Causal Inference under Interference through Designed Markets [*]

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Abstract

Equilibrium effects make it challenging to evaluate the impact of an individual-level treatment on outcomes in a single market, even with data from a randomized trial. In some markets, however, a centralized mechanism allocates goods and imposes useful structure on spillovers. For a class of strategy-proof "cutoff" mechanisms, we propose an estimator for global treatment effects using individual-level data from one market, where treatment assignment is unconfounded. Algorithmically, we re-run a weighted and perturbed version of the mechanism. Under a continuum market approximation, the estimator is asymptotically normal and semi-parametrically efficient. We extend this approach to learn spillover-aware treatment rules with vanishing asymptotic regret. Empirically, adjusting for equilibrium effects notably diminishes the estimated effect of information on inequality in the Chilean school system.

Keywords: Econometrics, School Choice, Empirical Analysis of Markets

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1 Introduction

An individual-level intervention in an economic system rarely affects agents in isolation. Interactions among market participants leads to spillover effects, where the treatment of one individual affects the outcomes of others. Spillover effects make it challenging to estimate global treatment effects, such as the difference in expected outcomes when everyone is treated compared to when nobody is treated ($\bar{\tau}_{\text{GTE}}$). Existing approaches in the causal inference literature assume either partial interference, where there are no spillovers across clusters of agents (Baird et al. 2018, Hudgens & Halloran 2008), or that spillovers occur through an observed network where connections between agents are sparse (Aronow & Samii 2017, Leung 2020). Except for parametric model-based approaches, there has been little progress in estimating global treatment effects under complete interference, where the treatment of each individual may impact outcomes of anyone else. The main contribution of this paper is showing that in markets where spillover effects are mediated by a specific class of centralized allocation mechanisms, even though there is complete interference, semi-parametric estimation of global effects is possible.

Settings where a centralized mechanism allocates scarce items are increasingly common in practice. In the U.S., versions of the deferred acceptance algorithm allocate students to schools (Abdulkadiroğlu & Sönmez 2003), and medical school graduates to residency programs (Roth 2003). Auctions allocate advertisements to search queries (Varian & Harris 2014) and Treasury bonds to investors (McMillan 2003). Often, policymakers are interested in estimating how an intervention that affects the reported preferences of market participants will impact resulting allocations from the mechanism. For example, Allende et al. (2019) provide information about school quality to families in a randomized experiment in Chile, where a centralized mechanism determines allocations to schools. One of their target estimands is $\bar{\tau}_{\text{GTE}}$, where the treatment is the information intervention and the outcome is the allocation of low-income families to high quality schools.

Standard causal approaches, such as the differences-in-means estimator of the Average Treatment Effect, do not estimate $\bar{\tau}_{\text{GTE}}$, even when the treatment is randomly assigned. By increasing the number of applicants to schools with limited capacity, the treatment affects admissions probabilities, which introduces spillovers and violates the Stable Unit Treatment Value Assumption (SUTVA) (Heckman et al. 1998). To estimate $\bar{\tau}_{\text{GTE}}$, Allende et al. (2019) use data from the experiment to estimate a parametric structural model of reported preferences over schools, and simulate the relevant counterfactuals using this model and the centralized mechanism. This paper proposes a new approach for estimating $\bar{\tau}_{\text{GTE}}$, which does not require correctly specifying a parametric model of individual behavior, and can be used with a variety of mechanisms and treatments. The estimator runs a re-weighted and perturbed version of the centralized mechanism on the observed submissions to the mechanism, where weights and perturbations are estimated using flexible machine learning methods.

We begin with a potential outcomes model that allows for complete interference. We then make three major restrictions under which $\bar{\tau}_{\text{GTE}}$ is identified for any mechanism: we assume that SUTVA holds at the level of individual reports to the mechanism, outcomes can be computed from the mechanism, and treatments follow selection-on-observables. Although weakening these assumptions is possible in our framework, it requires targeting a restricted version of $\bar{\tau}_{\text{GTE}}$ that is valid only for certain subgroups of market participants. For the unrestricted $\bar{\tau}_{\text{GTE}}$, our identification result suggests a plug-in approach for estimation: estimate the distribution of counterfactual submissions to the mechanism non-parametrically, and then run the mechanism on samples drawn from these distributions. Depending on the properties of the allocation mechanism, however, this approach may have an unacceptably large variance in finite samples.

For an estimator with better properties, we restrict attention to strategy-proof mechanisms that have a cutoff representation (Azevedo & Leshno 2016, Agarwal & Somaini 2018). This class of mechanisms has an equilibrium which is defined by a finite-vector of market-clearing cutoffs, and includes the uniform price auction, deferred acceptance, and top trading cycles. Under an asymptotic framework where a finite-sized market with n-participants converges to a continuum market with infinite participants (Azevedo & Leshno 2016), we show that the finite-market $\bar{\tau}_{\text{GTE}}$ converges at a $\sqrt{n}$ rate to a continuum market counterfactual, τ^*_{GTE}. The continuum-market counterfactual has a simple representation in terms of a set of moment conditions defined on the distribution of submissions to the mechanism.

This moment representation suggests a two-step doubly-robust estimator based on the theory in Kallus et al. (2019) and Chernozhukov et al. (2018). In the first-step, we use a propensity-score approach to estimate counterfactual market-clearing cutoffs. Then, in a second-stage, a debiased estimate of counterfactuals is computed by running a re-weighted and perturbed version of the mechanism, where the perturbations are estimated using a simple set of of machine learning regressions on the first-stage estimates. Data-splitting is used to control bias, allowing for weak conditions on the convergence rates of the machine learning estimators. Using techniques from the theory of empirical processes and the estimation of moment condition models, we show that this estimator is asymptotically normal, and that inference that is valid for the continuum market counterfactual is conservative for the finite-market estimand. Furthermore, the variance of the estimator meets the semi-parametric efficiency bound for the continuum market counterfactual.

When treatment effects are heterogeneous, a policymaker can improve outcomes by assigning treatment to a subset of individuals, depending on their pre-treatment covariates. There is a large literature on policy learning under SUTVA, but the problem is much more complex when there are spillover effects, see Viviano (2024) for an approach in the network setting based on mixed integer linear programming. In this paper, we provide the first asymptotic regret results for policy learning with equilibrium effects, employing a two-step doubly-robust approach for empirical welfare maximization. This yields an asymptotic regret bound in the finite market that is of the same order as lower bounds in the literature on policy learning without spillover effects (Athey & Wager 2021). A critical step in this result, which is the most challenging technical result of the paper, is demonstrating uniform convergence of estimated and finite market-clearing cutoffs to the continuum market-clearing cutoffs.

In two simulations, we show that our doubly-robust estimator has a variety of desirable properties in finite samples. First, we use a simple simulation of a uniform price auction to illustrate the robustness properties of our preferred estimator, in contrast to approaches based on parametric structural modeling. Next, we use a simulation of a school market with three schools to show that our confidence intervals for $\bar{\tau}_{\text{GTE}}$ perform well in finite samples and are narrower than doubly-robust intervals that target a partial equilibrium treatment effect.

Finally, we apply our methods in a real-world setting using data from Chile, where a centralized mechanism (based on deferred acceptance) allocates most children in the country to schools. We compile a dataset from the Ministry of Education that replicates many of the features of the data in Allende et al. (2019), except that the treatment is self-reported receipt of government-provided information on school quality, rather than an explicitly randomized intervention. We estimate $\bar{\tau}_{\text{GTE}}$, where the outcome measures the allocation of low-income families to good-quality schools. We find that if equilibrium effects are ignored, then the estimate of the impact of the treatment is large and significant, raising access of low-income families to good schools by nearly 1.5 percentage points. However, an estimate of the true impact of the intervention that takes into account the impact on the equilibrium of the school market is significantly smaller at 0.5 percentage points. The large bias of the average treatment effect comes from over-estimating the access of treated families to good schools relative to the all-treated counterfactual and under-estimating the access of control families to good schools relative to the all-control counterfactual. There is also substantial heterogeneity in treatment effects in the data. A rule that approximates the optimal targeting rule in equilibrium raises access of low-income families to good schools by 1.8 percentage points, substantially outperforming a uniform rule that allocates the intervention to all families.

1.1 Related Work

There is a body of existing work that estimates different types of causal effects in designed markets. Abdulkadiroğlu et al. (2017) estimate causal effects of allocations on future outcomes, such as test scores or income, using randomness in the matching mechanism for identification. Abdulkadiroğlu et al. (2022), Chen (2021), and Bertanha et al. (2023) extend this work to settings where individual scores are non-random but the cutoff structure of the mechanism allows an RDD analysis. Bertanha et al. (2023) also considers partial identification of preferences from strategic reports when mechanisms are not strategy proof. In contrast to this body of work, our paper focuses on an earlier step in the causal chain of events, which is the effect of a pre-allocation intervention on some function of allocations.

There is a small literature that considers settings with complete interference, where the treatment of each individual can impact the outcomes of any other individual in the sample. Miles et al. (2019) studies a model where interference occurs only through the proportion treated. They only consider estimands that are local, so restrict attention to counterfactual treatment policies that have the same proportion treated as in the observed data. Our paper estimates global causal effects, where the proportion treated is different from that observed in the data. Bright et al. (2022) characterize the bias of an RCT in a parametric model of a matching market, where a linear program computes the matching, so there is complete interference. They propose a simulation-based estimator of $\bar{\tau}_{\text{GTE}}$ that requires estimating their model using maximum likelihood estimation. Our paper studies markets with a different class of matching mechanisms, which are truthful and have a cutoff structure; in this class of mechanisms, we estimate causal effects without imposing a parametric model of the market.

Similar to this paper, Munro et al. (2023) assumes that spillovers occur exclusively through market prices, and observes outcomes from a single market. However, Munro et al. (2023) estimates treatment effects that are local to the current equilibrium. For the current paper, the ability to re-run a centralized mechanism in the market allows us to extrapolate from the observed market, so that estimating global counterfactuals is possible without making any additional parametric assumptions. Furthermore, observing submissions to the centralized mechanism means we can use data from a standard randomized experiment to estimate spillover-aware treatment effects, which is not possible in Munro et al. (2023), where an augmented randomized experiment is required.

To analyze the properties of the estimators in the paper, we use an asymptotic framework where the allocation mechanism operates on a distribution of agents, rather than a discrete number of agents. Using large-sample approximations for marketplaces is helpful in characterizing bias and variance of estimators of treatment effects, see Johari et al. (2022), Bright et al. (2022) and Liao & Kroer (2023), as well as Munro et al. (2023), for an analysis of A/B testing in various markets in equilibrium.

2 Finite and Continuum-Market Counterfactuals

We start by reviewing a general potential outcomes model that allows for spillover effects, where n participants in a two-sided market are drawn i.i.d. from a population. In this model, each market participant $i \in \{1, \dots, n\}$ has pre-treatment covariates $X_i \in \mathcal{X}$, 2^n potential outcomes, defined as $\{Y_i(\boldsymbol{w}) : \boldsymbol{w} \in \{0,1\}^n\}$ and binary treatment $W_i \in \{0,1\}$. When the n-length vector of treatments in the market is $\boldsymbol{W}$, then the researcher observes $Y_i = Y_i(\boldsymbol{W})$.

In this paper, we will estimate and maximize the value of counterfactual treatment rules. A candidate treatment rule is a function $\pi : \mathcal{X} \to \{0,1\}$, where $\pi \in \Pi$. Treatment allocation under the counterfactual rule is $W_i \sim \text{Bernoulli}(\pi(X_i))$. Throughout, we assume that the class of candidate rules Π includes π_1, which allocates treatment with probability 1 to everyone, and π_0, which allocates treatment with probability 0 to everyone. The finite-market value of a counterfactual treatment rule is

$$\bar{V}_n(\pi) = \mathbb{E}_\pi \left[\frac{1}{n} \sum_{i=1}^{n} Y_i(\boldsymbol{W}), \right]$$

where $E_\pi[\cdot]$ is the expectation with respect to random treatment allocation, holding potential outcomes and covariates fixed. Estimating $\bar{V}_n(\pi)$ will allow us to estimate policy-relevant average treatment effects (Section 3), and to learn outcome-maximizing policies (Section 4). Although the theory in Section 3 allows us to estimate the difference in average value between any two candidate policies, we pay particular attention to the Global Treatment Effect ($\bar{\tau}_{\text{GTE}}$), which is a random quantity defined as:

$$\bar{\tau}_{\text{GTE}} = \frac{1}{n} \sum_{i=1}^{n} [Y_i(\mathbf{1}_n) - Y_i(\mathbf{0}_n)],$$

where $\mathbf{1}_n$ and $\mathbf{0}_n$ are n-length vectors of 1s and 0s. Estimating this effect is useful for deciding whether or not to treat everyone in the market. Without spillovers, in settings where SUTVA holds at the outcome level, then $\bar{\tau}_{\text{GTE}}$ is equivalent to the familiar Average Treatment Effect, $\tau_{\text{ATE}} = \frac{1}{n} \sum_{i=1}^{n} Y_i(1) - Y_i(0)$.

For a single realization of the treatment vector $\boldsymbol{W}$ in a single market, under a baseline treatment policy $e : \mathcal{X} \to \{0,1\}$, the researcher observes an outcome $Y_i(\boldsymbol{W})$, a set of pre-treatment covariates $X_i \in \mathcal{X} \subseteq \mathbb{R}^m$ and the treatment W_i for each unit. With this data only, and in the absence of additional assumptions, we can't identify $\bar{V}_n(\pi)$. We next introduce some additional data and assumptions that apply to settings where outcomes are generated from a centralized mechanism. This will allow us to identify $\bar{\tau}_{\text{GTE}}$ without making parametric assumptions on individual choices, and characterize the relationship between finite-market and continuum-market counterfactuals.

2.1 Model

In the market observed by the researcher, n participants are allocated using a centralized mechanism to some subset of J items, which each have limited capacity. Observed allocations are a vector $D_i \in \mathbb{R}^J$. In addition to outcomes, allocations, a treatment, and covariates, the researcher also observes each unit's submission to the mechanism $B_i = B_i(W_i)$. $\{B_i(1), B_i(0)\}$ are the potential submissions to the mechanism, which we sometimes refer to as bids for conciseness. We start by providing a series of assumptions under which we can identify the value of counterfactual treatment rules using the joint distribution of (B_i, X_i, W_i). Let $\boldsymbol{B}(\boldsymbol{w})$ be the n-length vector of submissions to the mechanism under treatment vector $\boldsymbol{w}$.

Assumption 1. ***Identification***

1. *Given an n-length vector of submissions to the mechanism $\boldsymbol{B}(\boldsymbol{w})$, potential allocations $D_i(\boldsymbol{w}) = d_i(\boldsymbol{B}(\boldsymbol{w}))$, and outcomes $Y_i(\boldsymbol{w}) = y_i(\boldsymbol{B}(\boldsymbol{w}))$, where $d_i(\cdot)$ and $y_i(\cdot)$ are known for $i \in \{1, \dots, n\}$.*

2. *SUTVA holds for submissions to the mechanism: $B_i(\boldsymbol{W}) = B_i(\boldsymbol{W}')$ if $W_i = W_i'$.*

3. *Unconfoundedness and overlap hold, so $\{B_i(1), B_i(0)\} \perp\!\!\!\perp W_i | X_i$, and, letting $e(x) = P(W_i = 1 | X_i = x)$, for all $x \in \mathcal{X}$, $0 < e(x) < 1$.*

In the first part of Assumption 1, we assume that the mechanism is known and computable, so that an individual's allocation $D_i(\boldsymbol{w}) \in \mathbb{R}^J$ at a given treatment vector $\boldsymbol{w} \in \{0, 1\}^n$ can be computed given the n-length submissions to the mechanism $\boldsymbol{B}(\boldsymbol{w})$. This assumption holds for any market that is cleared by an auction or matching mechanism. We also assume outcomes can be computed from $\boldsymbol{B}(\boldsymbol{w})$, which is a stronger assumption that is met by bidder-level surplus in strategy-proof auctions and the measure of inequality in school allocations studied in Section 6. Weakening this assumption to handle outcomes that are unknown functions of allocations is also possible by combining the approach in this paper with Abdulkadiroğlu et al. (2017). However, this comes at the cost of identifying a restricted version of $\bar{\tau}_{\text{GTE}}$ that is valid only for certain subgroups, see a discussion in Appendix D.3.

The second part of Assumption 1 assumes that SUTVA holds for submissions to the mechanism, which means that bidding behavior does not depend on other market participants' treatments. Because of this, our method is best applied in settings with strategy-proof mechanisms. Furthermore, it rules out spillovers that occur outside the mechanism, such as sharing information received in a treatment through a social network. For many treatments, including subsidies and information received shortly before the mechanism is run, network-type spillovers are infeasible or very small. If they are expected to be large, then further work to combine network and market-spillover approaches is needed.[1]

Under these two assumptions, $\mathbb{E}[\bar{V}_n(\pi)]$ is a known functional of the treatment rule $\pi(\cdot)$, the marginal distribution of $B_i(1)$ and $B_i(0)$ and the market size. The last part of Assumption 1 identifies the marginal distribution of $B_i(1)$ and $B_i(0)$ by assuming that the treatment is randomly assigned conditional on covariates.[2] A natural next step is plug-in estimation: first, we estimate the distribution of counterfactual submissions to the mechanism non-parametrically, and then repeatedly re-run the mechanism on samples from these counterfactual distributions. Consistency of this estimator depends on the continuity and large-sample properties of the mechanism. Even if the estimator is consistent, it may converge slowly. For example, if the space of submissions to the mechanisms is high-dimensional,[3] or the space of pre-treatment covariates is high-dimensional, then estimating the marginal distributions of $B_i(1)$ and $B_i(0)$ directly in finite samples can be infeasible.

Rather than pursuing the plug-in approach further, we instead specify a general class of economic mechanisms where a $\sqrt{n}$ convergent and computationally efficient estimator for $\bar{\tau}_{\text{GTE}}$ is available. This class, formalized in Assumption 2, is made up of mechanisms for which an individual's allocation depends only on their own submission to the mechanism and a set of market-clearing cutoffs. This is a similar type of interference as in Munro et al. (2023), but because the market is cleared by a known mechanism, we can estimate counterfactuals that are not local to the observed equilibrium, which was not possible in the previous paper. A variety of commonly-used mechanisms have a cutoff structure, including the uniform price auction, deferred acceptance (Azevedo & Leshno 2016), and top trading cycles (Leshno & Lo 2021).

Assumption 2. *Structural Assumption (**Cutoff Mechanism**) For each $\boldsymbol{w} \in \{0,1\}^n$, allocations and outcomes for market participant i depend only on $B_i(w_i)$ and a fixed length*

[1]A partial identification approach is briefly discussed in Section 6.

[2]It is possible to use an IV-type assumption as an identifying condition instead at the cost of only identifying a restricted version of $\bar{\tau}_{\text{GTE}}$, see Appendix D.2.

[3]In the school choice setting, the space of possible submissions to the mechanism is equal to all possible rankings over all schools, which is grows exponentially in the number of schools.

vector of cutoffs $P_\pi = P_n(\boldsymbol{W}) \in \mathcal{S}$.

$$D_i(\boldsymbol{w}) = d(B_i(w_i), P_n(\boldsymbol{w})), \qquad Y_i(\boldsymbol{w}) = y(B_i(w_i), P_n(\boldsymbol{w})),$$

The cutoffs approximately clear the market with fractional capacity $s^* \in [0,1]^J$.[4] *Specifically, there exists a sequence* a_n *with* $\lim_{n\to\infty} a_n \sqrt{n} = 0$ *and constants* b, $c > 0$ *such that, for every* $\boldsymbol{w} \in \{0,1\}^n$,

$$\mathcal{C}_{\boldsymbol{w}} = \left\{ p \in \mathbb{R}^J : \left\| \sum_{i=1}^{n} \frac{1}{n} d(B_i(w_i), p) - s^* \right\|_2 \leq a_n \right\} \tag{1}$$

is non-empty with probability at least $1 - e^{-cn}$ *for all* n. *On the event where it is non-empty, the market price is in this set, so* $P_n(\boldsymbol{w}) \in \mathcal{C}_{\boldsymbol{w}}$.

These cutoffs are computed by the mechanism. Formally, there exists a function $m : \mathcal{B}^n \times \Delta^{n-1} \times [0,1]^J$ that maps the n-length vector of bids $\boldsymbol{B}(\boldsymbol{w})$, an n-length vector of weights for each bid $\boldsymbol{\gamma}$, and capacities for each item to a market-clearing cutoff, so

$$\left\| \sum_{i=1}^{n} \gamma_i d(B_i(w_i), m(\boldsymbol{B}(\boldsymbol{w}), \boldsymbol{\gamma}, s^*)) - s^* \right\|_2 \leq a_n, \tag{2}$$

and we can write $P_n(\boldsymbol{w}) = m(\boldsymbol{B}(\boldsymbol{w}), \frac{1}{n} \cdot \mathbf{1}_n, s^*)$, where Δ^k is the k-dimensional simplex. We next introduce two examples of mechanisms that are regularly used in practice and have a cutoff structure.

Example 1. ***Uniform Price Auction.*** *In a uniform price auction with a single good, unit demand, a supply of* m *units, and independent private values,* n *market participants bid their value* $B_i(w) \sim F_w$, *and the winning* m *bidders pay the* $(m+1)$*th highest bid. This auction has a cutoff structure, in that* $d(B_i(W_i), p) = \mathbb{1}(B_i(W_i) > p)$, *and* $\frac{1}{n} \sum_{i=1}^{n} d(B_i(W_i), P(\boldsymbol{W})) - s^* = 0$, *where* $s^* = m/n$.

