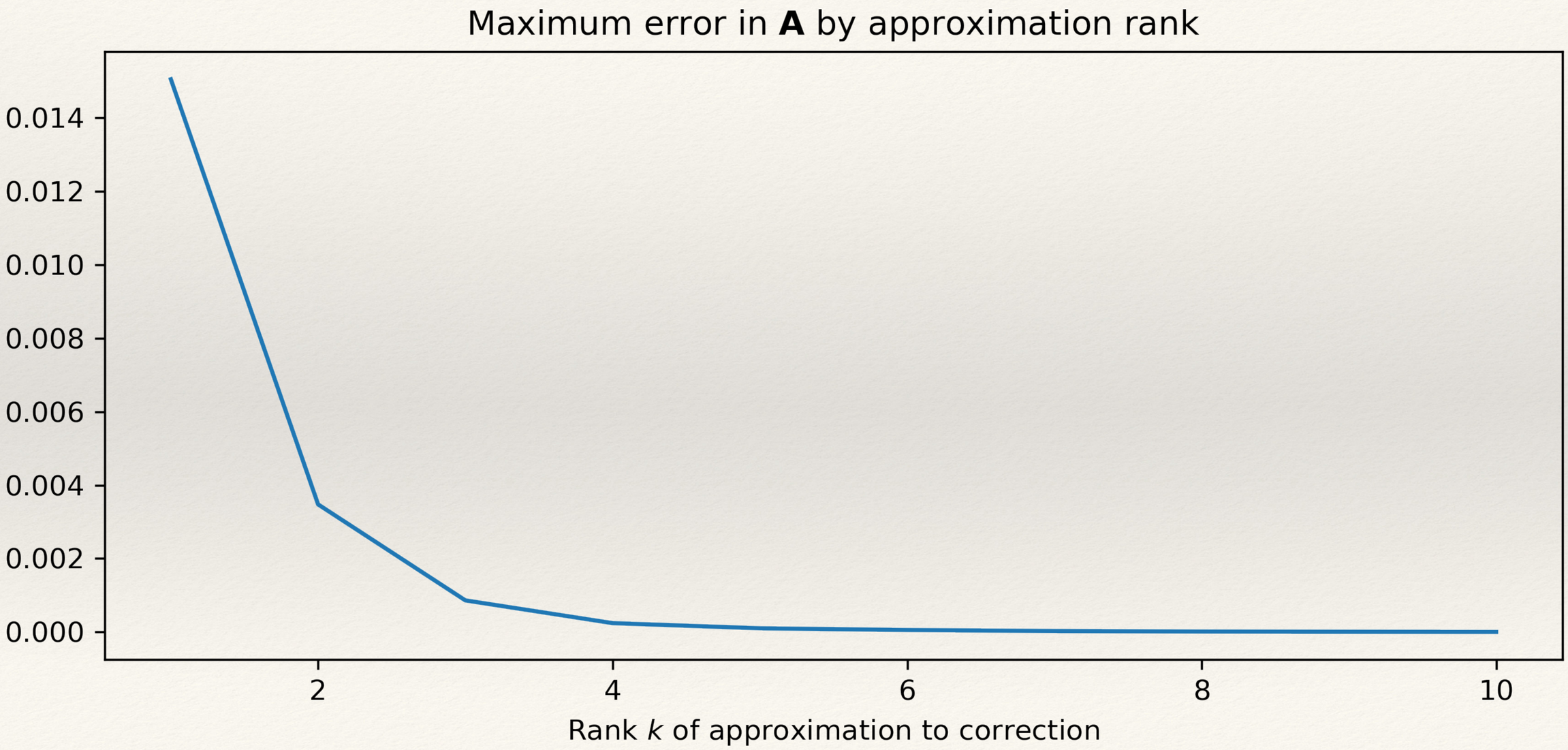
- Working directly with quasi-Toeplitz (infinite!), not truncated matrices

How well can we approximate **A**?



