Efficient Estimation of Causal Effects under Interference through Designed Markets *

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Abstract

It is often useful for a decision-maker to evaluate the effect of an intervention that affects agent preferences on allocations from a centralized mechanism. For example, if a policymaker aims to decrease socio-economic segregation in schools, they might evaluate whether informing low-income families about school performance raises their access to good quality schools. We define this type of counterfactual as a Global Treatment Effect (GTE) in a general model of causal inference under interference. When interference operates through an allocation mechanism that is truthful and has a cutoff structure, the target estimand can be defined in the asymptotic limit as a moment condition model with missing data. Under a selection-on-observables assumption, we propose a two-step doubly-robust estimator for the GTE and show that it is asymptotically normal with variance that meets the semi-parametric efficiency bound. We derive a theory of heterogeneous treatment effects in this setting, including an estimation method for the optimal targeting rule. We use these methods to analyze the effect of an information intervention in the Chilean school assignment system.

Keywords: Market Design, Causal Inference, Interference

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1 Introduction

A growing number of institutions allocate scarce items using a centralized mechanism. In the U.S., versions of the deferred acceptance algorithm allocate students to schools (Abdulkadiroğlu and Sönmez, 2003), and medical school graduates to residency programs (Roth, 2003). Auctions allocate advertisements to search queries (Varian and Harris, 2014), wireless spectrum to telecommunications companies (Binmore and Klemperer, 2002), and Treasury bonds to investors (McMillan, 2003). In many settings, policymakers are interested in the estimating the effect of an intervention that affects the preferences of participants in an allocation mechanism on resulting allocations. For example, in Chile, despite the existence of a priority quota for low-income families in the deferred acceptance mechanism used to allocate families to schools, lower-income families are admitted to high-quality schools at a much lower rate than high-income families. One hypothesis that has been explored in the literature is that better information about schools might reduce this socio-economic segregation by encouraging low-income families to apply to better schools (Allende et al., 2019). An information intervention has a direct effect on individual rankings over schools submitted to deferred acceptance, but also an indirect effect, in that aggregate changes in preferences affect the admissions standards at schools, when schools have a fixed capacity.

The estimand of interest is defined as a Global Total Effect (GTE), which is the effect when every agent in a market receives an intervention, compared to when none receive the intervention. To estimate this counterfactual when preferences, treatments, and allocations are observed directly, even when treatment is randomly assigned, it is not sufficient to simply compare the average allocations for treated families compared to control families. This is because there is interference; the differences-in-means estimate does not capture the indirect impact of the intervention through changes in the market equilibrium (Heckman et al., 1998).

This paper uses a model of causal inference under interference to define and estimate the GTE in markets where the allocation mechanism is known, truthful and follows a cutoff form (Agarwal and Somaini, 2018). When a mechanism has a cutoff form, an individual's allocations depend only on a finite vector of market "prices" and an individual's submissions to the mechanism. The assumption of truthfulness rules out strategizing by agents. Examples of mechanisms that are truthful and have a cutoff form include a uniform price auction with single-unit demand and deferred acceptance, as shown in Azevedo and Leshno (2016). We also restrict our attention to settings where interference operates only through the market mechanism, so SUTVA holds at the level of individual preferences, which rules out network-type informational spillovers. The "prices" of the market mechanism are thus an exposure mapping in the sense of Sävje et al. (2021), as in Munro et al. (2023).

To characterize τ_{GTE} in a form that is amenable to estimation, we use an asymptotic framework that takes the number of applicants/buyers to infinity, but holds the number of items fixed. Then, in the limit, we work with continuous approximations to allocation mechanisms, such as the fractional version of deferred acceptance studied by Azevedo and Leshno (2016) and also used in Agarwal and Somaini (2018) and Bertanha et al. (2023).

The first main result of the paper is that τ_{GTE}, which is defined at the sample level, is asymptotically normal and converges to τ^*_{GTE} in this limit. The limit τ^*_{GTE} has a representation as a moment condition model with data missing at random, see Wooldridge (2007), Chen et al. (2008) and Graham et al. (2012). Although the outcomes that we observe are not i.i.d. due to interference, the preferences that individuals submit to the mechanism are i.i.d. The global treatment effect can then be defined asymptotically in terms of moments of the distribution of these submissions. To identify the global treatment effect using observed data on treatments, preferences, allocations, and covariates, we assume strong overlap and unconfoundedness.

Under these assumptions, we derive the semi-parametric efficiency bound for the estimation of τ^*_{GTE}. Compared to the average treatment effect without interference, the efficiency bound includes additional terms due to the variation in the equilibrium when individuals are sampled from a population. However, we find that these additional terms can reduce variance, so the general equilibrium effect is often estimated more precisely than the partial equilibrium effect. We then derive an estimator that meets the semi-parametric efficiency bound under weak assumptions on the estimation of nuisance parameters.

We first show that two intuitive ways of estimating τ_{GTE} through simulation are equivalent to an IPW estimator and an outcome-modeling estimator. From the existing literature on moment condition models with missing data, we know that efficiency and asympotic normality of the IPW estimator relies on very specific rates of convergence for the propensity score estimator (Hirano et al., 2003; Wooldridge, 2007), and the outcome modeling approach requires correct specification and certain rates of convergence of an estimator of the distribution of preferences. In practice, it is usually difficult to satisfy these when the distribution of individual preferences is unknown. Estimation methods based on AIPW score, in contrast, have a desirable doubly-robust property, in that the asymptotic normality of the estimator requires that the product of the rates of the outcome model and the propensity score estimators meet a certain mild condition. In ATE problems, this allows the usage of a wide range of flexible machine learning estimators for the estimation of the propensity score and conditional outcome regressions, which converge slower than a simple parametric model, but are likely to capture the true generating process, see Chernozhukov et al. (2018). However, the standard double machine learning framework of Chernozhukov et al. (2018) applied to the estimation of the GTE requires estimating the entire distribution of treated and control preferences, conditional on covariates. Finding flexible models for both the propensity score and this distribution that meets the required rate conditions is infeasible.

Our preferred estimator is based on the related Localized Debiased Machine Learning (LDML) approach of Kallus et al. (2019). This is a two-step method that relies on data splitting into at least three folds. In the first stage, an IPW approach is used to estimate counterfactual market-clearing cutoffs. Then, a modified AIPW approach is used to estimate the GTE. The conditional mean functions used in the second step rely on a single set of regressions of allocations on covariates. Meeting the rate conditions for this set of regressions is straightforward for a variety of flexible machine learning estimators. We show that the LDML approach leads to an asymptotically normal and efficient estimator of τ_{GTE}.

In Section 4, we extend our approach to allow for heterogeneous allocation of the treatment. We derive the outcome-maximizing targeting rule, which takes into account equilibrium effects. Then, we introduce an empirical welfare maximization-based approach for estimating the outcome-maximizing rule, which also uses the doubly-robust scores derived in the previous section. Section 5 uses a simple model of a uniform price auction to illustrate the robustness properties of the LDML estimator, in contrast to the IPW and outcome modeling approaches. Then, we use a simulation of a school market with three schools to show that the asympotically valid confidence intervals for the GTE perform well in finite samples, and are over 20% more narrow than doubly-robust intervals for a partial equilibrium treatment effect.

In Section 6, we analyze data from Chilean schools. In Chile, a centralized mechanism based on deferred acceptance, with additional priorities and quotas, allocates most children in the country to schools. Despite the existence of a priority system for low income families, those families apply to and attend high quality schools at a much lower rate than higher income families. One potential reason for this disparity explored in the literature is information frictions, see Allende et al. (2019) for a randomized experiment. We use observational data to evaluate the effect of school information on allocations of low-income families. In a parent survey for 8th graders in 2019, some parents report that they are aware of the quality score for the school that their child attends. There is evidence that parents who are aware of the quality score make different choices when applying to high schools for 9th grade. We analyze the effect of everyone having the school quality information, compared to nobody having the information, on the probability that a low-income family is assigned to a good quality school, under an unconfoundedness assumption. We find that if equilibrium effects are ignored, then the estimate on the impact of the treatment is positive and quite large. However, the estimated effect, taking into account the effect of the information intervention on the market equilibrium, is significantly smaller, and is not far from zero.

1.1 Related Work

There is existing work on analyzing different types of causal effects in designed markets. Abdulkadiroğlu et al. (2017) estimate causal effects of allocations on noisy outcomes using randomness in the mechanism for identification. Abdulkadiroğlu et al. (2022), Chen (2021), and Bertanha et al. (2023) extend this work to settings where individual scores are non-random but the cutoff structure of the mechanism allows an RDD analysis. Bertanha et al. (2023) also considers partial identification of preferences from strategic reports when mechanisms are not strategy proof. In contrast to this body of work, our paper focuses on an earlier step in the causal chain of events, which is the effect of a pre-allocation intervention on some function of allocations.

In a model of randomized experiments in general equilibrium, Munro et al. (2023) showed that market prices can be considered an exposure mapping (Sävje et al., 2021). This structure allows Munro et al. (2023) to derive an estimator of causal effects including an equilibrium effect, but this equilibrium effect is local to the current equilibrium, and requires an experimental design with randomization of prices. A centralized market mechanism that has a competitive equilibrium representation provides additional structure beyond a market-clearing condition, which allows us to estimate a global treatment effect under interference under a selection-on-observables assumptions when covariates, the treatments, and submissions to the mechanism are observed.

To analyze the properties of the estimators in the paper, we use an asymptotic framework where the allocation mechanism operates on a distribution of agents, rather than a discrete number of agents. Using mean-field approximations for marketplaces is helpful in characterizing bias and variance of estimators of treatment effects, see Johari et al. (2022), Bright et al. (2022) and Liao and Kroer (2023), as well as Munro et al. (2023), for an analysis of A/B testing in various markets in equilibrium.

2 Defining Global Effects in Designed Markets

There is a two-sided market with n individuals on one side of the market, and J items on the other side of the market. Individuals with observed characteristics $X_i \in \mathcal{X}$ submit reports $B_i(W_i) \in \mathcal{B}$ to a centralized market mechanism, which assigns individuals to items. Each individual receives a binary treatment $W_i \in \{0, 1\}$, which can impact an individual's report to the mechanism. In an auction setting, an example intervention is a new predictive model that changes an individual's bid for certain advertising slots. In a school setting, an example intervention is information about school quality that changes a parent's ranking over schools. The potential outcomes $\{B_i(1), B_i(0)\}$ allow the effects of the treatment to differ by individual, depending on how the treatment affects their preferences. We do not need to assume a specific model for how the treatment affects an individual's choices. We assume that each individual $\{B_i(1), B_i(0), X_i\} \sim F$ is drawn from some population.

Each item j is associated with a fractional capacity q_j, so that for a sample size n, there are $n \cdot q_j$ slots available. In a designed market, the allocations of an individual to items, $D_i \in \{0, 1\}^J$ is computed by a centralized mechanism from the vector of reports $B(\boldsymbol{W}) \in \mathcal{B}^n$.

$$D_i(\boldsymbol{W}) = m(B(\boldsymbol{W}), q)$$

$D_{ij} = 1$ if individual is allocated a unit of item j. In a general mechanism, the effect of the report of individual j on the allocation of individual i is unrestricted. A large class of matching and auction mechanisms, however, have a competitive equilibrium representation, see Azevedo and Leshno (2016) and Agarwal and Somaini (2018) for an analysis of matching mechanisms. In a competitive equilibrium, the report of individual j only has an impact on the allocation of individual i through a set of market clearing cutoffs $P \in \mathcal{S}$. We restrict our attention to cutoff mechanisms:

Definition 1. Cutoff Mechanism. Individual allocations are determined by the function d :

$\mathcal{B} \times S \mapsto \{0,1\}^J$:

$$D_i = D_i(\boldsymbol{W}) = d(B_i(W_i), P(\boldsymbol{W}))$$
$$o_p(n^{-1/2}) = \frac{1}{n}\sum_{i=1}^{n} d(B_i(W_i), P(\boldsymbol{W})) - q$$

An individual's allocation depends only on their own submission to the mechanism and a set of market-clearing cutoffs. The cutoffs P are set so that demand is (approximately) equal to capacity for each of the J goods. We consider outcomes that are a known function of an individual's report, the market-clearing cutoffs, and their characteristics, $y : \mathcal{B} \times S \times \mathcal{X} \mapsto \mathbb{R}$.

$$Y_i = Y_i(\boldsymbol{W}) = y(B_i(W_i), P(\boldsymbol{W}), X_i).$$

An example of an outcome function is one that assigns a match value $u_j(X_i)$ for the allocation of individual i to option j: $Y_i(\boldsymbol{W}) = \sum_{j=1}^{J} u_j(X_i) D_{ij}(\boldsymbol{W})$. This rules out outcomes that are a noisy function of allocations, such as test scores or future income. The estimand of interest is the global treatment effect, which is the average effect on outcomes of treating everybody compared to treating nobody:

$$\tau_{GTE} = \frac{1}{n}\sum_{i=1}^{n} Y_i(\mathbf{1}) - Y_i(\mathbf{0}).$$

Without interference, in settings where SUTVA holds at the outcome level, then τ_{GTE} is equivalent to the familiar Average Treatment Effect:

$$\tau_{ATE} = \frac{1}{n}\sum_{i=1}^{n} Y_i(1) - Y_i(0).$$

When allocations are determined by a centralized mechanism, there is interference at the outcome level, so SUTVA does not hold. The treatment of individual i affects the outcome of individual j since the mechanism computes allocations based on the reports of all individuals. This means that even when treatment is randomly assigned, then estimators for τ_{ATE}, such as the differences in average outcomes between treated and control individuals, do not estimate τ_{GTE} (Sävje et al., 2021). However, the cutoff structure of the mechanism, knowledge of the outcome function, and some weak regularity conditions will allow us to derive asymptotically normal and efficient estimators of τ_{GTE} even in the presence of interference. Assumption 1 formalizes our restrictions on the finite sample properties of the mechanism.

Assumption 1. ***Assumptions on Mechanism Structure***

1. *The mechanism is a cutoff mechanism, following Definition 1*

2. *At the report level, SUTVA holds. For $\boldsymbol{W}$ and $\boldsymbol{W}'$, if $W_i = W_i'$, then $B_i(\boldsymbol{W}) = B_i(\boldsymbol{W}')$.*

3. *The outcome function* $y : \mathcal{B} \times S \times \mathcal{X} \mapsto \mathbb{R}$ *is bounded*

4. *For each* p*,* $y(B_i(w), p, X_i)$ *and* $d(B_i(w), p)$ *are continuous almost everywhere in* p*. The function classes* $\mathcal{F}_d = \{d_j(b, p) : j \in [J], p \in \mathcal{S}\}$ *and* $\mathcal{F}_y = \{y(b, p) : p \in \mathcal{S}\}$ *are suitably measurable and their uniform entropy number obeys, for all* $0 \leq \epsilon \leq 1$*,*

$$\sup_Q \log N(\epsilon ||\bar{F}_d||_{Q_d,2}, \mathcal{F}_d, L_2(Q_d)) \leq v \log(a/\epsilon),$$

$$\sup_Q \log N(\epsilon ||\bar{F}_y||_{Q_y,2}, \mathcal{F}_y, L_2(Q_y)) \leq v \log(a/\epsilon),$$

where the supremum is taken over all probability measures Q_d *and* Q_y *for which the classes* $\mathcal{F}_d$ *and* $\mathcal{F}_y$ *are not identically zero and* $\bar{F}_d$ *and* $\bar{F}_y$ *are each a given envelope function.*

5. $P \in \mathcal{S}$*, where* $\mathcal{S}$ *is a compact set*

The first assumption indicates that the allocation rule of the mechanism takes a competitive equilibrium form. When the mechanism takes a cutoff form, there is still interference, in that the treatment of individual j impacts individual i through the counterfactual market-clearing cutoffs, but the interference is structured in that it only occurs through the aggregate statistic $P(\boldsymbol{w})$. This is an example of the type of interference studied in the context of experiments in markets in Munro et al. (2023). The second rules out spillover effects at the report level. This rules out standard network-type spillovers, such as a treated individual sharing information with an untreated neighbor. The assumption also rules out strategic choices of B_i that depend on the market-clearing cutoffs $P(\boldsymbol{W})$. Mechanisms that are strategy-proof, such as Vickrey auctions, or deferred acceptance, lead to choices of B_i that meet the SUTVA condition. Assuming that the outcomes of interest are finite is reasonable in most cases. The fourth part of the assumption makes a Donsker assumption on the demand and outcome functions in p. The structure of many mechanisms lead to demand functions that are made up of indicator functions, which are not continuous everywhere. This framework allows us to handle settings where the demand and outcome functions have some discontinuity. Lastly, we assume that prices lie within a compact set. At the end of this section we will provide examples showing that the uniform price auction and deferred acceptance meet these assumptions.