Example 2. ***Deferred Acceptance.*** *In many cities, students are matched to schools using a version of the deferred acceptance algorithm with lottery scores. This mechanism is another example of a strategy-proof mechanism with a cutoff structure, as shown in Azevedo & Leshno (2016);* $p \in \mathcal{S}$ *is a vector of score cutoffs for each school. The submission to the mechanism*

[4] In a finite-sized market with $m \in \mathbb{R}^{J+}$ items available, then $s^* = m/n$. It is convenient to write the capacity constraint in fractional form for the continuum market approximation, where s^* is fixed as n grows large. In the finite-sized market, the mechanism allocates a fraction of the empirical distribution of market participants to each item. In the continuum market, it allocates an equivalent fraction of the population distribution of market participants to each item.

is a ranking over schools $R_i(W_i)$, where $jR_i(W_i)j'$ is 1 if school j is ranked above j', and zero otherwise, and an independent item-specific lottery number $S_i \in \mathbb{R}^J$. The allocation function is:

$$d(B_i(w), p) = \mathbb{1}\{S_{ij} > p_j, jR_i(W_i)0\} \prod_{j \neq j'} \mathbb{1}(jR_i(W_i)j' \text{ or } S_{ij'} < p_{j'}).$$

On the supply-side $s_j^* = m_j/n$, where m_j is the number of seats available in school j, and n is the total number of students.

Under Assumption 2, holding the n market participants fixed, the expected outcomes for a policy $\pi \in \Pi$ are :

$$\bar{V}_n(\pi) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_\pi \left[\pi(X_i) y(B_i(1), P_n(\boldsymbol{W})) + (1 - \pi(X_i)) y(B_i(0), P_n(\boldsymbol{W})\right],$$

where $\mathbb{E}_\pi[\cdot] = \mathbb{E}[\cdot|(B_i(1), B_i(0), X_i)_{i=1}^n]$ is the expectation over random treatments, holding all other sources of randomness in market fixed. By replacing empirical averages with their population counterparts, we can write expected outcomes in the continuum market as $V^*(\pi) = y_\pi(p_\pi^*)$, where $y_\pi(p) = \mathbb{E}[\pi(X_i)y(B_i(1), p) + (1-\pi(X_i))y(B_i(0), p)]$. The large-market cutoffs are defined as $\mathbb{E}[z_\pi(p_\pi^*) = 0]$, and $z_\pi(p) = \mathbb{E}[\pi(X_i)d(B_i(1), p) + (1-\pi(X_i))d(B_i(0), p)] - s^*$. Similarly, we can write $\tau^*_{\text{GTE}} = \mathbb{E}[y(B_i(1), p_1^*)] - \mathbb{E}[y(B_i(0), p_0^*)]$, and for $w \in \{0, 1\}$, $\mathbb{E}[d(B_i(w), p_w^*) - s^*] = 0$.

In the finite-sized market, there may be many market-clearing prices that satisfy Assumption 2, and $\bar{V}_n(\pi)$ is defined as an average of dependent terms. In the continuum market, the equilibrium effect of a policy has a simple representation in terms of a set of moment conditions.

In the next section, we show that although the continuum market has a simpler structure, it is a good approximation to the finite market. Specifically, counterfactuals in the finite-sized market converge to counterfactuals in the continuum market. Our notion of convergence follows the related economic theory literature, as in Azevedo & Leshno (2016), and grows n large but keeps J fixed.

To do this, we first impose a set of regularity conditions that ensure that the continuum market has unique and well-defined counterfactual equilibria, and that the finite market is sufficiently well-behaved. In Assumption 3, the weak continuity assumption and metric entropy condition allows for individual-level allocation functions that have some discontinuity in market-clearing cutoffs. However, at the population-level, expected allocations and outcomes must be smooth. The last part of the assumption ensures that Jacobian of expected demand has a well-behaved inverse, which is important for quantifying spillover effects through market-clearing cutoffs.

Assumption 3. ***Regularity of Outcomes.***

1. *There are constants $h_d, h_y, C > 0$ such that for each $j \in \{1, \ldots, J\}$, and $w \in \{0, 1\}$ the function classes $\mathcal{F}_{d,j} = \{B(w) \mapsto d_j(B(w), p) : p \in \mathcal{S}\}$ and $\mathcal{F}_y = \{B(w) \mapsto y(B(w), p) : p \in \mathcal{S}\}$ have uniform covering number obeying, for every $0 < \epsilon < 1$, $\sup_{Q_y} N(\epsilon, \mathcal{F}_{d,j}, L_2(Q_y)) \leq C(1/\epsilon)^{h_d}$, and $\sup_{Q_d} N(\epsilon, \mathcal{F}_y, L_2(Q_d)) \leq C(1/\epsilon)^{h_y}$.*

2. *The outcome function $y : \mathcal{B} \times S \mapsto \mathbb{R}$ is uniformly bounded and demand and outcomes are weakly continuous. There is a constant $L > 0$ such that for all pairs of prices p, p', all w, and all j, we have $\mathbb{E}[(d_j(B_i(w), p) - d_j(B_i(w), p'))^2] \leq L||p - p'||_2$ and $\mathbb{E}[(y(B_i(w), p) - y(B_i(w), p'))^2] \leq L||p - p'||_2$.*

3. *For all $w \in \{0, 1\}$ and $x \in \mathcal{X}$, $\mu^d_w(p, x) = \mathbb{E}[d(B_i(w), p)|X_i = x]$ and $\mu^y_w(p, x) = \mathbb{E}[y(B_i(w), p, X_i)|X_i = x]$ are twice continuously differentiable in p with first and second derivatives bounded uniformly by c'.*

4. *For each $\pi \in \Pi$, the singular values of the $J \times J$ Jacobian matrix $\nabla_p z_\pi(p^*_\pi)$ are bounded between c_3 and c_4.*

In Assumption 4, we assume that the equilibrium price in the population is unique and well-separated. Under regularity conditions on the distribution of values, Assumptions 2 - 4 are satisfied by the uniform price auction in Example 1, when bidder surplus is the outcome of interest, as shown in Appendix D.1. This result can also be extended to Example 2 under regularity conditions on the distribution of lottery numbers.

Assumption 4. ***Regularity of Equilibrium.*** *$\mathcal{S}$ is a compact set. For all $\pi \in \Pi$, $\mathcal{S}$ contains a ball of radius $c_1 > 0$ centered at p^*_π, and p^*_π is unique and well-separated, so for any $p \in \mathcal{S}$ with $||p - p^*_\pi|| \geq \frac{c_3}{2Jc'}$, there is a $c_2 > 0$ so that $2||z_\pi(p)|| \geq c_2$.*

2.2 Convergence of Finite-Market to Continuum Market

Our first result investigates finite-market counterfactuals as n grows large. The nature of spillovers, and the lack of uniqueness of equilibria in finite-sized markets makes analyzing estimators in a finite-sample framework challenging. Asymptotic approximations lead to more tractable characterizations of market equilibria, and will allow us to provide a variety of theoretical guarantees on the estimators introduced in this paper.

Theorem 1. *Under Assumption 1- 4,* $\sqrt{n}(\bar{V}(\pi) - V^*(\pi)) = O_p(1)$. $\bar{\tau}_{GTE}$ *has the following asymptotically linear form:*

$$\bar{\tau}_{GTE} - \tau^*_{GTE} = \frac{1}{n}\sum_{i=1}^{n}\Big(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*)\Big) - \tau^*_{GTE} + o_p(n^{-0.5}), \tag{3}$$

where $q_w(b,p) = y(b,p) - \nu_w^*(d(b,p) - s^*)$, *and* $\nu_w^* = \nabla_p^\top \mathbb{E}[y(B_i(w), p_w^*)](\nabla_p \mathbb{E}[d(B_i(w), p_w^*)])^{-1}$.

We prove Theorem 1 in Appendix A.2. Counterfactuals defined in the finite market converge at a $1/\sqrt{n}$ rate to counterfactuals defined in the continuum market. In the finite market, the effect of the treatment through individual choices on the equilibrium introduces spillovers. As the market size grows large, the equilibrium effect of the treatment remains, but its representation converges to a set of moment conditions defined on the population distribution. We now describe an estimation strategy that targets the moment representation of $V^*(\pi)$, but still retains guarantees for the finite-market estimand.

3 Estimating Counterfactual Values

In this section, we introduce an estimator that is $\sqrt{n}$-consistent for both the finite and continuum market value of counterfactual treatment rules. Algorithmically, this estimator runs a perturbed and re-weighted version of the allocation mechanism on the observed data, where the weights and perturbations are estimated using flexible machine learning methods, and three-way data splitting is used to control bias. This estimator is closely related to the more general theory in Kallus et al. (2019) for quantile-like treatment effects, but aspects of its design and analysis are unique to the problem studied in this paper.

Combining the moment representation of $V^*(\pi)$ and the overlap and unconfoundedness assumptions of Assumption 1, we can identify $V^*(\pi)$ using $J+1$ moment conditions and doubly-robust scores:

$$\begin{aligned} &\mathbb{E}[\pi(X_i)\Gamma_{1i}^{*y}(p_\pi^*) + (1-\pi(X_i))\Gamma_{0i}^{*y}(p_\pi^*)] = V^*(\pi), \\ &\mathbb{E}[\pi(X_i)\Gamma_{1i}^{*z}(p_\pi^*) + (1-\pi(X_i))\Gamma_{0i}^{*z}(p_\pi^*)] = 0, \end{aligned} \tag{4}$$

where doubly-robust scores combine the propensity score $e(x) = P(W_i = 1|X_i = x)$ and conditional mean functions $\mu_w^d(x,p) = \mathbb{E}[d(B_i(w), p)|X_i = x]$ and $\mu_w^y(x,p) = \mathbb{E}[y(B_i(w), p)|X_i =$

$x]$ for $w \in \{0,1\}$:

$$\begin{aligned}\Gamma^{*y}_{wi}(p) &= \mu^y_w(X_i,p) + \frac{\mathbb{1}(W_i = w)}{P(W_i = w|X_i = x)}(y(B_i(w),p) - \mu^y_w(X_i,p)),\\ \Gamma^{*z}_{wi}(p) &= \mu^d_w(X_i,p) + \frac{\mathbb{1}(W_i = w)}{P(W_i = w|X_i = x)}(d(B_i(w),p) - \mu^d_w(X_i,p)) - s^*.\end{aligned} \tag{5}$$

(4) is not the only set of moment conditions that identify τ^*_{GTE} under unconfoundedness and overlap. For example, it is possible to identify and estimate τ^*_{GTE} using the propensity score only. We prefer the doubly-robust approach since it requires much weaker assumptions on propensity scores for results on inference and semi-parametric efficiency. For a more detailed discussion of the benefits and drawbacks of the propensity score approach, we defer to the large related literature; see, for example, Bang & Robins (2005) and Graham et al. (2012). Another alternative, which is popular in the applied economics literature and discussed in more detail in Section 5, is to use a parametric structural model of bidding behavior for identification and estimation. Our approach avoids specifying a parametric model of bidding behavior.

The simplest doubly-robust estimator would solve for an empirical version of (4), as in Chernozhukov et al. (2018). However, this requires inverting the estimated conditional mean function, since it is a function of p, which implies estimating the entire bid distribution conditional on covariates. When the bid or covariate dimension is high, a flexible estimator of this conditional distribution will converge too slowly for the theory in Chernozhukov et al. (2018). Instead, we adapt the localization approach of Kallus et al. (2019), which solves an empirical version of (4) that fixes a single estimate of the conditional mean functions at a first-step estimator of counterfactual market-clearing cutoffs. An application of this approach that uses the centralized mechanism $m(\cdot)$ to find a solution to the empirical moment condition is in Definition 1.

Definition 1. Localized Doubly-Robust Estimator

1. Randomly split the dataset into $K = 3$ folds. Let $k(i)$ be the fold of observation i, for $i \in \{1,\dots n\}$. Let $\mathcal{I}_k$ denote the indices of data in fold k, and $\mathcal{I}_{-k}$ the data that is not in fold k. In addition, for each fold, randomly split $\mathcal{I}_{-k}$ into two disjoint subsets $\mathcal{H}_{-k}$ and $\mathcal{G}_{-k}$. For each fold $k \in \{1,2,3\}$,

    - On data in fold $\mathcal{H}_{-k}$, compute a first step cutoff estimate $\tilde{P}_\pi = m(\boldsymbol{B}, \tilde{\boldsymbol{\gamma}}_\pi, s^*)$, using estimated weights $\tilde{\gamma}_{\pi,i} = \pi(X_i)\frac{W_i}{|\mathcal{H}_{-k}|\tilde{e}(X_i)} + (1-\pi(X_i))\frac{1-W_i}{|\mathcal{H}_{-k}|(1-\tilde{e}(X_i))}$. $\tilde{e}(X_i)$ is estimated using (W_i, X_i) in fold $\mathcal{H}_{-k}$.
    - On data in fold $\mathcal{G}_{-k}$, estimate the propensity score $\hat{e}^k(X_i)$ using (W_i, X_i).

- On data in fold $\mathcal{G}_{-k}$, estimate the conditional mean functions using a flexible regression:
  - Estimate $\hat{\mu}_w^{y,k}(X_i)$ for $w \in \{0,1\}$ by regressing $y(B_i, \tilde{P}_\pi)$ on (X_i, W_i),
  - Estimate $\hat{\mu}_w^{d,k}(X_i)$ for $w \in \{0,1\}$ by regressing $d(B_i, \tilde{P}_\pi)$ on (X_i, W_i).

2. Using the full sample, compute a second-step estimate of cutoffs $\hat{P}_\pi = m(\boldsymbol{B}, \hat{\gamma}_\pi, \hat{s}_\pi)$, where the weights and perturbed capacities are:

$$\hat{\gamma}_{\pi,i} = \pi(X_i)\frac{W_i}{n\hat{e}^{k(i)}(X_i)} + (1-\pi(X_i))\frac{1-W_i}{n(1-\hat{e}^{k(i)}(X_i))},$$

$$\hat{s}_\pi = s^* + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{W_i}{\hat{e}^{k(i)}(X_i)} - 1\right)\pi(X_i)\hat{\mu}_1^{d,k(i)}(X_i) + (1-\pi(X_i))\left(\frac{1-W_i}{1-\hat{e}^{k(i)}(X_i)} - 1\right)\hat{\mu}_0^{d,k(i)}(X_i).$$

3. Using the full sample, estimate $\hat{V}_n(\pi)$ using doubly-robust scores:

$$\begin{aligned}
\hat{V}_n(\pi) &= \frac{1}{n}\sum_{i=1}^{n}\hat{\Gamma}_{1i}^{y}(\hat{P}_\pi) + (1-\pi(X_i))\hat{\Gamma}_{0i}^{y}(\hat{P}_\pi),\\
\hat{\Gamma}_{1i}^{y}(p) &= \hat{\mu}_1^{y,k(i)}(X_i) + \frac{W_i}{\hat{e}^{k(i)}(X_i)}(y(B_i,p) - \hat{\mu}_1^{y,k(i)}(X_i)), \qquad (6)\\
\hat{\Gamma}_{0i}^{y}(p) &= \hat{\mu}_0^{y,k(i)}(X_i) + \frac{1-W_i}{1-\hat{e}^{k(i)}(X_i)}(y(B_i,p) - \hat{\mu}_0^{y,k(i)}(X_i)).
\end{aligned}$$

Data are split three ways. For each split of data, doubly-robust scores are computed using nuisance functions estimated on the other two splits of data. One of these is used for a first stage inverse propensity-score estimate of the market-clearing cutoffs under treatment and control. The other is used for estimates of the propensity score and a single set of conditional mean functions. These estimated conditional mean functions are constructed via flexible regressions of outcomes and allocations computed at the first-step cutoff estimates. Then, the treatment effect is estimated in two steps. First, using conditional mean functions for allocations and the estimated propensity score, we run a perturbed and re-weighted version of the centralized allocation mechanism to estimate counterfactual market-clearing cutoffs. Then, the global treatment effect is estimated using a doubly-robust score evaluated at these counterfactual cutoffs. The key insight of the algorithm in Design 1 compared to the more general estimator in Kallus et al. (2019) is the use of the mechanism algorithms $m(\cdot)$ to find the market-clearing cutoffs. In the school choice application in Section 6, there are thousands of moment conditions, and finding the market-clearing cutoffs using a general root-finding approach rather than deferred acceptance would be extremely slow.

For this procedure to lead to an asymptotically normal and semi-parametric efficient estimator, we require the following restrictions on the nuisance function estimation:

Assumption 5. ***Assumptions on Nuisance Estimation:*** *For each $k \in \{1, \ldots, K\}$, let $\hat{\mu}_w(x)$ be a $(J+1)$ vector of functions that concatenates $\hat{\mu}_w^y(x)$ and $\hat{\mu}_w^d(x)$, estimated on a training set of size n/K. X_i is an independently drawn test observation. $\mathbb{E}_T[\cdot]$ is an expectation over random test data, conditional on the training data $(X_i, B_i, W_i)_{i=1}^n$, where the test data is drawn from the same distribution as the training data.*

1. *The estimated propensity score satisfies strong overlap: almost surely, $\hat{e}(X_i) \in (\kappa, 1-\kappa)$ for $\kappa > 0$.*

2. *The estimated conditional mean functions are uniformly bounded. There is a constant $M < \infty$ such that*
$$\sup_{w \in \{0,1\}, x \in \mathcal{X}, p \in \mathcal{S}} ||\hat{\mu}_w(x,p)||_\infty \leq M.$$

3. *For each $\pi \in \Pi$, there is a finite c such that with probability $1 - e^{-cn}$,*
$$\left(\mathbb{E}_T\left[||\hat{\mu}_w(X_i, \tilde{P}_\pi) - \mu(X_i, \tilde{P}_\pi)||^2\right]\right)^{1/2} \leq \rho_{\mu,n}, \tag{7}$$
$$\left(\mathbb{E}_T[(\hat{e}(X_i) - e(X_i))^2]\right)^{1/2} \leq \rho_{e,n}, \tag{8}$$
$$\left(||\tilde{P}_\pi - p_\pi^*||^2\right)^{1/2} \leq \rho_{\theta,n}, \tag{9}$$
where $\rho_{e,n} = o(1)$, $\rho_{\mu,n} + \rho_{\theta,n} = o(1)$, $\rho_{e,n}\rho_{\mu,n} = o(n^{-1/2})$, and $\rho_{e,n}\rho_{\theta,n} = o(n^{-1/2})$.

4. *Assumption on the error in the market-clearing rate. $\rho_{g,n} = o(n^{-1/2})$ and with probability at least $1 - e^{-cn}$, $\mathcal{C}(p)$ is non-empty, and $\hat{P}_n \in \mathcal{C}(p)$, where*
$$\mathcal{C}(p) = \left\{ p \in \mathcal{S} : \left|\left| \frac{1}{n}\sum_{i=1}^n \pi(X_i)\Gamma_{1i}^d(p;\hat{\eta}) + (1-\pi(X_i))\Gamma_{0i}^d(p;\hat{\eta}) \right|\right| \leq \rho_{g,n} \right\},$$
$\Gamma_{1i}^d(p;\hat{\eta}) = \hat{\mu}_1^{d,k(i)}(X_i) + \frac{W_i}{\hat{e}^{k(i)}(X_i)}(d(B_i(w),p) - \hat{\mu}_1^{d,k(i)}(X_i))$, and $\Gamma_{0i}^d(p;\hat{\eta}) = \hat{\mu}_0^{d,k(i)}(X_i) + \frac{1-W_i}{1-\hat{e}^{k(i)}(X_i)}(d(B_i(w),p) - \hat{\mu}_0^{d,k(i)}(X_i))$ and $\hat{\eta}$ collects the estimated nuisances.

Assumption 5 requires that the pairwise product of the rates of mean-square-consistency of the initial estimator of the counterfactual cutoffs, the propensity score, and the conditional mean functions are $o(n^{-1/2})$ and that each nuisance parameter is also consistent. This means that for a fixed p, the estimator for expected outcomes and allocations conditional on X_i can have a slow rate. The uniform guarantee on the performance of the estimators over

$\pi \in \Pi$ can be dropped for the point-wise results on the value function in this section, but is required for the regret guarantee in the next section. The main result of this section is that the algorithm described leads to an asymptotically normal estimator of counterfactuals of interest:

Theorem 2. *Under Assumptions 1 - 5,* $\hat{V}_n(\pi) = \frac{1}{n}\sum_{i=1}^{n}\Gamma^{*q}_{\pi i}(p^*_\pi) + o_p(n^{-1/2})$, *where*

$$\Gamma^{*q}_{\pi i}(p) = \pi(X_i)\Gamma^{*y}_{1i}(p) + (1-\pi(X_i))\Gamma^{*y}_{0i}(p) - \nu^*_\pi\Big(\pi(X_i)\Gamma^{*d}_{1i}(p) + (1-\pi(X_i))\Gamma^{*d}_{0i}(p^*_0) - s^*\Big),$$

and $\nu^*_\pi = \nabla_p^\top y_\pi(p^*_\pi)[\nabla_p z_\pi(p^*_\pi)]^{-1}$.

Corollary 3. *Under Assumptions 1 - 5,*

$$\hat{\tau}_{GTE} - \tau^*_{GTE} = \frac{1}{n}\sum_{i=1}^{n}\Gamma^{*q}_{1i}(p^*_1) - \Gamma^{*q}_{0i}(p^*_0) - \tau^*_{GTE} + o_p(n^{-1/2}),$$

where $\Gamma^{*q}_{wi}(p) = \Gamma^{*y}_{1i}(p) - \nu^*_w(\Gamma^{*d}_{1i}(p) - s^*)$. *And,*

$$\sqrt{n}(\hat{\tau}_{GTE} - \tau^*_{GTE}) \to_D N(0,\sigma^2),$$

where $\sigma^2 = \mathbb{E}[(\Gamma^q_{1i}(p^*_1) - \Gamma^q_{0i}(p^*_0) - \tau^*_{GTE})^2]$.