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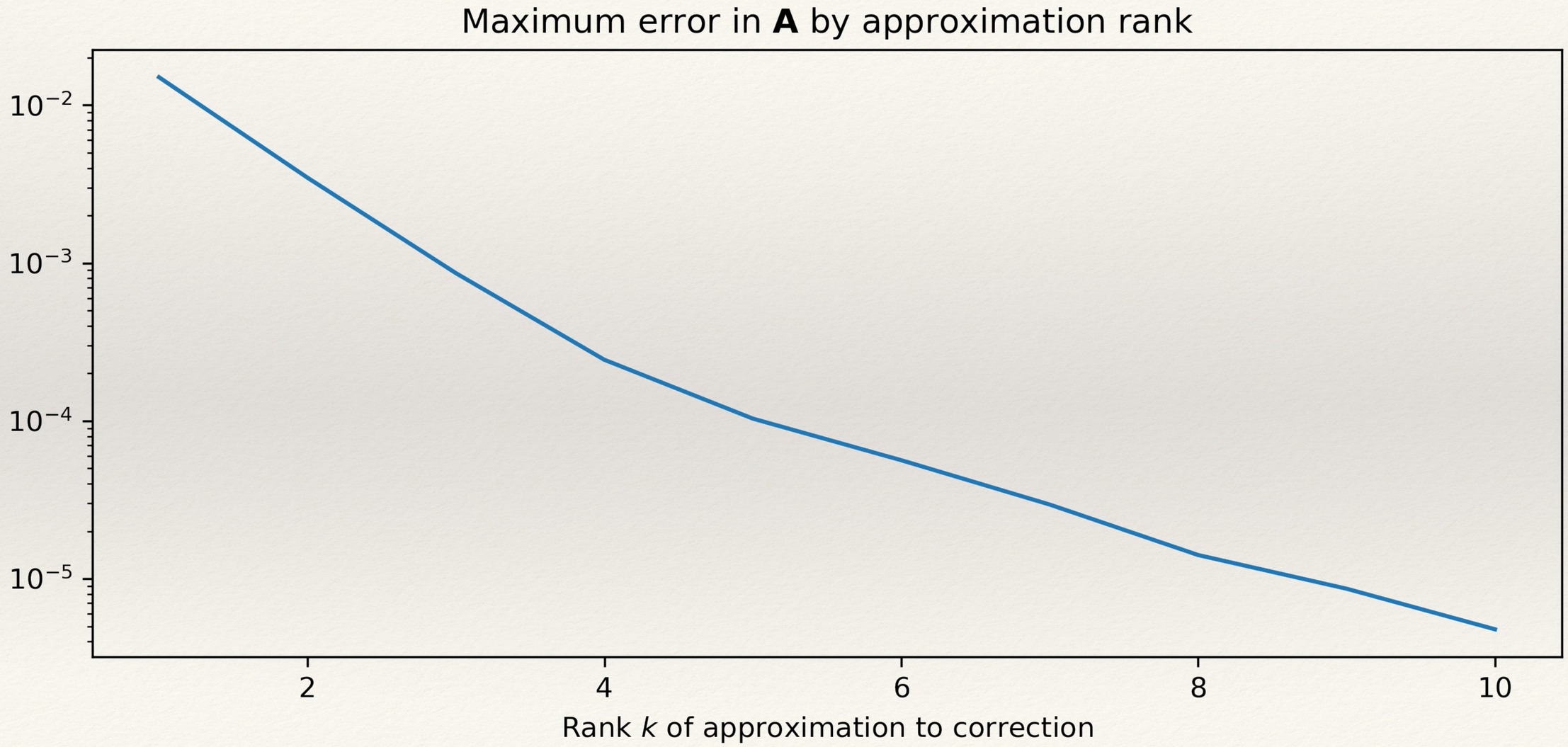