We can define the market-clearing cutoffs under treatment and control for a given sample as:

$$o_p(n^{-1/2}) = \frac{1}{n}\sum_{i=1}^{n} d(B_i(1), P(\mathbf{1})) - q$$
$$o_p(n^{-1/2}) = \frac{1}{n}\sum_{i=1}^{n} d(B_i(0), P(\mathbf{0})) - q$$

Under Assumption 1, we can write τ_{GTE} in terms of these counterfactual cutoffs.

$$\tau_{GTE} = \frac{1}{n}\sum_{i=1}^{n} y(B_i(1), P(\mathbf{1}), X_i) - y(B_i(0), P(\mathbf{0}), X_i)$$

To estimate τ_{GTE}, we can't observe both $P(\mathbf{1})$ and $P(\mathbf{0})$ simultaneously in a single sample of data. Instead, under assumptions on the data generating process, we will show that it is possible to estimate these cutoffs using observational data satisfying an unconfoundedness assumption. To evaluate the performance of various estimators, and define a variance-minimizing estimator, we choose to make an asymptotic approximation that is common in the theory literature on comparative statics for mechanisms. We consider an asymptotic framework where n grows but J remains fixed. In the limit, the mechanism allocates a fraction of a population-level distribution to a fixed number of items.

We can define counterfactual market-clearing cutoffs for the population as follows:

$$\mathbb{E}[d(B_i(1), p_1^*) - q] = 0$$
$$\mathbb{E}[d(B_i(0), p_0^*) - q] = 0,$$

where the expectation is taken over the population F from which the individual potential reports and characteristics $\{B_i(1), B_i(0), X_i\}$ are sampled. Assumption 2 provides regularity conditions at the population level.

Assumption 2. ***Mean-Field Assumptions***

1. *p_w^* is the unique solution to $\mathbb{E}[d(B_i(w), p_w^*)] = q$, for $w \in \{0, 1\}$.*

2. *$p_0^* \in int(S)$ and $p_1^* \in int(S)$*

3. *For $w \in \{0, 1\}$, $\mu_w^d(p, x) = \mathbb{E}[d(B_i(w), p)|X_i = x]$ and $\mu_w^y(p, x) = \mathbb{E}[y(B_i(w), p, X_i)|X_i = x]$ are twice continuously differentiable in p with bounded first and second derivatives.*

4. *$\nabla_p \mathbb{E}[d(B_i(w), p)]|_{p=p_w^*}$ is non-singular for $w \in \{0, 1\}$.*

The first assumption requires that the counterfactual cutoffs are the unique solution to the population market-clearing condition. This rules out fractional mechanisms with multiple equilibria. The assumption that the cutoffs are in the interior of the compact set $\mathcal{S}$ is straightforward to satisfy when the reports are bounded, see the examples in Section 2.1 and 2.2. The third assumption imposes smoothness assumptions on the demand and outcome functions. Although at an individual level we allow for some discontinuity such as step functions, at a population-level the demand and outcome functions conditional on X_i must be sufficiently smooth.

Under these regularity assumptions, we can analyze the asymptotic behavior of τ_{GTE}.

Proposition 1. *Under Assumption 1- 2,*

$$\sqrt{n}(\tau_{GTE} - \tau_{GTE}^*) \to_d N(0, \Sigma),$$

τ^*_{GTE} and the counterfactual cutoffs p^*_0 and p^*_1 are defined by a set of $2J+1$ moment conditions:

$$
\begin{aligned}
0 &= \mathbb{E}[y(B_i(1), p^*_1, X_i) - y(B_i(0), p^*_0, X_i)] - \tau^*_{GTE} \\
0 &= \mathbb{E}[d(B_i(1), p^*_1) - q] \\
0 &= \mathbb{E}[d(B_i(0), p^*_0) - q]
\end{aligned}
\tag{1}
$$

The proof of Proposition 1, and a formula for Σ is in Appendix A.2. τ_{GTE}, a global causal effect defined under interference, converges at a $\sqrt{n}$ rate to τ^*_{GTE}. τ^*_{GTE} can be defined in terms of a moment condition based on the unobserved joint distribution of $\{B_i(1), B_i(0)\}$, where the moments are evaluated at the limiting counterfactual cutoffs p^*_0 and p^*_1. This connection will allow us to use the literature on moment conditions with missing data to construct estimators and an inference strategy for τ_{GTE}. Before moving to an estimation strategy that builds on this moment representation of τ^*_{GTE}, we first provide some examples of centralized allocation mechanisms that meet Assumptions 1 and 2.

2.1 Uniform Price Auction

In the uniform price auction, with a single good and unit demand, then the report to the mechanism is a bid, which may be affected by the treatment. $B_i(W_i) = V_i(W_i)$, and $V_i(w) \sim F_{v,w}$. In a uniform price auction with capacity $q \cdot n$, the winning bidders pay the $n \cdot q + 1$ highest bid. Formally the allocation rule is $d(B_i(W_i), P) = \mathbb{1}(B_i(W_i) > P)$, and the market-clearing price satisfies

$$
o_p(n^{-1/2}) = \frac{1}{n}\sum_{i=1}^{n} d(B_i(W_i), P) - q
$$

We assume that individuals have an independent private value for the item, which may be affected by the treatment. Under this assumption, the optimal strategy is to bid that value, and the first part of Assumption 1 at the report level holds.

Proposition 2 verifies that the remaining assumptions on the mechanism are satisfied by $d(b,p) = \mathbb{1}(b > p)$, under restrictions on the distribution of the value under treatment and control.

Proposition 2. *Assume that $0 < q < 1$. Let $V_i(W_i) \in [V^-, V^+] \subset \mathbb{R}$ where V^- and V^+ are finite.*

1. *For all $x \in \mathcal{X}$, the conditional CDF of the value distribution, $F_{v(w),x}(v|x)$, is twice continuously differentiable in v for $w \in \{0,1\}$, with bounded first and second derivatives.*

2. *The unconditional distribution $F_{v(w)}(v)$ is strictly monotonic on $[V^-, V^+]$.*

3. *Assume that the outcome function $y(b,p,x)$ meets part 3 and 4 of Assumption 1 and part 3 of Assumption 2.*

Then, Assumption 1 and 2 hold when allocations are determined by a Uniform Price Auction.

The proof is in Appendix A.5.

2.2 Deferred Acceptance

In this example, we consider the deferred acceptance algorithm when it is used to assign students to schools. The discussion here also applies to other settings, such as the residency match, where a version of deferred acceptance is used. Students submit a ranking over schools, $R_i(W_i)$. There is a treatment, such as an information intervention, that affects the rankings that a student submits. We consider a version of deferred acceptance where students have an independent lottery number $S_{ij} \sim F_s$ drawn for each school. $j = 1, \ldots, J$ index the capacity-constrained schools. We augment this with an additional J' schools which are not capacity constrained under any possible treatment allocation. The rankings submitted include preferences over both capacity and non-capacity constrained schools. The J-dimensional market-clearing cutoffs for the capacity-constrained schools are determined by a J-dimensional market-clearing condition:

$$o_p(n^{-1/2}) \approx \frac{1}{n}\sum_{i=1}^{n} d(B_i(W_i), P) - q$$

The cutoffs for the non-constrained schools are 0. The demand function that formalizes the deferred acceptance mechanism in a competitive equilibrium framework (Azevedo and Leshno, 2016; Agarwal and Somaini, 2018) is :

$$d(B_i(w), p) = \mathbb{1}\{S_{ij} > p_j, jR_i(W_i)0\} \prod_{j \neq j'} \mathbb{1}(jR_i(W_i)j' \text{ or } S_{ij'} < p_{j'})$$

where the report to the mechanism $B_i(W) = \{R_i(W_i), S_i\}$ includes both the rankings submitted by the individual, which are affected by the treatment, as well as the lottery numbers, which are not controlled by the individual. The non-constrained schools are represented by the index 0. The SUTVA assumption on $B_i(W_i)$ implies that a student's submissions to the mechanism do not depend on the treatment status of other students. This holds, for example, under conditions where deferred acceptance is strategy-proof, and students submit complete lists over schools. In settings with informational spillovers or where the list submitted to DA is very short, then it may be that there is also interference at the report level.

An example of an outcome function that is useful to evaluate is one that assigns a match value to each student-school pair, depending on the characteristics of the student and the school:

$$y(B_i(w), p) = \sum_{j=0}^{J} d_j(B_i(w), p) u_j(X_i)$$

Deferred Acceptance can be considered as a multiple-good extension of the uniform price auction. Under restrictions on the distribution of the lottery numbers, then we can extend Proposition 2 to show that deferred acceptance also meets Assumption 1 and 2, see Proposition 3.

Proposition 3. *Assume that* $\sum_{j=1}^{J} q_j < 1$. *For each* j, *let* $S_{ij}(W_i) \in [S^-, S^+] \subset \mathbb{R}$ *where* S^- *and* S^+ *are finite.*

1. *For all $x \in \mathcal{X}$, the conditional CDF of the score distribution, $F_{s,x}(s|x)$, is twice continuously differentiable in s for $w \in \{0, 1\}$, with bounded first and second derivatives.*

2. *The unconditional distribution $F_s(s)$ is strictly monotonic on $[S^-, S^+]$.*

3. *Assume that the outcome function $y(b, p, x)$ meets part 3 and 4 of Assumption 1 and part 3 of Assumption 2.*

Then, Assumption 1 and 2 hold when allocations are determined by Deferred Acceptance

The proof is in Appendix A.6.

3 Estimating the Global Treatment Effect

3.1 Identification and Efficiency

The moment conditions of Equation 1 are infeasible to estimate directly in that they depend on $B_i(1)$ and $B_i(0)$. Under an unconfoundedness and overlap assumption, we can identify τ^*_{GTE} using conditions that depend only on the observed data $Z_i = (B_i(W_i), W_i, X_i)$.

Assumption 3. ***Selection on Observables***

1. *Overlap holds. Let $e(x) = Pr(W_i = 1|X_i = x)$. For all $x \in \mathcal{X}$, $0 < e(x) < 1$.*

2. *Unconfoundedness holds. $\{B_i(1), B_i(0)\} \perp\!\!\!\perp W_i|X_i$,*

It is possible to use an IV-type assumption as an identifying condition instead, see Appendix B for a brief discussion. Under Assumption 3, there are many different estimating equations that can be constructed from Equation 1. We discuss three approaches based on an IPW score, conditional mean functions, and a doubly-robust approach. First, define the conditional mean and unconditional mean functions as

$$\mu^d_w(p, x) = \mathbb{E}[d(B_i(w), p)|X_i = x], \qquad \mu^d_w(p) = \mathbb{E}[d(B_i(w), p)],$$
$$\mu^y_w(p, x) = \mathbb{E}[y(B_i(w), p, X_i|X_i = x], \quad \mu^y_w(p) = \mathbb{E}[y(B_i(w), p, X_i)].$$

We can identify τ^*_{GTE} using conditional mean functions only:

$$0 = \mathbb{E}[\mu^y_1(p^*_1, X_i) - \mu^y_0(p^*_0, X_i)] - \tau^*_{GTE}$$
$$0 = \mathbb{E}[\mu^d_1(p^*_1, X_i) - q], \qquad 0 = \mathbb{E}[\mu^d_0(p^*_0, X_i) = q]$$

Another identification approach is using the propensity score:

$$\begin{aligned} 0 &= \mathbb{E}\left[\frac{W_i y(B_i, p^*_1, X_i)}{e(X_i)} - \frac{(1 - W_i)y(B_i, p^*_0, X_i)}{1 - e(X_i)}\right] - \tau^*_{GTE} \\ 0 &= \mathbb{E}\left[\frac{W_i d(B_i, p^*_1)}{e(X_i)}\right], \qquad 0 = \mathbb{E}\left[\frac{(1 - W_i)d(B_i, p^*_0)}{1 - e(X_i)}\right] \end{aligned} \tag{2}$$

We use the notation $\mathbb{E}[\psi^{IPW}(Z_i; p^*, e(X_i))] = v(\tau^*_{GTE})$, where the observed data is $Z_i = (B_i, W_i, X_i)$, the market-clearing cutoffs are $p^* = (p^*_0, p^*_1)$, and $v(\tau^*_{GTE}) = \begin{bmatrix}\tau^*_{GTE} & 0 & 0\end{bmatrix}^\top$, to collect these $2J+1$ moment conditions into a single score vector.

We can also a define a moment condition based on a doubly-robust score, which is Neyman-orthogonal with respect to the propensity score and the conditional mean functions.

$$
\begin{aligned}
\tau^*_{GTE} &= \mathbb{E}\left[\mu^y_1(p^*_1, X_i) - \mu^y_0(p^*_1, X_i) + \frac{W_i(y(B_i, p^*_1, X_i) - \mu^y_1(p^*_1, X_i))}{e(X_i)} - \frac{(1-W_i)(y(B_i, p^*_0, X_i) - \mu^y_0(p^*_0, X_i))}{1-e(X_i)}\right] \\
0 &= \mathbb{E}\left[\mu^d_1(p^*_1, X_i) + \frac{W_i(d(B_i, p^*_1) - \mu^d_1(p^*_1, X_i))}{e(X_i)}\right] \\
0 &= \mathbb{E}\left[\mu^d_0(p^*_0, X_i) + \frac{(1-W_i)(d(B_i, p^*_0) - \mu^d_0(p^*_0, X_i))}{1-e(X_i)}\right]
\end{aligned}
\tag{3}
$$

We use the notation $\mathbb{E}[\psi^{DR}(Z_i; p^*, e(X_i), \mu(X_i; p^*)] = v(\tau^*_{GTE})$, where the conditional mean nuisance parameters are $\mu(X_i; p^*) = \Big(\mu^d_0(p^*_0, X_i), \quad \mu^d_1(p^*_1, X_i), \quad \mu^y_0(p^*_0, X_i), \quad \mu^y_1(p^*_1, X_i)\Big)$ to collect the $2J+1$ conditions of Equation 3 into a single score vector. Our estimator will combine an initial estimate of the market-clearing cutoffs based on ψ^{IPW}, with a second step based on ψ^{DR} to derive an estimator for τ^*_{GTE}.

Under our identifying assumptions, in order to evaluate the performance of our estimator, it is helpful to derive the semi-parametric efficiency bound for the estimation of τ^*_{GTE}. In the next section, we will specify a computationally simple estimator that meets this efficiency bound with weak assumptions on nuisance parameter estimation.

Theorem 4. ***Semi-Parametric Efficiency*** *Under the assumptions of Proposition 1 and Assumption 3, the semi-parametric efficiency bound for τ^*_{GTE} is equal to*

$$
V^* = Var[q(X_i)] + \mathbb{E}\left[\frac{\sigma^2_0(X_i)}{1-e(X_i)}\right] + \mathbb{E}\left[\frac{\sigma^2_1(X_i)}{e(X_i)}\right],
$$

where

$$
\begin{aligned}
q(X_i) &= \mu^y_1(p^*_1, X_i) - \boldsymbol{\gamma}_1^\top(\mu^d_1(p^*_1, X_i) - q) - \mu^y_0(p^*_0, X_i) + \boldsymbol{\gamma}_0^\top(\mu^d_0(p^*_0, X_i) - q) \\
\sigma^2_0(X_i) &= \mathbb{E}\Big[\big(y(B_i(0), p^*_0, X_i) - \mu^y_0(p^*_0, X_i) - \boldsymbol{\gamma}_0^\top(d(B_i(0), p^*_0) - \mu^d_0(p^*_0, X_i)\big)^2 | X_i\Big] \\
\sigma^2_1(X_i) &= \mathbb{E}\Big[\big(y(B_i(1), p^*_1, X_i) - \mu^y_1(p^*_1, X_i) - \boldsymbol{\gamma}_1^\top(d(B_i(1), p^*_1) - \mu^d_1(p^*_1, X_i)\big)^2 | X_i\Big] \\
\boldsymbol{\gamma}_0 &= [\nabla_p \mathbb{E}[d(B_i(0), p^*_0)]]^{-1} \nabla_p \mathbb{E}[y(B_i(0), p^*_0, X_i)] \\
\boldsymbol{\gamma}_1 &= [\nabla_p \mathbb{E}[d(B_i(1), p^*_1)]]^{-1} \nabla_p \mathbb{E}[y(B_i(1), p^*_1, X_i)]
\end{aligned}
$$

The proof of this theorem is in Appendix A.3. The proof follows uses the methodology presented in Bickel et al. (1993) and Newey (1990). The organization and notation of the proof is similar to other papers that apply this methodology to related estimands, including Hahn (1998) and Hirano et al. (2003) for average treatment effects, Firpo (2007) for quantile treatment effects, and Chen and Ritzwoller (2021) for long-run treatment effects. Our presentation and notation is closest to that of Firpo (2007). This bound applies whether or not the propensity score is known, so it also applies in settings where the data is generated from a randomized experiment.

