Linear Evolution: Analytical Pipeline

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October 17, 2018

Overview

In this write up, we will outline the pipeline taking biased $\tilde{\Phi}(\bar{\chi})$ samples to final $\delta\rho$, δ_m, δ_U, δ_R values at the end of linear evolution.

Finding Initial Conditions

Recall that the values incoming from Zander's code are

$$\tilde{\Phi}(\bar{\chi}) = \frac{\vec{\phi}(\bar{\chi}, t_0) \cdot \vec{\phi}(\bar{\chi}, t_0)}{N\sigma_0^2}, \tag{1}$$

where $\sigma_0^2 = \phi_{rms}^2$ and t_0 is the time at the end of inflation. $\bar{\chi}$ is the dimensionless version of χ, which is a physical distance coordinate at the waterfall transition. They are related as such:

$$\bar{\chi} = H\chi. \tag{2}$$

Previously, we've found that an appropriate coordinate transformation will take $\tilde{\Phi}(\bar{\chi})$ to the coordinates of $\delta_{U_0}(\bar{A})$, where $\bar{A}$ is the dimensionless Misner-Sharp comoving coordinate. Our results can be summarized by the equations

$$\delta_{U_0}(\bar{A}) = -\frac{\beta e^{-n^*}}{2\bar{r}(\bar{A})} \frac{\partial\tilde{\Phi}(\bar{\chi}(\bar{r}(\bar{A})))}{\partial\bar{\chi}} [\tilde{\Phi}(\bar{\chi}(\bar{r}(\bar{A})))]^{\beta-1} \tag{3}$$

for $\bar{A} \neq 0$, and

$$\delta_{U_0}(\bar{r} = 0) = -\frac{\beta e^{-2n^*}}{2} \frac{\partial^2\tilde{\Phi}(\bar{\chi}(0))}{\partial\bar{\chi}^2} [\tilde{\Phi}(\bar{\chi}(0)]^{\beta-1} \tag{4}$$

for $\bar{A} = 0$. The initial conditions that we're looking for are values for $\delta\rho(\xi = 0)$ and $\delta\dot{\rho}(\xi = 0)$, aka the mass density perturbation and its derivative w.r.t. ξ. Recall that we chose our initial timeslice to be one of uniform density, so $\delta_m(\xi = 0) = 0 \implies \delta\rho(\xi = 0) = 0$. As for $\delta\dot{\rho}$, we borrow an equation from the Bloomfield, Bulhosa, and Face paper:

$$\partial_\xi\tilde{\rho}(\xi, \bar{A}) = \delta\dot{\rho}(\xi, \bar{A}) = 2\tilde{\rho} - \alpha e^{\phi}\tilde{\rho}(1 + \omega)\left(3\tilde{U} + \frac{\bar{A}R\tilde{U}'}{(\bar{A}R)'}\right). \tag{5}$$

where $\tilde{U} = 1 + \delta_U$, $R = a\bar{A}\tilde{R}$, $\tilde{\rho} = 1 + \delta\rho$, $a(t_0) = 1$, dots indicate derivatives w.r.t. ξ and primes indicate derivatives w.r.t. $A = \bar{A}/H$. Then, we can then find our initial conditions by plugging in $\delta\rho = 0 \implies \tilde{\rho} = 1$, $\phi \propto \delta\rho = 0$, $R = A$ and $\alpha = 1/2 \implies \omega = 1/3$ for our radiation-dominated era:

$$\delta\dot{\rho}(\xi, A) = -2\delta_U - \frac{A}{3}\delta_U'. \tag{6}$$

Mode Decomposition

Recall our linear perturbation ODE for δ_m:

$$\partial_\xi^2 \delta_m - (3-5\alpha)\partial_\xi \delta_m + [(1-2\alpha)3w - 1]\alpha\delta_m = w\alpha^2 e^{2(1-\alpha)\xi}\left(\frac{4\delta_m'}{\bar{A}} + \delta_m''\right), \tag{7}$$