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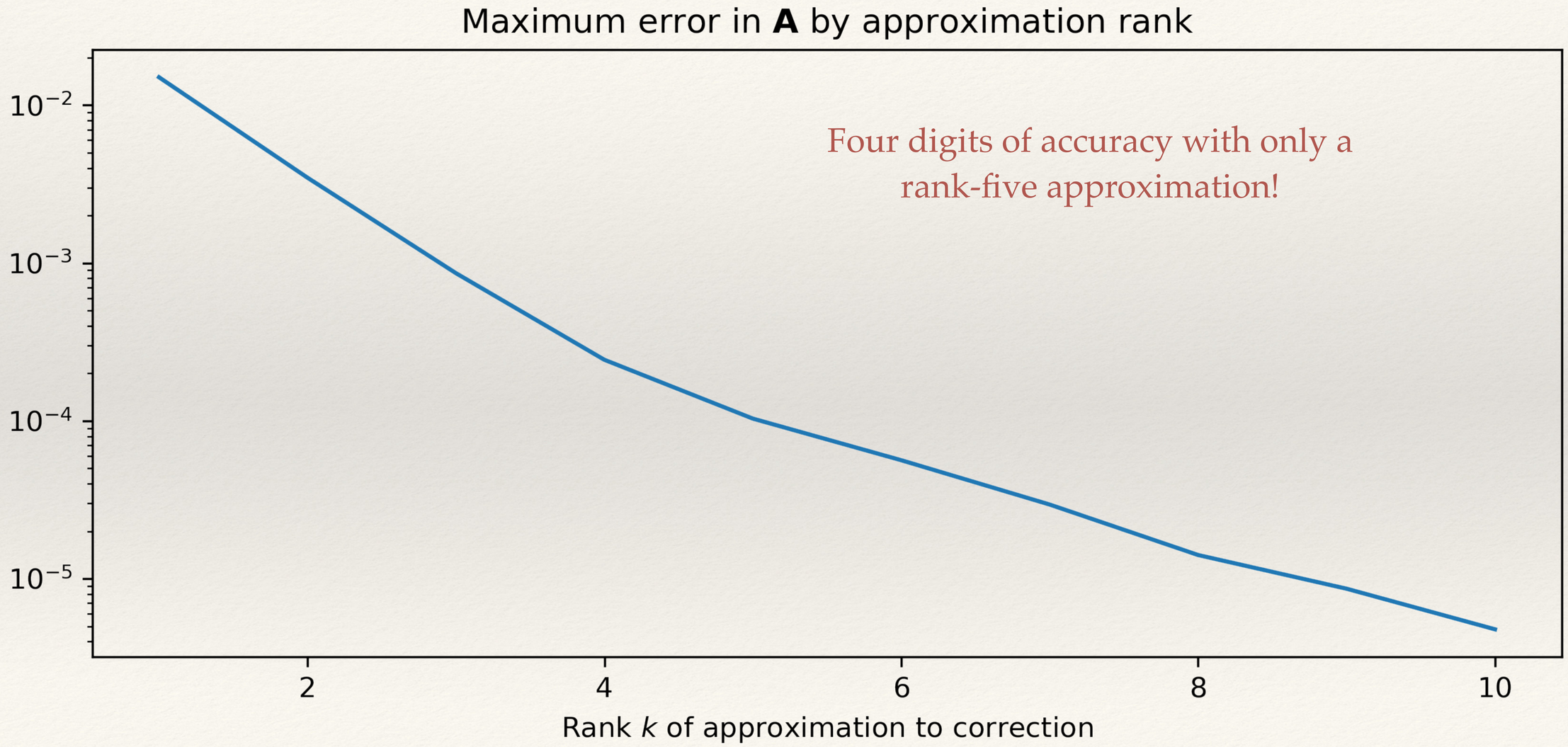
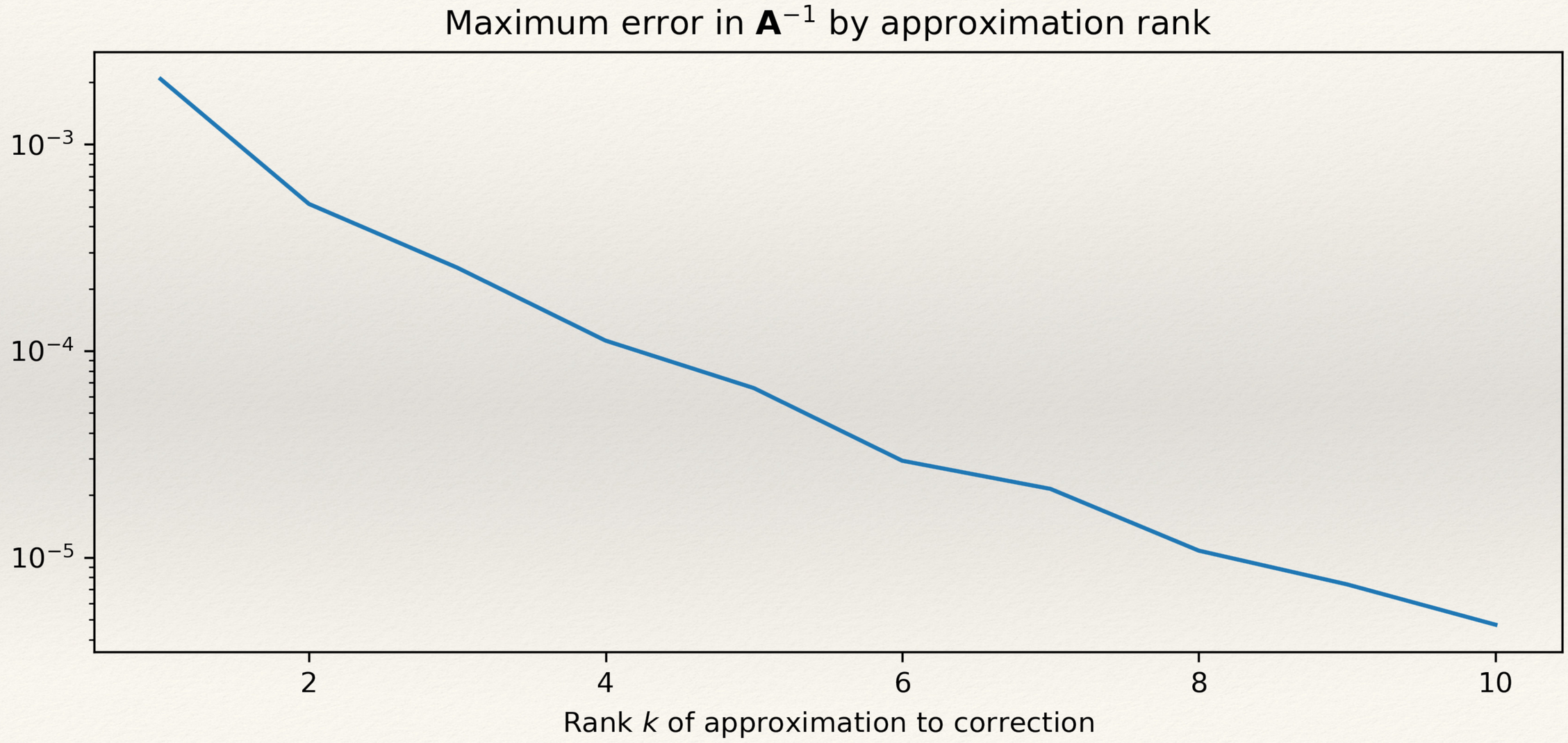


