Value Function Iteration

January 2019

Some Stuff on VFI

Value Function Iteration is an application of the Contraction Mapping Theorem to get the fixed point of the operator:

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}.$$

Why? The CMT implies two things relevant to this problem:

1. For some metric space (S, d) and contraction $T : S \to S$, there's a unique fixed point $v^* \in S$ (i.e. $v^* = Tv^*$)
2. From any v_0 the sequence $\{v_n\}_{n=1}^{\infty}$ defined by $v_n = Tv_{n-1}$ converges geometrically to v^* by rate of the modulus β.

So, that's cool. What that tells us is if we take some starting v_0, any starting v_0, and keep applying the contraction T (satisfying Blackwell's Sufficient Conditions for a contraction) we'll get to the unique fixed point that we want. That's the basic justification for VFI and it gives you the algorithm, which is basically:

1. Apply $F(x,y) + \beta v^{n-1}(y)$ over a grid of values.
2. Take the max over the grid of y and repeat until you get close enough.

A Problem

In this problem, you will use brute force to solve for the value function in the neoclassical growth model. Suppose that the household's utility function is $\frac{c^{1-\sigma}}{1-\sigma}$ and that the production function is k^{α}. Assume parameter values of $\beta = 0.96, \alpha = 0.3, \sigma = 3, \delta = 0.08$.

1. Compute the maximum sustainable level of capital $\bar{k}$ and the steady state level of capital k^*.
2. Create a grid of $N = 25$ equally spaced points $\{k_i\}_{i=1}^{25}$ in the interval $[\bar{k}, \underline{k}]$ with $\underline{k} = 0.1k^*$
3. Guess $\{V_{0,i}\}_{i=1}^{N} = 0$ at these grid points and iteratively update these values according to the following procedure:
   (a) For all pairs i, j compute:
   $$V_{ij}^{n} = u(f(k_i) + (1-\delta)k_i - k_j) + \beta V_j^{n-1}$$
   (b) Update by the rule
   $$V_i^n = \max_j V_{ij}^n$$

(c) Compute $R = \|V_i^n - V_i^{n-1}\|$.

(d) If $R < 1e-5$ or some other stopping condition, then end the algorithm. Otherwise, continue.

4. Plot the value and policy functions. Comment.
5. Plot the transition paths starting from a value of k below k^* and for an initial value above k^*. Do the paths converge to the steady state?
6. Now solve for the value function with grids of size $N = 50, 100, 200, 400, 800$. Plot all the value functions on one graph and all the policy functions on another. Does doubling the number of grid points substantially increase accuracy?
7. Make a table containing the number of iterations and time to convergence in each case. Explain why you see the patterns that emerge.