With known nuisance functions, standard techniques for method-of-moments estimators, as in Munro (2023), can be used to prove Theorem 2. With an unknown propensity score, the main challenge is to show that the error in the nuisance functions does not have a first order impact on the error of the estimator. Under weaker entropy conditions than in Assumption 3, the main result in Kallus et al. (2019) can be used to prove Theorem 2. However, the stronger conditions that we impose, which are met by economic mechanisms used in practice, lead to a more concise proof of Theorem 2, and are useful for the regret results in Section 4.

Corollary 3 follows directly from Theorem 2. Due to the market-clearing cutoffs, the asymptotic variance of $\hat{\tau}_{\mathrm{GTE}}$ depends on the variance of a linear combination of treatment effects on outcomes and treatment effects on allocations. The first component is the standard sampling variation in direct treatment effects, and the second is due to the variation in the equilibrium that is reached in the allocation mechanism. In many cases, the variance of $\hat{\tau}_{\mathrm{GTE}}$ is less than an estimator for the Average Treatment Effect, making confidence intervals that account for noise in the equilibrium effect will be tighter than those that ignore equilibrium effects.[5] This is the case both in the simulations in Section 5 and in the empirical example

[5]For example, assume a binary treatment raises the values of bidders in a Uniform Price Auction, and the outcome is bidder surplus. The variance in individual treatment effects contributes directly to the variance

of Section 6. Furthermore, the variance in Corollary 3 meets the semi-parametric efficiency bound for τ^*_{GTE}.

Theorem 4. ***Semi-Parametric Efficiency*** *Under the assumptions of Theorem 2, the semi-parametric efficiency bound for τ^*_{GTE} is equal to σ^2.*

The proof of this theorem is in Appendix A.5. The proof follows uses the methodology presented in Bickel et al. (1993) and Newey (1990), and is closely related to the bound for quantile treatment effects in Firpo (2007).

By computing a plug-in estimator of σ^2, we can perform asymptotically valid inference on the continuum market counterfactual τ^*_{GTE}. Consistency of a plug-in estimator for σ^2 follows from the existing assumptions, as shown in Theorem 4 of Kallus et al. (2019). Theorem 2 and 4 focus on the continuum market counterfactual τ^*_{GTE}. Although it is a convenient approximation, in many settings the true target of interest is the finite-market counterfactual $\bar{\tau}_{\text{GTE}}$. Combining Theorem 1 and Theorem 2, we have that $\sqrt{n}(\hat{\tau}_{\text{GTE}} - \bar{\tau}_{\text{GTE}}) \Rightarrow N(0, \bar{\sigma}^2)$, where $\bar{\sigma}^2 = \mathbb{E}[(\Gamma^{*q}_{1i}(p^*_1) - \Gamma^{*q}_{0i}(p^*_0) - q_1(B_i(1), p^*_1) + q_0(B_i(0), p^*_0))^2]$. Proposition 5 shows that inference that is valid for the continuum market estimand is conservative for the finite-market estimand.

Proposition 5. *Under the Assumptions of Theorem 2, $\sigma^2 \geq \bar{\sigma}^2$.*

Without spillovers, it is well-known that the variance of the population ATE is an upper bound for the variance of the sample ATE. Proposition 5 extends this result to the Global Treatment Effect with spillovers through a centralized market.

4 Policy Learning

In Section 3, we provide a doubly-robust estimator for the value of various counterfactual treatment rules; our primary application estimates the difference between the value of two treatment rules that treat everyone uniformly ($\bar{\tau}_{\text{GTE}}$). When treatment effects are heterogeneous, then a targeting rule can substantially improve outcomes compared to a uniform rule. In this section, we consider the problem of choosing $\pi \in \Pi$ to maximize finite-market or continuum-market expected outcomes. Under SUTVA, in the absence of additional constraints, the benefit of treating a group of individuals depends only on the average treatment effect of that group. When there are spillover effects, the problem is much more complex.

of a partial equilibrium treatment effect estimator. However, a GTE estimator also estimates the equilibrium price at treatment and control. To respect the capacity constraint in the auction, a sample with a higher average treatment effect will also have a higher estimated market price under treatment, which can dampen the impact of variance in treatment effects on the estimated GTE and reduce variance.

The benefit of treating a group of individuals depends on their direct response to the treatment as well as indirect effects on others, and the magnitude of both can vary depending on the treatment saturation in the sample. In this paper, however, restrictions on spillovers through market-clearing cutoffs provide enough structure that learning optimal treatment rules is possible.

We start by characterizing the optimal unrestricted treatment rule in the continuum market; although this leads to a useful description of the structure of the globally optimal rule, designing an estimator with good theoretical guarantees requires additional assumptions. We then consider the problem of estimating a treatment rule that is the member of a restricted class of rules, and maximizes outcomes in the finite market. We restrict Π to be a VC-class, and show that maximizing the estimated value function within this class using the algorithm in Section 3 has regret that decays at a $\sqrt{n}$ rate. This is a notable result; when spillover effects are mediated by the equilibrium of a cutoff mechanism, it is possible to learn the optimal policy at an asymptotic rate that matches the lower bound for the rate for policy learning without spillover effects (Athey & Wager 2021).

4.1 Unconstrained Class of Treatment Rules

Theorem 6 provides a score condition that any optimal rule must satisfy when Π is unconstrained.

Theorem 6. *Let Π be the class of all functions from $\mathcal{X}$ to $[0,1]$. Let*

$$\rho(x,\pi) = \mathbb{E}[q_\pi(B_i(1), p^*_\pi) - q_\pi(B_i(0), p^*_\pi)|X_i = x],$$

where $q_\pi(B_i(w), p) = y(B_i(w), p) - \nu^*_\pi(d(B_i(w), p) - s^*)$. For any optimal rule $\pi^* \in \arg\max V^*(\pi)$, for almost all $x \in \mathcal{X}$, $\pi^*(x) = 1$ when $\rho(x,\pi^*) > 0$, $\pi^*(x) = 0$ when $\rho(x,\pi^*) < 0$, and $\pi^*(x) \in [0,1]$ when $\rho(x,\pi^*) = 0$.

$\rho(x,\pi)$ is made up of two components. The first component is the average direct effect of treating market participants with $X_i = x$ on outcomes; holding market prices fixed. However, raising the treatment probability for a group of market participants also affects the market-clearing cutoffs. The second component measures the indirect effect of treating market-participants; treating more participants affects demand for certain items in the market, and the resulting change in p^*_π affects outcomes. If the sum of these two effects is positive then the treatment probability for the group is positive. This is in contrast to the globally optimal rule under SUTVA, where only sign of the conditional average direct effect of the treatment on outcomes matters. While this result is useful for understanding the structure of the optimal rule, the ultimate goal in this section is to characterize the regret of an estimator for the optimal treatment rule. Unfortunately, obtaining even consistency is challenging for the globally optimal rule; a plug-in estimator may not meet the condition of Theorem 6, since $\hat{\rho}(e, x)$ estimated at the treatment rule observed in the data may be very different from $\rho(\tilde{\pi}, x)$, where $\tilde{\pi}(x) = \mathbb{1}(\hat{\rho}(x, e) > 0)$. In the next section, we constrain Π to be a VC-class, which allows for an empirical welfare maximization approach that has asymptotic regret guarantees, even in the finite market. Furthermore, this constraint is often useful in practice, where "simple" treatment rules, such as linear threshold rules, are desirable.

4.2 Constrained Class of Treatment Rules

We now assume that Π is a VC-class of functions with dimension v. The estimator of the optimal value function maximizes the doubly-robust estimator of the value function from Section 3 over Π, specifically

$$\hat{\pi} \in \arg\max_{\pi \in \Pi} \hat{V}_n(\pi).$$

The main contribution of this section is formalizing how well the estimated rule performs compared to the oracle rule that maximizes the unobserved finite-market value $\bar{V}_n(\pi)$ directly. A key step in this result is to show that both $\bar{V}_n(\pi)$ and $\hat{V}_n(\pi)$ converge uniformly in $\pi \in \Pi$ as n grows large to the continuum market value $V^*(\pi)$. For this uniform convergence, we require an additional assumption on the nuisance functions in addition to Assumption 5, which we provide in Assumption 6.

Assumption 6. *With probability at least $1 - o(1)$, the function class $\mathcal{F}_{\hat{\mu}} = \{X \mapsto \hat{\mu}^y(X, p) : p \in \mathcal{S}\}$ and, for each $j \in \{1, \dots J\}$ the class $\mathcal{F}_{\hat{\mu},j} = \{X \mapsto \hat{\mu}_j^d(X, p) : p \in \mathcal{S}\}$ have uniform covering number obeying, for every $0 < \epsilon < 1$, $\sup_{Q_y} N(\epsilon, \mathcal{F}_{\hat{\mu}}, L_2(Q_y)) \leq C(1/\epsilon)^{h_y}$ and $\sup_{Q_d} N(\epsilon, \mathcal{F}_{\hat{\mu},j}, L_2(Q_d)) \leq C(1/\epsilon)^{h_d}$.*

Although we allow the estimated conditional mean functions to be complex functions of X_i, they must be relatively simple functions of p. Since we already impose a metric entropy condition on individual-level outcome functions in p, in some cases, such as for the K-nearest-neighbors estimator used in Section 6, this is automatically satisfied by Assumption 3. For more general machine learning estimators, verifying this type of condition may require additional effort. We can now prove Theorem 7.

Theorem 7. *Under the assumptions of Theorem 2 and Assumption 6, also assume that Π is a VC-class of dimension v. Then, regret in both the finite market and the continuum market* *from the empirical welfare maximization procedure decays asymptotically at a $1/\sqrt{n}$ rate:*

$$V^*(\hat{\pi}) - \arg\max_{\pi \in \Pi} V^*(\pi) = O_p\left(\frac{1}{\sqrt{n}}\right),$$
$$\bar{V}_n(\hat{\pi}) - \arg\max_{\pi \in \Pi} \bar{V}_n(\pi) = O_p\left(\frac{1}{\sqrt{n}}\right).$$

Characterizing the maximizer of the finite-market value of a treatment rule directly is challenging, since it is a quantity that depends on possibly non-unique market-clearing cutoffs and non-smooth allocation functions. By linking both the finite-market value and estimated market-value to the continuum-market value instead, where the equilibria is unique and aggregate responses are smooth, then we manage to obtain asymptotic regret results for the finite-sized market. The constants in the asymptotic regret bound depend on the VC-class dimension v, the number of items J, as well as the coefficients h_d and h_y in the covering number bounds for the allocation and outcome functions. The dependence on n implies that the estimated maximizer converges quickly to the oracle maximizer of either the finite or continuum market value. This rate matches the lower bound for policy learning without SUTVA, and upper bounds for regret with network spillovers in Viviano (2024). This strong result is possible in the centralized market setting because all spillovers occur through a finite-vector of market clearing cutoffs. A key step in the proof is showing $\sqrt{n}$- uniform convergence of the estimated-market clearing cutoffs to the continuum market-clearing cutoffs under weak assumptions on the convergence of nuisance functions. The proof technique used for the market-clearing cutoffs can be extended to any M or Z-estimator. This could allow for policy learning results in other semi-parametric models with heterogeneity and interactions between units that can be described using a set of moment conditions in the population.

5 Simulations

In this section, we illustrate the theoretical results in Section 3 using two simple simulations. In the first simulation, we illustrate the robustness properties of the doubly-robust estimators using a simulation of a uniform price auction where bidders values are generated from different distributions. In the second simulation, which is of a market for schools with three schools, we show that asymptotically valid confidence intervals for τ^*_{GTE} built on the LDML estimator have good coverage for $\bar{\tau}_{\mathrm{GTE}}$ in finite samples.

5.1 Comparison to Structural Modeling Approaches

We simulate data generated from a uniform price auction and compare the LDML estimator of $\bar{\tau}_{\mathrm{GTE}}$ to alternative approaches, especially those relying on parametric structural models. In the simulation, treatment affects bids to the auction. There is a 20-dimensional set of covariates that is correlated with the bids and affects the probability of selecting the treatment. The auction has a fractional capacity of 0.5, so that the top half of the bids in the auction receive a single unit of the good. The treatment affects outcomes through a shift in the distribution of bids submitted to the auction, and through a shift in the equilibrium market-clearing price. The outcome of interest is the observed average surplus for bidders in the auction, assuming that the bids submitted to the auction are equal to the values for the bidders. The data-generating process is explicitly described in Appendix B. For each bidder, we observe the bid B_i, the treatment W_i and pre-treatment covariates X_i. We compute RMSE and bias of a variety of estimators when the target estimand is $\bar{\tau}_{\mathrm{GTE}}$ by repeatedly sampling a finite-sized market of size $n = 100$, $n = 1000$ and $n = 10,000$. These estimators take as input $(Y_i, B_i, W_i, X_i)_{i=1}^n$.

The estimators are as follows:

1. A doubly-robust estimator of the Average Treatment Effect using generalized-random forests (`DR-ATE`). This estimator compares observed surplus for treated and control market participants at the observed equilibrium. It adjusts for selection-on-observables, but does not account for spillover effects.

2. A structural model based estimator of $\bar{\tau}_{\mathrm{GTE}}$ (`SM-GTE`). The estimator assumes that $B_i(w) \sim \mathrm{LogNormal}(\mu_w(X_i), \sigma)$. For $w \in \{0, 1\}$, $\hat{\mu}_w(X_i)$ and $\hat{\sigma}$ are estimated using a linear regression of $\log(B_i)$ on X_i for individuals with $W_i = w$. Then, $\hat{\tau}^{SM}_{\mathrm{GTE}}$ is computed by simulating the difference in average surplus in an n-sized market with bids drawn from $\hat{F}_1(X_i)$ and one with bids drawn from $\hat{F}_0(X_i)$.

3. Bias-corrected structural model estimator (`SMDR-GTE`). We solve an empirical version of (4) using the DML algorithm of Chernozhukov et al. (2018), where propensity scores are estimated using a random forest and conditional mean functions are computed as in `SM-GTE` with the lognormal assumption.

4. A doubly-robust estimator following the localization approach in Definition 1 (`LDML-GTE`). Both propensity scores and conditional mean functions estimated using random forests.

With only 100 datapoints, then the noise in the estimation for methods that rely on estimating the distribution of bids directly is high. As the number of datapoints increases,

	n=100		n=1,000		n=10,000	
	Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
`DR-ATE`	0.29	0.30	0.26	0.26	0.242	0.243
`SM-GTE`	-0.17	0.39	0.0019	0.021	0.000	**0.005**
`SMDR-GTE`	-0.17	0.39	-0.0016	0.031	-0.0003	0.008
`LDML-GTE`	0.034	0.09	0.0017	0.028	-0.0008	**0.008**

Table 1: Bids follow a log-normal distribution. Metrics averaged over 100 simulations of each sample size from the data-generating process.

the model-based estimator, which makes the correct parametric assumption on the bid distribution, converges the fastest. The bias-corrected structural model also performs well, although has increased variance since the bias correction adds noise when the model is correct. The LDML estimator does not make any parametric assumptions, and instead uses flexible-machine learning estimators for nuisance parameter estimation. It has an asymptotic distribution that does not depend on the estimation errors of the nuisance functions. The ATE estimator, which ignores the equilibrium effect of the treatment, has a large bias even as the sample size increases.

In the second set of simulations, we generate bids from a truncated normal distribution rather than a lognormal distribution. Otherwise, the data-generating process is the same. We compute the set of estimators, where we continue to use a random-forest based approach for the nuisance functions for the LDML estimators, and a log-normal based approach for the structural modeling estimators.

	n=100		n=1,000		n=10,000	
	Bias	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
`GRF-ATE`	0.10	0.08	0.094	0.096	0.093	0.093
`SM-GTE`	0.14	0.29	0.068	0.10	0.078	0.080
`SMDR-GTE`	0.04	0.22	0.0004	0.018	0.0000	**0.0049**
`LDML-GTE`	-0.01	0.05	0.0004	0.015	0.0004	**0.0047**

Table 2: Truncated Normal Distribution for Bids. Metrics averaged over 100 simulations of each sample size from the data-generating process.

This time, the structural modeling approach performs very poorly. The parametric assumption is incorrect, and as a result the outcome model is asymptotically biased. The SMDR estimator uses the propensity score to successfully remove the bias from the structural model. The LDML estimator does not make any parametric assumptions on the bid distribution and continues to perform very well here.

If a parametric model is correctly specified, then a maximum-likelihood estimator of that model is asymptotically linear and efficient. In addition, once the primitives of the model are specified and estimated, a variety of counterfactuals can often be evaluated, including those that are more complex than the estimand considered in this paper. The downside of this approach is if the model is not correctly specified, then the estimator of τ^*_{GTE} will be asymptotically biased. Unfortunately, it can be challenging to specify a parametric model that captures the complexity and heterogeneity of individual choice behavior, especially in settings where possible submissions to the mechanism are high-dimensional. The localized doubly-robust estimator performs well, without requiring correct specification of a parametric model of submissions to the mechanism.

5.2 Analysis of Coverage and Confidence Interval Width

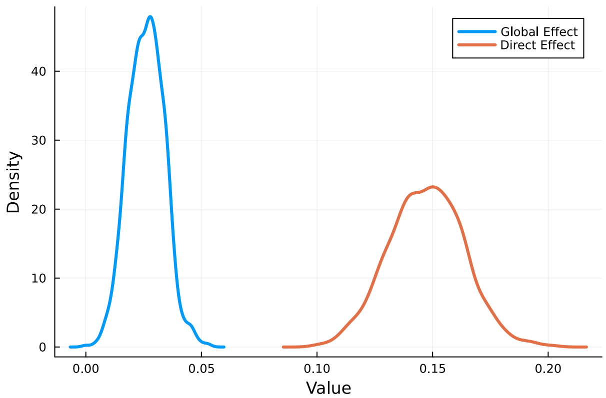
We next construct a simulation of a schools market, where individuals rank schools according to a random utility model, and the treatment affects a subgroup of students' preferences for a high quality school. There are three schools, with fractional capacity of 25%, 25% and 100%, respectively. Only the first two are high quality. The outcome is average match-value, where the planner has a higher value for a certain subgroup of students attending a high quality school. The data-generating process is described in detail in Appendix B.

The distribution of the ground truth for two estimands defined on a sample of n individuals is plotted in Figure 1a. Theorem 1 indicates that distribution of $\sqrt{n}(\bar{\tau}_{\text{GTE}} - \tau^*_{\text{GTE}})$ is asymptotically normal, and we see in the plot that the density for $\bar{\tau}_{\text{GTE}}$ roughly corresponds to a normal density. We also plot the distribution of the estimand $\bar{\tau}_{\text{DTE}}$ in repeated samples from the data-generating process. $\bar{\tau}_{\text{DTE}}$ is the average direct treatment effect, which is defined in Hu et al. (2022) as

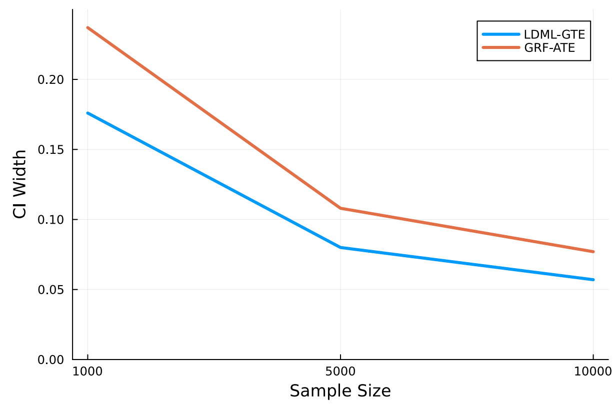
$$\bar{\tau}_{DTE} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\pi[Y_i(W_i = 1; \boldsymbol{W}_{-i})] - \mathbb{E}_\pi[Y_i(W_i = 0; \boldsymbol{W}_{-i})].$$

This estimand is relevant, because estimators for the average treatment effect are consistent for $\bar{\tau}_{\text{DTE}}$ when used in settings with interference (Sävje et al. 2021). With samples of data drawn from the data-generating process, we construct estimates and conservative confidence intervals for $\bar{\tau}_{\text{DTE}}$ by using methods for the averaged treatment effect based on generalized random forests, as described in Athey et al. (2019), and implemented in the R package `grf`. The results in Munro et al. (2023) suggest that for this simulation, using confidence intervals for the average treatment effect will be slightly conservative for $\bar{\tau}_{\text{DTE}}$. For the confidence intervals for $\bar{\tau}_{\text{GTE}}$, we use the LDML estimator and confidence intervals for τ^*_{GTE} that are described in Section 3. These are conservative for the finite market estimand $\bar{\tau}_{\text{GTE}}$.

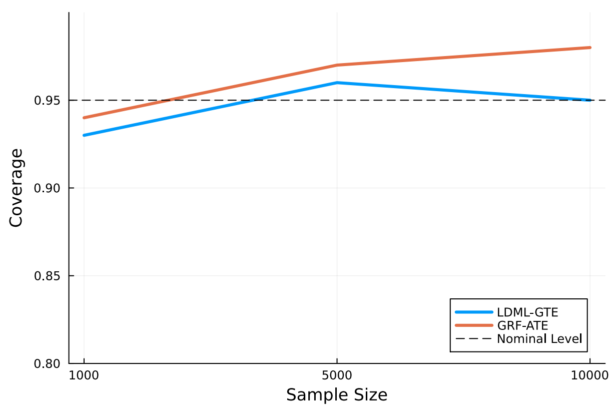
We see in Figure 1c that both the ATE and GTE confidence intervals are near the nominal coverage level for their respective estimands, with the GRF-derived confidence intervals slightly over-covering. However, since the partial equilibrium effect $\bar{\tau}_{\mathrm{DTE}}$ varies more than the general equilibrium effect, the confidence interval width for the estimate of $\bar{\tau}_{\mathrm{GTE}}$ is substantially more narrow than the width for the estimate of the $\bar{\tau}_{DTE}$.