Low-rank approximation to $\mathbf{A}^{-1}$ also close!



Alternative: use structure for iterative solutions (and solve giant models in the process!)

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  - [conceptually, $\mathbf{a}^{-1}$ is inverse for infinitely-well-anticipated shocks]

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- Iterative method (GMRES) **very** good at solving $(\mathbf{I} + \mathbf{C})^{-1}d\mathbf{y}$ if $\mathbf{C}$ low-rank

What can we do with $\mathbf{a}^{-1}$?

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- Cheap, doesn't require explicitly forming new matrices like $\mathbf{C}$

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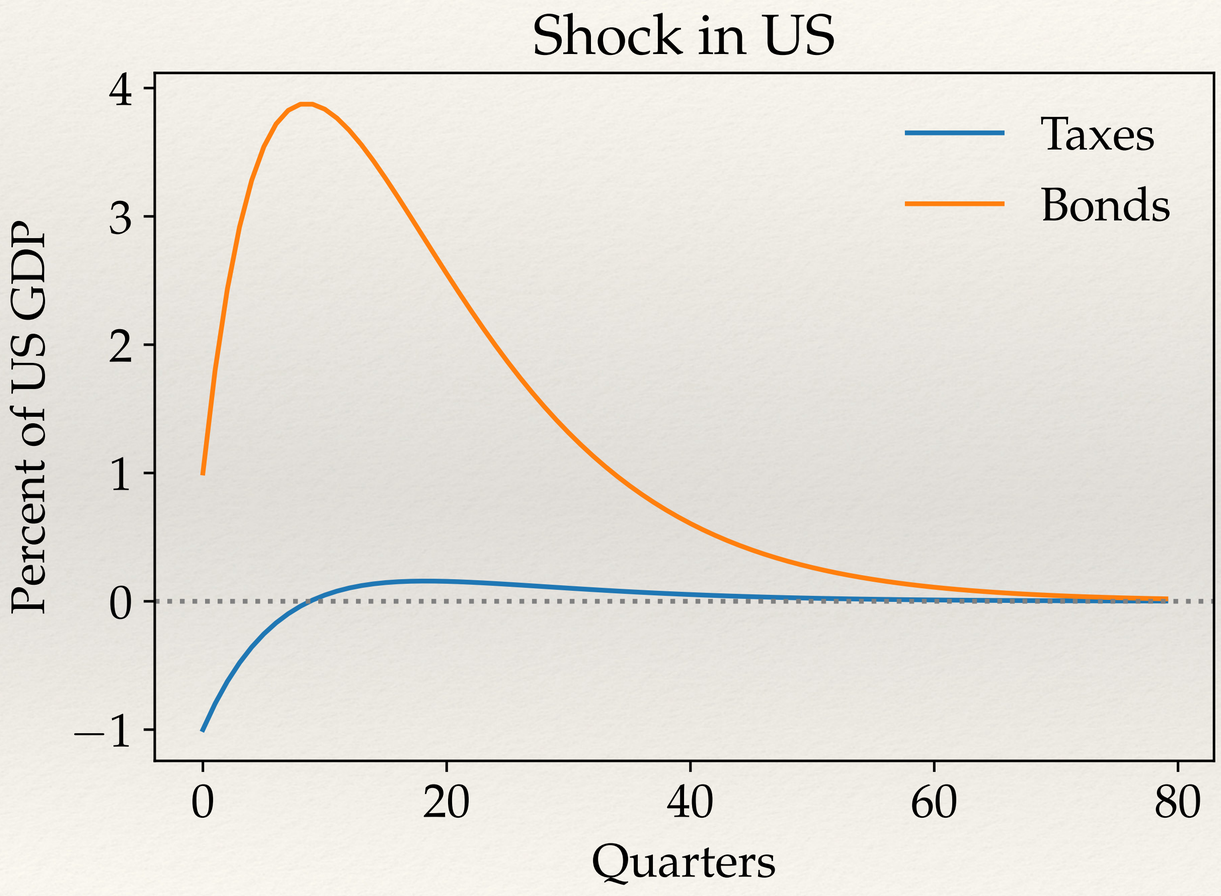
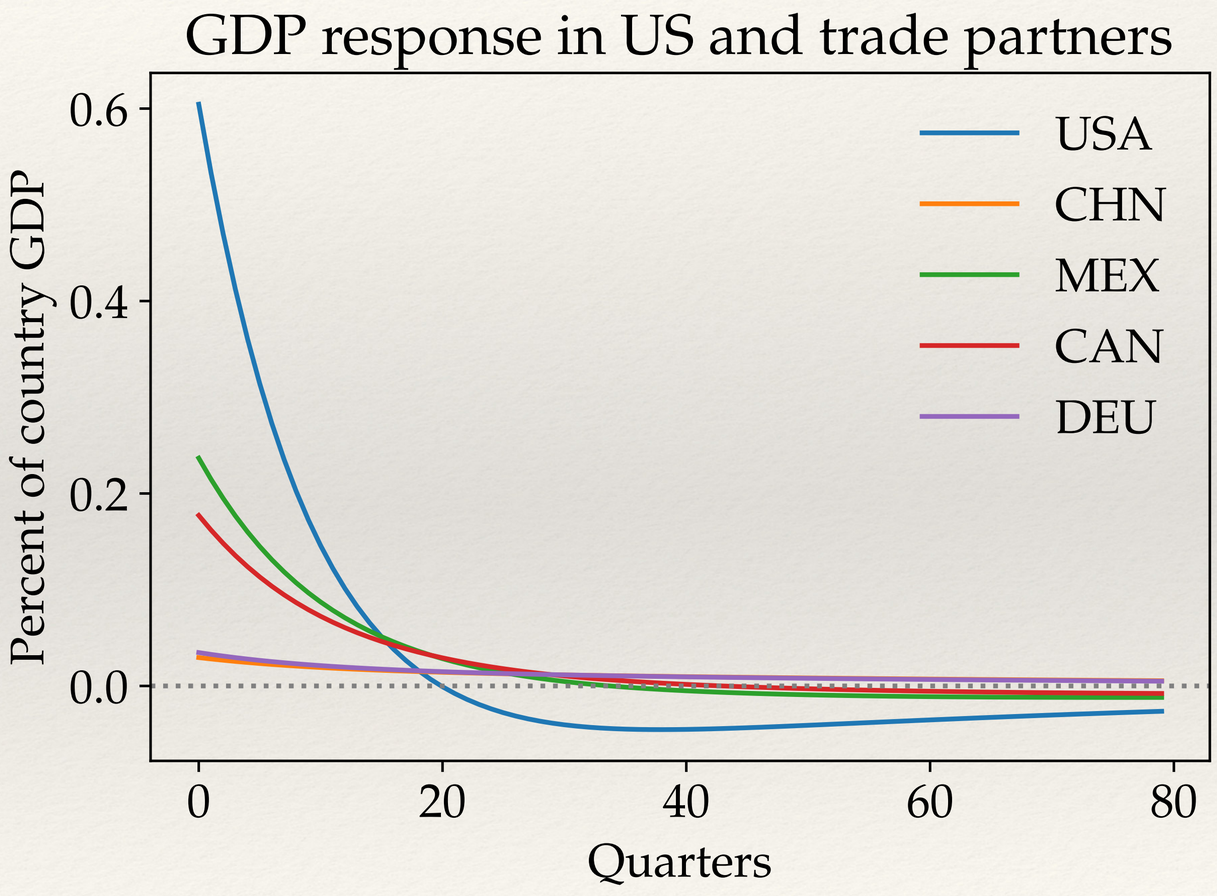
- With iterative approach, solves in a few seconds on laptop!