In our previous write up, we found that the solutions to this ODE with a single spatial boundary condition of $\delta_m'(u=0) = 0$ are of the form

$$\delta_m(s,u) = \frac{s}{u} j_1(ku)(B_k j_1(ks) + C_k y_1(ks)), \tag{8}$$

with B and C as constants determined by the initial conditions and $s = e^{\xi/2}$ and $u = \sqrt{3}\bar{A}$ as temporal and spatial coordinates respectively. Borrowing the relation between $\delta\rho$ and δ_m from the Bloomfield, Bulhosa, and Face paper, we see

$$\delta\rho = \delta_m + \frac{\bar{A}}{3}\delta_m' \tag{9}$$

We can compute that

$$\frac{d}{dx}j_1(x) = \frac{1}{2}\left(j_0(x) - \frac{j_1(x) + xj_2(x)}{x}\right) \implies \frac{d}{dx}\left(\frac{j_1(x)}{x}\right) = \frac{xj_0(x) - 3j_1(x) - xj_2(x)}{2x^2} \tag{10}$$

$$\delta_m + \frac{\bar{A}}{3}\delta_m = \frac{j_1(ku)}{ku} + \frac{ku\sqrt{3}}{3\sqrt{3}}\left(\frac{kuj_0(ku) - 3j_1(ku) - kuj_2(ku)}{2(ku)^2}\right) = \frac{j_1(ku)}{ku} + \frac{j_0(ku)}{3} - \frac{j_1(ku)}{ku} = \frac{j_0(ku)}{3} \tag{11}$$

Hence, we find that the spatial part of $\delta\rho \propto j_0(ku)$. As zeroth order Bessel functions of the first kind are an orthocomplete set, we can conclude that j_0's form the basis of the space of possible $\delta\rho(u)$. The time dependence of δ_m and $\delta\rho$ is the same, so we conclude that the modes of $\delta\rho$ solutions take the form

$$\delta\rho(s,u) = sj_0(ku)(B_k j_1(ks) + C_k y_1(ks)). \tag{12}$$

We then impose our second boundary condition that $\delta\rho(L) = 0$ for some length L. Recalling $j_0(n\pi) = 0$, our quantization condition is then:

$$\lambda = 2L/n \implies k = \frac{2\pi}{\lambda} = \frac{2\pi}{2L/n} = \frac{n\pi}{L},\ n = 1, 2, ... \tag{13}$$

$$\implies \delta\rho(s,u) = Rsj_0\left(\frac{n\pi}{L}u\right)\left(B_n j_1\left(\frac{n\pi}{L}s\right) + C_n y_1\left(\frac{n\pi}{L}s\right)\right), R \text{ a normalization constant} \tag{14}$$

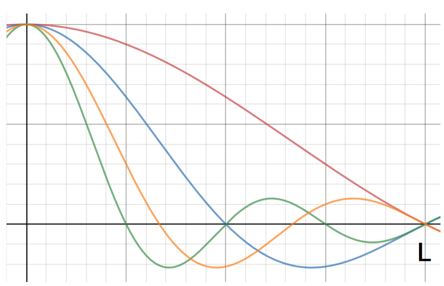

Figure 1: The first four modes for the spatial component of $\delta\rho$

Our next step is then to decompose our initial $\delta\rho$ and $\delta\dot{\rho}$ into spatial modes so that we can find the coefficients B and C for each n. Pick a sufficiently large even integer N. We know $\delta\rho(\xi = 0) = 0$, so very simply

$$\delta\rho(\xi = 0, u) = \sum_{n=1}^{N} a_n j_0\left(\frac{n\pi}{L}u\right), \quad a_n = 0 \ \forall n = 1, ..., N. \tag{15}$$

For $\delta\dot{\rho}$, however, we'll have to use spectral methods. Recall from Fourier spectral methods that for $n \leq N-1$

$$\int_{-\pi}^{\pi} \cos nx dx = \frac{2\pi}{N}\sum_{m=1}^{N} \cos\left(-n\pi + \frac{2\pi mn}{N}\right) \tag{16}$$

which we can rewrite as

$$\int_{0}^{L} \cos\frac{n\pi}{L}x dx = \frac{L}{N}\sum_{m=1}^{N} \cos\left(-n\pi + \frac{2\pi mn}{N}\right) \tag{17}$$

Then we can develop a system of interpolation for spherical Bessel functions similar to that of the Fourier series. Recall that the orthogonality of the j_0 implies

$$\int_0^L x^2 j_0\left(\frac{n\pi}{L}x\right) j_0\left(\frac{n'\pi}{L}x\right) dx = \frac{L^3}{2}[j_1(n\pi)]^2\delta_{n,n'} = \frac{L^3}{2n^2\pi^2}\delta_{n,n'}. \tag{18}$$

We define a discrete inner product

$$(f(x), g(x))_G \equiv \frac{L}{N}\sum_{m=1}^{N} x_m^2 f(x_m)g(x_m), \ x_m = -L + \frac{2Lm}{N} \tag{19}$$