Due to the market-clearing cutoffs, the efficiency bound of τ^*_{GTE} looks different than that of τ_{ATE} without interference. The bound for τ_{ATE} matches the bound for τ^*_{GTE} when both $\boldsymbol{\gamma}_0 = 0$ and $\boldsymbol{\gamma}_1 = 0$, which occurs if the outcomes do not depend on the market-clearing cutoffs. The minimum asymptotic variance of an estimator includes components due to variation in treatment effects when a sample is drawn from a population, but also due to variation in the equilibrium that is reached in the allocation mechanism. When these components are negatively correlated with the noise from the sampling of outcomes, then confidence intervals that account for noise in the equilibrium effect will be tighter than those that ignore equilibrium effects. We see that this is the case both in the simulations in Section 5 and in the empirical example of Section 6.

3.2 Standard Estimators

Proposition 1 shows that asymptotically, the global treatment effect can be represented as a moment condition model with missing data. In this section, we consider three standard estimators used frequently in the average treatment effect setting, which can also be used to estimate parameters of moment conditional models with missing data. These three estimators are based on outcome modeling, inverse-propensity score weighting, and the double robust score. We interpret how the existing results in the literature on the performance of these methods for moment problems with missing data can be interpreted in the context of the specific model of designed markets in this paper.

The first standard causal estimator is based on a modeling approach. First, assume a parametric model of an individual's report to the mechanism $B_i(1)|X_i$ and $B_i(0)|X_i$, and estimate this choice model using the observed data. Under a parametric assumption on the distribution for treated and control reports, where the vector of parameters is θ_w for $w \in \{0,1\}$, the estimated distributions are defined as : $F^B(b|x;\hat{\theta}_1)$ and $F^B(b|x;\hat{\theta}_0)$. Next, estimate $\hat{\mu}^d_w(p,x) = \int d(b,p)dF^B(b|x;\hat{\theta}_w)$ and $\hat{\mu}^y_w(p,x) = \int y(b,p,x)dF^B(b|x;\hat{\theta}_w)$. Then, it is straightforward to estimate $\hat{\tau}_{GTE}$ using this model:

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\mu^y_1(\hat{p}^M_1, X_i) - \hat{\mu}^y_0(\hat{p}^M_0, X_i) = \hat{\tau}^M_{GTE} \\
&\frac{1}{n}\sum_{i=1}^{n}\mu^d_1(\hat{p}^M_1, X_i) - q = 0, \qquad \frac{1}{n}\sum_{i=1}^{n}\mu^d_0(\hat{p}^M_0, X_i) - q = 0.
\end{aligned}
\tag{4}
$$

One way of implementing this estimator is the commonly used approach of estimating a complete model of individual choices using observed data, and simulating from that model to evaluate counterfactuals, see for example Allende et al. (2019) in the school choice setting. The advantage of this approach is once the model is specified and estimated, a variety of counterfactuals can be evaluated, including those that are more complex than the estimand considered in this paper. The downside of this approach is if the model is not correctly specified, then estimates of the counterfactuals will be asymptotically biased. It can be challenging to specify a parametric model that captures the complexity of individual choice behavior. In this paper, we will explore approaches that do not require the specification of a correct parametric model for individual choices. Instead flexible non-parametric or semi-parametric models of certain conditional means of choices can be used, which are able to capture a wide range of individual behavior.

The IPW approach, on the other hand, does not require specifying a correct model of an individual's choices of report to the mechanism. Instead, it estimates the market-clearing cutoffs and treatment effect based on reweighted empirical distribution of reports conditional on the treatment, where the weights are (estimated) propensity scores. Even when the propensity score is known, then weights conditional on X_i should be estimated and used. An estimate of τ^*_{GTE} using the IPW score is as follows:

$$\frac{1}{n}\sum_{i=1}^{n}\psi^{IPW}(Z_i;(\hat{p}_1^{IPW},\hat{p}_0^{IPW}),\hat{e}(X_i)) = v(\hat{\tau}^{IPW}). \tag{5}$$

For the IPW estimate of τ^*_{GTE} to be asymptotically efficient requires the estimation of $e(X_i)$ to meet a fairly restrictive set of conditions. See Hirano et al. (2003) for the average treatment effect and Firpo (2007) for the quantile treatment effect, and a broader discussion of the limitations of the IPW approach in moment condition models with missing data in Graham et al. (2012).

Our third alternative is to use the doubly robust score alone to estimate τ^*_{GTE}.

$$\frac{1}{n}\sum_{i=1}^{n}\psi^{DR}(Z_i;\hat{p}^{DR},\hat{e}(X_i),\hat{\mu}(\hat{p}^{DR},X_i)) = v(\hat{\tau}^{DR}).$$

Double-robustness implies that for consistency of τ^*_{GTE}, only one of $\hat{e}$ and $\hat{\mu}$ has to be consistent. For asymptotic normality, the rate condition of Chernozhukov et al. (2018) means that the product of the rates of the conditional mean estimator and the propensity score estimator must be $o(n^{-1/2})$, when both are consistent. In many problems, this allows flexible and computationally efficient machine learning estimators to be used for the nuisance parameters. However, for the moment condition given in Equation 1, estimating the counterfactual cutoffs requires an estimate of the nuisance parameter $\mu^d(p,x)$ for all $p \in \mathcal{S}$. As discussed for the modeling estimator, estimating the entire distribution of $B_i(w)$ is required to estimate this conditional mean function for all $p \in \mathcal{S}$. Estimating the entire distribution of reports using a flexible semi-parametric or non-parametric model that also meets the rate conditions of Chernozhukov et al. (2018) is often impractical. The two-step procedure introduced in the next section leads to a procedure that maintains doubly-robust properties, without requiring the estimation of the entire distribution of $B_i(w)$.

3.3 Two-Step Doubly-Robust Estimator

Kallus et al. (2019) recognized that in certain moment condition problems with missing data, such as quantile treatment effects, conditional mean functions may depend on an estimated parameter. In order to apply the double machine learning framework of Chernozhukov et al. (2018), nuisance functions must be estimated for a wide range of possible parameters. Kallus et al. (2019) introduce a two-step methodology that only estimates a single conditional mean function, evaluated at a first-step estimate of the parameter. We show that this methodology is useful for estimating the causal effects under interference, when interference occurs through a centralized mechanism with a cutoff structure.

Definition 2. Two-Step Estimator

1. Randomly split the dataset into five subsets. For subset k, label the indexes for subset k as I_0^k and label the other four subsets of data as I_1^k, I_2^k, I_3^k and I_4^k. For each subset $k \in \{1, \dots 5\}$:
   - Using dataset I_1^k and I_2^k, estimate $\hat{p}_{IPW,1}^{(k)}$ and $\hat{p}_{IPW,0}^{(k)}$ using Equation 5.
   - Using dataset $I_1^k, I_2^k, I_3^k, I_4^k$, estimate the propensity score $\hat{e}^{(k)}(X_i)$ using (W_i, X_i)
   - Using dataset I_3^k, I_4^k, estimate the conditional mean functions using regressions
     - For $w \in \{0, 1\}$, estimate $\hat{\mu}_w^{(k),y}(\hat{p}_{IPW,w}^{(k)}, X_i)$ by regressing $y(B_i, \hat{p}_{IPW,w}^{(k)}, X_i)$ on X_i for $i \in I_3^k, I_4^k$ with $W_i = w$
     - For $w \in \{0, 1\}$, estimate $\hat{\mu}_w^{(k),d}(\hat{p}_{IPW,w}^{(k)}, X_i)$ by regressions of $d(B_i, \hat{p}_{IPW,w}, X_i)$ on X_i for $i \in I_3^k, I_4^k$ with $W_i = w$.

2. Estimate $\hat{\tau}^{LDML}$ using ψ^{DR}:

$$\frac{1}{n} \sum_{k=1}^{5} \sum_{i \in I_0^k} \psi^{DR}(Z_i; \hat{p}^{LDML}, \hat{e}^{(k)}(X_i), \hat{\mu}^{(k)}(\hat{p}_{IPW}^{(k)}, X_i)) = v(\hat{\tau}^{LDML}).$$

Data is split at least three ways, rather than two-ways as in the double machine learning approach of Chernozhukov et al. (2018). On one of the splits, a first stage IPW estimate of the market-clearing cutoffs under treatment and control is estimated, and outcomes and demand evaluated at those market-clearing cutoffs are computed. On a second split, the propensity score, and a single set of conditional mean functions evaluated at the IPW-estimate of the cutoffs are estimated. Finally, on a third split of the data, the empirical doubly robust score with estimated nuisance parameters is computed. This is repeated for different splits of the data and the results are averaged for the final estimator. For this to lead to an asympotically normal and efficient estimator, we require the following restrictions on the nuisance function estimation.

Assumption 4. ***Assumptions on Nuisance Estimation:*** *With probability $1 - \Delta_n$, where $\Delta_n = o(1)$, then for each split $k \in \{1, 2, 3, 4, 5\}$.*

1. *The estimated propensity score is bounded away from 0 and 1. For* $\epsilon > 0$, $\sup_{x\in\mathcal{X}} ||\hat{e}^{(k)}(x) - 0.5|| \leq 0.5 - \epsilon$.

2. *For any sequence of constants* $\Delta_n \to 0$, *the nuisance estimates* $(\hat{\mu}^{(k)}(\cdot;\hat{p}^{(k)}_{IPW}), \hat{e}^{(k)}(\cdot))$ *belong to the realization set* $\mathcal{T}_n$ *with probability at least* $1-\Delta_n$. *For* $w \in \{0,1\}$,

$$
\begin{aligned}
&||(\mathbb{E}(\hat{\mu}^{(k)}(X_i;\hat{p}^{(k)}_{IPW}) - \mu^*(X_i;\hat{p}^{(k)}_{IPW}))^2)^{1/2}|| \leq \rho_{\mu,n} \\
&(\mathbb{E}(\hat{e}^{(k)}(X_i) - e(X_i))^2)^{1/2} \leq \rho_{e,n} \\
&||\hat{p}^{(k)}_{IPW} - p^*|| \leq \rho_{\theta,n},
\end{aligned}
$$

where $\rho_{e,n}(\rho_{\mu,n} + C\rho_{\theta_n}) \leq \frac{\epsilon^3}{3}\delta_n n^{-1/2}$, $\rho_{e,n} \leq \frac{\delta_n^3}{\log n}$, $\rho_{\mu,n} + C\rho_{\theta,n} \leq \frac{\delta_n^2}{\log n}$, $\delta_n \leq \frac{4C^2\sqrt{d}+2\epsilon}{\epsilon^2}$, *and* $\delta_n \leq \min\left\{\frac{\epsilon^2}{8C^2 d}\log n, \sqrt{\frac{\epsilon^3}{2C\sqrt{d}}}\log^{1/2} n\right\}$. *Furthermore, the nuisance realization set contains the true nuisance parameters* $(\mu^*(\cdot;p_1^*,p_0^*), e(\cdot))$.

Up to factors that are polynomials of logarithms in n, then Assumption 4 requires that the pairwise product of the rates of the initial estimator of the counterfactual cutoffs, the propensity score, and the outcome function are $o(n^{-1/2})$ and that each nuisance parameter is also consistent. When the initial estimator of the counterfactual cutoffs is an IPW estimator using the same propensity score estimator as the treatment effect estimator, then this assumption requires that the estimators of the propensity score and counterfactual cutoffs each have rate $o(n^{-1/4})$[1]. For the conditional mean functions, we are no longer required to estimate the entire conditional distribution of reports $B_i(w)|X_i$ for $w \in \{0,1\}$. Instead, we estimate $2J+1$ regressions of outcomes and allocations evaluated at the first-stage estimate of the market-clearing cutoffs on treatment and covariates. In this setting, it is reasonable that a flexible machine learning method, such as neural networks or random forests, would meet an $o(n^{-1/4})$ rate condition. We now prove that the algorithm described leads to an asymptotically normal and semi-parametrically efficient estimator, when data is generated from an intervention in a designed market, and the estimand of interest is the global treatment effect.

Theorem 5. *Under Assumptions 1 - 4,* $\hat{\tau}^{LDML}_{GTE}$ *is asymptotically normal with variance that matches the semi-parametric efficiency bound derived in Theorem 4:*

$$\sqrt{n}(\hat{\tau}^{LDML}_{GTE} - \tau^*_{GTE}) \to_D N(0, V^*).$$

The proof, in Appendix A.4, verifies the conditions of the main theorem in Kallus et al. (2019). We can use the analytical form of the variance in Theorem 4 to compute a plug-in variance estimator. Alternatively, it is possible to use the bootstrap to compute variance estimates.

[1] See Remark 3 and Appendix C of Kallus et al. (2019)

4 Optimal Targeting in Designed Markets

So far, this paper has considered the identification and estimation of τ_{GTE}, which is the effect of treating everybody in a sample compared to treating nobody. This is relevant for deciding whether or not to rollout the treatment to everybody. However, a policymaker may have finer control over the treatment allocation, and have interest in specifying the optimal $\pi : \mathcal{X} \to [0,1]$, where $\pi(x) = Pr(W_i = 1|X_i = x)$. There is a large literature on optimal targeting, which largely focuses on settings without interference. In the absence of interference, and without any constraints on the targeting rule, the optimal rule assigns treatment to those with positive Conditional Average Treatment Effect (CATE), where the CATE is defined as $\mathbb{E}[Y_i(1) - Y_i(0)|X_i = x]$.

Under interference, the CATE is not well-defined, so Munro et al. (2023) introduces definitions of the Conditional Average Direct Effect (CADE) and Conditional Average Indirect Effect (CAIE) instead. The CADE is the expected effect of treating individuals with a certain covariate value on their own outcomes. The CAIE is the expected effect of treating individuals with a certain covariate value on everyone else's outcomes. Theorem 9 of Munro et al. (2023) implies that in the large sample limit, when treatments are allocated according to $\pi(\cdot)$, the CADE and the CAIE take the following form in our model:

$$\begin{aligned}\tau^*_{CADE}(x,\pi) &= \mu^y_1(p^*_\pi, x) - \mu^y_0(p^*_\pi, x)\\ \tau^*_{CAIE}(x,\pi) &= -\gamma_\pi^\top(\mu^d_1(p^*_\pi, x) - \mu^d_0(p^*_\pi, x)).\end{aligned}$$

where p^*_π is the asymptotic market clearing cutoffs under the treatment rule:

$$\mathbb{E}[\pi(X_i)d(B_i(1), p^*_\pi) + (1-\pi(X_i))d(B_i(0), p^*_\pi)] = q,$$

and $\gamma_\pi = \nabla_p\mathbb{E}[d(B_i(W_i), p^*_\pi)]^{-1}\nabla_p\mathbb{E}[y(B_i(W_i), p^*_\pi, X_i)]$ when $W_i \sim \text{Bernoulli}(\pi(X_i))$.

Munro et al. (2023) derive the outcome-maximizing treatment rule under the constraint that the rule induces the same equilibrium as the observed data. In this paper, we are interested in characterizing and estimating the treatment rule that is outcome-maximizing without any constraints on the equilibrium.

Define the space of candidate treatment rules as Π. For example, the space of treatment rules might be defined as the set of possible logistic functions from $\mathcal{X}$ to $[0,1]$. Let the optimal rule maximize outcomes in the large sample limit:

$$\pi^* = \arg\max_{\pi\in\Pi} V(\pi) = \arg\max_{\pi} \mathbb{E}[\pi(X_i)(y(B_i(1), p^*_\pi, X_i) - y(B_i(0), p^*_\pi, X_i))],$$

Proposition 6. *Assume Π is a vector space. Any optimal rule π^* meets the following score condition almost surely for $x \in \mathcal{X}$:*

1. $\pi^*(x) = 1$, *and* $\tau^*_{CADE}(x,\pi^*) + \tau^*_{CAIE}(x,\pi^*) > 0$, *or*
2. $\pi^*(x) = 0$, *and* $\tau^*_{CADE}(x,\pi^*) + \tau^*_{CAIE}(x,\pi^*) < 0$, *or*

3. $0 \leq \pi^*(x) \leq 1$, *and* $\tau^*_{CADE}(x, \pi^*) + \tau^*_{CAIE}(x, \pi^*) = 0$.

When X_i is discrete, $v_\pi = \tau^*_{CADE}(x, \pi) + \tau^*_{CAIE}(x, \pi)$ is the derivative of the objective function with respect to $\pi(x)$. Proposition 6 indicates that any optimal rule must satisfy a set of necessary conditions that depend on the constraints of the rule and the gradient of the objective function with respect to the targeting rule. The first component of the derivative takes into account the direct impact of raising $\pi(x)$ through changes in the reports to the mechanism for individuals with covariates $X_i = x$. The second component of the derivative takes into account the indirect impact of raising $\pi(x)$ that occurs through changes in the market-clearing cutoffs that result from aggregate changes in submissions to the mechanism.