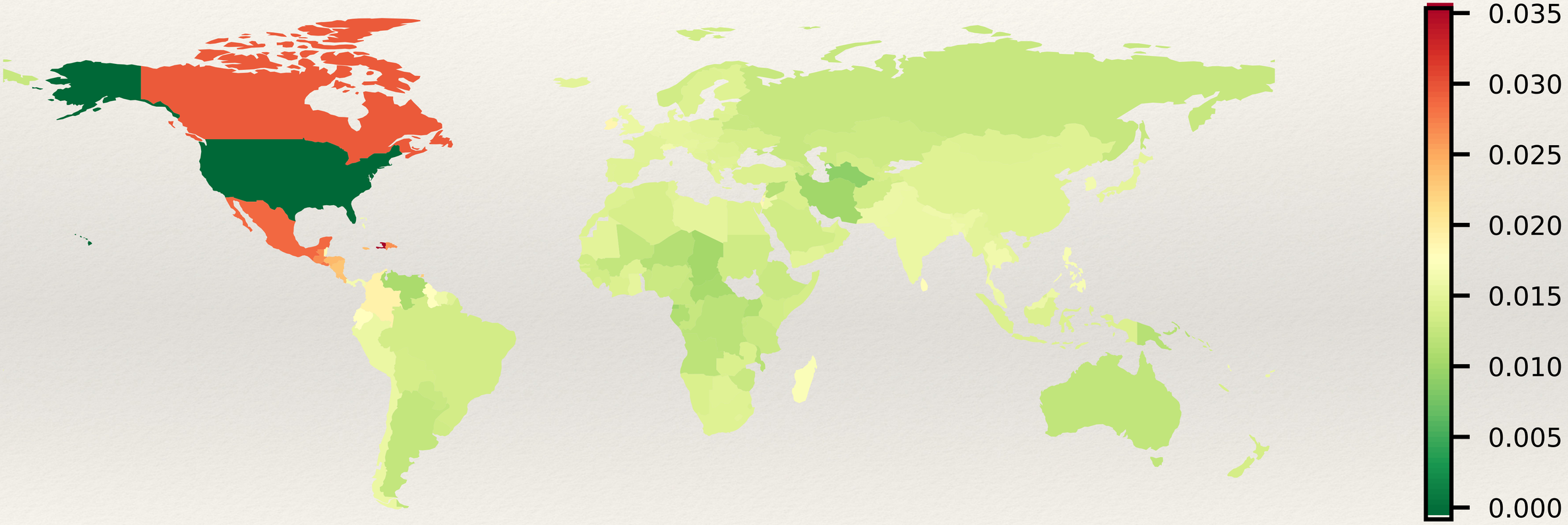
Peek at solution: selected countries over time