Let's consider a reduced basis of $j_0(n\pi x/L)$ with $0 \leq n \leq N/2 - 1$. We can show that the reduced basis if orthogonal under the discrete inner product. For $n, n' > 0$, we see

$$x^2 j_0\left(\frac{n\pi}{L}x\right) j_0\left(\frac{n'\pi}{L}x\right) = \frac{L^2}{nn'\pi^2}\sin\left(\frac{n\pi}{L}x\right) sin\left(\frac{n'\pi}{L}x\right) \tag{20}$$

$$= \frac{L^2}{2nn'\pi^2}\left(\cos\left(\frac{(n-n')\pi}{L}x\right) - \cos\left(\frac{(n+n')\pi}{L}x\right)\right). \tag{21}$$

Then using the discrete to exact integration identity borrowed from the Fourier series spectral methods since $n + n' < N$, we see that

$$n \neq n' \implies \left(j_0\left(\frac{n\pi}{L}x\right), j_0\left(\frac{n'\pi}{L}x\right)\right)_G = 0 \tag{22}$$

$$n = n' \implies \left(j_0\left(\frac{n\pi}{L}x\right), j_0\left(\frac{n'\pi}{L}x\right)\right)_G = \frac{L^2}{2(n\pi)^2}\int_0^L \cos(0)dx = \frac{L^3}{2(n\pi)^2}. \tag{23}$$

This matches our orthogonality relation for spherical Bessel functions from above. The case when $n = 0$ or $n' = 0$ also follows straightforwardly. Thus we have successfully constructed a discrete inner product on the reduced basis of $j_0(n\pi x/L)$ with $0 \leq n \leq N/2 - 1$. We then can construct an interpolating function $S_N(x)$ for our function f on this basis such that

$$S_N(x_m) = f(x_m), \ m = 1, ..., N, \ x_m = -L + \frac{2Lm}{N}. \tag{24}$$

The the interpolant can be expressed as

$$S_N(x) = \sum_{n=0}^{N/2-1} b_n j_0\left(\frac{n\pi}{L}x\right) = b_0 j_0(0) + \sum_{n=1}^{N/2-1} b_n j_0\left(\frac{n\pi}{L}x\right) \tag{25}$$

where

$$b_n = \frac{2n^2\pi^2}{L^3}\left(j_0\left(\frac{n\pi}{L}x\right), S_N(x)\right)_G = \frac{2n^2\pi^2}{L^3}\left(j_0\left(\frac{n\pi}{L}x\right), f(x)\right)_G \tag{26}$$

Then turning back to the problem at hand, we can decompose the spatial component of $\delta\dot{\rho}$ with the formula

$$\delta\dot{\rho}_N(\xi = 0, u) = \sum_{n=1}^{N/2-1} b_n j_0\left(\frac{n\pi}{L}u\right), \quad b_n = \frac{2n^2\pi^2}{L^3}\left(j_0\left(\frac{n\pi}{L}x\right), \delta\dot{\rho}\right)_G \tag{27}$$

Finding $\delta\rho$

Now that we have a decomposition of $\delta\dot{\rho}$ into b_n's, to find the sets of coefficients B_n and C_n, for each n we need to solve the set of equations at $\xi = 0 \implies s = 1$:

$$a_n = 0 = B_n j_1\left(\frac{n\pi}{L}\right) + C_n y_1\left(\frac{n\pi}{L}\right) \tag{28}$$

$$b_n = B_n\left(j_1\left(\frac{n\pi}{L}\right) + \frac{L}{n\pi}j_1'\left(\frac{n\pi}{L}\right)\right) + C_n\left(y_1\left(\frac{n\pi}{L}\right) + \frac{L}{n\pi}y_1'\left(\frac{n\pi}{L}\right)\right). \tag{29}$$

Then we will have the full expression for $\delta\rho$:

$$\boxed{\delta\rho(\xi, \bar{A}) = e^{\xi/2}\sum_{n=1}^{N/2-1} j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)\left(B_n j_1\left(\frac{n\pi}{L}e^{\xi/2}\right) + C_n y_1\left(\frac{n\pi}{L}e^{\xi/2}\right)\right).} \tag{30}$$

To find the appropriate value of ξ_0 to stop our time evolution at, we figure out when $\delta\rho(\xi_0, \bar{A} = 0) \approx 0.05$. Note that if the $n = 1$ mode peaks and $\delta\rho$ still hasn't reached 0.05 yet, we can the halt linear evolution. Our model doesn't account for things outside of $u = L$ propagating inwards, and by the time the lowest mode peaks, these external influences will have reached the center; if the perturbation has not evolved out of the linear regime at that point, it never will. As the contribution from y_1 will be negligible, we know our ξ_{max} must be the first root of the derivative of the $n = 1$ mode:

$$\frac{d}{d\xi}\delta\rho_1(\xi, 0)) \sim \frac{d}{d\xi}\left[e^{\xi/2}j_1\left(\frac{\pi}{L}e^{\xi/2}\right)\right]. \tag{31}$$