To find a treatment rule that meets the conditions of Proposition 6 we use an empirical welfare maximization approach with doubly-robust scores. We assume, as in Section 3 that the observed data is $\{B_i(W_i), W_i, X_i\}$, with treatment selection following the possibly unknown $e(X_i) = Pr(W_i = 1 | X_i = x)$, and the assumptions of Theorem 5 hold. A consistent estimate of the objective value for a given rule π using doubly-robust scores can be computed as:

$$\begin{aligned}
&\hat{V}(\pi) = \frac{1}{n} \sum_{i=1}^{n} \pi(X_i) \hat{\Gamma}^y_i(\hat{P}_\pi), \\
&0 = \pi(X_i) \Gamma^d_{1,i}(\hat{P}_\pi) + (1 - \pi(X_i)) \hat{\Gamma}^d_{0,i}(\hat{P}_\pi) - q, \\
&\hat{\Gamma}^y_i(p) = \hat{\mu}^{y,(k)}_1(X_i) - \hat{\mu}^{y,(k)}_0(X_i) + \frac{W_i(y(B_i(W_i), p) - \hat{\mu}^{y,(k)}_1(X_i))}{\hat{e}^{(k)}(X_i)} - \frac{(1 - W_i)(y(B_i(W_i), p) - \hat{\mu}^{y,(k)}_0(X_i))}{1 - \hat{e}^{(k)}(X_i)}, \\
&\hat{\Gamma}^d_{1,i}(p) = \hat{\mu}^{d,(k)}_1(X_i) + \frac{W_i(d(B_i(W_i), p) - \hat{\mu}^{d,(k)}_1(X_i))}{\hat{e}^{(k)}(X_i)}, \\
&\hat{\Gamma}^d_{0,i}(p) = \hat{\mu}^{d,(k)}_0(X_i) + \frac{(1 - W_i)(d(B_i(W_i), p) - \hat{\mu}^{d,(k)}_0(X_i))}{1 - \hat{e}^{(k)}(X_i)},
\end{aligned} \tag{6}$$

where $\hat{\mu}^{(k)}(x) = \hat{\mu}^{(k)}(X_i; \hat{p}^{(k)}_{IPW})$ and all the nuisance estimates are constructed according to the cross-fitting procedure in Section 3. The optimal rule is estimated by maximizing the estimated objective function $\hat{V}(\pi)$. The algorithm for doing so depends on the class of allocation rules Π and the properties of $V(\pi)$. Consider, for example a parametric class of allocation rules $\pi(x; \beta)$, where $\pi(x; \beta)$ is differentiable in β. When the estimated objective is strongly convex in β, we can use gradient descent to find a maximum. With convexity rather than smoothness assumptions on the objective, as long as the class of allocation rules is compact, then an approximately optimal solution can be found using a grid search.

An important question is how a maximizer of $\hat{V}(\pi)$ performs compared to a maximizer of the true objective $V(\pi)$. Although it is straightforward to show that for a single value of π, $\hat{V}(\pi)$ is a consistent and asymptotically normal estimate of $V(\pi)$, deriving results that are uniform over Π is more challenging. Assuming that $\hat{V}(\pi)$ has a unique maximizer $\hat{\pi}^* = \arg\max_{\pi \in \Pi} \hat{V}(\pi)$, the regret is defined as:

$$R(\hat{\pi}^*) = \max\{V(\pi) : \pi \in \Pi\} - V(\hat{\pi}^*)$$

The approach in Athey and Wager (2021) can be extended to show that expected regret is of order $\sqrt{VC(\Pi)/n}$, where $VC(\Pi)$ is the Vapnik-Chervonenkis dimension of the class of treatment rules Π. This implies maximizing the empirical value function well-approximates maximizing the true value function.

5 Simulations

In this section, we illustrate the theoretical results in Section 3 using two simple simulations. In the first, we illustrate the robustness properties of the LDML estimator, in contrast to the outcome modeling estimator, using a simulation of a uniform price auction where bidders values are generated from different distributions. In the second simulation, of a market for schools with three schools, we show that asymptotically valid confidence intervals for τ^*_{GTE} built on the LDML estimator have good coverage for τ_{GTE} in finite samples.

5.1 Auction Simulation

In this section, we simulate data generated from a uniform price auction for a single good, and use it to illustrate some of the properties of the LDML, outcome modeling, and IPW estimators discussed in Section 3. A treatment affects bids to the auction. There is a 20-dimensional set of covariates that is correlated with the bids and affects the probability of selecting the treatment. The auction has a fractional capacity of 0.5, so that the top half of the bids in the auction receive a single unit of the good. The treatment affects outcomes through a shift in the distribution of bids submitted to the auction, and through a shift in the equilibrium market-clearing price. The outcome of interest is the observed average surplus for bidders in the auction, assuming that the bids submitted to the auction are equal to the values for the bidders.

The data-generating process is follows, where $\Phi(\cdot)$ is the standard normal CDF:

$$\begin{aligned}
&B_i(1) \sim F_1(X_i), \qquad B_i(0) \sim F_0(X_i), \qquad X_i \sim \text{Uniform}(0,1)^p, \\
&W_i \sim \text{Bernoulli}(\Phi(X_{1i} - 0.5X_{2i} + 0.5X_{3i})), \qquad D_i(W_i, p) = \mathbb{1}(B_i(W_i) \geq p), \\
&Y_i(\boldsymbol{W}) = (B_i(W_i) - P(\boldsymbol{W}))\mathbb{1}(B_i(W_i) > P(\boldsymbol{W})), \qquad \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(B_i(W) > P(\boldsymbol{W})) = \frac{1}{2}.
\end{aligned}$$

In the simulation, we compute the RMSE, bias and standard deviation of a variety of estimators when the target estimand is $\tau_{GTE} = \frac{1}{n}\sum_{i=1}^{n} Y_i(\mathbf{1}) - Y_i(\mathbf{0})$. If the bid distributions $F_1(X_i)$ and $F_0(X_i)$ take a known parametric form, then the outcome-modeling approach is the consistent and efficient estimator of τ^*_{GTE}. In the first set of simulations, we generate

$$B_i(0) \sim \text{LogNormal}(0.8X_{1i} - 0.3X_{2i} - 0.2X_{3i}, 0.3), \qquad B_i(1) = 1.5B_i(0)$$

On samples from a uniform price auction run on these bids, we compute three estimators:

- A model-based estimator following Equation 4. $\hat{F}_1(X_i)$ and $\hat{F}_0(X_i)$ are LogNormal$(\hat{\mu}_w(X_i), \hat{\sigma})$, where $\hat{\mu}_w(X_i)$ is estimated using a linear regression of $\log(B_i(w))$ on X_i for individuals with $W_i = w$.
- An IPW-based estimator following Equation 5, with two-way data splitting and propensity scores estimated using a random forest.
- An LDML estimator following Definition 2, with data splitting, and both propensity scores and conditional mean nuisance functions estimated using random forests.

	n=100		n=1,000		n=10,000	
	Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
$\hat{\tau}^M$	-0.16	0.32	0.0038	0.019	-0.0002	0.004
$\hat{\tau}^{IPW}$	0.045	0.116	0.028	0.046	0.014	0.017
$\hat{\tau}^{LDML}$	0.033	0.106	0.012	0.040	-0.002	0.010

Table 1: Lognormal Distribution for Bids

With only 100 datapoints, then the noise in the estimation for all methods is high, and τ_{GTE} is not estimated precisely. As the number of datapoints increases, the model-based estimator, which makes the correct parametric assumption on the bid distribution, converges the fastest. The LDML and IPW estimators, however, do not make any parametric assumptions, and instead use flexible-machine learning estimators for nuisance parameter estimation. Depending on the convergence properties of the propensity score, the IPW estimator may have bias that decays slowly, and an asymptotic variance that is not efficient. The LDML estimator has an asymptotic distribution that does not depend on the estimation errors of the nuisance functions. We see for this simulation, the RMSE of the LDML estimator does decrease at a faster rate than that of the IPW estimator. However, the outcome modeling estimator, which makes a correct parametric assumption, performs best.

In the second set of simulations, we generate bids from a truncated normal distribution rather than a lognormal distribution. Otherwise, the data-generating process is the same. We compute the same three estimators, where we continue to use a random-forest based approach for the nuisance functions for the IPW and LDML estimators, and a log-normal based approach for the outcome modeling estimator.

This time, the outcome modeling approach performs very poorly. The parametric assumption is incorrect, and as a result the outcome model is asymptotically biased. On the other hand, the IPW and LDML estimators, which use flexible models for certain statistics of the observed data, rather than making a parametric assumption on the bid distribution, perform equally well here.

	n=100		n=1,000		n=10,000	
	Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
$\hat{\tau}^M$	0.13	0.085	0.077	0.096	0.080	0.082
$\hat{\tau}^{IPW}$	0.01	0.073	-0.0004	0.020	-0.0003	0.00063
$\hat{\tau}^{LDML}$	0.018	0.083	0.004	0.021	-0.0004	0.00065

Table 2: Truncated Normal Distribution for Bids

5.2 Simulation of a Market for Schools

We next construct a simulation of a schools market, where individuals rank schools according to a random utility model, and the treatment affects a subgroup of students' preferences for a high quality school. There are three schools, with fractional capacity of 25%, 25% and 100%, respectively. Only the first two are high quality. The outcome is average match-value, where the planner has a higher value for a certain subgroup of students attending a high quality school. The data-generating process is described in detail in Appendix C.

The distribution of the ground truth for two estimands defined on a sample of n individuals is plotted in Figure 1a. Proposition 1 indicates that distribution of $\sqrt{n}(\tau_{GTE} - \tau^*_{GTE})$ is asymptotically normal, and we see in the plot that the density for τ_{GTE} roughly corresponds to a normal density. We also plot the distribution of the estimand τ_{DTE} in repeated samples from the data-generating process. τ_{DTE} is the direct treatment effect, which is defined in Hu et al. (2022) as

$$\tau_{DTE} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[Y_i(W_i = 1; \boldsymbol{W}_{-i})|Y_i(\cdot)] - \mathbb{E}[Y_i(W_i = 0; \boldsymbol{W}_{-i})|Y_i(\cdot)]$$

Estimators for the average treatment effect in settings without interference are consistent for the direct treatment effect when used in settings with interference (Sävje et al., 2021). With samples of data drawn from the data-generating process, we construct estimates and conservative confidence intervals for τ_{DTE} by using methods for the averaged treatment effect based on generalized random forests, as described in Athey et al. (2019), and implemented in the R package `grf`. The results in Munro et al. (2023) indicate that for this simulation, using confidence intervals for the average treatment effect will be slightly conservative for the direct treatment effect.

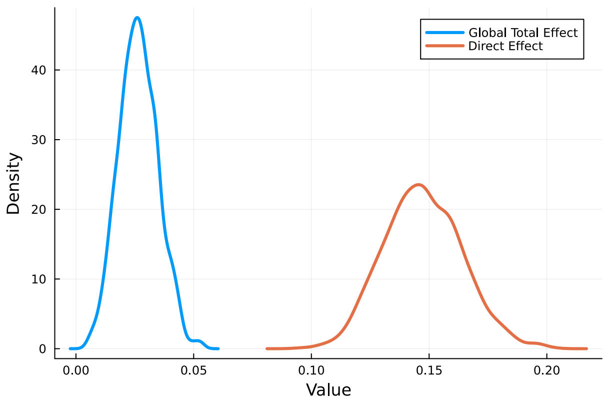
We construct confidence intervals for the LDML estimator using a consistent plug-in estimate of the variance in Theorem 5:

$$\hat{V} = \hat{\text{Var}}[\hat{Q}_{1,i} - \hat{Q}_{0,i}], \qquad \hat{Q}_{w,i} = \hat{\Gamma}^y_{w,i}(\hat{P}_w) - \hat{\gamma}_w^\top[\hat{\Gamma}^{d,w}_i(\hat{P}_w) - q].$$

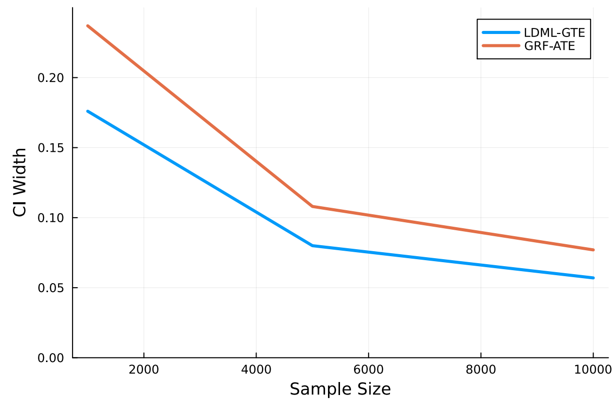
The estimated doubly-robust scores for outcomes and demand are defined in Equation 6. The estimate of $\gamma_w = [\nabla_p \mathbb{E}[d(B_i(w), p^*_w)]]^{-1}\nabla_p \mathbb{E}[y(B_i(w), p^*_w, X_i)]$ can be computed using a finite-differencing approach by perturbing the cutoffs around $\hat{P}_1$ and $\hat{P}_0$ and observing how a doubly-robust estimate of average allocations and outcomes changes. Under weak regularity conditions, the consistency of $\hat{V}$ and asymptotic validity of confidence intervals based on $\hat{V}$ is established in

Theorem 4 of Kallus et al. (2019).

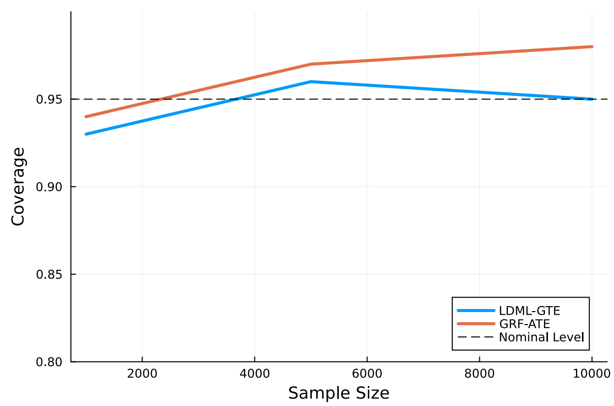
We see in Figure 1c, that both the GRF-derived confidence intervals and the LDML-derived confidence intervals are near the nominal coverage level for their respective estimands, with the GRF-derived confidence intervals slightly over-covering. However, since the partial equilibrium effect τ_{DTE} varies more than the general equilibrium effect, the confidence interval width for the estimate of the GTE is substantially more narrow than the estimate of the DTE. The noise in the counterfactual cutoff estimation is negatively correlated with noise from the variance in outcomes evaluated at a single cutoff, which makes τ_{GTE} a lower variance target at a given sample size.



(a) The distribution of τ_{GTE} and τ_{DTE} for a repeated sample of $n = 1000$ agents over $S = 1000$ samples



(b) Confidence interval width for treatment effect estimators, averaged over $S = 100$ samples



(c) Coverage for treatment effect estimators, averaged over $S = 100$ samples

6 Evaluating Interventions in the Chilean School Market

Historically, the Chilean school system has had a high level of socio-economic segregation (Bellei, 2013; Valenzuela et al., 2014). In 2015, the Chilean government passed the Inclusion Law with the goal of improving the education quality for lower-income families. The law had multiple components affecting admissions criteria and subsidies for schools that receive government funding in the country. One of the major components of the law was eliminating school-specific admissions criteria in favor of a centralized admission system based on deferred acceptance, see Correa et al. (2019) for a detailed description of the mechanism in Chile. After an initial rollout of the centralized admission system in the region of Magallanes in 2017, by 2020 the system was implemented in all regions in Chile. Along with other changes in the Inclusion law, the centralized admission system is intended to reduce socio-economic segregation in schools by removing school-specific admissions criteria which may have discriminated against low-income families and implementing a quota that reserves some proportion of seats in each school for lower-income families. However, as of data from 2019, a significant gap remains in place. Figure 2 shows the distribution of the quality of the school that a family ranks first, separated by whether or not the family has priority in the admissions system due to low income. The school quality measure is based on the average 10th grade student score in math and reading from 2018. The applications data is from 2019, for prospective 9th graders. Lower-income families rank higher quality schools first at a much lower rate than higher-income families.