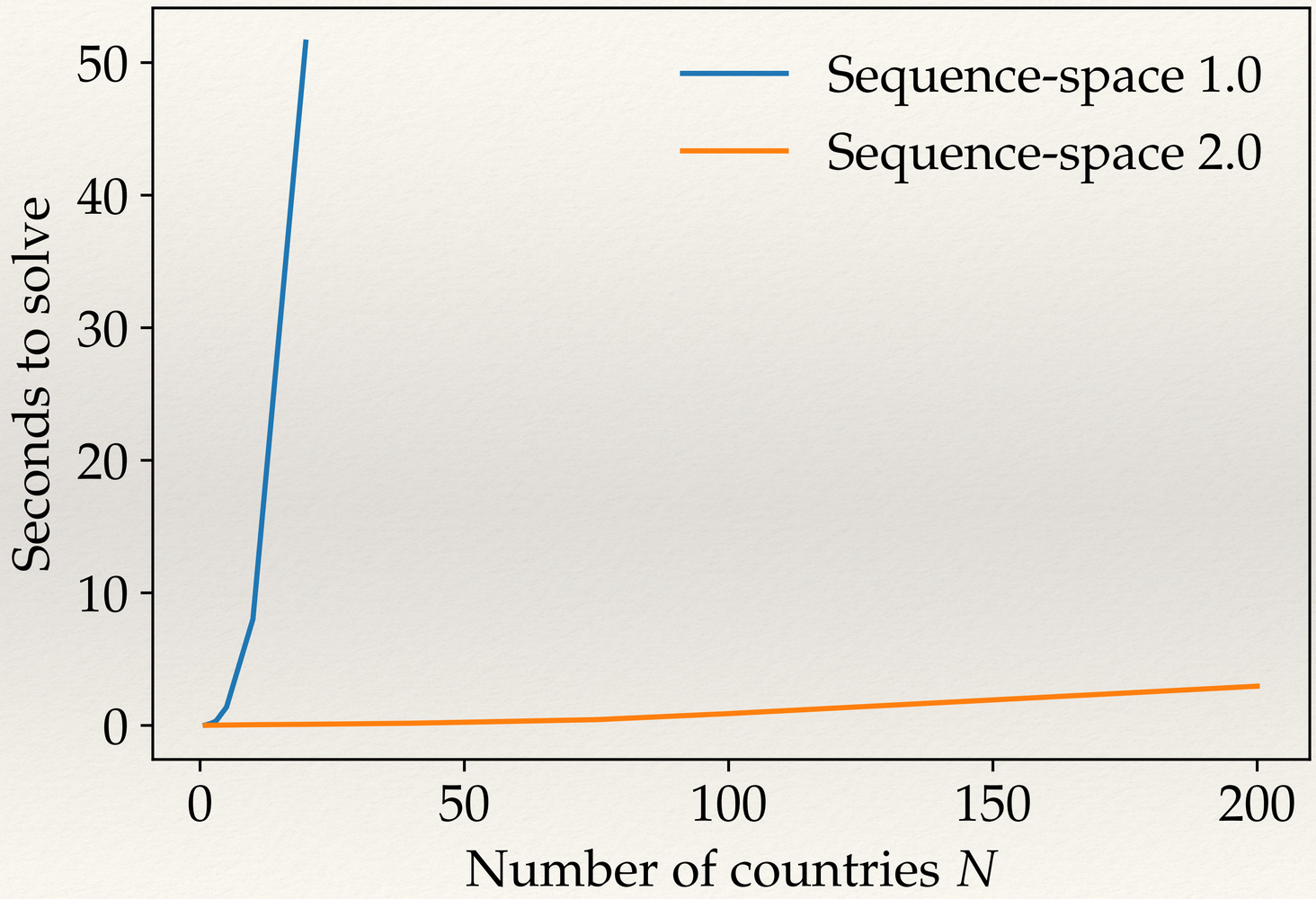


Peek at solution: on impact across countries