From Mathematica, we know that the first root of $f(x) = \frac{d}{dx}(xj_1(x))$ is at $x = 2.7437072699922695$, so we conclude that

$$\xi_{max} = 2\log\left(2.7437072699922695\frac{L}{\pi}\right). \tag{32}$$

Finding δ_m, δ_U, and δ_R

Once we obtain $\delta\rho(\xi_0, \bar{A})$, we can use it to find δ_m, δ_U, and δ_R. First let's tackle δ_m. Recall from above that

$$\delta\rho(\xi, \bar{A}) = \delta_m(\xi, \bar{A}) + \frac{\bar{A}}{3}\delta_m'(\xi, \bar{A}) \tag{33}$$

where the prime is a derivative with respect to $\bar{A}$. Thus, we find

$$\delta\rho = \frac{[\bar{A}^3\delta_m]'}{3\bar{A}^2} \implies \boxed{\delta_m(\xi_0, \bar{A}) = \frac{3}{\bar{A}^3}\int_0^{\bar{A}} \bar{A}^2\delta\rho(\xi_0, \bar{A})d\bar{A}} \tag{34}$$

We can find the solution to this integral analytically. Mathematica gives us

$$\sigma_n(\bar{A}) = \int_0^{\bar{A}} \bar{A}^2 j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)d\bar{A} = \frac{L^2}{9\pi^3 n^3}\left(\sqrt{3}L\sin\left(\frac{\sqrt{3}n\pi\bar{A}}{L}\right) - 3n\pi\bar{A}\cos\left(\frac{\sqrt{3}n\pi\bar{A}}{L}\right)\right) \tag{35}$$

Then we obtain

$$\boxed{\delta_m(\xi_0,\bar{A}) = \frac{3}{\bar{A}^3}\int_0^{\bar{A}} \bar{A}^2\delta\rho(\xi_0,\bar{A})d\bar{A} = \frac{3e^{\xi_0/2}}{\bar{A}^3}\sum_{n=1}^{N/2-1}\sigma_n(\bar{A})\left(B_n j_1\left(\frac{n\pi}{L}e^{\xi_0/2}\right) + C_n y_1\left(\frac{n\pi}{L}e^{\xi_0/2}\right)\right)}. \tag{36}$$

As for δ_U, we know $\alpha = 1/2, \omega = 1/3$, and the Bloomfield, Bulhosa, Face paper informs us that

$$\dot{\delta_m} = 3\omega\alpha\delta_m - 2\delta_U = \frac{\delta_m}{2} - 2\delta_U \tag{37}$$

$$\implies \boxed{\delta_U(\xi_0,\bar{A}) = -\frac{\dot{\delta}_m(\xi_0,\bar{A})}{2} + \frac{\delta_m(\xi_0,\bar{A})}{4}}. \tag{38}$$

To find $\dot{\delta_m}$, we need to calculate some derivatives analytically:

$$\dot{\delta_m}(\xi_0,\bar{A}) = \frac{3}{\bar{A}^3}\int_0^{\bar{A}} \bar{A}^2\dot{\delta\rho}(\xi_0,\bar{A})d\bar{A} \tag{39}$$

Using Mathematica, we find that

$$\beta_n = \left.\frac{d}{d\xi}\right|_{\xi=\xi_0} e^{\xi/2}j_1\left(\frac{n\pi e^{\xi/2}}{L}\right) = \frac{1}{2}\left(e^{\xi_0/2}j_1\left(\frac{n\pi e^{\xi_0/2}}{L}\right) + \frac{e^{\xi_0}n\pi}{L}j_1'\left(\frac{n\pi e^{\xi_0/2}}{L}\right)\right) \tag{40}$$

$$\gamma_n = \left.\frac{d}{d\xi}\right|_{\xi=\xi_0} e^{\xi/2}y_1\left(\frac{n\pi}{L}e^{\xi/2}\right) = \frac{1}{2}\left(e^{\xi_0/2}y_1\left(\frac{n\pi e^{\xi_0/2}}{L}\right) + \frac{e^{\xi_0}n\pi}{L}y_1'\left(\frac{n\pi e^{\xi_0/2}}{L}\right)\right) \tag{41}$$