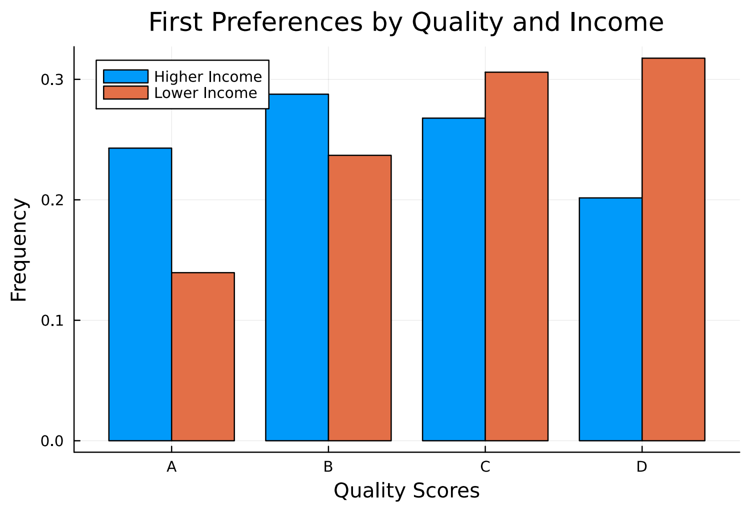


Figure 2: The distribution of quality of first-ranked schools, for families applying in 2019 for 9th grade.

There are variety of reasons why the gap might remain after the broad changes to the school system beginning in 2015. Lower income families may live further from higher-quality schools, and furthermore, may prefer to attend closer schools due to budget or time constraints. Another reason is that some families may lack information about school quality, or the returns to schooling. If they were better informed, they would apply to more high-quality schools. This hypothesis was explored using a randomized trial in Allende et al. (2019). The randomized intervention was a video and report card that provided information on nearby schools and a higher-level message on why it is important to choose a good school. The authors find that the intervention increases the proportion of lower-income families that apply to high-quality schools. By embedding their randomized trial in a structural model of school demand, supply, and centralized admissions, they find that the effect on allocations, taking into account capacity constraints, is substantially reduced, holding school capacity, prices, and quality fixed.

Data from the existing paper is not available. Instead, we estimate and perform inference on the effect of having informed parents on the access to quality education for low-income students using the methodology developed in this paper, and observational data on Chilean students and the centralized admission process available from the Ministry of Education from 2018-2020. We also find that information affects choices positively, but that capacity constraints in the school system at high-quality schools reduce the effect of the intervention significantly.

6.1 Data

We combine two datasets from the Ministry of Education for 2018 - 2020. For the admissions system, we use publicly available data on the centralized admissions process (SAE) for 2020 for those applying to the 9th grade in Chile. The process for school assignment in Chile occurs as follows. First, families apply to schools and the assignment algorithm is run. In 2018, over 80% of students accepted their assignment after the first round (Correa et al., 2019). Then, there is a second round of deferred acceptance for those who reject their assignment, where only schools with excess capacity are offered. Students unassigned in the second round are assigned to the nearest school without a copay with an available seat. Since most students are allocated in the first round (over 80%, in 2018), we focus on treatment effects where the outcome is the first round allocation. The data on the admissions process for the first round includes:

- The ranking of programs for each school that each student submits to the centralized mechanism
- Information on the priority of each student according to the rules of the admission system, including whether they have low-income priority [2]
- The location with error of each student and the location of each school
- The actual assignment of the student after the school assignment algorithm

For demographic data on students and school quality, we use student-level data collected for the standardized test in Chile, known as SIMCE. This data is available from the Ministry of Education for researchers by request. For school quality for the 9th grade admissions process, we use a rough measure which is the average student math and reading score for the school in 2018 amongst 10th graders. For demographic information, students in the admissions process for 2020 completed a standardized test in 2019 as 8th graders. The parents of over 80% of these students filled out an optional parent survey, which includes information on education level of the parents, their attitudes towards education, their income and household size, and their knowledge about their children's school quality. We are able to link these datasets using an anonymized identifier provided by the Ministry of Education.

[2] See Correa et al. (2019) for full details on the priorities and quotas in the Chilean admission system.

6.2 Treatment Effect Estimates

For our analysis on the effect of information on access to education, the treatment is the third question in the 30th section of the parent survey, which asks:
Do you know the following information about your child's school? Performance category of this school. [3]

$W_i = 1$ if the response to this question is Yes, and $W_i = 0$ if it is No or if the response or survey for the family is missing. The observed confounders are location (available for all applicants), and household size, mother and father education level, whether or not the mother and father are indigenous and the income of the family (available for those whose parents fulled out the SIMCE survey in 8th grade). Table 4 in Appendix D includes the mean and standard deviation for each of the variables. 53% of the sample of 114,749 applicants to 9th grade has $W_i = 1$.

Table 3 includes an estimate of treatment effects, where the outcomes are realized before schools are allocated, and so are not impacted by interference in our framework. We use an IPW approach with propensity score estimated using logistic regression to estimate these treatment effects. This table provides some evidence that information helps low income families submit applications that improve their chances at being allocated to better-quality schools. In the sample, 13.9% of low income families with $W_i = 0$ rank a top-quartile school first. Controlling for selection using the variables in Table 4, the estimated treatment effect of the school quality information on this ranking metric is 0.8 percentage points, which is a meaningful 5.5% increase.

The admissions system allows families to apply to as many schools as they want, so there are families in the dataset applying to up to 35 schools. However, the average number of schools ranked by low income families is only 3.5. If these families ranked additional schools, the allocations of the admissions system may improve. We find that the estimated ATE when the outcome is the number of schools in a families ranked list is positive, but small and insignificant.

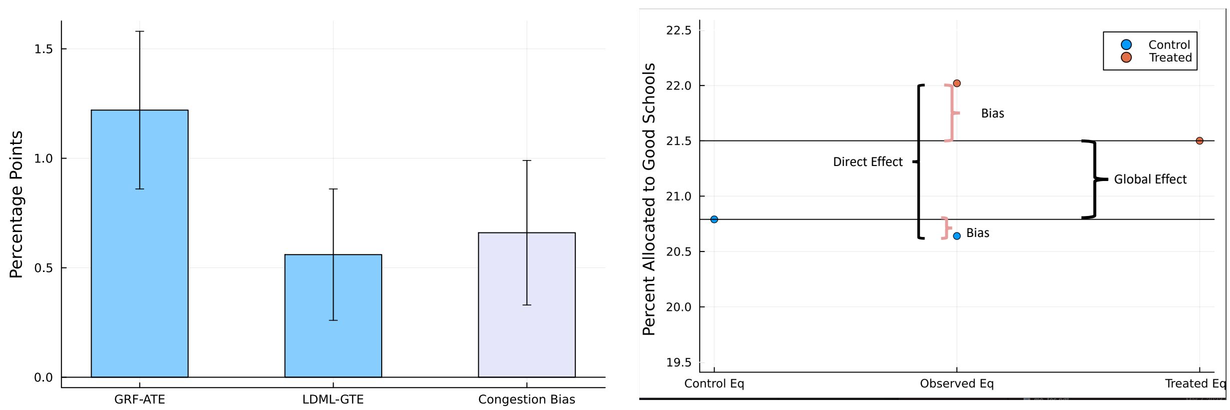
	Top Quartile School Ranked First	**Length of Application List**
$\hat{\tau}_{ATE}$	0.788%	0.018
	(0.282)	(0.022)

Table 3: IPW estimates of the Average Treatment Effect, on outcomes that are not affected by interference.

Figure 3a shows an estimate of treatment effects, when the outcome is whether a low income family is accepted to an above-average school in Chile. We see that the GRF-ATE estimator, which corrects for selection, but not equilibrium effects, estimates a nearly 1.5 percentage point increase in the allocation of low-income families to good quality schools. However, the LDML estimate of 0.46 percentage points is much lower, since it corrects for how much more difficult it is to access good quality schools in the counterfactual where everybody is treated, compared to when no families

[3]The survey language (in Spanish) is: ¿Conoce usted la siguiente información del colegio de su hijo(a)? Categoría de desempeño de este colegio.

receive information. Figure 3b provides a breakdown of the bias of the GRF-ATE estimator. At the observed equilibrium, the probability of admission to a good-quality school is higher than at the 100% treated equilibrium, and lower than that of the 0% treated equilibrium. Estimating τ_{GTE} accurately requires estimating the access of treated families at the treated equilibrium, and control families at the control equilibrium.



(a) Estimates of the treatment effect of informing parents about school quality on allocation of low-income families to good quality schools.

(b) The GRF-ATE estimator of the direct effect over-estimates the access of treated families to good-quality schools and under-estimates the access of control families.

Qualitatively, the estimates that we recover in observational data using the methodology in this paper match those reported in Allende et al. (2019), and computed using a randomized experiment in combination with a structural model.

7 Discussion

Without some structure, estimating general causal effects under interference requires data that is infeasible to observe. Under a fully specified and point-identified parametric model of individuals interacting in a market, any counterfactual can be simulated, but the model must be specified correctly. In this paper, in order to estimate treatment effects under interference, we use the structure implied by the existence of a centralized allocation mechanism, but remain non-parametric about individual choices, which can be difficult to specify correctly. This leads to a computationally simple estimator for the GTE that is doubly-robust and semi-parametrically efficient.

However, there are a variety of counterfactuals of interest that go beyond the estimands considered in this paper. These include settings with supply-side responses, outcomes that are a non-deterministic function of allocations, and mechanisms with strategic behavior, where individuals make choices conditional on their expectations of the market equilibrium. For these problems, exploring whether it is possible to derive robust estimators that combine non-parametric causal methodology with economic structure imposed by design will be an interesting avenue for future work.

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A Proofs of Main Results

A.1 Building Blocks

Lemma 7. ***Convergence of Counterfactual Cutoffs*** *Under Assumptions 1 - 2, then the market clearing cutoffs when $W_i = w$ for all $i \in [n]$ and $w \in \{0, 1\}$ converges in probability to p_w^*:*

$$P_w \to_p p_w^*.$$

Proof. We prove this lemma by verifying the conditions Theorem 5.9 of van der Vaart (1998).

First, the uniform convergence

$$\sup_{p \in \mathcal{S}} \left|\left| \frac{1}{n} \sum_{i=1}^{n} d(B_i(w), p) - \mathbb{E}[d(B_i(w), p)] \right|\right| \to_p 0$$

follows from Lemma 2.4 of Newey and McFadden (1994), since by assumption $d(B_i(w), p)$ is weakly continuous in p and bounded, and $\mathcal{S}$ is compact. Since we have that $\mathbb{E}[d(B_i(w), p)]$ is continuous in p, $\mathcal{S}$ is compact, and p_w^* is unique by assumption, then the second required condition holds (see for example Problem 5.27 of van der Vaart (1998)), for all $\epsilon > 0$:

$$\inf_{p: d(p, p_w^*) \geq \epsilon} ||\mathbb{E}[d(B_i(w), p) - q|| > 0 = ||\mathbb{E}[d(B_i(w), p_w^*)]||.$$

$\square$

Lemma 8. ***Asymptotic Normality of Counterfactual Cutoffs*** *Under Assumptions 1 - 2, then the market-clearing cutoffs when $W_i = w$ for all $i \in [n]$, which we call P_w for $w \in \{0, 1\}$, are asymptotically linear:*

$$\sqrt{n}(P_w - p_w^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (d(B_i(w), p_w^*) - \mathbb{E}[d(B_i(w), p_w^*)])$$

This implies that P_w is asymptotically normal:

$$\sqrt{n}(P_w - p_w^*) \to_D N(0, \Omega_w),$$

where $\Omega_w = \mathbb{E}[\nabla_p \mathbb{E}[d(B_i(w), p_w^*)]^{-1} (d(B_i(w), p_w^*) - q)(d(B_i(w), p_w^*) - q)^\top \nabla_p \mathbb{E}[d(B_i(w), p_w^*)]^{-1}]$.

Proof. We verify the conditions of Theorem 3.3.1 of van der Vaart and Wellner (1997) to prove this Lemma.

- By Lemma 7, $P_w \to_p p_w^*$.
- The finite sample market place approximately clears the market: $\frac{1}{n} \sum_{i=1}^{n} d(B_i(w), P_w) - q = o_p(n^{-1/2})$.

- From Lemma 9, we have the expansion,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} d(B_i(w), P(w)) - \mathbb{E}[d(B_i(w), P(w))] - (d(B_i(w), p_w^*) - \mathbb{E}[d(B_i(w), p_w^*)]) = o_p(1).$$

- By the CLT,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} d(B_i(w), p) - \mathbb{E}[d(B_i(w), p)] \to_D N(0, V^2)$$

  where $V^2 < \infty$ because $d(B_i(w), p)$ is bounded.

- $\mathbb{E}[d(B_i(w), p)])$ is twice continuously differentiable in p

- $\nabla_p \mathbb{E}[d(B_i(w), p_w^*)]$ is invertible by assumption

Now that the conditions are verified, the result of the Lemma follows directly from Theorem 3.3.1 of van der Vaart and Wellner (1997). $\square$

Lemma 9. *Let $F_i(p) = f(p, \theta_i)$ for random θ_i be a bounded random vector-valued function. The class of $f(p, \theta)$ indexed by $p \in \mathcal{S}$ is a Donsker function class. $\mathbb{E}[F_i(p)]$ is twice continuously differentiable in p with bounded derivatives, and $F_i(p)$ is continuous in p with probability 1. Let P_n be some random variable such that $P_n \to_p p^*$. Then, we have the following quadratic mean convergence:*

$$\mathbb{E}[(F_i(P_n) - F_i(p^*))^2] \to_p 0$$

The following expansion holds:

$$\frac{1}{n}\sum_{i=1}^{n} F_i(P_n) = \frac{1}{n}\sum_{i=1}^{n} F_i(p^*) + \mathbb{E}[F_i(P_n)] - \mathbb{E}[F_i(p^*)] + o_p(n^{-0.5}).$$

And, if we also have that $P_n = p^* + O_p(\sqrt{n})$ then the following expansion holds:

$$\frac{1}{n}\sum_{i=1}^{n} F_i(P_n) = \frac{1}{n}\sum_{i=1}^{n} F_i(p^*) + (P_n - p^*)^\top \nabla_p \mathbb{E}[F_i(p^*)] + o_p(n^{-0.5}).$$

Proof. This Lemma also appears in Munro et al. (2023). First, for the quadratic mean convergence, we show the function $\gamma(p) = \mathbb{E}[(F_i(p) - F_i(p^*))^2]$ for any p is continuous. Then, the result holds from the continuous mapping theorem. Write $F_i(p) = f(p, \theta_i)$ for random θ_i. Let the set of θ such that $f(p,\theta)$ is discontinuous at p be DC_p. The set of θ such that the function is continuous is C_p.

$$\begin{aligned}
\mathbb{E}[(F_i(p) - F_i(p^*))^2] &= \int_\theta [f(p,\theta) - f(p^*,\theta)]^2 p(\theta) d\theta \\
&= \int_{\theta \in C_p} [f(p,\theta) - f(p^*,\theta)]^2 p(\theta) d\theta - \int_{\theta \in DC_p} [f(p,\theta) - f(p^*,\theta)]^2 p(\theta) d\theta \\
&= \int_{\theta \in C_p} [f(p,\theta) - f(p^*,\theta)]^2 p(\theta) d\theta \\
&= \gamma(p)
\end{aligned}$$

We can drop the integral over the discontinuous functions because it is a sum of bounded terms that happen with zero probability. By the dominated convergence theorem, which lets us exchanged the limit and the integral since $f(p,\theta)$ is bounded, then we have that $\gamma(p)$ is continuous, since it is equal to the integral of functions each of which are continuous. We have now proved that the desired quadratic mean convergence holds.

Next, for the expansion. Given we have shown the quadratic mean convergence and that the function class is Donsker, we can use Lemma 19.24 of van der Vaart (1998) to show the first expansion directly.

$$\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n} F_i(P_n) - \mathbb{E}[F_i(P_n)]\right) - \left(\frac{1}{n}\sum_{i=1}^{n} F_i(p^*) - \mathbb{E}[F_i(p^*)]\right)\right] \to_p 0,$$

which is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n} F_i(P_n) = \frac{1}{n}\sum_{i=1}^{n} F_i(p^*) + \mathbb{E}[F_i(p)] - \mathbb{E}[F_i(p^*)] + o_p(n^{-0.5}).$$

For the final expansion. Since we have that $\mathbb{E}[F_i(p)]$ is twice continuously differentiable in p, we can use a Taylor expansion to write that

$$\mathbb{E}[F_i(P_n)] = \mathbb{E}[F_i(p^*)] + (P_n - p^*)\nabla_p \mathbb{E}[F_i(p^*)] + R_n$$

$R_n = o_p(n^{-0.5})$ since derivatives are bounded and $P_n - p^* = O_p(n^{-0.5})$. Plugging this into the first expansion, we have now shown that the second expansion holds:

$$\frac{1}{n}\sum_{i=1}^{n} F_i(P_n) = \frac{1}{n}\sum_{i=1}^{n} F_i(p^*) + (P_n - p^*)^\top \nabla_p \mathbb{E}[F_i(p^*)] + o_p(n^{-0.5}).$$

□

A.2 Proof of Proposition 1

To prove this proposition, we first prove Lemma 10.