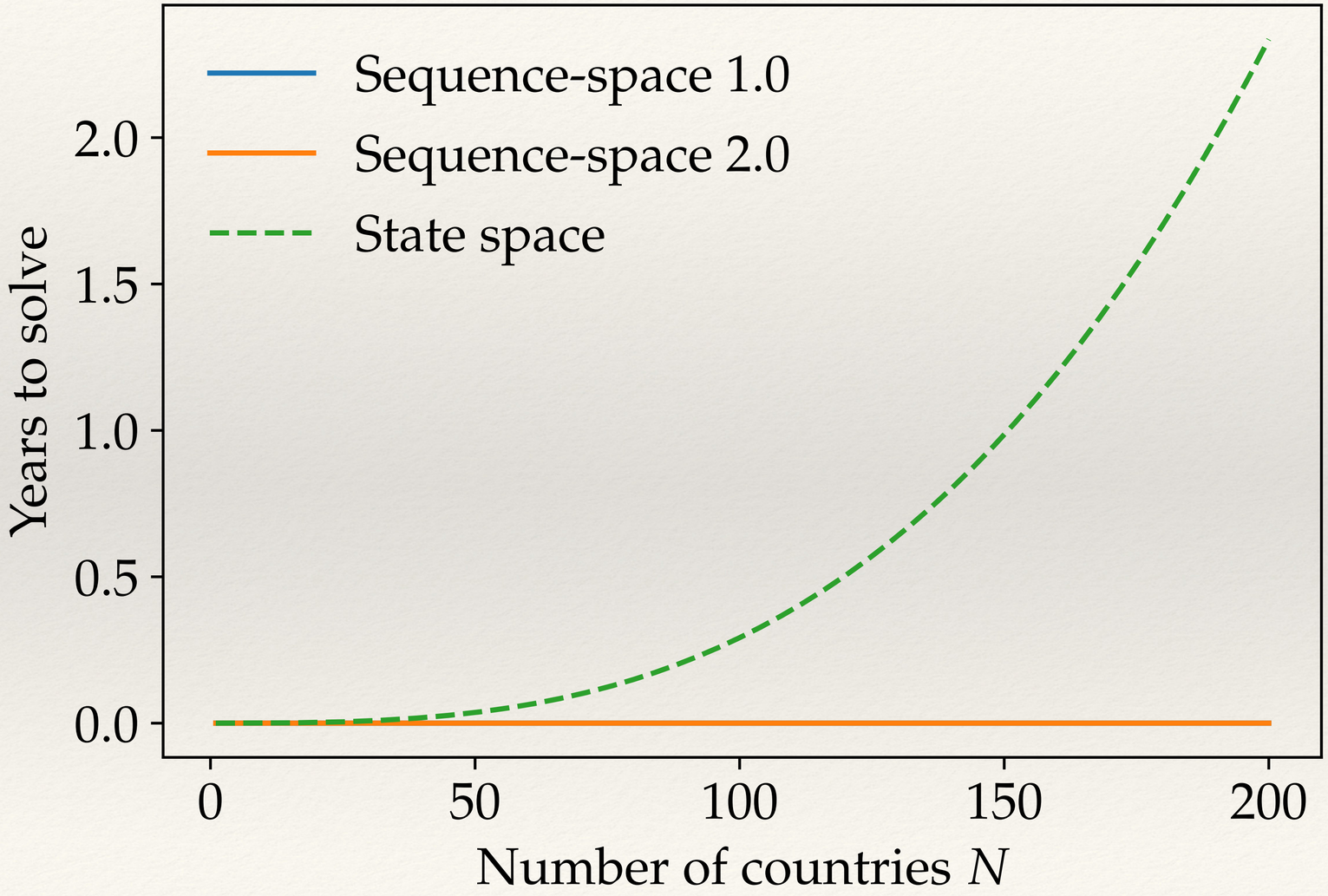
Peek at solution: after 20 quarters across countries



How fast is this?