(a) The distribution of $\bar{\tau}_{\mathrm{GTE}}$ and $\bar{\tau}_{DTE}$ for a repeated sample of $n = 1000$ agents over $S = 1000$ samples



(b) Confidence interval width for treatment effect estimators, averaged over $S = 100$ samples



(c) Coverage for treatment effect estimators, averaged over $S = 100$ samples

Figure 1: Monte Carlo Simulation Results

6 Impact Evaluation in the Chilean School Market

In 2015, the Chilean government passed the Inclusion Law, which, among many other changes, eliminated school-specific admissions criteria in favor of a centralized admission system (Correa et al. 2019). The centralized admission system is based on deferred acceptance, and was intended partly to reduce socio-economic segregation in the Chilean school system, by removing discriminatory admissions criteria and reserving some seats for low-income families. Despite these changes, low-income families attend good-quality schools at a much lower rate than high-income families.

There are variety of reasons why the gap might remain after the broad changes to the school system beginning in 2015. Lower income families may live further from higher-quality schools, and may prefer to attend closer schools due to budget or time constraints. Another reason is that some families may lack information about school quality, or the returns to schooling. Allende et al. (2019) explore this hypothesis using an RCT that randomized a video and report card providing information on nearby school quality. They found that the intervention increases applications of low-income families to high-quality schools. However, by simulating from a parametric model of demand for schools, they find that the effect on allocations in equilibrium is substantially less, due to capacity constraints.

The data from the previous paper, where unconfoundedness holds due to the randomized experimental design, is not available, so the approach in this paper, which avoids making a parametric assumption on individual choices, can't be compared directly. Instead, we estimate and perform inference on the effect of information on income inequality by constructing a similar observational dataset on Chilean students using data from the Ministry of Education in Chile. We also find that information affects choices positively, and that capacity constraints reduce the effect of the intervention on allocations significantly.

For this application, we combine two datasets from the Ministry of Education for 2018 - 2020. For the admissions system, we use publicly available data on the centralized admissions process (SAE) for 2020 for those applying to the 9th grade in Chile. This data includes the rankings each student submits to the algorithm, their priority, location, and actual assignment. We link this to another student-level dataset collected as part of the SIMCE[6] standardized test system in Chile. This data includes additional demographic information on parents and students collected through a survey, and is part of a private dataset that can be requested from the Ministry of Education in Chile. For school quality for the 9th grade admissions process, we use a rough measure which is the average student math and reading score for the school in 2018 amongst 10th graders.

The treatment we analyze is a proxy for the receipt of information on government school quality. $W_i = 1$ if a parent responds "Yes" to the following question:
Do you know the following information about your child's school? Performance category of this school. [7] 53% of the sample of 114,749 applicants to 9th grade have $W_i = 1$. The

[6] Sistema de Medición de la Calidad de la Educación

[7] The survey language (in Spanish) is: ¿Conoce usted la siguiente información del colegio de su hijo(a)? Categoría de desempeño de este colegio. It is the third question in the thirtieth section of the parent survey in the SIMCE dataset.

observed pre-treatment covariates are location (available for all applicants), and household size, mother and father education level, whether or not the mother and father are indigenous and the income of the family (available for those whose parents filled out the SIMCE survey in 8th grade). Missing covariates are imputed using a k-nearest neighbors approach. Table 5 in Appendix C includes the mean and standard deviation for each of the variables.

6.1 Treatment Effect Estimates

We first check that the treatment impacts the rankings that low-income families submit to the allocation mechanism, before we examine the effect on allocations. Submitted rankings are not subject to spillover effects through the allocation mechanism, since deferred acceptance is strategy-proof. So, we use `DR-ATE` to estimate the average treatment effect on two outcomes for low-income families, in Table 3. The first outcome is an indicator if the family ranks a top 50% school first, and second is the length of the application list that a family submits. Note that the length of the submitted rankings is unrestricted in the Chilean mechanism. The estimated treatment effect on ranking a high-quality school is 2.3%.[8] The effect on list length is positive, but small. So, there is evidence that the information intervention encourages low-income families to apply to better-quality schools.

	Top 50% School Ranked First	**Length of Application List**
`DR-ATE`	2.3%	0.03
	(0.40)	(0.01)

Table 3: `DR-ATE` estimates of the effect of information on applications of low-income families to 9th grade in 2019.

Because of capacity constraints, not all families that rank a high-quality school first are admitted to that school. Estimating treatment effects on allocations is more challenging due to interference that occurs through the allocation mechanism. Table 4 shows an estimate of treatment effects, when the outcome is whether a low income family is accepted to an above-average school in Chile. We see that the `DR-ATE` estimator, which corrects for selection, but not equilibrium effects, estimates a 1.3 percentage point increase in the allocation of low-income families to good quality schools. However, the LDML estimate of the GTE is 0.5 percentage points, which is much lower. Figure 2 provides a breakdown of the bias of the `DR-ATE` estimator. At the observed equilibrium, the probability of admission to a good-quality school is higher than at the 100% treated equilibrium, and lower than that of the 0%

[8]In the market, 36% of low income families with $W_i = 0$ rank a top-50% school first.

Estimator	Treatment Effect Estimate (s.e.)
`LDML-GTE`	0.54% (0.36)
`DR-ATE`	1.30% (0.32)
ATE-Bias	0.76% (0.38)

Table 4: Estimates of the treatment effect of informing parents about school quality on allocation of low-income families to good quality schools.

treated equilibrium. Estimating $\bar{\tau}_{\mathrm{GTE}}$ accurately requires estimating the access of treated families at the all-treated equilibrium, and control families at the all-control equilibrium.

We briefly discuss a possible source of bias in the `LDML-GTE` estimate. There are two possible sources of interference from an information treatment; the first is spillovers through the mechanism due to capacity constraints, and the second is network spillovers. The estimates in Table 4 only account for the first type of spillover. Even if a family does not report receiving school quality information, they may make choices that are correlated with their treated neighbors' choices. If the network spillovers are positive, so that increasing the number of treated neighbors always increases the probability that a family raises the rank of a high-quality school, then the effect estimate in Table 4 is a lower bound on the Global Treatment Effect under both network and congestion interference. If network spillovers are also sometimes positive and sometimes negative, then further work is needed to account for both types of spillovers. Regardless, we expect that congestion-related spillovers dominate network spillovers in this setting.

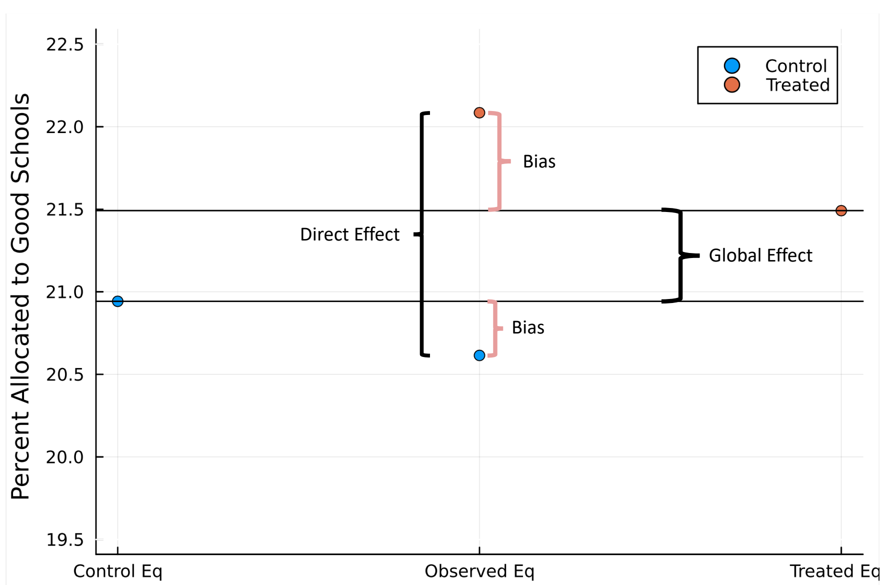


Figure 2: The `DR-ATE` estimator of the direct effect over-estimates the access of treated families to good-quality schools and under-estimates the access of control families.

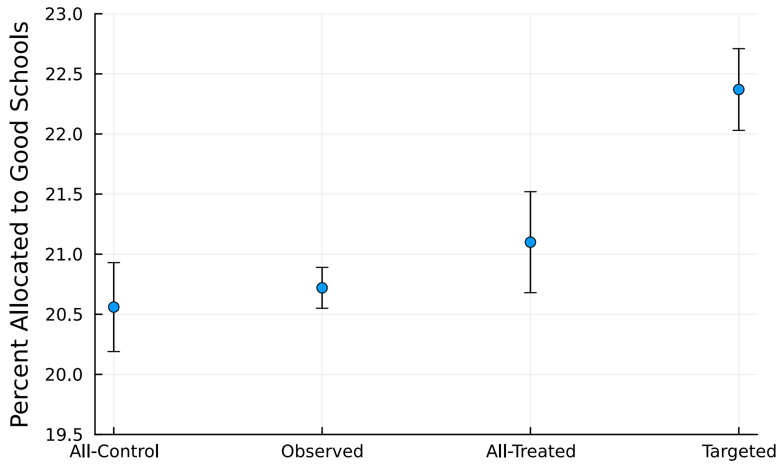


Figure 3: The estimated percentage of low-income families assigned to a good-quality school for different treatment rules. Error bars are standard errors

By using a non-parametric causal framework to analyze treatment effects in this setting, heterogeneity at an individual level is not restricted. There may be heterogeneity in whether or not individuals respond positively to the information, as well as heterogeneity in spillover effects. As discussed in Section 4 we can choose and evaluate treatment rules that treat only a subset of the sample defined by pre-treatment covariates.

Figure 3 estimates the outcomes for a variety of treatment rules. All-Control assigns nobody to treatment and All-Treated assigns everybody to treatment. The Observed rule is the treatment pattern observed in the data. The targeting rule approximates a version of the globally optimal rule in Section 4.1 through plug-in estimation and the value of the rule is estimated on a hold-out sample of the data.

The gain of the targeting rule over a rule that treats everyone is large, at 1.27% with an estimated standard error computed using the bootstrap of 0.46%. It also significantly outperforms a simple rule that assigns treatment only to low income families. This indicates that there is substantial heterogeneity in treatment response in the data.

It is not clear that in practice it would be desirable or fair to target the basic information on school quality considered in this specific example. However, the presence of significant heterogeneity in treatment response suggests that targeted policies may be of interest in school choice settings.

7 Discussion

Without some structure, estimating general causal effects under interference is infeasible. Under a fully specified and point-identified parametric model of individuals interacting in a market, any counterfactual can be simulated, but the model must be specified correctly. In this paper, we instead use the structure implied by the existence of a centralized allocation mechanism, but remain non-parametric about individual choices, which can be difficult to specify correctly. Using continuum market approximations to finite-sized markets, we show that equilibrium effects in finite samples are well-approximated by a set of moment conditions. This leads to a computationally simple and doubly-robust estimator for the value of counterfactual policies.

With data from the school market in Chile, we show that correcting for congestion effects substantially reduces the estimated effect of an information intervention on inequality in school allocations. Furthermore, there is significant heterogeneity in the effect of the information intervention, so a targeting rule performs much better than a policy that provides information to everybody.

There are a variety of counterfactuals of interest that go beyond the estimands considered in this paper. These include settings with supply-side responses, outcomes that are a non-deterministic function of allocations, and mechanisms with strategic behavior, where individuals make choices conditional on their expectations of the market equilibrium. For these problems, exploring whether it is possible to derive robust estimators that combine non-parametric causal methodology with economic structure imposed by design will be an interesting avenue for future work.

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A Proofs of Main Results

A.1 Notation

We first introduce notation which will be used throughout the proofs. The norm $||\cdot||$ is the $L2$-norm.

Similar to how we defined $\Gamma_{d,i}(p;\hat{\eta})$ in Assumption 5, we can define doubly-robust scores on outcomes with estimated nuisances:

$$\Gamma^{y}_{1i}(p;\hat{\eta}) = \hat{\mu}^{y,k(i)}_1(X_i) + \frac{W_i}{\hat{e}^{k(i)}(X_i)}(y(B_i(w),p) - \hat{\mu}^{y,k(i)}_1(X_i))$$
$$\Gamma^{y}_{0i}(p;\hat{\eta}) = \hat{\mu}^{y,k(i)}_0(X_i) + \frac{1-W_i}{1-\hat{e}^{k(i)}(X_i)}(y(B_i(w),p) - \hat{\mu}^{y,k(i)}_0(X_i))$$

Let $\Gamma^{y}_{n,\pi}(p;\eta) = \frac{1}{n}\sum_{i=1}^{n}\Big(\pi(X_i)\Gamma^{y}_{1i}(p;\eta) + (1-\pi(X_i))\Gamma^{y}_{0i}(p;\eta)\Big)$ and $y(p;\eta) = \mathbb{E}_T[\Gamma^{y}_{n,\pi}(p;\eta)]$. Similarly, $\Gamma^{z}_{n,\pi}(p;\eta) = \frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\Gamma^{d}_{1i}(p;\eta)+(1-\pi(X_i))\Gamma^{d}_{0i}(p;\eta)-s^*$ and $z(p;\eta) = \mathbb{E}_T[\Gamma^{z}_{n,\pi}(p;\eta)]$. Note that $\Gamma^{*y}_{wi}(p) = \Gamma^{y}_{wi}(p;\eta^*)$ and $\Gamma^{*d}_{wi}(p) = \Gamma^{d}_{wi}(p;\eta^*)$ for $w\in\{0,1\}$, where η^* collects the true propensity score and conditional mean functions. Similarly, we have $y_\pi(p;\eta^*) = y_\pi(p)$ and $z_\pi(p;\eta^*) = z_\pi(p)$. For empirical averages of actual outcomes and allocations rather than doubly robust-scores, we also define:

$$Y_{n,\pi}(p) = \frac{1}{n}\Big(\pi(X_i)y(B_i(1),p) + (1-\pi(X_i))y(B_i(0),p)\Big),$$
$$Z_{n,\pi}(p) = \frac{1}{n}\Big(\pi(X_i)d(B_i(1),p) + (1-\pi(X_i))d(B_i(0),p)\Big) - s^*.$$

A.2 Proof of Theorem 1

The first part of the Theorem holds by Lemma 11. For the asymptotically linear expansion, we next need to prove that for any $\pi\in\Pi$,

$$Y_{n,\pi}(P_\pi) = Y_{n,\pi}(p^*_\pi) - \nu_\pi Z_{n,\pi}(p^*_\pi) + o_p(n^{-1/2}). \tag{10}$$

Since $\bar{\tau}_{\text{GTE}} = Y_{n,1}(P_1)-Y_{n,0}(P_0)$, where the subscript 1 and 0 refers to a treatment rule where everybody and nobody is treated, respectively, then the following argument completes the proof:

$$\begin{aligned}\bar{\tau}_{\text{GTE}} - \tau^*_{\text{GTE}} &= Y_{n,1}(p^*_1) - \nu_1 Z_{n,1}(p^*_1) + \nu_0 Z_{n,0}(p^*_0) - Y_{n,0}(p^*_0) - \tau^*_{\text{GTE}} + o_p(n^{-1/2}),\\ &= \frac{1}{n}\sum_{i=1}^{n} q_1(B_i(1),p^*_1) - q_0(B_i(0),p^*_0) - \tau^*_{\text{GTE}} + o_p(n^{-1/2}),\end{aligned}$$

Since outcomes and net demand are bounded, then the variance of the term in the expansion is finite, and the CLT also applies to this expansion. Thus, to finish the proof, we show (10).

$$\begin{aligned} Y_{n,\pi}(P_\pi) &= Y_{n,\pi}(p^*_\pi) + y_\pi(P_\pi) - y_\pi(p^*_\pi) + o_p(n^{-1/2}), \\ &= Y_{n,\pi}(p^*_\pi) - \nu_\pi Z_{n,\pi}(p^*_\pi) + o_p(n^{-1/2}). \end{aligned}$$

The first line is by Lemma 14. The second line is by a combination of a first-order taylor expansion and Lemma 8. As a last step for this proof, we prove Lemma 8.

Lemma 8. ***Asymptotic Normality of Counterfactual Cutoffs*** *Under the Assumptions of Theorem 1, then the market-clearing cutoffs under treatment rule $\pi \in \Pi$, which we call P_π, are asymptotically linear:*

$$\sqrt{n}(P_\pi - p^*_\pi) = -(\nabla_p z_\pi(p^*_\pi))^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (W_i d(B_i(1), p^*_\pi) + (1 - W_i) d(B_i(0, p^*_\pi) - s^*)$$

Proof. First, by Lemma 18, we have that $P_\pi = p^*_\pi + O_p(n^{-1/2})$. To strengthen this to an asymptotic linearity result, we use Theorem 3.3.1 of van der Vaart & Wellner (1997). By Assumption 2, we have the required market-clearing condition, $Z_{n,\pi}(P_\pi) = o_p(n^{-1/2})$. By Lemma 14, we have that $Z_{n,\pi}(P_\pi) - z_\pi(P_\pi) - Z_{n,\pi}(p^*_\pi) + z_\pi(p^*_\pi) = o_p(n^{-1/2})$. By Assumption 3, $\nabla_p z_\pi(p)$ is twice continuously differentiable in p and $\nabla_p z_\pi(p)$ is positive definite at p^*_π. Since allocations are bounded, $\mathbb{E}[(\pi(X_i)d(B_i(1), p) + (1 - \pi(X_i))d(B_i(0), p) - s^*)^2]$ is bounded. By Theorem 3.3.1 of van der Vaart & Wellner (1997), verifying these conditions is enough to prove the theorem:

$$(P_\pi - p^*_\pi) = -[\nabla_p z_\pi(p^*_\pi)]^{-1} Z_{n,\pi}(p^*_\pi) + o_p(n^{-1/2}).$$

□

A.3 Proof of Theorem 2 and Corollary 3

The proof of Theorem 2 follows some of the structure and ideas in Kallus et al. (2019). For Theorem 2, we start with the following expansion:

$$\begin{aligned} \hat{V}_n(\pi) &= \Gamma^y_{n,\pi}(\hat{P}_\pi; \hat{\eta}_\pi) \\ &= \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) + y_\pi(\hat{P}_\pi; \hat{\eta}_\pi) - y_\pi(p^*_\pi, \eta^*_\pi) + R_{1n} \\ &= \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) + y_\pi(\hat{P}_\pi; \eta^*_\pi) - y_\pi(p^*_\pi; \eta^*_\pi) + R_{1n} + R_{2n} \\ &= \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) - \nu^*_\pi Z_n(p^*_\pi; \eta^*_\pi) + R_{1n} + R_{2n} + R_{3n} \end{aligned}$$

To finish the proof, we need to show that each of the remainder terms are $o_p(n^{-1/2})$.

$$R_{1n} = \Gamma^y_{n,\pi}(\hat{P}_\pi; \hat{\eta}_\pi) - \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) - y_\pi(\hat{P}_\pi; \hat{\eta}_\pi) + y_\pi(p^*_\pi, \eta^*_\pi)$$

By Lemma 15, $R_{1n} = o_p(n^{-1/2})$. $R_{2n} = y_\pi(\hat{P}_\pi; \hat{\eta}_\pi) - y_\pi(\hat{P}_\pi; \eta^*_\pi)$. By Lemma 17, Assumption 5 and the rate for $\hat{P}_\pi$ in Lemma 9, $R_{2n} = o_p(n^{-1/2})$. For R_{3n}, by a Taylor expansion, we have

$$\begin{aligned} y_\pi(\hat{P}_\pi; \eta^*_\pi) - y_\pi(p^*_\pi; \eta^*_\pi) &= \nabla_p^\top [y_\pi(p^*_\pi; \eta^*_\pi)](\hat{P}_\pi - p^*_\pi) + O(||\hat{P}_\pi - p^*_\pi||^2) \\ &\overset{(1)}{=} -\nu^*_\pi Z_n(p^*_\pi; \eta^*_\pi) + o_p(n^{-1/2}) + O(||\hat{P}_\pi - p^*_\pi||^2) \\ &\overset{(2)}{=} -\nu^*_\pi Z_n(p^*_\pi; \eta^*_\pi) + o_p(n^{-1/2}). \end{aligned}$$

(1) and (2) are both by Lemma 9. We have now shown that $\hat{V}_n(\pi) = \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) - \nu^*_\pi \Gamma^z_{n,\pi}(p^*_\pi; \eta^*_\pi) + o_p(n^{-1/2})$. We can now apply this expansion to $\hat{\tau}_{\text{GTE}} = \hat{V}_n(\mathbf{1}_n) - \hat{V}_n(\mathbf{0}_n)$.

$$\hat{\tau}_{\text{GTE}} = \frac{1}{n} \sum_{i=1}^n \Gamma^q_{1,i}(p^*_\pi; \eta^*_\pi) - \Gamma^q_{0,i}(p^*_\pi; \eta^*_\pi) + o_p(n^{-1/2}).$$

Centering at τ^*_{GTE}, we have an average of mean-zero and i.i.d. terms with finite variance:

$$\hat{\tau}_{\text{GTE}} - \tau^*_{\text{GTE}} = \frac{1}{n} \sum_{i=1}^n \Gamma^q_{1,i}(p^*_\pi; \eta^*_\pi) - \mathbb{E}[\Gamma^q_{1,i}(p^*_\pi; \eta^*_\pi)] - (\Gamma^q_{0,i}(p^*_\pi; \eta^*_\pi) - \mathbb{E}[\Gamma^q_{1,i}(p^*_\pi; \eta^*_\pi)]) + o_p(n^{-1/2}).$$

So, the CLT now applies:

$$\sqrt{n}(\hat{\tau}_{\text{GTE}} - \tau^*_{\text{GTE}}) \Rightarrow_d N(0, \sigma^2),$$

where $\sigma^2 = \text{Var}(\Gamma^q_{1,i}(p^*_\pi; \eta^*_\pi) - \Gamma^q_{0,i}(p^*_\pi; \eta^*_\pi))$.