Then $\dot{\delta_m}$ at ξ_0 is

$$\dot{\delta\rho}(\xi_0,\bar{A}) = \sum_{n=1}^{N/2-1} j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)(B_n\beta_n + C_n\gamma_n) \tag{42}$$

$$\boxed{\dot{\delta_m}(\xi_0,\bar{A}) = \frac{3}{\bar{A}^3}\sum_{n=1}^{N/2-1}\sigma_n(\bar{A})(B_n\beta_n + C_n\gamma_n)}. \tag{43}$$

The last quantity we want to find is $\delta_R(\xi_0,\bar{A})$. From Bloomfield, Bulhosa, and Face, we have

$$\dot{\delta_R} = \alpha(\delta_U + \delta_\phi) \tag{44}$$

$$\delta_\phi = -\frac{3\alpha\omega}{2}\delta\rho \implies \dot{\delta_R} = \frac{1}{2}\left(\delta_U - \frac{1}{4}\delta\rho\right) \tag{45}$$

Recalling that

$$\delta_U = -\frac{\dot{\delta}_m}{2} + \frac{\delta_m}{4},\ \delta\rho = \delta_m + \frac{\bar{A}}{3}\delta_m' \tag{46}$$

we find that

$$\dot{\delta_R} = -\frac{\dot{\delta_m}}{4} - \frac{\bar{A}}{24}\delta_m' \implies \boxed{\delta_R(\xi_0,\bar{A}) = -\frac{\delta_m(\xi_0,\bar{A})}{4} - \frac{\bar{A}}{24}\frac{d}{d\bar{A}}\int_0^{\xi_0}\delta_m(\xi_0,\bar{A})d\xi} \tag{47}$$

We need to calculate the integral for δ_m analytically:

$$\int_0^{\xi_0}\delta_m(\xi_0,\bar{A}) = \frac{3}{\bar{A}^3}\int_0^{\bar{A}}\bar{A}^2 d\bar{A}\int_0^{\xi_0} d\xi\delta\rho(\xi_0,\bar{A}). \tag{48}$$

Using Mathematica, we find that

$$\phi_n = \int_0^{\xi_0} d\xi e^{\xi/2} j_1\left(\frac{n\pi e^{\xi/2}}{L}\right) = -\frac{2e^{-\xi_0/2}L^2}{n^2\pi^2}\sin\left(\frac{e^{\xi_0/2}n\pi}{L}\right) \tag{49}$$

$$\psi_n = \int_0^{\xi_0} d\xi e^{\xi/2} y_1\left(\frac{n\pi e^{\xi/2}}{L}\right) = \frac{2e^{-\xi_0/2}L^2}{n^2\pi^2}\cos\left(\frac{e^{\xi_0/2}n\pi}{L}\right). \tag{50}$$

Then we obtain

$$\int_0^{\xi_0} \delta\rho(\xi_0, \bar{A})d\xi = \sum_{n=1}^{N/2-1} j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)(B_n\phi_n + C_n\psi_n) \tag{51}$$

$$\implies \boxed{\frac{d}{d\bar{A}}\int_0^{\xi_0} \delta_m(\xi_0, \bar{A})d\xi = \frac{3}{\bar{A}}\sum_{n=1}^{N/2-1}(B_n\phi_n + C_n\psi_n)\left(j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right) - \frac{3}{\bar{A}^3}\sigma_n(\bar{A})\right).} \tag{52}$$

Notice that in many of the above expressions, the quantity $\sigma_n/\bar{A}^3$ appears. This quantity is slightly problematic at the origin, $\bar{A} = 0$, as both $\sigma_n(\bar{A} = 0)$ evaluate to zero. Using L'Hôpital's rule, we find that

$$\left.\frac{\sigma_n(\bar{A})}{\bar{A}^3}\right|_{\bar{A}=0} = \frac{\frac{d}{d\bar{A}}\int_0^{\bar{A}} \bar{A}^2 j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)d\bar{A}}{\frac{d}{d\bar{A}}\bar{A}^3} = \frac{\bar{A}^2 j_0\left(\frac{n\pi}{L}\sqrt{3}\bar{A}\right)}{3\bar{A}^2} = \frac{j_0(0)}{3}. \tag{53}$$

Finally, after computing all the necessary quantities, we hand our results for $\delta\rho$, δ_m, δ_U, and δ_R at $\xi = \xi_0$ to the numerical nonlinear evolution code.

References

[1] J. Bloomfield, D. Bulhosa, S. Face, "Formalism for Primordial Black Hole Formation in Spherical Symmetry," arXiv:1504.02071 [gr-qc].

[2] M. Xu and M. Gull, "Linear Evolution Pipeline."

[3] J. Bloomfield, S. Face, M. Xu, "Linear Evolution: Analytical Solution."

[4] J. Bloomfield, "Lecture Notes: Spectral Methods."