How fast is this? Compare to state space



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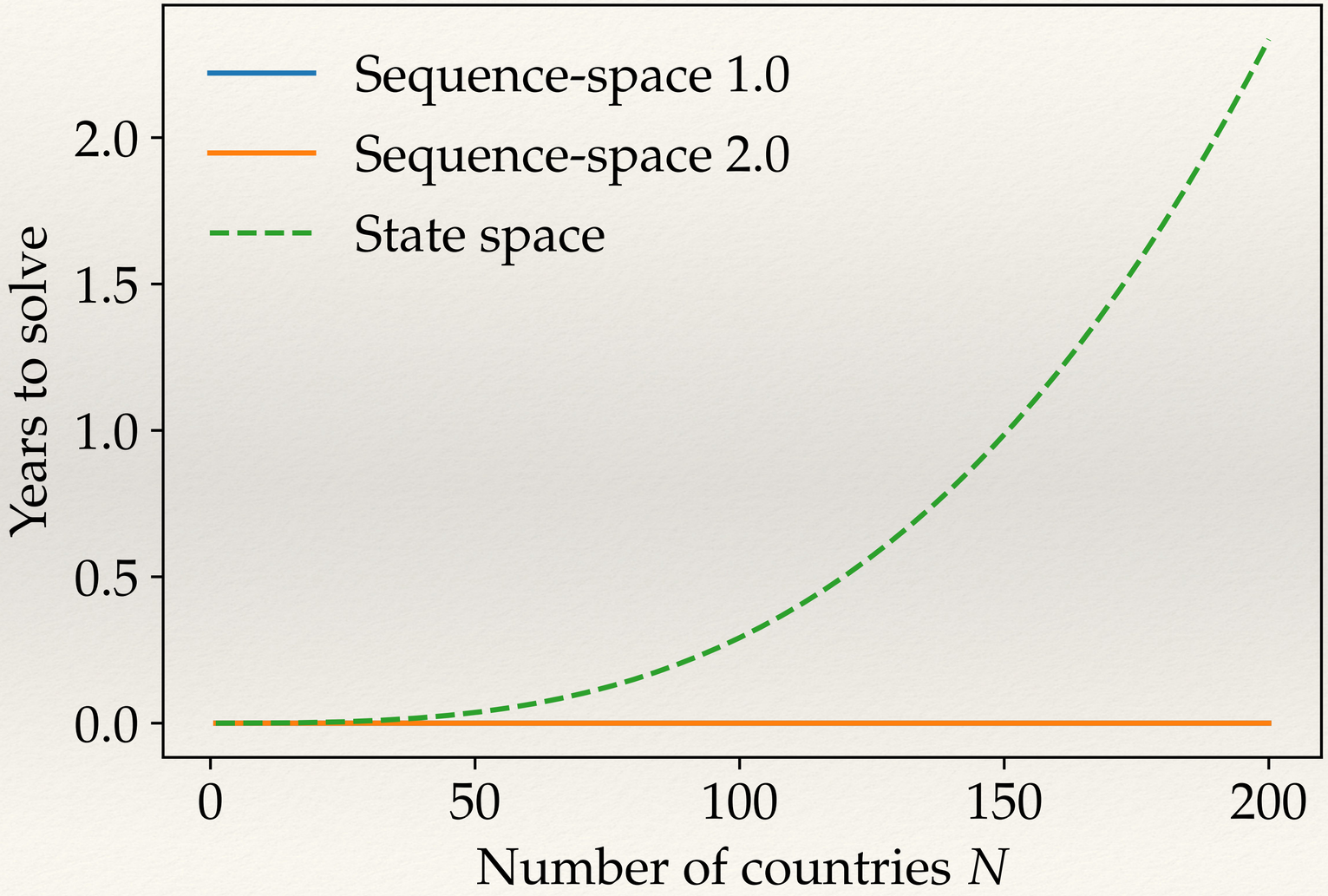
(Extrapolation using cubic cost of solving state-space system!)



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  - **winding number test** for determinacy
  - **truncation-free computations** with low-rank correction
  - extremely fast **iterative computations, even in huge models**
    - solves 177-country HANK in 3 seconds!!