Lemma 9. ***Central Limit Theorem for $\hat{P}_\pi$:*** *Under the Assumptions of Theorem 2, for each in $\pi \in \Pi$,*

$$\sqrt{n}(\hat{P}_\pi - p^*_\pi) = -[\nabla_p z_\pi(p^*_\pi)]^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \Big(\pi(X_i)\Gamma^d_{1i}(p^*_\pi; \eta^*_\pi) + (1 - \pi(X_i))\Gamma^d_{0i}(p^*_\pi; \eta^*_\pi) \Big) + o_p(1).$$

Proof. By Lemma 19, $\hat{P}_\pi - p^*_\pi = O_p(n^{-1/2})$. We now strengthen this to a central limit theorem that applies for arbitrary $\pi \in \Pi$. By Lemma 15,

$$\Gamma^z_{n,\pi}(\hat{P}_\pi; \hat{\eta}_\pi) - \Gamma^z_{n,\pi}(p^*_\pi; \eta^*_\pi) = z_\pi(\hat{P}_\pi; \hat{\eta}_\pi) - z_\pi(p^*_\pi; \eta^*_\pi) + o_p(n^{-1/2}).$$

By Lemma 17,

$$\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) = z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi) + o_p(n^{-1/2}).$$

Recalling that by Assumption 5, $\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) = o_p(n^{-1/2})$, we can now use a Taylor expansion:

$$\begin{aligned}
-\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) &= \nabla_p z_\pi(p^*_\pi)(\hat{P}_\pi - p^*_\pi) + O(||\hat{P}_\pi - p^*_\pi||^2) + o_p(n^{-1/2}) \\
-\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) &= \nabla_p z_\pi(p^*_\pi)(\hat{P}_\pi - p^*_\pi) + o_p(n^{-1/2}), \\
\hat{P}_\pi - p^*_\pi &= -[\nabla_p z_\pi(p^*_\pi)]^{-1}\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi)
\end{aligned}$$

where the second line is by Lemma 19. This now completes the proof, since $\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) = \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\Gamma^d_{1i}(p^*_\pi;\eta^*_\pi) + (1-\pi(X_i))\Gamma^d_{0i}(p^*_\pi;\eta^*_\pi)$.

□

A.4 Proof of Proposition 5

The two main expansions used here are:

$$\begin{aligned}
\bar{\tau}_{\text{GTE}} &= \frac{1}{n}\sum_{i=1}^{n}[q_1(B_i(1),p^*_1) - q_0(B_i(0),p^*_0)] + o_p(n^{-0.5}), \\
\hat{\tau}_{\text{GTE}} &= \frac{1}{n}\sum_{i=1}^{n}\left[\Gamma^{*q}_{1i}(p^*_1) - \Gamma^{*q}_{0i}(p^*_0)\right] + o_p(n^{-1/2}),
\end{aligned}$$

Notice that $\mathbb{E}[\Gamma^{*q}_{1i}(p^*_1) - \Gamma^{*q}_{0i}(p^*_0)|X_i,B_i(1),B_i(0)] = q_1(B_i(1),p^*_1) - q_0(B_i(0),p^*_0)$. Combining these, we have that

$$\hat{\tau}_{\text{GTE}} - \bar{\tau}_{\text{GTE}} = \frac{1}{n}\sum_{i=1}^{n}\Gamma^{*q}_{1i}(p^*_1) - \Gamma^{*q}_{0i}(p^*_0) - [q_1(B_i(1),p^*_1) - q_0(B_i(0),p^*_0)].$$

Then, using the CLT,

$$\begin{aligned}
\sqrt{n}\,(\hat{\tau}_{\text{GTE}} - \bar{\tau}_{\text{GTE}}) &\Rightarrow N(0,\bar{\sigma}^2), \\
\sqrt{n}\,(\hat{\tau}_{\text{GTE}} - \tau^*_{\text{GTE}}) &\Rightarrow N(0,\sigma^2),
\end{aligned}$$

with $\bar{\sigma}^2 = \mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))^2]$ and $\sigma^2 = \mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - \tau^*_{\text{GTE}})^2]$. Working with σ^2:

$$\begin{aligned}
\sigma^2 &= \mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - \tau^*_{\text{GTE}})^2] \\
&= \mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*) + q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})^2] \\
&= \mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))^2] + \mathbb{E}[(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})^2] \\
&\qquad + 2\mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})] \\
&\overset{(1)}{=} \bar{\sigma}^2 + \mathbb{E}[(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})^2]
\end{aligned}$$

(1) comes from the law of iterated expectations, with the details shown below:

$$\begin{aligned}
&\mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})] \\
&= \mathbb{E}[\mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))(q_1(B_i(1), p_1^*) \\
&\qquad - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})|X_i, B_i(1), B_i(0)]] \\
&= \mathbb{E}[\mathbb{E}[(\Gamma^{*q}_{1i}(p_1^*) - \Gamma^{*q}_{0i}(p_0^*) - q_1(B_i(1), p_1^*) + q_0(B_i(0), p_0^*))|X_i, B_i(1), B_i(0)](q_1(B_i(1), p_1^*) \\
&\qquad - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})] \\
&= 0.
\end{aligned}$$

This implies that $\bar{\sigma}^2 = \sigma^2 - \mathbb{E}[(q_1(B_i(1), p_1^*) - q_0(B_i(0), p_0^*) - \tau^*_{\text{GTE}})^2]$. Since the second term in the right hand side is weakly positive, $\bar{\sigma}^2 \leq \sigma^2$, which proves the corollary.

A.5 Proof of Theorem 4

The proof follows uses the methodology presented in Bickel et al. (1993) and Newey (1990). The organization and notation of the proof is similar to other papers that apply this methodology to related estimands, including Hahn (1998) and Hirano et al. (2003) for average treatment effects, Firpo (2007) for quantile treatment effects, and Chen & Ritzwoller (2021) for long-run treatment effects. The presentation and notation is closest to that of Firpo (2007).

Deriving the Score Function

Under Assumption 1, the density of the data $(B_i(1), B_i(0), W_i, X_i)$ can be factorized as:

$$\phi(b(1), b(0), w, x) = f(b(1), b(0), |x)e(x)^w(1 - e(x))^{1-w}f(x)$$

Under Assumption 1, the density of the observed data (B, W, X) can be factorized as:

$$\phi(b, w, x) = [f_1(b|x)e(x)]^w [f_0(b|x)(1 - e(x))]^{1-w} f(x).$$

where $f_1(b|x) = \int f(b_1, b_0|x)db_0$ and $f_0(b|x) = \int f(b_1, b_0|x)db_1$. We define a regular parametric submodel of the observed data density indexed by θ:

$$\phi(b, w, x; \theta) = [f_1(b|x; \theta)e(x; \theta)]^w [f_0(b|x; \theta)(1 - e(x; \theta))]^{1-w} f(x; \theta)$$

We can now derive the score of the parametric submodel:

$$s(b, w, x; \theta) = w \cdot s_1(b|x; \theta) + (1 - w) \cdot s_0(b|x; \theta) + \frac{w - e(x)}{e(x)(1 - e(x))} e'(x) + s_x(x; \theta)$$

where

$$s_1(b|x; \theta) = \frac{\partial}{\partial \theta} \log f_1(b|x; \theta), \qquad s_0(b|x; \theta) = \frac{\partial}{\partial \theta} \log f_0(b|x; \theta), \qquad e'(x; \theta) = \frac{\partial}{\partial \theta} \log e(x; \theta),$$
$$s_x(x; \theta) = \frac{\partial}{\partial \theta} \log f(x; \theta).$$

The tangent space of this model is defined as the set of functions

$$g(r, w, x) = wg_1(b|x) + (1 - w)g_0(b|x) + (w - e(x))g_2(x) + g_3(x)$$

such that g_1 through g_3 range through all square integrable functions satisfying

$$\mathbb{E}[g_1(B_i|X_i)|X_i = x, W_i = 1] = 0$$
$$\mathbb{E}[g_0(B_i|X_i)|X_i = x, W = 0] = 0$$
$$\mathbb{E}[g_3(X_i)] = 0$$

Pathwise Differentiability

We derive a Fréchet derivative of $\tau^*_{\text{GTE}} = \tau^*_1 - \tau^*_0$, where $\tau^*_1 = \mathbb{E}[y(B_i(1), p^*_1)]$ and $\tau^*_0 = \mathbb{E}[y(B_i(0), p^*_0)]$. We go through the details for τ^*_1, and then state the result for τ^*_0, since the derivation follows the same steps.

$$\tau'_1 = \nabla_p \mathbb{E}[y(B_i(1), p^*_1)]^\top p'_1 + \frac{\partial}{\partial \theta} \int \int y(b, p^*_1) f_1(b|x; \theta) f(x; \theta) db dx \tag{11}$$

The next step is to derive p'_1. By the uniqueness of Assumption 4, p^*_1 is defined implicitly by $\mathbb{E}[d(B_i(1), p_1^*) - s^*] = 0$. By the implicit function theorem, we can write

$$p_1' = -\nabla_p \mathbb{E}[d(B_i(1), p_1^*) - s^*]^{-1} \frac{\partial}{\partial \theta} \int \int (d(b, p_1^*) - s^*) f_1(b|x;\theta) f(x;\theta) dbdx.$$

The derivative of the moment conditions, evaluated at θ_0, are as follows, where we write $f(x;\theta_0) = f(x)$ and $f_1(b|x;\theta_0) = f_1(b|x)$.

$$\begin{aligned}
\frac{\partial}{\partial \theta} \int \int y(b, p_1^*) f_1(b|x;\theta) f(x;\theta) dbdx &= \int \int y(b, p_1^*) s_1(b|x) f_1(b|x) f(x) dbdx \\
&\quad + \int \int y(b, p_1^*) f_1(b|x) s_x(x) f(x) dbdx \\
\frac{\partial}{\partial \theta} \int \int (d(b, p_1^*) - s^*) f_1(b|x;\theta) f(x;\theta) dbdx &= \int \int (d(b, p_1^*) - s^*) s_1(b|x) f_1(b|x) f(x) dbdx \\
&\quad + \int \int (d(b, p_1^*) - s^*) f_1(b|x) s_x(x) f(x) dbdx
\end{aligned}$$

Plugging these into the Equation 11,

$$\tau_1' = \int \int q_1(b) s_1(b|x) f_1(b|x) f(x) dbdx + \int \int q_1(b) f_1(b|x) s_x(x) f(x) dbdx,$$

where $q_1^*(b) = y(b, p_1^*) - \nu_1^*(d(b, p_1^*) - s^*)$. Let $q_0^*(b) = y(b, p_0^*) - \nu_0^*(d(b, p_0^*) - s^*)$. After the same procedure for τ_0', we can write

$$\begin{aligned}
\tau_{\text{GTE}}' =& \int \int q_1^*(b) s_1(b|x) f_1(b|x) f(x) dbdx + \int \int q_1^*(b) f_1(b|x) s_x(x) f(x) dbdx \\
&- \int \int q_0^*(b) s_0(b|x) f_0(b|x) f(x) dbdx - \int \int q_0^*(b) f_0(b|x) s_x(x) f(x) dbdx. \\
=& \mathbb{E}[q_1^*(B_i(1)) s_1(B_i(1)|X_i)] + \mathbb{E}[\mu_1^q(X_i) s_x(X_i)],
\end{aligned}$$

where $\mu_w^q(X_i) = \mathbb{E}[q_w^*(B_i)|X_i, W_i = w]$ for $w \in \{0, 1\}$.

Conjectured Efficient Influence Function

A function that is in the tangent space is:

$$\begin{aligned}
\psi(B_i, W_i, X_i) =& \mathbb{E}[q_1^*(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0^*(B_i)|X_i, W_i = 0] - \tau \\
&+ \frac{W_i(q_1^*(B_i) - \mathbb{E}[q_1^*(B_i)|X_i, W_i = 1])}{e(x)} - \frac{(1 - W_i)(q_0^*(B_i) - \mathbb{E}[q_0^*(B_i)|X_i, W_i = 0])}{1 - e(x)}.
\end{aligned}$$

We can verify it is in the tangent space.

1. $g_1(b|x) = \frac{q_1^*(b) - \mathbb{E}[q_1^*(B_i)|X_i=x, W_i=1]}{e(x)}$. For any x,

$$\mathbb{E}[g_1(B_i|X_i)|X_i = x, W_i = 1)] = \frac{\mathbb{E}[q_1^*(B_i)|X_i = x, W_i = 1] - \mathbb{E}[q_1^*(B_i)|X_i = x, W_i = 1]}{e(x)} = 0.$$

2. $g_0(b|x) = \frac{q_0^*(b) - \mathbb{E}[q_0^*(B_i)|X_i=x, W_i=0]}{1-e(x)}$. For any x,

$$\mathbb{E}[g_0(B_i|X_i)|X_i = x, W_i = 0] = \frac{\mathbb{E}[q_0^*(B_i)|X_i = x, W_i = 0] - \mathbb{E}[q_0^*(B_i)|X_i = x, W_i = 0]}{1 - e(x)} = 0.$$

3. $g_2(x) = 0$

4. $g_3(x) = \mathbb{E}[q_1^*(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0^*(B_i)|X_i, W_i = 0] - \tau$

$$\mathbb{E}[g_3(X_i)] = \mathbb{E}[\mu_1^q(X_i)] - \mathbb{E}[\mu_0^q(X_i)] - \mathbb{E}[\mu_1^q(X_i)] + \mathbb{E}[\mu_0^q(X_i)] = 0.$$

Given it is an element of the tangent space, if it is an influence function it is efficient. To verify that is an influence function, we must show that $\mathbb{E}[\psi(B_i, W_i, X_i)s(B_i, W_i, X_i)] = \tau'$. We can divide $\psi(B_i, W_i, X_i) = \psi_1(B_i, W_i, X_i) - \psi_0(B_i, W_i, X_i)$, where

$$\psi_1(B_i, W_i, X_i) = \mathbb{E}[q_1^*(B_i)|X_i, W_i = 1] - \mathbb{E}[q_1^*(B_i)|W_i = 1] + \frac{W_i(q_1^*(B_i) - \mathbb{E}[q_1^*(B_i)|X_i, W_i = 1])}{e(x)}$$

$$\begin{aligned}\psi_0(B_i, W_i, X_i) = {} & \mathbb{E}[q_0^*(B_i)|X_i, W_i = 1] - \mathbb{E}[q_0^*(B_i)|W_i = 0] \\ & + \frac{(1 - W_i)(q_0^*(B_i) - \mathbb{E}[q_0^*(B_i)|X_i, W_i = 0])}{1 - e(x)}\end{aligned}$$

We work through the details for $\psi_1(\cdot)$, since the process is the same for $\psi_0(\cdot)$.

$$\begin{aligned}
& \mathbb{E}[\psi_1(B_i, W_i, X_i)s(B_i, W_i, X_i)] \\
& = \mathbb{E}\left[(q_1^*(B_i(1)) - \mu_1^q(X_i))s_1(B_i(1)|X_i) + s_x(X_i)(q_1^*(B_i(1)) - \mu_1^q(X_i))\right] \\
& \qquad + \mathbb{E}[W_i s_1(B_i(1)|X_i) \cdot \mu_1^q(X_i) + (1 - W_i)s_0(B_i(0)|X_i) \cdot \mu_1^q(X_i) + s_x(X_i)\mu_1^q(X_i)] \\
& = \mathbb{E}[q_1^*(B_i(1))s_1(B_i(1)|X_i)] + \mathbb{E}[s_x(X_i)\mu_1^q(X_i)] \\
& \qquad + \mathbb{E}[(1 - e(X_i))\mathbb{E}[s_1(B_i(1)|X_i) - s_0(B_i(0)|X_i)|X_i = x]\mu_1^q(X_i)] \\
& \overset{(1)}{=} \mathbb{E}[q_1^*(B_i(1))s_1(B_i(1)|X_i)] + \mathbb{E}[s_x(X_i)\mu_1^q(X_i)] \\
& = \tau_1'
\end{aligned}$$

(1) is because $\mathbb{E}[s_w(B_i(w)|X_i)|X_i = x] = 0$ for each $x \in \mathcal{X}$ and $w \in \{0, 1\}$.

Similarly, we can show that $\mathbb{E}[\psi_0(B_i, W_i, X_i)s(B_i, W_i, X_i)] = \tau_0'$. We have shown that the function $\psi(B_i, W_i, X_i)$ is an efficient influence function. The semi-parametric efficiency bound is thus:

$$\begin{aligned} V^* &= \mathbb{E}[\psi(B_i, W_i, X_i)^2], \\ &= \mathbb{E}[(\Gamma_{1i}^{q*}(p_1^*) - \Gamma_{0i}^{q*}(p_0^*) - \tau_{\mathrm{GTE}}^*)^2]. \end{aligned}$$

A.6 Theorem 6

The first step is to show that the Fréchet derivative of $V^*(\pi)$ at π is the linear functional defined by

$$\partial V^*(\pi)h = \int h(x)\mathbb{E}[q_1(B_i(1), p_\pi^*) - q_0(B_i(0), p_\pi^*)|X_i = x]dF_x(x).$$

where $h : \mathcal{X} \to [0, 1]$ and $F_x(\cdot)$ is the distribution of $X_i \in \mathcal{X}$. First, we write $V^*(\pi)$ as an integral over x:

$$\begin{aligned} V^*(\pi) &= \mathbb{E}[\pi(X_i)(y(B_i(1), p_\pi^*) + (1 - \pi(X_i))y(B_i(0), p_\pi^*))], \\ &= \int \Big(y_1(x, \pi) \cdot \pi(x) + y_0(x, \pi) \cdot (1 - \pi(x))\Big)dF(x), \end{aligned}$$

where $y_w(x, \pi) = \mathbb{E}[y(B_i(w), p_\pi^*)]$. We next derive the Fréchet derivative of $V^*(\pi)$ using the product rule, where $\tau^y(x, \pi) = y_1(x, \pi) - y_0(x, \pi)$.

$$\begin{aligned} \partial V(\pi)h &= \int \tau^y(x, \pi) \cdot h(x)dF(x) + \int \partial y_1(x, \pi)h \cdot \pi(x) + \partial y_0(x, \pi)h \cdot (1 - \pi(x)) \\ &\overset{(1)}{=} \int \tau^y(x, \pi) \cdot h(x)dF(x) - \nu_\pi^* \cdot \int h(x) \cdot \tau^d(x, \pi)dF(x) \\ &= \int h(x) \cdot (\tau^y(x, \pi) - \nu_\pi^* \tau^d(x, \pi))dF(x) \end{aligned}$$

Step (1) is from the chain rule, since

$$\int \pi(x) \cdot \partial y_1(x, \pi)h + \partial y_0(x, \pi)h \cdot (1 - \pi(x))dF(x) = \nabla_p^\top y_\pi(p_\pi^*)\partial p^*(\pi)h$$

and, by the implicit function theorem,

$$\partial p^*(\pi)h = -\nabla_p z_\pi(p_\pi^*)^{-1} \cdot \int h(x)(d_1(x, \pi) - d_0(x, \pi))dF(x),$$

where we can swap derivatives and expectation since the derivatives of conditional expectations are bounded. Since all functions from $\mathcal{X}$ to $[0,1]$ is a convex subset of a vector space, Theorem 2 of Chapter 7 of Luenberger (1969) indicates that a necessary condition for a local maximum π^* is that for all $\pi \in \Pi$,

$$\partial V(\pi)(\pi - \pi^*) \leq 0$$

Let $\rho(\pi, x) = (\tau^y(x,\pi) - \nu_\pi^* \tau^d(x,\pi))$. We can prove by contradiction that the optimal targeting policy must meet the conditions in the theorem. If there is some $\bar{\pi}$ that is optimal but does not meet the conditions in the theorem, then, one of the following must be true:

1. For x in some set Q that occur with non-zero probability, $\rho(\bar{\pi}, x) < 0$ but $\bar{\pi}(x) > 0$. But then choose π such that $\pi(x) = \bar{\pi}(x)$ for $x \notin Q$ and $\pi(x) = 0$ for $x \in Q$. We have that

$$\partial V(\pi)(\pi - \pi^*) = \int_{x \in Q} \rho(\bar{\pi}, x)(0 - \bar{\pi}(x)) dF(x) > 0,$$

which contradicts the optimality of $\bar{\pi}$.

2. Or, for x in some set P that occurs with non-zero probability, $\rho(\bar{\pi}, x) > 0$ but $\bar{\pi}(x) < 1$. Choose π such that $\pi(x) = \bar{\pi}(x)$ for $x \notin P$ and $\pi(x) = 1$ for $x \in P$. We have that

$$\partial V(\pi)(\pi - \pi^*) = \int_{x \in Q} \rho(\bar{\pi}, x)(1 - \bar{\pi}(x)) dF(x) > 0,$$

which contradicts the optimality of $\bar{\pi}$.