Lemma 10. *Under Assumptions 1 - 2, then*

$$\sqrt{n}(\tau_w - \tau_w^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} y(B_i(W_i), p_w^*, X_i) - \mathbb{E}[y(B_i(W_i), p_w^*, X_i)] - \boldsymbol{\gamma}_w^\top (d(B_i(W_i), p_w^*) - q) + o_p(1)$$

Proof.

$$\begin{aligned}
\tau_w - \tau_w^* &= \frac{1}{n} \sum_{i=1}^{n} y(B_i(W_i), P_w, X_i) - \mathbb{E}[y(B_i(W_i), p_w^*, X_i)] \\
&\overset{(1)}{=} \frac{1}{n} \sum_{i=1}^{n} y(B_i(W_i), p_w^*, X_i) - \mathbb{E}[y(B_i(W_i), p_w^*, X_i)] + (P_w - p_w^*)^\top \nabla_p \mathbb{E}[y(B_i(W_i), p_w^*, X_i)] + o_p(n^{-0.5}) \\
&\overset{(2)}{=} \frac{1}{n} \sum_{i=1}^{n} y(B_i(W_i), p_w^*, X_i) - \mathbb{E}[y(B_i(W_i), p_w^*, X_i)] - \boldsymbol{\gamma}_w^\top (d(B_i(W_i), p_w^*) - q) + o_p(n^{-0.5})
\end{aligned}$$

(1) comes from Lemma 9 and (2) follows from Lemma 8.

□

Now, using Lemma 10 for both $\tau_1 - \tau_1^*$ and $\tau_0 - \tau^*$, we can expand $\tau_{GTE} - \tau_{GTE}^*$. Let $S_i^w = y(B_i(W_i), p_w^*, X_i) - \boldsymbol{\gamma}_w^\top (d(B_i(W_i), p_w^*))$.

$$\sqrt{n}(\tau_{GTE} - \tau_{GTE}^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (S_i^1 - S_i^0) - (\mathbb{E}[S_i^1] - \mathbb{E}[S_i^0])$$

By the CLT, we have that $\sqrt{n}(\tau_{GTE} - \tau_{GTE}^*) \to_D N(0, \mathbb{E}[Q_i^2])$ and $Q_i = (S_i^1 - S_i^0) - (\mathbb{E}[S_i^1] - \mathbb{E}[S_i^0])$.

A.3 Proof of Theorem 4

The proof follows uses the methodology presented in Bickel et al. (1993) and Newey (1990). The organization and notation of the proof is similar to other papers that apply this methodology to related estimands, including Hahn (1998) and Hirano et al. (2003) for average treatment effects, Firpo (2007) for quantile treatment effects, and Chen and Ritzwoller (2021) for long-run treatment effects. Our presentation and notation is closest to that of Firpo (2007).

Deriving the Score Function

Under Assumption 3, the density of the data $(B(1), B(0), W, X)$ can be factorized as:

$$\phi(b(1), b(0), w, x) = f(b(1), b(0), |x) e(x)^w (1 - e(x))^{1-w} f(x)$$

Under Assumption 3, the density of the observed data (B, W, X) can be factorized as:

$$\phi(b, w, x) = [f_1(b|x)e(x)]^w [f_0(b|x)(1 - e(x))]^{1-w} f(x).$$

where $f_1(b|x) = \int f(b_1, b_0|x)db_0$ and $f_0(b|x) = \int f(b_1, b_0|x)db_1$. We define a regular parametric submodel of the observed data density indexed by θ:

$$\phi(b, w, x; \theta) = [f_1(b|x; \theta)e(x; \theta)]^w [f_0(b|x; \theta)(1 - e(x; \theta))]^{1-w} f(x; \theta)$$

We can now derive the score of the parametric submodel:

$$s(b, w, x; \theta) = w \cdot s_1(b|x; \theta) + (1 - w) \cdot s_0(b|x; \theta) + \frac{w - e(x)}{e(x)(1 - e(x))} e'(x) + s_x(x; \theta)$$

where

$$\begin{aligned}
s_1(b|x; \theta) &= \frac{\partial}{\partial \theta} \log f_1(b|x; \theta) \\
s_0(b|x; \theta) &= \frac{\partial}{\partial \theta} \log f_0(b|x; \theta) \\
e'(x; \theta) &= \frac{\partial}{\partial \theta} \log e(x; \theta) \\
s_x(x; \theta) &= \frac{\partial}{\partial \theta} \log f(x; \theta)
\end{aligned}$$

The tangent space of this model is defined as the set of functions

$$g(r, w, x) = wg_1(b|x) + (1 - w)g_0(b|x) + (w - e(x))g_2(x) + g_3(x)$$

such that g_1 through g_3 range through all square integrable functions satisfying

$$\begin{aligned}
&\mathbb{E}[g_1(B_i|X_i)|X_i = x, W_i = 1] = 0 \\
&\mathbb{E}[g_0(B_i|X_i)|X_i = x, W = 0] = 0 \\
&\mathbb{E}[g_3(X_i)] = 0
\end{aligned}$$

Pathwise Differentiability

We next derive the pathwise derivative of $\tau_{GTE} = \tau_1 - \tau_0$, where $\tau_1 = \mathbb{E}[y(B_i(1), p_1)]$ and $\tau_0 = \mathbb{E}[y(B_i(0), p_0)]$. We go through the details for τ_1, and then state the result for τ_0, since the derivation follows the same steps.

$$\tau_1' = \nabla_p \mathbb{E}[y(B_i(1), p_1)]^\top p_1' + \frac{\partial}{\partial \theta} \int \int y(b, p_1) f_1(b|x; \theta) f(x; \theta) db dx \tag{7}$$

The next step is to derive p_1'. By Assumption 1, p_1 is defined implicitly by $\mathbb{E}[d(B_i(1), p_1) - q] = 0$. By the implicit function theorem, we can write

$$p_1' = -\nabla_p \mathbb{E}[d(B_i(1), p_1) - q]^{-1} \frac{\partial}{\partial \theta} \int \int (d(b, p_1) - q) f_1(b|x; \theta) f(x; \theta) dbdx.$$

The derivative of the moment conditions, evaluated at θ_0, are as follows, where we write $f(x; \theta_0) = f(x)$ and $f_1(b|x; \theta_0) = f_1(b|x)$.

$$\begin{aligned}
\frac{\partial}{\partial \theta} \int \int y(b, p_1) f_1(b|x; \theta) f(x; \theta) dbdx &= \int \int y(b, p_1) s_1(b|x) f_1(b|x) f(x) dbdx \\
&\quad + \int \int y(b, p_1) f_1(b|x) s_x(x) f(x) dbdx \\
\frac{\partial}{\partial \theta} \int \int (d(b, p_1) - q) f_1(b|x; \theta) f(x; \theta) dbdx &= \int \int (d(b, p_1) - q) s_1(b|x) f_1(b|x) f(x) dbdx \\
&\quad + \int \int (d(b, p_1) - q) f_1(b|x) s_x(x) f(x) dbdx
\end{aligned}$$

Plugging these into the Equation 7,

$$\tau_1' = \int \int q_1(b) s_1(b|x) f_1(b|x) f(x) dbdx + \int \int q_1(b) f_1(b|x) s_x(x) f(x) dbdx,$$

where $q_1(b) = y(b, p_1) - \gamma_1^\top (d(b, p_1) - q)$. Let $q_0(b) = y(b, p_0) - \gamma_0^\top (d(b, p_0) - q)$. After the same procedure for τ_0', we can write

$$\begin{aligned}
\tau_{GTE}' =& \int \int q_1(b) s_1(b|x) f_1(b|x) f(x) dbdx + \int \int q_1(b) f_1(b|x) s_x(x) f(x) dbdx \\
& - \int \int q_0(b) s_0(b|x) f_0(b|x) f(x) dbdx - \int \int q_0(b) f_0(b|x) s_x(x) f(x) dbdx. \\
=& \mathbb{E}[q_1(B_i(1)) s_1(B_i(1)|X_i)] + \mathbb{E}[\mu_1^q(X_i) s_x(X_i)]
\end{aligned}$$

Conjectured Efficient Influence Function

Let $\mu_1^q(X_i) = \mathbb{E}[q_1(B_i)|X_i, W_i = 1]$ and $\mu_0^q(X_i) = \mathbb{E}[q_1(B_i)|X_i, W_i = 0]$. First, a reminder that

$$\begin{aligned}
\boldsymbol{\gamma}_0 &= [\nabla_p \mathbb{E}[d(B_i(0), p_0)]]^{-1} \nabla_p \mathbb{E}[y(B_i(0), p_0, X_i)] \\
\boldsymbol{\gamma}_1 &= [\nabla_p \mathbb{E}[d(B_i(1), p_1)]]^{-1} \nabla_p \mathbb{E}[y(B_i(1), p_1, X_i)]
\end{aligned}$$

A function that is in the tangent space is:

$$\begin{aligned}
\psi(B_i, W_i, X_i) =& \mathbb{E}[q_1(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0(B_i)|X_i, W_i = 0] - \tau + \frac{W_i(q_1(B_i) - \mathbb{E}[q_1(B_i)|X_i, W_i = 1])}{e(x)} \\
& - \frac{(1 - W_i)(q_0(B_i) - \mathbb{E}[q_0(B_i)|X_i, W_i = 0])}{1 - e(x)}
\end{aligned}$$

We can verify it is in the tangent space.

1. $g_1(b|x) = \frac{q_1(b) - \mathbb{E}[q_1(B_i)|X_i = x, W_i = 1]}{e(x)}$. For any x,

$$\mathbb{E}[g_1(B_i|X_i)|X_i = x, W_i = 1)] = \frac{\mathbb{E}[q_1(B_i)|X_i = x, W_i = 1] - \mathbb{E}[q_1(B_i)|X_i = x, W_i = 1]}{e(x)} = 0.$$

2. $g_0(b|x) = \frac{q_0(b) - \mathbb{E}[q_0(B_i)|X_i = x, W_i = 0]}{1 - e(x)}$. For any x,

$$\mathbb{E}[g_0(B_i|X_i)|X_i = x, W_i = 0] = \frac{\mathbb{E}[q_0(B_i)|X_i = x, W_i = 0] - \mathbb{E}[q_0(B_i)|X_i = x, W_i = 0]}{1 - e(x)} = 0.$$

3. $g_2(x) = 0$

4. $g_3(x) = \mathbb{E}[q_1(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0(B_i)|X_i, W_i = 0] - \tau$

$$\begin{aligned}\mathbb{E}[g_3(X_i)] &= \mathbb{E}[\mu_1^q(X_i)] - \mathbb{E}[\mu_0^q(X_i)] - \mathbb{E}[\mu_1^q(X_i)] + \mathbb{E}[\mu_0^q(X_i)] \\ &= 0\end{aligned}$$

Given it is an element of the tangent space, if it is an influence function it is efficient. To verify that is an influence function, we must show that

$$\mathbb{E}[\psi(B_i, W_i, X_i)s(B_i, W_i, X_i)] = \tau'$$

We can divide $\psi(B_i, W_i, X_i) = \psi_1(B_i, W_i, X_i) - \psi_0(B_i, W_i, X_i)$, where

$$\psi_1(B_i, W_i, X_i) = \mathbb{E}[q_1(B_i)|X_i, W_i = 1] - \mathbb{E}[q_1(B_i)|W_i = 1] + \frac{W_i(q_1(B_i) - \mathbb{E}[q_1(B_i)|X_i, W_i = 1])}{e(x)}$$
$$\psi_0(B_i, W_i, X_i) = \mathbb{E}[q_0(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0(B_i)|W_i = 0] + \frac{(1 - W_i)(q_0(B_i) - \mathbb{E}[q_0(B_i)|X_i, W_i = 0])}{1 - e(x)}$$

We work through the details for $\psi_1(\cdot)$, since the process is the same for $\psi_0(\cdot)$.

$$\begin{aligned}
&\mathbb{E}[\psi_1(B_i, W_i, X_i)s(B_i, W_i, X_i)] \\
&= \mathbb{E}\left[(q_1(B_i(1)) - \mu_1^q(X_i))s_1(B_i(1)|X_i) + s_x(X_i)(q_1(B_i(1)) - \mu_1^q(X_i))\right] \\
&\qquad + \mathbb{E}[W_i s_1(B_i(1)|X_i) \cdot \mu_1^q(X_i) + (1 - W_i)s_0(B_i(0)|X_i) \cdot \mu_1^q(X_i) + s_x(X_i)\mu_1^q(X_i)] \\
&= \mathbb{E}[q_1(B_i(1))s_1(B_i(1)|X_i)] + \mathbb{E}[s_x(X_i)\mu_1^q(X_i)] + \mathbb{E}[(1 - e(X_i))\mathbb{E}[s_1(B_i(1)|X_i) - s_0(B_i(0)|X_i)|X_i = x]\mu_1^q(X_i)] \\
&\overset{(1)}{=} \mathbb{E}[q_1(B_i(1))s_1(B_i(1)|X_i)] + \mathbb{E}[s_x(X_i)\mu_1^q(X_i)] \\
&= \tau_1'
\end{aligned}$$

(1) is because $\mathbb{E}[s_w(B_i(w)|X_i)|X_i = x] = 0$ for each $x \in \mathcal{X}$ and $w \in \{0, 1\}$.

Similarly, we can show that $\mathbb{E}[\psi_0(B_i, W_i, X_i)s(B_i, W_i, X_i)] = \tau_0'$. We have now shown that

$$\mathbb{E}[\psi(B_i, W_i, X_i)s(B_i, W_i, X_i)] = \tau'.$$

Semi-Parametric Efficiency Bound

We have shown that the function $\psi(B_i, W_i, X_i)$ is an efficient influence function. The semi-parametric efficiency bound is

$$V^* = \mathbb{E}[\psi(B_i, W_i, X_i)^2],$$

We can show this matches the form in Theorem 4: Let $\mu_w^q(X_i) = \mathbb{E}[q_w(B_i)|X_i, W_i = w]$,

$$\begin{aligned}
V^* &= \mathbb{E}\left[\left(\mu_1^q(X_i) - \mu_0^q(X_i) - \tau_{GTE}^* + \frac{W_i(q_1(B_i) - \mu_1^q(X_i)}{e(X_i)} - \frac{(1-W_i)(q_0(B_i) - \mu_0^q(X_i)}{1-e(X_i)}\right)^2\right] \\
&= \mathrm{Var}\left[\mu_1^q(X_i) - \mu_0^q(X_i) + \frac{W_i(q_1(B_i) - \mu_1^q(X_i))}{e(X_i)} - \frac{(1-W_i)(q_0(B_i) - \mu_0^q(X_i))}{1-e(X_i)}\right] \\
&= \mathrm{Var}[\mu_1^q(X_i) - \mu_0^q(X_i)] + \mathbb{E}\left[\left(\frac{W_i(q_1(B_i) - \mu_1^q(X_i))}{e(X_i)}\right)^2\right] + \mathbb{E}\left[\left(\frac{(1-W_i)(q_0(B_i) - \mu_0^q(X_i))}{1-e(X_i)}\right)^2\right] \\
&= \mathrm{Var}[\mu_1^q(X_i) - \mu_0^q(X_i)] + \mathbb{E}\left[\frac{\mathbb{E}[W_i^2|X_i]}{e(X_i)^2}\mathbb{E}\left[\left(q_1(B_i(1)) - \mu_1^q(X_i)\right)^2|X_i\right]\right] \\
&\qquad - \mathbb{E}\left[\frac{\mathbb{E}[(1-W_i)^2|X_i]}{(1-e(X_i))^2}\mathbb{E}\left[\left(q_0(B_i(0)) - \mu_0^q(X_i)\right)^2|X_i\right]\right] \\
&= \mathrm{Var}[\mu_1^q(X_i) - \mu_0^q(X_i)] + \mathbb{E}\left[\frac{\mathbb{E}[\left(q_1(B_i(1)) - \mu_1^q(X_i)\right)^2|X_i]}{e(X_i)}\right] - \mathbb{E}\left[\frac{\mathbb{E}[\left(q_0(B_i(0)) - \mu_0^q(X_i)\right)^2|X_i]}{1-e(X_i)}\right]
\end{aligned} \tag{8}$$

The last equality is because $\mathbb{E}[W_i^2|X_i] = \mathbb{E}[W_i|X_i] = e(X_i)$. Since we have that $q_w(b) = y(b, p_0) - \gamma_1^\top(d(b, p_w) - q)$, this expression now matches the one in Theorem 4.

A.4 Proof of Theorem 5

The following assumption matches the assumptions of Theorem 3 of Kallus et al. (2019) under pointwise convergence, and with the notation slightly modified to better match the conventions in this paper.

Assumption 5. ***Assumptions of Theorem 3 of Kallus et al. (2019)*** *There exist positive constants c', C, and c_1 to c_7 such that for probability distribution $\mathbb{P}$, the following conditions hold:*

1. *We can write the true parameter vector θ^* as the solution to:*

$$\mathbb{E}[U(Y_i(w); \theta_1) + V(\theta_2)] = 0, \tag{9}$$

or as the difference in solutions to moment conditions of the form of Equation 9 for $w = 1$ and

$w = 0$.[4]. Under Assumption 3, showing the score function for $w = 1$, this can be transformed into the moment condition:

$$\mathbb{E}[\psi(Z_i; \theta_1, e(X_i), \mu^*(Z_i; \theta_2))] = 0$$

where

$$\begin{aligned}
&\psi(Z_i; \theta; e(X_i), \mu^*(X_i; \theta_1)) = \mu^*(X_i; \theta_1) + \frac{\mathbb{1}(W_i = 1)(U(Y_i; \theta_1) - \mu^*(X_i; \theta_1))}{e(X_i)} + V(\theta_2) \\
&\mu^*(X_i; \theta_1) = \mathbb{E}[U(Y_i(1); \theta_1)|X_i = x] \\
&e(X_i) = Pr(W_i = 1|X_i = x)
\end{aligned}$$

2. *(Strong Overlap). Assume that there exists a positive constant $\epsilon > 0$ such that $e(X_i) \geq \epsilon$ and $1 - e(X_i) \geq \epsilon$ almost surely.*

3. *For any sequence of constants $\Delta_n \to 0$, the nuisance estimates $(\hat{\mu}^{(k)}(\cdot; \hat{\theta}^{(k)}_{1,init}), \hat{e}^{(k)}(\cdot))$ belong to the realization set $\mathcal{T}_n$ for all $k = 1, \dots K$ with probability at least $1 - \Delta_n$. The estimated propensity $\hat{e}(X_i)$ satisfies strong overlap almost surely. For $w \in \{0, 1\}$,*

$$\begin{aligned}
&||(\mathbb{E}(\hat{\mu}^{(k)}(X_i; \hat{\theta}^{(k)}_{1,init}) - \mu^*(X_i; \hat{\theta}^{(k)}_{1,init}))^2)^{1/2}|| \leq \rho_{\mu,n} \\
&(\mathbb{E}(\hat{e}^{(k)}(X_i) - e(X_i))^2)^{1/2} \leq \rho_{e,n} \\
&||\hat{\theta}^{(k)}_{1,init} - \theta^*_1|| \leq \rho_{\theta,n},
\end{aligned}$$

where $\rho_{e,n}(\rho_{\mu,n} + C\rho_{\theta_n}) \leq \frac{\epsilon^3}{3}\delta_n n^{-1/2}$, $\rho_{e,n} \leq \frac{\delta_n^3}{\log n}$, $\rho_{\mu,n} + C\rho_{\theta,n} \leq \frac{\delta_n^2}{\log n}$, $\delta_n \leq \frac{4C^2\sqrt{d}+2\epsilon}{\epsilon^2}$, and $\delta_n \leq \min\left\{\frac{\epsilon^2}{8C^2 d}\log n, \sqrt{\frac{\epsilon^3}{2C\sqrt{d}}}\log^{1/2} n\right\}$. Furthermore, the nuisance realization set contains the true nuisance parameters $(\mu^*(\cdot; \theta^*_1), e(\cdot))$.

4. *The solution approximation error for the estimating equation satisfies $v_n \leq \delta_n n^{-1/2}$*

5. *Θ is a compact set and θ^* is in the interior of Θ*

6. *The map $(\theta, a, b) \mapsto \mathbb{E}[\psi(Z; \theta, a, b)]$ is twice continuously Gateaux-differentiable on $\theta \times T$.*

7. *The singular values of the covariance matrix Σ are bounded between constants c_5 and c_6:*

$$\Sigma = \mathbb{E}\left[J^{*-1}\psi(Z_i; \theta^*, \mu^*(Z_i; \theta^*_1), e(X_i))\psi(Z_i; \theta^*, \mu^*(Z_i; \theta^*_1), e(X_i))^\top J^{*-1}\right]$$

8. *For each $(\mu, e) \in T_n$ the function class $\mathcal{F}_{1,\eta} = \{z \mapsto \psi_j(z; \theta, \mu, e); j = 1, \dots, d_\theta, \theta \in \Theta\}$ is suitably measurable and its uniform covering entropy obeys*

$$\sup_Q \log n(\epsilon||\bar{F}_{1,\eta}||_{Q,2}, \mathcal{F}_{1,\eta}, ||\cdot||_{Q,2}) \leq v\log(a/\epsilon)$$

[4] See Remark 2 of Kallus et al. (2019)

for all $0 < \epsilon \leq 1$, where $\bar{F}_{1,\eta}$ is a measurable envelope for $\mathcal{F}_{1,\eta}$, that satisfies $||\bar{F}_{1,\eta}||_{P,q} \leq c_1$.

9. *For $j = 1, \dots, d$, $\theta \mapsto \mathbb{E}[U_j(Y(w); \theta_1) + V(\theta_2)]$ is differentiable at any θ in a compact set Θ, and each component of its gradient is c'-Lipschitz continuous at θ^*. Moreover, for any $\theta \in \Theta$ with $||\theta - \theta^*|| \geq \frac{c_3}{2\sqrt{d}c'}$, we have that $2||\mathbb{E}[U(Y(w); \theta_1) + V(\theta_2)]|| \geq c_2$.*

10. *The singular values of $\partial_{\theta^\top}\mathbb{E}[U(Y(w); \theta_1) + V(\theta_2)]|_{\theta=\theta^*}$ are bounded between c_3 and c_4*

11. *For any $\theta \in \mathcal{B}\left(\theta^*; \frac{4C\sqrt{d}\rho_{e,n}}{\delta_n \epsilon}\right) \cap \Theta$, $r \in (0,1)$ and $j = 1, \dots, d$, there exist $h_1(w, x; \theta_1$, $h_2(w, x; \theta_1)$ such that $\mathbb{E}[h_1(w, X; \theta_1)] < \infty$, and $\mathbb{E}[h_2(w, X; \theta_1)] < \infty$ and almost surely*

$$|\partial_r \mu_j^*(X; \theta_1^* + r(\theta_1 - \theta_1^*))| \leq h_1(w, X; \theta_1)$$
$$|\partial_r^2 \mu_j^*(X; \theta_1^* + r(\theta_1 - \theta_1^*))| \leq h_2(w, X; \theta_1)$$

12. *For $j = 1, \dots, d$ and any $\theta \in \Theta$, we have $(\mathbb{E}(\mu_j^*(X_i; \theta_1))^2)^{1/2} \leq C$*

13. *For $j = 1, \dots, d$ and any $\theta \in \mathcal{B}\left(\theta^*; \frac{4C\sqrt{d}\rho_{e,n}}{\delta_n \epsilon}\right) \cap \Theta$:*

$$\begin{aligned}
&\left\{\mathbb{E}[\mu_j^*(X_i; \theta_1) - \mu_j^*(X_i; \theta_1^*)]^2\right\}^{1/2} \leq C||\theta_1 - \theta_1^*|| \\
&|| \left\{\mathbb{E}[\partial_{\theta_1} \mu_j^*(X_i; \theta_1)]^2\right\}^{1/2} || \leq C \\
&\sigma_{\max}(\mathbb{E}[\partial_{\theta_1}\partial_{\theta_1^\top} \mu_j^*(X_i; \theta_1)]) \leq C \\
&\sigma_{\max}(\partial_{\theta_2}\partial_{\theta_2^\top} V_j(\theta_2)) \leq C
\end{aligned}$$

Theorem 11. ***Theorem 3 of Kallus et al. (2019)*** *Under Assumption 5, then*

$$\sqrt{n}(\hat{\theta} - \theta^*) = \frac{-1}{\sqrt{n}} \sum_{i=1}^{n} J^{*-1}\psi(Z_i; \theta^*, \mu^*, e) \to_D N(0, \Omega)$$

where $\Omega = J^{*-1}\mathbb{E}[\psi(Z_i; \theta^*, \mu^*, e)\psi(Z_i; \theta^*, \mu^*, e)'](J^{*-1})'$ and $J^* = \nabla_\theta \mathbb{E}[\psi(Z_i; \theta_1^*, e(X_i), \mu^*(Z_i; \theta_2^*))]]_{\theta=\theta^*}$.

Proof. First, we show that each component of Assumption 5 is implied by Assumptions 1 - 4.

1. Mapping to the notation of Kallus et al. (2019), we have $\theta_2^* = \tau_{GTE}^*$ and $\theta_1^* = (p_0^*, p_1^*)$. The observed data is $Z_i = (B_i(W_i), X_i, W_i)$. We can write $\tau_{GTE}^* = \tau_1^* - \tau_0^*$.

$$0 = \mathbb{E}[y(B_i(w), p_w^*, X_i)] - \tau_w^*$$
$$0 = \mathbb{E}[d(B_i(w), p_w^*) - q]$$

τ_w^* takes the form of Equation 9, with $V(\theta_2) = \left(-\tau_w^*, 0\right)$ and

$$U(Y_i(w); \theta_1) = \left(y(B_i(w), p_w, X_i), d(B_i(w), p_w, X_i) - q\right)$$

2. This holds by Assumption 3

3. This holds by Assumption 4, with $\delta_n = o(1)$

4. This holds by the second part of Assumption 1, with $\delta_n = o(1)$.

5. Since Y_i is bounded, we can define compact set on the real line that includes the maximum and minimum possible value of the outcome in the interior of that set. We assume that $\mathcal{S}$ is compact, and that p_0 and p_1 lie in the interior of that set in Assumption 2.

6. For the derivative with respect to p_0 and p_1, we show for p_1 only. The first element of $\mathbb{E}[\psi_1(Z_i; p_1, a, b)]$ is:

$$\mathbb{E}\left[b + \frac{W_i(y(B_i(1), p_1, X_i) - b)}{a}\right]$$
$$= b + \frac{\mathbb{E}[e(X_i)\mu^y(p_1, X_i)]}{a}$$

This term is linear in b, so is twice continuously differentiable in b. It is also twice continuously differentiable in a when $a \neq 0$. We can swap derivatives and expectations by the dominated convergence theorem since Y_i is bounded. The expectation is twice differentiable in p since in Assumption 2, we have assumed that $\mu^y(p, x)$ is twice continuously differentiable in p. The argument is the same for the other components of $\psi_1(Z_i; p_1, a, b)$ and $\psi_0(Z_i; p_0, a, b)$.

7. Since we are analyzing non-uniform convergence, following Remark 2 of Kallus et al. (2019) we can relax this assumption to just ensuring that c_6 is finite. This is the case when $d(B_i(w), p)$ and $y(B_i(w), p, X_i)$ are bounded.

8. Since the score function is a linear combination of $y(B_i(w), p, X_i)$ and $d(B_i(w), p)$, by the composition rules of Donsker classes, then this holds by the Donsker assumption in Assumption 1.

9. The first part of this holds by the continuous differentiability of $\mu_w^d(p, x)$ and $\mu_w^y(p, x)$ in p with bounded first derivative. The second part holds by uniqueness of p_0^* and p_1^*. If we have have $||\tilde{\theta} - \theta^*|| > 0$, but $\mathbb{E}[U(Y_i(w), \tilde{\theta}_1) + V(\tilde{\theta}_2)] = 0$, then it must be that $\tilde{\theta}$ also satisfies the score condition. This means that p_0^* and p_1^* do not uniquely satisfy $\mathbb{E}[d(B_i(1), p)] - q = 0$ and $\mathbb{E}[d(B_i(0), p)] - q = 0$.

10.
$$J^* = \nabla_\theta \mathbb{E}[\psi(Z_i; \theta, \mu(\theta), e(X_i))]_{\theta=\theta^*} = \begin{bmatrix} 1 & -\nabla_p \mu_0^y(p) & \nabla_q \mu_1^y(q) \\ 0 & 0 & \nabla_q \mu_1^d(q) \\ 0 & \nabla_p \mu_0^d(p) & 0 \end{bmatrix} \Bigg|_{p=p_0^*, q=p_1^*}$$

Inverting this,

$$J^{*-1} = \begin{bmatrix} -1 & \nabla_q \mu_1^y(q)[\nabla_q \mu_1^d(q)]^{-1} & -\nabla_p \mu_0^y(p)[\nabla_p \mu_0^d(p)]^{-1} \\ 0 & 0 & [\nabla_p \mu_0^d(p)]^{-1} \\ 0 & [\nabla_q \mu_1^d(p)]^{-1} & 0 \end{bmatrix} \Bigg|_{p=p_0^*, q=p_1^*} \tag{10}$$

The inverted matrices exist under the invertibility condition in Assumption 2.

11. This condition holds since we have assumed that the first and second derivatives of $\mu_w^d(p,x)$ and $\mu_w^y(p,x)$ are bounded in Assumption 2.

12. This condition holds since $y(b(w), p, x)$ and $d(b(w), p)$ is bounded.

13. (a) This holds since $\mu_j^d(p,x)$ and $\mu^y(p,x)$ are continuously differentiable in p with a bounded first derivative.

    (b) This holds since $\mu_j^d(p,x)$ and $\mu^y(p,x)$ are continuously differentiable in p with a bounded first derivative.

    (c) For the third condition, this holds since $\mu_j^d(p,x)$ and $\mu^y(p,x)$ are twice continuously differentiable in p with a bounded second derivative.

    (d) For the last condition, $V(\theta_2) = -\theta_2$. The second derivative is 0, so this holds for any $C > 0$.

Now that we have verified Assumption 5, then Theorem 11 holds, for $\hat{\tau}_{GTE}$.This assumption implies that the asymptotic variance of $\hat{\tau}_{GTE}$ is equal to the $[1,1]$-th element of the matrix Ω

$$\Omega = J^{*-1}\mathbb{E}\begin{bmatrix} [\phi_{\tau,i}]^2 & \phi_{\tau,i}\phi_{1,i} & \phi_{\tau,i}\phi_{0,i} \\ \phi_{1,i}\phi_{\tau,i} & \phi_{1,i}\phi_{1,i}^\top & \phi_{1,i}\phi_{0,i}^\top \\ \phi_{0,i}\phi_{\tau,i} & \phi_{0,i}\phi_{1,i}^\top & \phi_{0,i}\phi_{0,i}^\top \end{bmatrix} [J^{*-1}]^\top$$

where J^{*-1} is given in Equation 10 and $\phi_{\tau,i}$, $\phi_{1,i}$ and $\phi_{0,i}$ are the first, second, and third elements of the vector $\psi^{DR}(Z_i; p^*, e(X_i), \mu(X_i; p^*)] - v(\tau^*_{GTE})$. By evaluating the matrix product, the $[1,1]th$ element of this matrix is equal to the V^* given in Theorem 4. Showing this explicitly, we can write the first row of $[J^*]^{-1}$ as $\begin{bmatrix} -1 & a_2 & a_3 \end{bmatrix}$ and the symmetric matrix is $\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$.

$$\begin{aligned}
\Omega_{1,1} &= \mathbb{E}[b_{11} - 2a_2 b_{12} - 2a_3 b_{13} + a_2^2 b_{22} + a_3^2 b_{33} + 2a_2 a_3 b_{23}] \\
&= \mathbb{E}[\phi_{\tau,i}^2 - 2\gamma_1^\top \phi_{1,i}\phi_{\tau,i} + 2\gamma_0^\top \phi_{0,i}\phi_{\tau_i} + (\gamma_1^\top \phi_{1,i})^2 + (\gamma_0^\top \phi_{0,i})^2 - 2\gamma_1^\top \phi_{1,i}\gamma_0^\top \phi_{0,i}] \\
&= \mathbb{E}[(\phi_{\tau,i} - \gamma_1^\top \phi_{1,i} + \gamma_0^\top \phi_{0,i})^2] \\
&= \mathbb{E}\left[\left(\mu_1^q(X_i) - \mu_0^q(X_i) - \tau^*_{GTE} + \frac{W_i(q_1(B_i) - \mu_1^q(X_i)}{e(X_i)} - \frac{(1-W_i)(q_0(B_i) - \mu_0^q(X_i)}{1-e(X_i)}\right)^2\right]
\end{aligned}$$

where $q_w(b) = y(b, p_0) - \gamma_1^\top(d(b, p_w^*) - q)$ and $\mu_w^q(X_i) = \mathbb{E}[q_w(B_i)|X_i, W_i = w]$. This now matches the first line of Equation 8 in the proof of the efficiency result, which shows that the variance of the $LDML$ estimator matches that of the efficient score.

An analytical characterization for other elements of the covariance of the vector $\begin{bmatrix} \hat{\tau} & \hat{p}_1 & \hat{p}_0 \end{bmatrix}^\top$ are also available by computing the matrix product for other elements of Ω.