A.7 Proof of Theorem 7

First, we review some notation. Let $\pi \in \Pi$. We have estimated, oracle, finite-market and population versions of the value function.

$$\begin{aligned}
\hat{V}_n(\pi) &= \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\Gamma_{1i}^y(\hat{P}_\pi; \hat{\eta}_\pi) + (1 - \pi(X_i))\Gamma_{0i}^y(\hat{P}_\pi; \hat{\eta}_\pi) \\
V_n(\pi) &= \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\Gamma_{1i}^y(p_\pi^*; \eta_\pi^*) + (1 - \pi(X_i))\Gamma_{0i}^y(p_\pi^*; \eta_\pi^*), \\
\bar{V}_n(\pi) &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\pi\left[y(B_i(W_i), P_\pi)\right] \\
V^*(\pi) &= y_\pi(p_\pi^*).
\end{aligned}$$

Then, we follow the argument in Kitagawa & Tetenov (2018). For any $\tilde{\pi} \in \Pi$,

$$\begin{aligned} V^*(\tilde{\pi}) - V^*(\hat{\pi}) &= V^*(\tilde{\pi}) - \hat{V}_n(\hat{\pi}) + \hat{V}_n(\hat{\pi}) - V^*(\hat{\pi}) \\ &\leq V^*(\tilde{\pi}) - \hat{V}_n(\tilde{\pi}) + \hat{V}_n(\hat{\pi}) - V^*(\hat{\pi}) \\ &\leq 2 \sup_{\pi \in \Pi} |V^*(\pi) - \hat{V}_n(\pi)| \end{aligned}$$

For the finite-market regret bound, we have a similar argument. For any $\tilde{\pi} \in \Pi$, we have that

$$\begin{aligned} \bar{V}_n(\tilde{\pi}) - \bar{V}_n(\hat{\pi}) &= \bar{V}_n(\tilde{\pi}) - \hat{V}_n(\hat{\pi}) + \hat{V}_n(\hat{\pi}) - \bar{V}_n(\hat{\pi}) \\ &\leq \bar{V}_n(\tilde{\pi}) - \hat{V}_n(\tilde{\pi}) + \hat{V}_n(\hat{\pi}) - \bar{V}_n(\hat{\pi}) \\ &\leq 2 \sup_{\pi \in \Pi} |\hat{V}_n(\pi) - \bar{V}_n(\pi)| \\ &\leq 2 \sup_{\pi \in \Pi} |\hat{V}_n(\pi) - V^*(\pi)| + 2 \sup_{\pi \in \Pi} |\bar{V}_n(\pi) - V^*(\pi)| \end{aligned} \tag{12}$$

In addition, we have that

$$\sup_{\pi \in \Pi} |V^*(\pi) - \hat{V}_n(\pi)| \leq \sup_{\pi \in \Pi} |V^*(\pi) - V_n(\pi)| + \sup_{\pi \in \Pi} |\hat{V}_n(\pi) - V_n(\pi)| \tag{13}$$

Using notation from Section A.1, for the first term in (13),

$$\begin{aligned} \sup_{\pi \in \Pi} |V_n(\pi) - V^*(\pi)| &= \sup_{\pi \in \Pi} |\Gamma_n^y(p_\pi^*; \eta_\pi^*) - y_\pi(p_\pi^*; \eta_\pi^*)| \\ &\leq \sup_{\pi \in \Pi, p \in \mathcal{S}} |\Gamma_n^y(p; \eta_\pi^*) - y_\pi(p; \eta_\pi^*)| \\ &= O_p(1) \end{aligned}$$

where the conclusion that the term is $O_p(1)$ comes from Lemma 13. For the second term in (13), we use Lemma 12, so we can now conclude that

$$\sup_{\pi \in \Pi} \sqrt{n} |V^*(\pi) - \hat{V}_n(\pi)| = O_p(1).$$

This takes care of the regret bound for the continuum market and the first part of (12). For the second part of (12), to complete the regret bound for the finite-sized market, we use Lemma 11.

Online Appendix

B Simulation Details

The data-generating process for Section 5.1 is as follows, where $\Phi(\cdot)$ is the standard normal CDF:

$$
\begin{aligned}
&B_i(1) \sim F_1^B(X_i), \qquad B_i(0) \sim F_0^B(X_i), \qquad X_i \sim \text{Uniform}(0,1)^{20},\\
&W_i \sim \text{Bernoulli}(\Phi(X_{1i} - 0.5X_{2i} + 0.5X_{3i})), \qquad D_i(W_i, p) = \mathbb{1}(B_i(W_i) \geq p),\\
&Y_i(\boldsymbol{W}) = (B_i(W_i) - P(\boldsymbol{W}))\mathbb{1}(B_i(W_i) > P(\boldsymbol{W})), \qquad \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(B_i(W) > P(\boldsymbol{W})) = \frac{1}{2}.
\end{aligned}
$$

$F_1^B(x)$ and $F_0^B(x)$ are varied. In the simulation for Table 1, $B_i(0) \sim \text{LogNormal}(0.8X_{1i} - 0.3X_{2i} - 0.2X_{3i}, 0.3)$ and $B_i(1) = 1.5B_i(0)$.

We next describe the data generating process for the coverage simulation in Section 5.2. The fractional capacities of the schools are $q = [0.25, 0.25, 1.0]$. Schools 1 and 2 are high-quality, with $Q_j = 1$, and capacity constrained, but school 3, which is low quality, with $Q_j = 0$, is not. The subgroup of interest for the planner is denoted by $C_i \in \{0, 1\}$. The match value $V_{ij} = 2$ if $C_i = 1$ and $Q_j = 1$, and $V_{ij} = 1$ if $C_i = 0$ and $Q_j = 1$, otherwise it is 0. The covariates X_i that are observed for each individual are 5 standard normal variables, which are $X_{j,i}$ from $j = 1 \ldots 5$, and the indicator C_i. Let $\Phi(\cdot)$ be the standard Normal CDF. The subgroup indicator is

$$C_i \sim \text{Bernoulli}(\Phi(1 + X_{3,i}))$$

Those with $C_i = 1$ have a lower mean utility for quality in the absence of treatment. $\mu_L = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}^\top$ and $\mu_H = \begin{bmatrix} 1.0 & 0.5 & 0.0 \end{bmatrix}^\top$. The vector of utilities of individual i for the schools $j \in \{1, 2, 3\}$ is:

$$U_i = C_i\mu_L + (1 - C_i)\mu_H + C_iW_i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + X_{2,i}^\top \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix} + \epsilon_i$$

where ϵ_i is a three-dimensional vector of standard normal variables. The treatment raises the probability that an individual with $C_i = 1$ applies to a high-quality school. The students each submit a ranking $R_i(W_i)$ over the three schools to the mechanism based on the order of their utilities U_i. The score for each individual and each school is $S_{ij} \sim \text{Uniform}(0, 1)$, so in the notation of the general setup, $B_i(W_i) = \{R_i(W_i), S_i)\}$. Finally, the treatment allocation and outcome generation, which obeys selection-on-observables, follows $W_i \sim \text{Bernoulli}(0.5X_{3,i} - 0.5X_{2,i} + v_i)$ and $Y_i(\boldsymbol{W}) = \sum_{i=1}^{n} d(B_i(W_i), P(\boldsymbol{W}))V_{ij}$. The noise term $v_i \sim \text{Bernoulli}(0.5)$.

C Empirical Details

Variable	**Treated**	**Control**
income	4.22	4.77
	(3.32)	(3.82)
ma_educ	11.01	11.46
	(3.14)	(3.14)
pa_educ	10.99	11.45
	(3.45)	(3.45)
ma_indig	0.18	0.17
	(0.38)	(0.37)
pa_indig	0.15	0.14
	(0.35)	(0.35)
hhsize	2.45	2.46
	(1.29)	(1.27)
latitude	-34.36	-34.15
	(4.90)	(5.04)
longitude	-71.47	-71.37
	(1.02)	(1.03)

Table 5: Summary Statistics for $n = 114,749$ applicants to 9th grade in 2020. $W_i = 1$ indicates a parent reported they were aware of the performance category of the 8th grade school of their child. Income is in $100,000 pesos, and education is in years.

D Extensions

D.1 Verifying Regularity Conditions

Proposition 10. *Assume that $0 < s^* < 1$ and that market participants are bidding in a uniform price auction. We impose the following assumptions on the distribution of bids.*

- *$B_i(W_i) \in [V^-, V^+] \subset \mathbb{R}$ where V^- and V^+ are finite and strictly positive.*
- *For all $x \in \mathcal{X}$, the conditional CDF of the bid distribution, $F_{w,x}(b|x)$, is twice continuously differentiable in b for $w \in \{0, 1\}$, with the absolute value of the first and second derivatives uniformly bounded by finite constant b_1. In addition, the first derivative is bounded below by finite constant b_2.*

Then, Assumption 2 - 4 hold when outcomes are a surplus measure, so $y(B_i(w), p) = (B_i(w) - p)d(B_i(w), p)$.

The argument in Proposition 10 can also be extended to deferred acceptance; see Agarwal & Somaini (2018) for verification of many of the required conditions.

Proof. We start by verifying Assumption 4. It holds because we can choose some $c_1 > 0$ and then define $\mathcal{S}$ as $[V^- - c_1, V^+ + c_1]$. This is a compact set and the market clearing price $V^- < p^*_\pi < V^+$ (since capacity is strictly between 0 and 1) must always contain a ball of radius at least c_1. The unconditional distribution of $B_i(W_i)$ is

$$F_\pi(b) = \int \pi(x)F_{1|x}(b) + (1 - \pi(x))F_{0|x}(b)dF_x(x).$$

Since the first derivative of $F_\pi(b)$ is bounded below by b_2, then for any $s^* \in (0, 1)$, p^*_π is the unique solution defined as $p^*_\pi = F_\pi^{-1}(1-s^*)$. Furthermore, we have that $z_\pi(p) = 1-F_\pi(p)-s^*$. By the mean-value theorem, for some $c \in \mathcal{S}$, $z_\pi(p) - z_\pi(p') = z'_\pi(c)(p - p')$. Since the magnitude of $z'_\pi(c)$ is lower bounded by b_2, and $z_\pi(p^*_\pi) = 0$, we can write $|z_\pi(p)| \geq b_2|p - p'|$. This means if $|p-p^*_\pi| \geq c_3/2c'$, then $|z_\pi(p)|$ is always greater than $b_2c_3/2c'$, which is a strictly positive lower bound.

For Part 1 of Assumption 3, the class of $d(B_i(w), p)$ indexed by $p \in \mathcal{S}$ is a VC-class of functions (the class of indicator functions is a VC-class), so the covering number has the polynomial bound required. The class of linear functions $(B_i(w) - p)$ indexed by $p \in \mathcal{S}$ also has a polynomial bound, since the covering number of that class equals the covering number of $\mathcal{S}$, which is compact. Then, $y(B_i(w), p)$ is a Lipschitz combination of functions each with covering numbers that have a polynomial bound, so by Lemma 21, Part 1 holds.

For Part 2 of Assumption 3, outcomes are bounded because $B_i(w)$ is bounded by V^+. For the weak continuity assumption, we have the following argument, where $F_w(\cdot)$ is the CDF of $B_i(w)$.

$$
\begin{aligned}
\mathbb{E}[(d(B_i(w),p) - d(B_i(w),p'))^2] &= \mathbb{E}[(\mathbb{1}(B_i(w) > p) - \mathbb{1}(B_i(w) > p'))^2] \\
&= (F_w(p') - F_w(p))\mathbb{1}(p' > p) + (F_w(p) - F_w(p'))\mathbb{1}(p' \leq p) \\
&\leq b_1 ||p - p'||
\end{aligned}
$$

where the last step is because the CDF of $B_i(1)$ and $B_i(0)$ is differentiable with bounded first derivatives. For outcomes,

$$
\begin{aligned}
\mathbb{E}[(y(B_i(w),p) - y(B_i(w),p'))^2] &= \mathbb{E}[(B_i(w) - p)(d(B_i(w),p) - d(B_i(w),p') + (p - p')d(B_i(w),p'))^2] \\
&\leq 4V^+\mathbb{E}[(d(B_i(w),p) - d(B_i(w),p'))^2] + 2||p - p'||_2^2 \\
&(4V^+b_1 + 2JV^+)||p - p'||_2
\end{aligned}
$$

where we use the result for $d(\cdot)$ in the last step.

For Part 3, $\nabla_p \mu_w^d(x,p) = \nabla_p P(B_i(w) \geq p | X_i = x) = 1 - F_{w|x}(p|x)$. The conditional CDF is twice continuously differentiable in p, with first and second derivatives bounded by b_1. For outcomes, $\nabla_p \mu_w^y(p,x) = \nabla_p \mathbb{E}[(B_i(w) - p)d(B_i(w) > p)|X_i = x] = \nabla_p \int_p^{V+}(b)dF_w(b|x) - \nabla_p p \cdot (1 - F_{w|x}(p|x))$. By Leibniz's rule, and that p is bounded, this is also twice continuously differentiable in p, with bounded first and second derivatives, by the properties of the conditional distribution of $B_i(w)$. s

For the last part, we have that

$$
\nabla_p \mathbb{E}[\pi(X_i)\mu_1^d(X_i, p_\pi^* + (1 - \pi(X_i))\mu_0^d(X_i, p_\pi^*)] = -\mathbb{E}[\pi(X_i)f_{1|x}(p_\pi^*|X_i)] + (1 - \pi(X_i)f_{0|x}(p_\pi^*|X_i)]
$$

We can exchange the derivative and expectation by the dominated convergence theorem. To evaluate the derivative, notice that $\mu_1^d(x,p) = P(B_i(1) \geq p | X_i = x) = 1 - F_{1|x}(p|x)$. The RHS is bounded between b_2 and b_1, since $f_{w|x}(p|x)$ is uniformly bounded between b_2 and b_1 and $0 \leq \pi(X_i) \leq 1$.

We can finish by verifying the finite-market-clearing assumption in Assumption 2. Since $0 < s^* < 1$, then $Z_n(V^-) < 0$ and $Z_n(V^+) > 0$. So, with probability 1, $Z_n(p)$ crosses 0. Since $d(B_i(w),p)$ is bounded by 1, and the probability that any two bidders have the same value is 0, the magnitude of any jump in $Z_n(p)$ is bounded by $1/n$. This means with probability 1, $Z_n(P_\pi) \leq 1/n$. □

D.2 Using IV for Identification and Estimation

This section provides a brief discussion of how a restricted version of the Global Treatment Effect can be estimated when unconfoundedness does not hold, but there is a binary instrumental variable that affects take-up of a binary treatment. In an IV setting, we have potential treatments $W_i(1)$ and $W_i(0)$ that depend on an instrument $Z_i \in \{0,1\}$. Under a monotonicity assumption, $W_i(1) \geq W_i(0)$. Under interference, there are a variety of counterfactuals that can be defined. One relevant counterfactual when there may be control over the instrument, but not the treatment directly, is the intent-to-treat effect. This is the effect on average outcomes in the sample when all individuals receive the instrument, compared to a setting where no agents receive the instrument. It can be written in this setting with interference as

$$\begin{aligned}\bar{\tau}_{GITT} =& \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(W_i(1) > W_i(0))[y(B_i(1), Q_1) - y(B_i(0), Q_0)] \\ &+ \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(W_i(1) = W_i(0))[y(B_i(0), Q_1) - y(B_i(0), Q_0)]\end{aligned}$$

where Q_1 and Q_0 are defined as

$$\begin{aligned}0 =& \frac{1}{n}\sum_{i=1}^{n}[\mathbb{1}(W_i(1) > W_i(0))d(B_i(1), Q_1) + \mathbb{1}(W_i(1) = W_i(0))d(B_i(0), Q_1) - s^*] \\ 0 =& \frac{1}{n}\sum_{i=1}^{n}[d(B_i(0), Q_0) - s^*]\end{aligned}$$

When the market-clearing cutoffs are determined by the aggregate behavior of everyone, then outcomes of compliers are affected directly by the treatment and indirectly by the change in the equilibrium. The outcomes of those who do not take up the treatment, however, are also affected by the changes in preferences of the compliers, due to the equilibrium effect. Using the techniques in the proof of Theorem 1, we can show that this corresponds to the following moment condition problem with missing data. Let $C_i = \mathbb{1}(W_i(1) > W_i(0))$.

$$\begin{aligned}0 &= \tau^*_{GITT} - P(C_i = 1)\mathbb{E}[y(B_i(1), q_1^*) - y(B_i(0), q_0^*)|C_i = 1] - \\ &\qquad P(C_i = 0)\mathbb{E}[y(B_i(0), q_1^*) - y(B_i(0), q_0^*)|C_i = 0] \\ 0 &= P(C_i = 1)\mathbb{E}[d(B_i(1), q_1^*) - s^*|C_i = 1] + P(C_i = 0)\mathbb{E}[d(B_i(0), q_1^*) - s^*|C_i = 0] \\ 0 &= \mathbb{E}[d(B_i(0), q_0^*) - s^*]\end{aligned}$$

The Local Average Treatment Effect (Imbens & Angrist 1994) -type quantities in this moment equation can be identified and estimated using standard IV assumptions: overlap, instrumental relevance, and exogeneity. For example, $\mathbb{E}[y(B_i(1), q_1^*)|W_i(1) > W_i(0)]$ is a moment that matches the form of Equation 19 in Appendix A of Kallus et al. (2019). Under the IV identifying assumptions, including monotonicity, then a Neyman orthogonal estimation equation for this moment is given by Equation 22 of Appendix A of the paper.

D.3 Connecting to Research Design Meets Market Design

In this paper, we identify and estimate the effect of an individual-level treatment on allocations in a centralized market in equilibrium. Abdulkadiroğlu et al. (2017) consider a different type of causal effect. They are interested in the effect of allocations (e.g. attending a charter school) on some stochastic outcome, like test scores or future income. We briefly discuss how these two approaches can be combined, under an additional (major) assumption that the treatment only affects outcomes through some function $g : \{1, \ldots, J\} \mapsto \{1, \ldots, M\}$ that aggregates allocations, where $M << J$. For example, $g(D_i) \in \{0, 1\}$ could be an indicator of D_i is a charter school, or a good-quality school. In cases where J is small, then we can have that $g(\cdot)$ is the identity function, and no additional restriction on spillovers will be required.

Let $\{Y_i(g(d_i(w_i, p(\boldsymbol{w}))) : w_i \in \{0, 1\}, p(\boldsymbol{w}) \in \mathbb{R}^J, d_i \in \{1, \ldots, J\}\}$ define general potential outcomes for a market of size n, where the treatment affects outcomes only through some aggregation of an individual's allocation, such as whether they attended a charter school. An individual's observed outcome $Y_i = Y_i(g(D_i))$ depends on an individual's treatment W_i through their allocation $D_i = D_i(W_i, P(\boldsymbol{W}))$. An individual's type $\theta_i = (R_i(1), R_i(0), X_i)$.

Define the allocation-specific propensity score under the observed treatment rule the J-length vector $h_e(\theta_i) = P(g(D_i(W_i, P(\boldsymbol{W})))|R_i(W_i))$, where $R_i(W_i)$ is the ranking of schools that a students submits. We must have that M is small enough so that the lottery scores that are used for tie-breaking students in the same priority group ensure that for some non-negligible group , that $0 < h_e(\theta_i) < 1$. By comparing students with similar allocation-specific propensity scores who have different allocations, then Abdulkadiroğlu et al. (2017) identify the effect of some aggregation of allocations. In our notation, this type of causal effect is $\mathbb{E}[Y_i(j) - Y_i(k)|h_e(\theta_i) = h]$. Using the approach in Abdulkadiroğlu et al. (2017), we can use the observed data to identify $\mathbb{E}[Y_i(m)|0 < h_e(\theta_i) < 1]$ for all $m \in \{1, \ldots, M\}$. The approach in our paper, on the other hand, identifies the change in allocations in equilibrium from a counterfactual treatment allocation. The algorithm in Section 3 can be used to estimate $Pr\Big(g(D_i(1, P(\mathbf{1}))) = m\Big)$.

Thus, the two approaches can be linked to identify a restricted form of a global treatment effect, under the restrictive assumption that the treatment only impacts outcomes through attendance at some aggregate type of school.

$$\begin{aligned}\tau_{\text{LTET}} &= \mathbb{E}[Y_i(\mathbf{1})|0 < h_e(\theta_i) < 1] - \mathbb{E}[Y_i(\mathbf{0})|0 < h_e(\theta_i) < 1] \\ &= \sum_{m=1}^{M} \mathbb{E}[Y_i(m)|0 < h_e(\theta_i) < 1]\Big(Pr\Big(g(D_i(1, P(\mathbf{1}))) = m\Big) - Pr\Big(g(D_i(0, P(\mathbf{0}))) = m\Big)\Big).\end{aligned}$$

The approach in this paper estimates the effect of the treatment on allocations to certain types of schools in equilibrium. For certain subgroups with non-zero propensity score, we can link that effect on allocations to stochastic outcomes using the approach in Abdulkadiroğlu et al. (2017). Accommodating settings where outcomes depend on the treatment in more complex (and realistic) ways is a subject for future work.

E Concentration Results

Lemma 11. *Under the assumptions of Theorem 1,* $\sqrt{n}|\bar{V}_n(\pi) - V^*(\pi)| = O_p(1)$. *Under the assumptions of Theorem 7,* $\sup_{\pi\in\Pi} \sqrt{n}|\bar{V}_n(\pi) - V^*(\pi)| = O_p(1)$.

Proof. First, we make the following expansion:

$$\bar{V}_n(\pi) - V^*(\pi) = \mathbb{E}_\pi[Y_{n,\pi}(P_\pi) - Y_{n,\pi}(p^*_\pi) + Y_{n,\pi}(p^*_\pi)] - y_\pi(p^*_\pi).$$

Then, we work with expected outcome functions instead:

$$\sup_{\pi\in\Pi} |\bar{V}_n(\pi) - V^*(\pi)| \leq \sup_{\pi\in\Pi} |\mathbb{E}_\pi[y_\pi(P_\pi)] - y_\pi(p^*_\pi)| + 3 \sup_{\pi\in\Pi, p\in\mathcal{S}} |\mathbb{E}_\pi[Y_{n,\pi}(p)] - y_\pi(p^*_\pi)|.$$

For the first term, $\sup_{\pi\in\Pi} |\mathbb{E}_\pi[y_\pi(P_\pi)] - y_\pi(p^*_\pi)| \leq M \sup_{\pi\in\Pi} \mathbb{E}_\pi[||P_\pi - p^*_\pi||] = O_p(n^{-1/2})$, where the uniform bound on $\mathbb{E}_\pi[||P_\pi - p^*_\pi||]$ comes from Lemma 18, under the assumptions of Theorem 7. For the second term, Assumption 3 indicates that $\mathcal{F} = \{(B(1), B(0), X) \mapsto \pi(X)y(B(1), p) + (1 - \pi(X))y(B(0), p) : p \in \mathcal{S}\}$ has uniform ε-covering number that is bounded by a polynomial of $(1/\varepsilon)$, and Π is a VC-class of finite dimension, so by the composition rules of Lemma 21, and the tail bound of Lemma 20, we have that $\sup_{\pi\in\Pi, p\in\mathcal{S}} |\mathbb{E}_\pi[Y_{n,\pi}(p)] - y_\pi(p^*_\pi)| = O_p(n^{-1/2})$. Under the Assumptions of Theorem 1, the same argument can be used to show the bound pointwise in π, using the pointwise result in Lemma 18 rather than the uniform result.

□

Lemma 12. *Under the assumptions of Theorem 7,*

$$\sup_{\pi \in \Pi} \sqrt{n} |\hat{V}_n(\pi) - V_n(\pi)| = O_p(1).$$

Proof. First, we make the following expansion.

$$\begin{aligned}\hat{V}_n(\pi) - V_n(\pi) &= \Gamma^y_{n,\pi}(\hat{P}_\pi; \hat{\eta}_\pi) - \Gamma^y_{n,\pi}(p^*_\pi; \eta^*_\pi) \\ &= \Gamma^y_{n,\pi}(\hat{P}_\pi; \hat{\eta}_\pi) - \Gamma^y_{n,\pi}(\hat{P}_\pi; \eta^*_\pi) + \Gamma^y_{n,\pi}(\hat{P}_\pi; \eta^*_\pi) - \Gamma^y_{n,\pi}(p^*_\pi, \eta^*_\pi)\end{aligned}$$

Then, we work with expected outcome functions instead:

$$\begin{aligned}\sup_{\pi \in \Pi} |\hat{V}_n(\pi) - V_n(\pi)| \le \sup_{\pi \in \Pi} &|y_\pi(\hat{P}_\pi; \hat{\eta}_\pi) - y_\pi(\hat{P}_\pi; \eta^*_\pi)| + |y_\pi(\hat{P}_\pi; \eta^*_\pi) - y_\pi(p^*_\pi; \eta^*_\pi)| \\ &+ \sup_{p \in \mathcal{S}, \pi \in \Pi} 2|\Gamma^y_{n,\pi}(p; \hat{\eta}_\pi) - y_\pi(p; \hat{\eta}_\pi)| + \sup_{p \in \mathcal{S}, \pi \in \Pi} 2|\Gamma^y_{n,\pi}(p; \eta^*_\pi) - y_\pi(p; \eta^*_\pi)| \\ &= O_p(n^{-1/2})\end{aligned}$$

The first term is $O_p(n^{-1/2})$ by Lemma 17. The rate of the second term comes from a Taylor expansion and the uniform convergence rate for $\hat{P}_\pi$ in Lemma 19. The rate of the third term comes from Lemma 13 and the fourth term comes from Lemma 16.

□

Lemma 13. *Under the assumptions of Theorem 2,*

$$\begin{aligned}\sup_{p \in \mathcal{S}} |\Gamma^y_{n,\pi}(p; \eta^*_\pi) - y_\pi(p, \eta^*_\pi)| &= O_p\left(n^{-1/2}\right), \\ \sup_{p \in \mathcal{S}} ||\Gamma^z_{n,\pi}(p; \eta^*_\pi) - y_\pi(p, \eta^*_\pi)|| &= O_p\left(n^{-1/2}\right).\end{aligned}$$

Under the assumptions of Theorem 7,

$$\begin{aligned}\sup_{\pi \in \Pi, p \in \mathcal{S}} |\Gamma^y_{n,\pi}(p; \eta^*_\pi) - y_\pi(p, \eta^*_\pi)| &= O_p\left(n^{-1/2}\right), \\ \sup_{\pi \in \Pi, p \in \mathcal{S}} ||\Gamma^z_{n,\pi}(p; \eta^*_\pi) - y_\pi(p, \eta^*_\pi)|| &= O_p\left(n^{-1/2}\right).\end{aligned}$$

Proof.

$$\begin{aligned}\Gamma^y_{n,\pi}(p; \eta^*_\pi) - y_\pi(p; \eta^*_\pi) = &\frac{1}{n}\sum_{i=1}^n \pi(X_i)\Gamma^y_{1i}(p; \eta^*_\pi) - \mathbb{E}_T[\pi(X_i)\mu^y_1(X_i, p)] \\ &+ \sum_{i=1}^n (1 - \pi(X_i))\Gamma^y_{0i}(p; \eta^*_\pi) - \mathbb{E}_T[(1 - \pi(X_i))\mu^y_0(X_i, p)]\end{aligned}$$

To bound $\sup_{p \in \mathcal{S}, \pi \in \Pi} |\Gamma^y_{n,\pi}(p; \eta^*_\pi) - y_\pi(p; \eta^*_\pi|$, we will just bound the treated terms, since the argument for the control terms is the same. First, we expand the treated terms:

$$\begin{aligned}
&\sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\Gamma^y_{1i}(p;\eta^*_\pi) - \mathbb{E}[\pi(X_i)\mu^y_1(X_i,p)] \right| \\
&\leq \sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\frac{W_i}{e(X_i)}y(B_i(1),p) - \mathbb{E}[\pi(X_i)y(B_i(1),p)] \right| \\
&\qquad + \sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\mu^y_1(X_i,p)\left(1-\frac{W_i}{e(X_i)}\right) \right| \\
&\leq \sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\frac{W_i}{e(X_i)}y(B_i(1),p) - \mathbb{E}[\pi(X_i)y(B_i(1),p)] \right| \\
&\qquad + \sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\mu^y_1(X_i,p)\left(1-\frac{W_i}{e(X_i)}\right) \right|
\end{aligned}$$

Since Π is a VC-class of dimension v, by Theorem 2.6.7 of van der Vaart & Wellner (1997), it has uniform covering numbers that are bounded by $C(1/\epsilon)^{2v}$ for some constant C. Assumption 3 implies that the function class $\mathcal{F}_y = \{B(w) \mapsto y(B(w),p) : p \in \mathcal{S}\}$ has covering numbers that are bounded by $C(1/\epsilon)^{h_d}$. Then, by Lemma 21, the function class $\mathcal{G} = \{(W,X,B(1)) \mapsto \pi(X)\frac{W}{e(X)}y(B(1),p) : p \in \mathcal{S}\}$ has covering numbers that are bounded by $C(1/\varepsilon)^V$ for finite V that is of order $v + h_y$. By Lemma 20, we can now conclude that

$$\sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\Gamma^y_{1i}(p;\eta^*_\pi) - \mathbb{E}[\pi(X_i)\mu^y_1(X_i,p)] \right| = O_p\left(n^{-1/2}\right). \tag{14}$$

$\mu^y_1(X_i,p)$ is c'-Lipschitz in p. Since $p \in \mathcal{S}$, and $\mathcal{S}$ is a compact subset of $\mathbb{R}^J$, we can show the function class $\mathcal{F}_\mu = \{X \mapsto \mu^y_1(X,p) : p \in \mathcal{S}\}$ has uniform covering number that is bounded by $C\left(\frac{1}{\varepsilon}\right)^J$ for some constant $C > 0$. Theorem 2.7.11 of van der Vaart & Wellner (1997) shows that the $2\epsilon c'$ bracketing number of $\mathcal{F}_\mu$ is bounded by the covering number of $\mathcal{S}$, which in turn is bounded by $C(1/\epsilon)^J$ for some constant C (see, for example, Lemma 2.7 of Sen (2018)). Since the ε-uniform covering number of $\mathcal{F}_\mu$ is bounded by the 2ε-bracketing number (see Definition 2.1.6 of van der Vaart & Wellner (1997)), this is enough to bound the uniform covering number of $\mathcal{F}_\mu$. Again using the composition result of Lemma 21 and Lemma 20 (as above), we can now conclude that

$$\sup_{\pi\in\Pi, p\in\mathcal{S}} \left| \frac{1}{n}\sum_{i=1}^{n} \pi(X_i)\mu^y_1(X_i,p)\left(1-\frac{W_i}{e(X_i)}\right) \right| = O_p\left(n^{-1/2}\right). \tag{15}$$

With the same argument for the control terms, we have now concluded that:

$$\sup_{p\in\mathcal{S},\pi\in\Pi} |\Gamma^y_{n,\pi}(p;\eta^*_\pi) - y_\pi(p;\eta^*_\pi)| = O_p\left(n^{-1/2}\right).$$

Using the same argument, we can also bound each of $\sup_{p\in\mathcal{S},\pi\in\Pi} |Z_{j,n,\pi}(p;\eta^*_\pi) - z_{j,\pi}(p;\eta^*_\pi)|$ for $j \in \{1,\ldots,J\}$ and, using a union bound also conclude that:

$$\sup_{p\in\mathcal{S},\pi\in\Pi} ||\Gamma^z_{n,\pi}(p;\eta^*_\pi) - z_\pi(p;\eta^*_\pi)|| = O_p\left(n^{-1/2}\right).$$

For the first part of the Lemma, we can follow the same argument as above without taking the supremum over Π.

□

Lemma 14. ***Asymptotic Equicontinuity***

Under the assumptions of Theorem 1,

$$Y_{n,\pi}(P_\pi) - Y_{n,\pi}(p^*_\pi) - y_\pi(P_\pi) + y_\pi(p^*_\pi) = o_p(n^{-1/2}),$$
$$Z_{n,\pi}(P_\pi) - Z_{n,\pi}(p^*_\pi) - z_\pi(P_\pi) + z_\pi(p^*_\pi) = o_p(n^{-1/2}),$$

Proof. We prove this for $Y_{n,\pi}(\cdot)$ and the proof is the same for each element of the J-length vector $Z_{n,\pi}(\cdot)$. Let $\mathcal{F} = \{(X_i, B_i(W_i), W_i) \mapsto W_i y(B_i(1), p) + (1-W_i)y(B_i(0), p) : p \in \mathcal{S}\}$.

Notice that $\mathbb{E}[Y_{n,\pi}(p)] = y_\pi(p)$. By Assumption 3, for some finite C, the ε covering number of $\mathcal{F}$ is bounded by $C(1/\varepsilon)^{2h_y}$, for all $0 < \varepsilon < 1$. So, $\mathcal{F}$ is a Donsker-class of funcitons. Since we also have weak continuity of $W_i y(B_i(1), p) + (1-W_i)y(B_i(W_i), p)$ in the sense of Assumption 3, by Lemma 19.24 of van der Vaart (1998), we have that $Y_{n,\pi}(P_\pi) - Y_{n,\pi}(p^*_\pi) - y_\pi(P_\pi) + y_\pi(p^*_\pi) = o_p(n^{-1/2})$.

□

Lemma 15. ***Asymptotic Equicontinuity with Estimated Nuisances***

Under the assumptions of Theorem 2, we have the following asymptotic equicontinuity result:

$$\Gamma^y_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^y_{n,\pi}(p^*_\pi;\eta^*_\pi) - y_\pi(\hat{P}_\pi;\hat{\eta}_\pi) + y_\pi(p^*_\pi;\eta^*_\pi) = o_p(n^{-1/2}),$$
$$\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi) + z_\pi(p^*_\pi,\eta^*_\pi) = o_p(n^{-1/2}).$$

Proof. We prove this for $Y_n(\cdot)$ and the proof is the same for $Z_n(\cdot)$. We can decompose the empirical average by data-splitting, so we can treat the estimated nuisances as fixed:

$$
\begin{aligned}
&\Gamma^y_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^y_{n,\pi}(p^*_\pi;\eta^*_\pi) - y_\pi(\hat{P}_\pi;\hat{\eta}_\pi) + y_\pi(p^*_\pi,\eta^*_\pi) \\
&= \sum_{k=1}^K \frac{n_k}{n}\frac{1}{n_k}\sum_{i\in I_k}[\pi(X_i)(\Gamma^y_{1i}(\hat{P}_\pi;\hat{\eta}^k_\pi) - \Gamma^y_{1i}(p^*_\pi;\eta^*_\pi)) + (1-\pi(X_i))(\Gamma^y_{0i}(\hat{P}_\pi;\hat{\eta}^k_\pi) - \Gamma^y_{0i}(p^*_\pi;\eta^*_\pi)] \\
&\quad + \sum_{k=1}^K y_\pi(p^*_\pi,\eta^*_\pi) - y_\pi(\hat{P}_\pi;\hat{\eta}^k_\pi) \\
&= \sum_{k=1}^K \frac{n_k}{n} R^k_n,
\end{aligned}
$$

where $R^k_n = \frac{1}{n_k}\sum\limits_{i\in I_k}[\pi(X_i)(\Gamma^y_{1i}(\hat{P}_\pi;\hat{\eta}^k_\pi) - \Gamma^y_{1i}(p^*_\pi;\eta^*_\pi)) + (1-\pi(X_i))(\Gamma^y_{0i}(\hat{P}_\pi;\hat{\eta}^k_\pi) - \Gamma^y_{0i}(p^*_\pi;\eta^*_\pi)] +$ $y_\pi(p^*_\pi,\eta^*_\pi) - y_\pi(\hat{P}_\pi;\hat{\eta}^k_\pi)$. For the average within a single split, since the nuisance functions are estimated on a different split of data, we can treat them as fixed.

$$
\mathcal{F}_{\hat{\eta}_k} = \{(X_i, B_i(W_i), W_i) \mapsto \pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi) + (1-\pi(X_i))\Gamma^y_{0i}(p;\hat{\eta}^k_\pi) : p \in \mathcal{S}\}
$$

By Assumption 3 for some finite C, the ε covering number of $\mathcal{F}_{\hat{\eta}_k}$ is bounded by $C(1/\varepsilon)^{2h_y}$, for all $0 < \varepsilon < 1$. This means that $\mathcal{F}_{\hat{\eta}_k}$ is a Donsker class of functions. Since we also have weak continuity of $y(B_i(w), p)$ in the sense of Assumption 3, by Lemma 19.24 of van der Vaart (1998), for all $t > 0$, we have $\lim\limits_{n\to\infty} P(\sqrt{n}R^k_n > t|\hat{\eta}^k) \to 0$. Conditional convergence in probability implies unconditional convergence in probability, since $P(\sqrt{n}R^k_n > t) = \mathbb{E}[P(\sqrt{n}R^k_n > t|\hat{\eta}^k)]$, and the probability is bounded so we can swap the limit and the expectation. This means $R^k_n = o_p(n^{-1/2})$.

Since this argument applies to each split of the data, and there is a finite number of splits, we have now shown that

$$
\Gamma^y_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^y_{n,\pi}(p^*_\pi;\eta^*_\pi) - y_\pi(\hat{P}_\pi;\hat{\eta}_\pi) + y_\pi(p^*_\pi,\eta^*_\pi) = o_p(n^{-1/2}).
$$

The proof follows the same argument for $\Gamma^z_{n,\pi}(\cdot)$.

□

Lemma 16. *Under the assumptions of Theorem 2,*

$$
\sup_{p\in\mathcal{S}} |\Gamma^y_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p,\hat{\eta}_\pi)| = O_p\left(n^{-1/2}\right),
$$
$$
\sup_{p\in\mathcal{S}} ||\Gamma^z_{n,\pi}(p;\hat{\eta}_\pi) - z_\pi(p,\hat{\eta}_\pi)|| = O_p\left(n^{-1/2}\right),
$$

Under the assumptions of Theorem 7,

$$\sup_{\pi\in\Pi,p\in\mathcal{S}} |\Gamma^y_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p,\hat{\eta}_\pi)| = O_p\left(n^{-1/2}\right),$$
$$\sup_{\pi\in\Pi,p\in\mathcal{S}} ||\Gamma^z_{n,\pi}(p;\hat{\eta}_\pi) - z_\pi(p,\hat{\eta}_\pi)|| = O_p\left(n^{-1/2}\right),$$

Proof. We start with the second part of the Lemma. We can write these terms as a weighted sum of averages across each of the splits. Let I_k be the indexes of observations in split k and $\hat{\eta}^k_\pi$ the nuisance functions estimated on observations outside the split.

$$\begin{aligned}
\Gamma^y_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p;\hat{\eta}_\pi) &= \frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}_\pi) - \mathbb{E}_T[\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}_\pi)] \\
&\quad + \sum_{i=1}^{n}(1-\pi(X_i))\Gamma^y_{0i}(p;\hat{\eta}_\pi) - \mathbb{E}_T[(1-\pi(X_i))\Gamma^y_{0i}(p;\hat{\eta}_\pi] \\
&= \sum_{k=1}^{K}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in I_k}\frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi) - \mathbb{E}_T[\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi)] \\
&\quad + \sum_{k=1}^{K}\frac{n_k}{n}\frac{1}{n_k}\sum_{i\in I_k}(1-\pi(X_i))\Gamma^y_{0i}(p;\hat{\eta}^k_\pi) - \mathbb{E}_T[(1-\pi(X_i))\Gamma^y_{0i}(p;\hat{\eta}^k_\pi]
\end{aligned} \tag{16}$$

We show the details for the treated terms only since the argument for the control terms is the same. Note to keep the notation manageable, we drop the data-splitting notation for the estimated nuisance functions, but recall that there is three-way data-splitting, so we can treat the data in split k, $\tilde{P}_\pi$ and $\hat{e}(\cdot), \hat{\mu}(\cdot)$ as all mutually independent. For the average within a single split, we have the below expansion.

$$\begin{aligned}
&\sup_{\pi\in\Pi,p\in\mathcal{S}}\left|\frac{1}{n}\sum_{i\in I_k}\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi) - \mathbb{E}_T[\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi)]\right| \\
&\leq \sup_{\pi\in\Pi,p\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\frac{W_i}{\hat{e}^k(X_i)}y(B_i(1),p) - \mathbb{E}_T\left[\frac{W_i}{\hat{e}^k(X_i)}\pi(X_i)y(B_i(1),p)\right]\right| \\
&\quad + \sup_{\pi\in\Pi}\left|\frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\hat{\mu}^y_1(X_i,\tilde{P}_\pi)\left(1-\frac{W_i}{\hat{e}(X_i)}\right) - \mathbb{E}_T\left[\pi(X_i)\hat{\mu}^y_1(X_i,\tilde{P}_\pi)\left(1-\frac{W_i}{\hat{e}(X_i)}\right)\right]\right| \\
&\overset{(1)}{\leq} O_p(n^{-1/2}) + \sup_{\pi\in\Pi,p\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^{n}\pi(X_i)\hat{\mu}^y_1(X_i,p)\left(1-\frac{W_i}{\hat{e}(X_i)}\right) - \mathbb{E}_T\left[\pi(X_i)\hat{\mu}^y_1(X_i,p)\left(1-\frac{W_i}{\hat{e}(X_i)}\right)\right]\right| \\
&\overset{(2)}{=} O_p(n^{-1/2})
\end{aligned}$$

For term that we handle in (1), we can condition on $\hat{e}(\cdot)$ and treat it as fixed. Conditional on $\hat{e}(\cdot)$, this term is mean-zero. Then, because of the uniform overlap condition, the tail bound for this term constructed in the same way as in (14) does not depend on the estimated part of the nuisance function, so unconditionally, we also have that the term is $O_p(n^{-1/2})$.

For the next term, we rely on the additional assumption in Assumption 6 and the assumption that estimated conditional mean functions are uniformly bounded. Again, we can use the composition result and tail bound in Lemma 21 and Lemma 20 to construct a tail bound for the term that does not depend on the specific instance of the estimated function.

This argument applies for each of the K splits, and can be applied also to the control terms, and to each of the components of $Z_n(\cdot)$, so we can now conclude that:

$$\sup_{\pi\in\Pi, p\in\mathcal{S}} |\Gamma^y_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p,\hat{\eta}_\pi)| = O_p\left(n^{-1/2}\right),$$
$$\sup_{\pi\in\Pi, p\in\mathcal{S}} ||\Gamma^z_{n,\pi}(p;\hat{\eta}_\pi) - z_\pi(p,\hat{\eta}_\pi)|| = O_p\left(n^{-1/2}\right).$$

To finish the proof, without using Assumption 6, then under the assumptions of Theorem 2, we have

$$\begin{aligned}
&\sup_{p\in\mathcal{S}}\left|\frac{1}{n}\sum_{i\in I_k}\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi) - \mathbb{E}_T[\pi(X_i)\Gamma^y_{1i}(p;\hat{\eta}^k_\pi)]\right| \\
&\le \sup_{p\in\mathcal{S}}\left|\frac{1}{n}\sum_{i=1}^n \pi(X_i)\frac{W_i}{\hat{e}^k(X_i)}y(B_i(1),p) - \mathbb{E}_T\left[\frac{W_i}{\hat{e}^k(X_i)}\pi(X_i)y(B_i(1),p)\right]\right| \\
&\quad+\left|\frac{1}{n}\sum_{i=1}^n \pi(X_i)\hat{\mu}^y_1(X_i,\tilde{P}_\pi)\left(1-\frac{W_i}{\hat{e}(X_i)}\right) - \mathbb{E}_T\left[\pi(X_i)\hat{\mu}^y_1(X_i,\tilde{P}_\pi)\left(1-\frac{W_i}{\hat{e}(X_i)}\right)\right]\right| \\
&= O_p(n^{-1/2}),
\end{aligned}$$

where the first term is $O_p(n^{-1/2})$ by the same argument as above (when we also take the supremum over $\pi\in\Pi$). Conditional on the estimated nuisances, the second term is mean-zero with finite variance. By the CLT, then conditional on estimated nuisances, it is $O_p(n^{-1/2})$, where we can choose constants in the $O_p(n^{-1/2})$ definition that are uniform over all possible instances of the nuisance parameters, by the uniform boundedness of the estimated nuisances. So, the second term is $O_p(n^{-1/2})$ as well.