□

A.5 Proof of Proposition 2

Proof. The first two parts of Assumption 1 are discussed in the text. The third holds by assumption. The fourth holds for the demand function since the function class is a class of indicator functions, which is a Donsker class.

Since $V_i(W_i)$ is bounded, we can define $\mathcal{S}$ as $[V^- - \epsilon, V^+ + \epsilon]$ for some $\epsilon > 0$, where V^- is the minimum possible value of V_i and V^+ is the maximum. This is a compact set, and for any $P^+ > V^+$, $d(V_i(W_i), P^+) = 0$, and for any $P^- < V^-$, $d(V_i(W_i), P^-) = 1$. $D_n(p) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(V_i(W_i) > p)$ is weakly monotonic on p. As long as $0 < q < 1$, then any market-clearing price will be in $\mathcal{S}$.

For Assumption 2, we first derive the form of $\mu_w^d(p, x)$.

$$\mathbb{E}[\mathbb{1}(V_i(w) > p | X_i = x)] = 1 - F_{v(w)|x}(p|x)$$

The unconditional distribution $F_{v(w)} = \int F_{v(w)|x} dF_x(x)$. Under the strict monotonicity assumption, then for $q \in (0, 1)$ p_w^* is the unique solution defined as $p_w^* = F_{v(w)}^{-1}(1 - q)$. By the definition of $\mathcal{S}$, at the boundaries of $\mathcal{S}$, the distribution function is either 0 or 1. So, for $q \in (0, 1)$, then p_w^* is always in the interior of $\mathcal{S}$.

The third part of Assumption 2 is satisfied by assumption, given we can express $\mu_w^d(p, x)$ in terms of the conditional distribution $F_{v(w)|x}(p|x)$.

Last, we have that $\nabla_p \mathbb{E}[d(B_i(w), p)] = -f_{v(w)}(p)$. This is invertible by the strict monotonicity of $F_{v(w)}$, which implies that $f_{v(w)}(p_w^*) \neq 0$ for $w \in \{0, 1\}$.

□

A.6 Proof of Proposition 3

This is an extension of Proposition 2.

Proof. The first two parts of Assumption 1 are discussed in the text. The third holds by assumption. The fourth holds for the demand function since the function class is a class of indicator functions, which is a Donsker class.

Since S_i is bounded, we can define $\mathcal{S}$ as $[S^- - \epsilon, S^+ + \epsilon]^J$ for some $\epsilon > 0$, where S^- is the minimum possible value of S_i and S^+ is the maximum. This is a compact set, and for any $P^+ > S^+$, $d(V_i(W_i), P^+) = 0$, and for any $P^- < S^-$, $d(V_i(W_i), P^-) = 1$. $D_{jn}(p) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{S_{ij} > p_j, jR_i(W_i)0\} \prod_{j \neq j'} \mathbb{1}(jR_i(W_i)j' \text{ or } S_{ij'} < p_{j'})$ is weakly monotonic in p_j. Thus, as long as $0 < q < 1$, then any market-clearing price will be in $\mathcal{S}$.

For Assumption 2, we first derive the form of $\mu^d_{j,w}(p,x)$. For a given ranking r, let $a(r,j)$ define the set of schools ranked above school j.

$$\mathbb{E}[d_j(B_i(w),p)|X_i=x] = \sum_{r\in\mathcal{R}} Pr(r|w)(1-F_{s|x}(p_j)) \prod_{j'\in a(r,j)} F_{s|x}(p_{j'})$$

Since lottery numbers are assigned independently for each school, the probability that an individual is assigned to school j takes a simple form in terms of conditional distributions of the lottery number. The unconditional distribution $F_s = \int F_{s|x} dF_x(x)$.

For uniqueness, we use Proposition C.4 of Agarwal and Somaini (2018). Using this proposition requires showing that $\mu^d_j(p)$ is strictly decreasing in p^*_j. This is the case, since $\mu^d_j(p)$ depends on p^*_j only through $F_s(p^*)$, which is strictly monotonic.

The third part of Assumption 2 is satisfied by assumption, given we can express $\mu^d_w(p,x)$ in terms of products of the conditional distribution $F_{s|x}(s|x)$, and the conditional distribution meets the required smoothness assumptions.

Lastly, we can also use Proposition C.4 of Agarwal and Somaini (2018) for the invertibility assumption, as long as $\mu^d_j(p)$ is continuous in p. This holds since it is twice continuously differentiable in p, by the twice differentiability of $F_s(s)$. □

A.7 Proof of Proposition 6

Proof. The first step is to show that

$$\partial V(\pi;h) = \int h(x)(\tau^*_{CADE}(x,\pi) + \tau^*_{CAIE}(x,\pi))dF(x)$$

where $\partial V(\pi;h)$ is the Gateaux derivative of $V(\pi)$ in the direction of $h \in \Pi$. First, we write $V(\pi)$ as an integral over x:

$$\begin{aligned} V(\pi) &= \mathbb{E}[\pi(X_i)(Y_i(1,p^*_\pi) - Y_i(0,p^*_\pi))] \\ &= \int \tau^*_{CADE}(x,\pi)\cdot\pi(x)dF(x). \end{aligned}$$

We can derive the Gateaux derivative of $V(\pi)$ using the product rule:

$$\begin{aligned} \partial V(\pi;h) &= \lim_{\delta\to 0} \frac{\int \tau^*_{CADE}(x,\pi+\delta h)\cdot[\pi(x)+\delta h(x)]dF(x) - \int \tau^*_{CADE}(x,\pi)\cdot\pi(x)dF(x)}{\delta} \\ &= \lim_{\delta\to 0}\int \tau^*_{CADE}(x,\pi+\delta h)\cdot h(x)dF(x) + \lim_{\delta\to 0}\frac{\int(\tau^*_{CADE}(x,\pi+\delta h) - \tau^*_{CADE}(x,\pi))dF(x)}{\delta} \\ &\overset{(1)}{=} \int \tau^*_{CADE}(x,\pi)\cdot h(x)dF(x) - \int \nabla_p\tau^*_{CADE}(x,\pi)dF(x)\cdot\nabla_p\mu^d(p^*_\pi)^{-1}\cdot\int h(x)\tau^d_{CADE}(x,\pi)dF(x) \\ &= \int \tau^*_{CADE}(x,\pi)\cdot h(x)dF(x) + \int \tau^*_{CAIE}(x,\pi)\cdot h(x)dF(x) \\ &= \int h(x)(\tau^*_{CADE}(x,\pi) + \tau^*_{CAIE}(x,\pi))dF(x) \end{aligned}$$

Step (1) is from the chain rule, and since the Gateaux derivative

$$\begin{aligned}\partial p^*(\pi; h) &= -\nabla_p \mathbb{E}[\pi(X_i)d(B_i(1), p_\pi^*) - d(B_i(0), p_\pi^*)]^{-1} \int h(x)\mathbb{E}[\pi(X_i)d(B_i(1), p_\pi^*) - d(B_i(0), p_\pi^*)|X_i = x] \\ &= -\nabla_p \mu^d(p_\pi^*)^{-1} \cdot \int h(x)\tau_{CADE}^d(x, \pi)dF(x)\end{aligned}$$

Since the vector space Π is convex, Theorem 2 of Chapter 7 of Luenberger (1969) indicates that a necessary condition for a local maximum π^* is that for all $\pi \in \Pi$,

$$\partial V(\pi; \pi - \pi^*) \leq 0$$

The remaining steps in the proof follows the proof of Theorem 1 in Munro et al. (2023). Let $\rho(\pi, x) = f(x)(\tau_{CADE}^*(x, \pi) + \tau_{CAIE}^*(x, \pi))$. We can prove by contradiction that the optimal targeting policy must meet the conditions in the theorem. If there is some $\bar{\pi}$ that is optimal but does not meet the conditions in the theorem, then, one of the following must be true:

1. For x in some set Q that occur with non-zero probability, $\rho(\bar{\pi}, x) < 0$ but $\bar{\pi}(x) > 0$. But then choose π such that $\pi(x) = \bar{\pi}(x)$ for $x \notin Q$ and $\pi(x) = 0$ for $x \in Q$. We have that

$$\partial V(\pi; \pi - \pi^*) = \int_{x \in Q} \rho(\bar{\pi}, x)(0 - \bar{\pi}(x))d\mu(x) > 0,$$

which contradicts the optimality of $\bar{\pi}$.

2. Or, for x in some set P that occurs with non-zero probability, $\rho(\bar{\pi}, x) > 0$ but $\bar{\pi}(x) < 1$. Choose π such that $\pi(x) = \bar{\pi}(x)$ for $x \notin P$ and $\pi(x) = 1$ for $x \in P$. We have that

$$\partial V(\pi; \pi - \pi^*) = \int_{x \in Q} \rho(\bar{\pi}, x)(1 - \bar{\pi}(x))d\mu(x) > 0,$$

which contradicts the optimality of $\bar{\pi}$.

□

B Using IV for Identification and Estimation

This section provides a brief discussion of how the setting in the paper is affected when unconfoundedness does not hold, but there is a binary instrumental variable that affects take-up of a binary treatment. A more complete statistical analysis of treatment effects under equilibrium-type interference with instrumental variables is reserved for future work. In an IV setting, we have potential treatments $W_i(1)$ and $W_i(0)$ that depend on an instrument $Z_i \in \{0, 1\}$. Under a monotonicity assumption, $W_i(1) > W_i(0)$. Under interference, there are a variety of counterfactuals that can be defined. One relevant counterfactual when there may be control over the instrument, but not the treatment directly, is the intent-to-treat effect. This is the effect on average outcomes in the sample when all individuals receive the instrument, compared to a setting where no agents receive the instrument. It can be written in this setting with interference as

$$\begin{aligned}\tau_{LGTE} =& \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(W_i(1) > W_i(0))[y(B_i(1), \tilde{P}_1, X_i) - y(B_i(0), \tilde{P}_0, X_i)] \\ &+ \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(W_i(1) = W_i(0))[y(B_i(0), \tilde{P}_1, X_i) - y(B_i(0), \tilde{P}_1, X_i)]\end{aligned}$$

where $\tilde{P}_1$ and $\tilde{P}_0$ are defined as

$$\begin{aligned}o_p(n^{-1/2}) =& \frac{1}{n}\sum_{i=1}^{n}[\mathbb{1}(W_i(1) > W_i(0))d(B_i(1), \tilde{P}_1, X_i) + \mathbb{1}(W_i(1) = W_i(0))d(B_i(0), \tilde{P}_1, X_i) - q] \\ o_p(n^{-1/2}) =& \frac{1}{n}\sum_{i=1}^{n}[d(B_i(0), \tilde{P}_0, X_i) - q]\end{aligned}$$

When the market-clearing cutoffs are determined by the aggregate behavior of everyone, then outcomes of compliers are affected directly by the treatment and indirectly by the change in the equilibrium. The outcomes of those who do not take up the treatment, however, are also affected by the changes in preferences of the compliers, due to the equilibrium effect. Using the techniques in the proof of Proposition 1, we can show that this corresponds to the following moment condition problem with missing data. Let $C_i = W_i(1) > W_i(0)$.

$$\begin{aligned}0 &= \tau^*_{GITT} - Pr(C_i = 1)\mathbb{E}[y(B_i(1), \tilde{p}_1, X_i) - y(B_i(0), \tilde{p}_0, X_i)|C_i = 1] - \\ &\qquad Pr(C_i = 0)\mathbb{E}[y(B_i(0), \tilde{p}_1, X_i) - y(B_i(0), \tilde{p}_0, X_i)|C_i = 0] \\ 0 &= Pr(C_i = 1)\mathbb{E}[d(B_i(1), \tilde{p}_1, X_i) - q|C_i = 1] + Pr(C_i = 0)\mathbb{E}[d(B_i(0), \tilde{p}_1, X_i) - q|C_i = 0] \\ 0 &= \mathbb{E}[d(B_i(0), \tilde{p}_0, X_i) - q]\end{aligned}$$

The Local Average Treatment Effect (Imbens and Angrist, 1994) -type quantities in this moment equation can be identified and estimated using standard IV assumptions: overlap, instrumental relevance, and exogeneity. For example, $\mathbb{E}[y(B_i(1), \tilde{p}_1, X_i)|W_i(1) > W_i(0)]$ is a moment that matches the form of Equation 19 in Appendix A of Kallus et al. (2019). Under the IV identifying assumptions, including monotonicity, then a Neyman orthogonal estimation equation for this moment is given by Equation 22 of Appendix A of the paper. As shown in that Appendix, the LDML estimation approach with three-way data splitting can be used for an asymptotically normal estimate of this expectation, and nuisance parameter estimation only requires estimating a simple set of regressions using flexible machine-learning estimates, as in the case with unconfoundedness.

Another possibility, which requires a strong assumption, is to assume that the distribution of $B_i(1), B_i(0)|C_i = 1$ is equal to the distribution of $B_i(1), B_i(0)|C_i = 0$.[5] Then, τ^*_{GTE} can be

[5] This is true if compliance is random in the population. It is likely possible to weaken this assumption in favor of a treatment effect homogeneity assumption that holds conditional on X_i, see the discussion in Athey and Wager (2021).

estimated, rather than τ^*_{LGTE}.

$$
\begin{aligned}
0 &= \tau^*_{GTE} - \mathbb{E}[y(B_i(1), p_1^*, X_i) - y(B_i(0), p_0^*, X_i)|C_i = 1] \\
0 &= \mathbb{E}[d(B_i(1), p_1^*, X_i)|C_i = 1] - q \\
0 &= \mathbb{E}[d(B_i(0), p_0^*, X_i)|C_i = 1] - q
\end{aligned}
$$

This set of moment conditions fits directly into the framework of Appendix A of Kallus et al. (2019).

C Simulation Details

The data generating process for the coverage simulation in Section 5.2 is described in this section. The fractional capacities of the schools are $q = [0.25, 0.25, 1.0]$. Schools 1 and 2 are high-quality, with $Q_j = 1$, and capacity constrained, but school 3, which is low quality, with $Q_j = 0$, is not. The subgroup of interest for the planner is denoted by $C_i \in \{0, 1\}$. The match value $V_{ij} = 2$ if $C_i = 1$ and $Q_j = 1$, and $V_{ij} = 1$ if $C_i = 0$ and $Q_j = 0$, otherwise it is 0. The covariates X_i that are observed for each individual are 5 standard normal variables, which are $X_{j,i}$ from $j = 1 \dots 5$, and the indicator C_i. Let $\Phi(\cdot)$ be the standard Normal CDF. The subgroup indicator is

$$C_i \sim \text{Bernoulli}(\Phi(1 + X_{3,i}))$$

Those with $C_i = 1$ have a lower mean utility for the treatment in the absence of treatment. $\mu_L = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}^\top$ and $\mu_H = \begin{bmatrix} 1.0 & 0.5 & 0.0 \end{bmatrix}^\top$. The vector of utilities of individual i for the schools $j \in \{1, 2, 3\}$ is:

$$U_i = C_i \mu_L + (1 - C_i)\mu_H + C_i W_i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + X_{2,i}^\top \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix} + \epsilon_i$$

where ϵ_i is a three-dimensional vector of standard normal variables. The treatment raises the probability that an individual with $C_i = 1$ applies to a high-quality school. The students each submit a ranking $R_i(W_i)$ over the three schools to the mechanism based on the order of their utilities U_i. The score for each individual and each school is $S_{ij} \sim \text{Uniform}(0, 1)$, so in the notation of the general setup, $B_i(W_i) = \{R_i(W_i), S_i)\}$. The noise term $v_i \sim \text{Bernoulli}(0.5)$. Finally, the treatment allocation and outcome generation, which obeys selection-on-observables, is as follows

$$
\begin{aligned}
W_i &\sim \text{Bernoulli}(0.5X_{3,i} - 0.5X_{2,i} + v_i) \\
Y_i(\boldsymbol{W}) &= \sum_{i=1}^{n} d(B_i(W_i), \boldsymbol{P}) V_{ij}
\end{aligned}
$$

Where $d(\cdot)$ is according to the deferred acceptance, which determines the cutoffs $\boldsymbol{P}$ that clear the market for schools.

D Empirical Details

Variable	**Treated**	**Control**
income	4.22	4.77
	(3.32)	(3.82)
ma_educ	11.01	11.46
	(3.14)	(3.14)
pa_educ	10.99	11.45
	(3.45)	(3.45)
ma_indig	0.18	0.17
	(0.38)	(0.37)
pa_indig	0.15	0.14
	(0.35)	(0.35)
hhsize	2.45	2.46
	(1.29)	(1.27)
latitude	-34.36	-34.15
	(4.90)	(5.04)
longitude	-71.47	-71.37
	(1.02)	(1.03)

Table 4: Summary Statistics for $n = 114,749$ applicants to 9th grade in 2020. $W_i = 1$ indicates a parent reported they were aware of the performance category of the 8th grade school of their child. Income is in \$100,000 pesos, and education is in years.