By (16), (and since the same argument applies to $Z_n(\cdot)$), we have now shown that :

$$\sup_{p\in\mathcal{S}} |\Gamma^y_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p,\hat{\eta}_\pi)| = O_p\left(n^{-1/2}\right),$$
$$\sup_{p\in\mathcal{S}} ||\Gamma^z_{n,\pi}(p;\hat{\eta}_\pi) - y_\pi(p,\hat{\eta}_\pi)|| = O_p\left(n^{-1/2}\right).$$

□

Lemma 17. ***Uniform Nuisance Convergence.***

Under the assumptions of Theorem 2, there is a finite $C_1 > 0$ and $C_2 > 0$ such that with probability at least $1 - o(1)$,

$$\sup_{\pi\in\Pi}\sqrt{n}||z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi)|| \le C_1\sqrt{n}\sup_{\pi\in\Pi}||\hat{P}_\pi - p^*_\pi||\rho_{e,n} + \sqrt{n}\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n} + \sqrt{n}\frac{C_1}{\kappa}\rho_{e,n}\rho_{p,n},$$
$$\sup_{\pi\in\Pi}\sqrt{n}|y_\pi(\hat{P}_\pi;\hat{\eta}_\pi) - y_\pi(\hat{P}_\pi;\eta^*_\pi)| = C_2\sqrt{n}\sup_{\pi\in\Pi}||\hat{P}_\pi - p^*_\pi||\rho_{e,n} + \sqrt{n}\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n} + \sqrt{n}\frac{C_2}{\kappa}\rho_{e,n}\rho_{p,n}.$$

This type of inequality also holds pointwise, in that for the same C_1 and C_2, with probability at least $1 - o(1)$, for each $\pi \in \Pi$, we have:

$$\sqrt{n}||z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi)|| \le C_1\sqrt{n}||\hat{P}_\pi - p^*_\pi||\rho_{e,n} + \sqrt{n}\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n} + \sqrt{n}\frac{C_1}{\kappa}\rho_{e,n}\rho_{p,n},$$
$$\sqrt{n}|y_\pi(\hat{P}_\pi;\hat{\eta}_\pi) - y_\pi(\hat{P}_\pi;\eta^*_\pi)| = C_2\sqrt{n}||\hat{P}_\pi - p^*_\pi||\rho_{e,n} + \sqrt{n}\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n} + \sqrt{n}\frac{C_2}{\kappa}\rho_{e,n}\rho_{p,n}.$$

Proof. We prove this for $z_\pi(\cdot)$ and the argument for $y_\pi(\cdot)$ is the same.

$$z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi) = \mathbb{E}_T[\pi(X_i)(\Gamma^z_{1,i}(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{1,i}(\hat{P}_\pi;\hat{\eta}_\pi))] + \mathbb{E}_T[\pi(X_i)(\Gamma^z_{0,i}(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{0,i}(p;\hat{\eta}_\pi))].$$

We bound the treated terms and the argument for the control terms is the same.

$$\begin{aligned}\mathbb{E}_T[\pi(X_i)(\Gamma^z_{1,i}(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{1,i}(\hat{P}_\pi;\hat{\eta}_\pi))] &= \mathbb{E}_T\left[\pi(X_i)(d(B_i(1),p) - \mu^d_1(X_i,p^*_\pi))\left(\frac{W_i}{\hat{e}(X_i)} - \frac{W_i}{e(X_i)}\right)\right]_{p=\hat{P}_\pi} \\ &+ \mathbb{E}_T\left[\pi(X_i)(\hat{\mu}^d_1(X_i,\tilde{P}_\pi) - \mu^d_1(X_i,p^*_\pi))\left(\frac{W_i}{e(X_i)} - \frac{W_i}{\hat{e}(X_i)}\right)\right] \\ &+ \mathbb{E}_T\left[\pi(X_i)(\hat{\mu}^d_1(X_i,\tilde{P}_\pi) - \mu^d_1(X_i,p^*_\pi))\left(1 - \frac{W_i}{e(X_i)}\right)\right]\end{aligned} \tag{17}$$

The last term is equal to zero. For the first term, we can bound the absolute value of each element of the vector. With probability at least $1 - o(1)$,

$$
\begin{aligned}
&\left|\mathbb{E}_T\left[\pi(X_i)(d_j(B_i(1),p)-\mu^d_{1,j}(X_i,p^*_\pi))\left(\frac{W_i}{\hat{e}(X_i)}-\frac{W_i}{e(X_i)}\right)\right]\right|_{p=\hat{P}_\pi} \\
&=\left|\mathbb{E}_T\left[\pi(X_i)(\mu^d_1(X_i,p)-\mu^d_1(X_i,p^*_\pi))\left(\frac{\hat{e}(X_i)-e(X_i)}{\hat{e}(X_i)}\right)\right]\right|_{p=\hat{P}_\pi} \\
&\leq \mathbb{E}_T\left[\left|\frac{\pi(X_i)}{\hat{e}(X_i)}\right|\left|(\mu^d_{1,j}(X_i,p)-\mu^d_{1,j}(X_i,p^*_\pi))\right|\left|\hat{e}(X_i)-e(X_i)\right|\right] \\
&\leq \frac{1}{\kappa}\mathbb{E}_T\left[\left|(\mu^d_{1,j}(X_i,p)-\mu^d_{1,j}(X_i,p^*_\pi))\right|\left|\hat{e}(X_i)-e(X_i)\right|\right] \\
&\leq \frac{1}{\kappa}M||\hat{P}_\pi-p^*_\pi||\sqrt{\mathbb{E}_T\left[(\hat{e}(X_i)-e(X_i))^2\right]} \\
&\leq \frac{C}{\kappa}\rho_{e,n}||\hat{P}_\pi-p^*_\pi||
\end{aligned}
\tag{18}
$$

for finite C that does not depend on π. The second-last step is by the differentiability of $\mu^z_1(X_i,p)$ in p with uniformly bounded derivatives.

Similarly, we can show that

$$
\begin{aligned}
&\mathbb{E}_T\left[\pi(X_i)(\hat{\mu}_1(X_i,\tilde{P}_\pi)-\mu_1(X_i,p^*_\pi))\left(\frac{W_i}{e(X_i)}-\frac{W_i}{\hat{e}(X_i)}\right)\right] \\
&=\left[\pi(X_i)(\hat{\mu}_1(X_i,\tilde{P}_\pi)-\mu_1(X_i,p^*_\pi))\left(\frac{W_i}{e(X_i)}-\frac{W_i}{\hat{e}(X_i)}\right)\right] \\
&\leq \frac{1}{\kappa}\mathbb{E}_T\left[\left(\left|(\hat{\mu}^z_{1,j}(X_i,\tilde{P}_\pi)-\mu^z_{1,j}(X_i,\tilde{P}_\pi))\right|+\left|(\mu^z_{1,j}(X_i,\tilde{P}_\pi)-\mu^z_{1,j}(X_i,p^*_\pi))\right|\right)\left|\hat{e}(X_i)-e(X_i)\right|\right] \\
&\leq \frac{1}{\kappa}\sqrt{\mathbb{E}_T\left[(\hat{\mu}^z_{1,j}(X_i,\tilde{P}_\pi)-\mu^z_{1,j}(X_i,\tilde{P}_\pi))^2\right]}\sqrt{\mathbb{E}_T\left[(\hat{e}(X_i)-e(X_i))^2\right]} \\
&\quad+\frac{1}{\kappa}\sqrt{\mathbb{E}_T\left[(\mu^z_{1,j}(X_i,\tilde{P}_\pi)-\mu^z_{1,j}(X_i,p^*_\pi))^2\right]}\sqrt{\mathbb{E}_T\left[(\hat{e}(X_i)-e(X_i))^2\right]} \\
&\leq \frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n}+\frac{C}{\kappa}\rho_{e,n}\rho_{p_n}
\end{aligned}
\tag{19}
$$

We have now shown that with probability at least $1-o(1)$, that

$$
||z_\pi(\hat{P}_\pi;\eta^*_\pi)-z(p^*_\pi;\hat{\eta}_\pi)|| \leq \sqrt{J}\left(\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n}+\frac{C}{\kappa}\rho_{e,n}\rho_{p_n}+\frac{C}{\kappa}\rho_{e,n}||\hat{P}_\pi-p^*_\pi||\right),
$$

$$
\sup_{\pi\in\Pi}||z_\pi(\hat{P}_\pi;\eta^*_\pi)-z(p^*_\pi;\hat{\eta}_\pi)|| \leq \sqrt{J}\left(\frac{1}{\kappa}\rho_{\mu,n}\rho_{e,n}+\frac{C}{\kappa}\rho_{e,n}\rho_{p_n}+\frac{C}{\kappa}\rho_{e,n}\sup_{\pi\in\Pi}||\hat{P}_\pi-p^*_\pi||\right).
$$

□

Lemma 18. ***Concentration of finite-market cutoffs*** *Under the Assumptions of The-*

orem 1, $E_\pi[||P_\pi - p^*_\pi||] = O_p(n^{-1/2})$ and $||P_\pi - p^*_\pi|| = O_p(n^{-1/2})$. Under the Assumptions of Theorem 7, $\sup_{\pi\in\Pi} \mathbb{E}_\pi[||P_\pi - p^*_\pi||] = O_p(n^{-1/2})$.

Proof. By Jensen's inequality, $\sup_{\pi\in\Pi} \mathbb{E}_\pi[||P_\pi - p^*_\pi||] \leq \mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||P_\pi - p^*_\pi||\right]$. By (21), we have that

$$\sup_{\pi\in\Pi} \min\{c_3||P_\pi - p^*_\pi||, c_2\} \leq 2\sup_{\pi\in\Pi} ||z_\pi(P_\pi)||\,.$$

So, we can finish the proof by showing that $\mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||z_\pi(P_\pi)||\right] = O_p(n^{-1/2})$.

$$\begin{aligned}
\mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||z_\pi(P_\pi)||\right] &\leq \mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||z_\pi(P_\pi) - Z_{n,\pi}(P_\pi)||\right] + \mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||Z_{n,\pi}(P_\pi)||\right] \\
&\leq \mathbb{E}_\pi\left[\sup_{\pi\in\Pi, p\in\mathcal{S}} ||z_\pi(p) - Z_{n,\pi}(p)||\right] + \mathbb{E}_\pi\left[\sup_{\pi\in\Pi} ||Z_{n,\pi}(P_\pi)||\right] \qquad (20)\\
&= O_p(n^{-1/2})
\end{aligned}$$

The first term in (20) is $O_p(n^{-1/2})$ by the following argument. Theorem 7 indicates that Π is a VC-class and Assumption 3 indicates that $\mathcal{F}_{d,j} = \{B(w) \mapsto d_j(B(w), p) : p \in \mathcal{S}\}$ has uniform ε-covering number bounded by a polynomial in $(1/\varepsilon)$. So, by the composition rules in Lemma 21 and the tail bound in Lemma 20, then $\mathbb{E}\left[\sup_{\pi\in\Pi, p\in\mathcal{S}} \sqrt{n}||z_\pi(p) - Z_{n,\pi}(p)||\right] = O(1)$. By Markov's inequality, this means that $\mathbb{E}_\pi\left[\sup_{\pi\in\Pi, p\in\mathcal{S}} \sqrt{n}||z_\pi(p) - Z_{n,\pi}(p)||\right] = O_p(1)$.

For the second term, by Assumption 2, with probability exponentially small in n, then $\sqrt{n}\sup_{\pi\in\Pi} ||Z_{n,\pi}(P_\pi)||$ is at most $\sqrt{n}\cdot M$, and with probability at most 1, then $n\sup_{\pi\in\Pi} ||Z_{n,\pi}(P_\pi)||^2 = o(1)$. This means that $\mathbb{E}\left[\sup_{\pi\in\Pi} \sqrt{n}||Z_{n,\pi}(P_\pi)||\right] = o(1)$ and by Markov's inequality,

$$\mathbb{E}_\pi\left[\sup_{\pi\in\Pi} \sqrt{n}||Z_{n,\pi}(P_\pi)||\right] = o_p(1).$$

Also by Markov's inequality, following the argument in (20) pointwise for each π shows that $E_\pi[||P_\pi - p^*_\pi||] = O_p(n^{-1/2})$ and $||P_\pi - p^*_\pi|| = O_p(n^{-1/2})$, which is enough to prove the first part of the Lemma. ☐

Lemma 19. ***Concentration of estimated market-clearing cutoffs***

Under the Assumptions of Theorem 7,

$$\sup_{\pi \in \Pi} ||\hat{P}_\pi - p^*_\pi|| = O_p(n^{-1/2}).$$

Under the Assumptions of Theorem 2, for each $\pi \in \Pi$,

$$||\hat{P}_\pi - p^*_\pi|| = O_p(n^{-1/2}).$$

Proof. We start with a version of uniform consistency.

By the twice continuous differentiability of $z(p; \eta^*)$ in p with bounded derivatives, then $\xi_z(p) = \nabla_p z_\pi(p)$ is Lipschitz continuous in p with constant c'. Specifically, for any $\epsilon > 0$ and any p that is an element of the open ball $\mathcal{B}(p^*; \epsilon/c')$, then $||\xi_z(p) - \xi_z(p^*_\pi)|| \leq J\epsilon$. By the mean-value form of the Taylor expansion, there exists a $\bar{p}$ such that

$$\begin{aligned}
||z_\pi(p; \eta^*_\pi) - z_\pi(p^*_\pi; \eta^*_\pi)|| &= ||\xi_z(\bar{p})(p - p^*_\pi)|| \\
&\geq ||\xi_z(p^*_\pi)(p - p^*_\pi)|| - ||(\xi_z(\bar{p}) - \xi_z(p^*_\pi))(p - p^*_\pi)|| \\
&\overset{(1)}{\geq} ||\xi_z(p^*_\pi)(p - p^*_\pi)|| - \epsilon J||(p - p^*_\pi)|| \\
&\overset{(2)}{\geq} ||\xi_z(p^*_\pi)(p - p^*_\pi)|| - \frac{1}{2}||\xi_z(p^*_\pi)(p - p^*_\pi)|| \\
&= \frac{1}{2}||\xi_z(p^*_\pi)(p - p^*_\pi)|| \\
&\geq \frac{c_3}{2}||p - p^*_\pi||
\end{aligned} \tag{21}$$

So, for any $p \in \mathcal{B}(p^*_\pi; \frac{c_3}{2Jc'})$, $2||z_\pi(p; \eta^*)|| \geq c_3||p - p^*_\pi||$. In addition, by Assumption 4, for any $p \in \mathcal{S} \backslash \mathcal{B}(p^*_\pi; \frac{c_3}{2Jc'}), 2||z_\pi(p)|| \geq c_2$.

$$\sup_{\pi \in \Pi} \min\{c_3||\hat{P}_\pi - p^*_\pi||, c_2\} \leq 2 \sup_{\pi \in \Pi} \left|\left| z_\pi(\hat{P}_\pi; \eta^*_\pi) \right|\right|$$

To finish the proof of uniform consistency, then, we need to show that with probability $1 - o(1)$,

$$\sup_{\pi \in \Pi} ||z_\pi(\hat{P}_\pi; \eta^*_\pi)|| \leq g_n,$$

for $g_n = o(1)$. Since $c_3 > 0$ and $c_2 > 0$ are fixed constants, this implies for sufficiently large n, that with probability $1 - o(1)$, that $||\hat{P}_\pi - p^*_\pi|| \leq b_n$ for $b_n = o(1)$.

We proceed using the following decomposition:

$$\sup_{\pi\in\Pi}||z_\pi(\hat{P}_\pi;\eta_\pi)|| \leq \underbrace{\sup_{\pi\in\Pi}||z_\pi(\hat{P}_\pi;\eta_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi)||}_{(i)} + \underbrace{\sup_{\pi\in\Pi}||z_\pi(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi)||}_{(ii)}$$
$$+ \underbrace{\sup_{\pi\in\Pi}||\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi)||}_{(iii)}$$

Since $||p-p^*_\pi||$is bounded, (i) is $o_p(1)$ by Lemma 17. Lemma 16 indicates that (ii) is $O_p(n^{-1/2})$. For (iii), we use the last part of Assumption 5 which implies that $\sup_{\pi\in\Pi}||\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi)|| = o_p(n^{-1/2})$.

Combining the bounds for each of these terms, we have now shown that $\sup_{\pi\in\Pi}||\hat{P}_\pi - p^*_\pi|| = o_p(1)$. Next, we want to strengthen the uniform consistency result into a rate. We want to show that

$$\sup_{\pi\in\Pi}\sqrt{n}||\hat{P}_\pi - p^*_\pi|| \leq \sup_{\pi\in\Pi}||(\nabla_p z_\pi(p^*_\pi))^{-1}||||\sqrt{n}\Gamma^z_{n,\pi}(p^*_\pi,\eta^*_\pi)|| + \sqrt{n}MR_{1n}\sup_{\pi\in\Pi}||P_n - p^*_\pi|| + R_{2n}, \tag{22}$$

where $R_{1n} = o_p(1)$ and $R_{2n} = O_p(1)$. Once we have this, the proof is straightforward. Since the eigenvalues of $\nabla_p z_\pi(p^*_\pi)$ are uniformly bounded by c_3 from below and $z_\pi(p^*_\pi;\eta^*_\pi) = 0$.

$$\begin{aligned}
\sup_{\pi\in\Pi}\sqrt{n}||\hat{P}_\pi - p^*_\pi||(1 - MR_{1n}) &\leq \frac{1}{c_3}\sup_{\pi\in\Pi}||\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi)|| + R_{2n} \\
\sup_{\pi\in\Pi}\sqrt{n}||\hat{P}_\pi - p^*_\pi||(1 - MR_{1n}) &\leq \frac{1}{c_3}\sup_{\pi\in\Pi,p\in\mathcal{S}}||\Gamma^z_{n,\pi}(p;\eta^*_\pi) - z_\pi(p;\eta^*_\pi)|| + R_{2n}
\end{aligned}$$

Since $1/(1 - MR_{2n}) = O_p(1)$, $R_{2n} = O_p(1)$, and by Lemma 13 , $||\sup_{\pi\in\Pi,p\in\mathcal{S}}||\Gamma^z_{n,\pi}(p;\eta^*_\pi) - z_\pi(p;\eta^*_\pi)|| = O_p(1)$, then $\sup_{\pi\in\Pi}\sqrt{n}||P_n - p^*_\pi|| = O_p(1)$. So, to finish the proof, we must show (22), with the required convergence properties for R_{1n} and R_{2n}. We start with the following expansion:

$$\begin{aligned}
\Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) + U_{1n} &= z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi), \\
\Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - \Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) + U_{1n} + U_{2n} &= z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi), \\
-\Gamma^z_{n,\pi}(p^*_\pi;\eta) + U_{1n} + U_{2n} &= (\hat{P}_\pi - p^*_\pi)\nabla_p z_\pi(p^*_\pi) + \mathcal{O}(||\hat{P}_\pi - p^*_\pi||^2),
\end{aligned}$$

where $U_{1n} = \Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi) - \Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) + z_\pi(\hat{P}_\pi;\eta^*_\pi)$, $U_{2n} = -\Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) + \Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi)$, and the last step is by the mean-value form for a Taylor expansion.

By the mean-value form for a Taylor expansion of $z_\pi(\hat{P}_\pi) - z_\pi(p^*_\pi)$, for a fixed M that does not depend on π, then the previous step implies:

$$\sqrt{n}||\hat{P}_\pi - p^*_\pi|| \leq || - (\nabla_p z_\pi(p^*_\pi))^{-1}||||\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi)|| + M||\hat{P}_\pi - p^*_\pi||^2 + U_{1n} + U_{2n}. \tag{23}$$

where M is a fixed constant that does not depend on π, since the derivatives of $z_\pi(p;\eta^*_\pi)$ in p are uniformly bounded. To finish showing a version of (22), we examine U_{1n} and U_{2n} more closely.

$$\begin{aligned}
\sup_{\pi\in\Pi}\sqrt{n}||U_{1n}|| &= \sup_{\pi\in\Pi}\sqrt{n}||\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi) - z_\pi(p^*_\pi;\eta^*_\pi) - \Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\eta^*_\pi)|| \\
&\leq 2\sqrt{n}\sup_{p\in\mathcal{S},\pi\in\Pi}||\Gamma^z_{n,\pi}(p;\eta^*_\pi) - z_\pi(p;\eta^*_\pi)|| \\
&= O_p(1),
\end{aligned}$$

where the equality sign follows from Lemma 13.

$$\begin{aligned}
\sup_{\pi\in\Pi}||U_{2n}|| &= \sup_{\pi\in\Pi}||\Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi)|| \\
&\leq ||z_\pi(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi)|| + ||\Gamma^z_{n,\pi}(\hat{P}_\pi;\eta^*_\pi) - z_\pi(\hat{P}_\pi;\eta^*_\pi) - \Gamma^z_{n,\pi}(\hat{P}_\pi;\hat{\eta}_\pi) - z_\pi(\hat{P}_\pi;\hat{\eta}_\pi)||
\end{aligned}$$

For the second term, we rely on Lemma 13 and 16 yet again, which implies that $\sup_{\pi\in\Pi,p\in\mathcal{S}}||\Gamma^z_{n,\pi}(p;\hat{\eta}_\pi) - \Gamma^z_{n,\pi}(p;\hat{\eta}_\pi)|| = O_p(n^{-1/2})$ and $\sup_{\pi\in\Pi,p\in\mathcal{S}}||\Gamma^z_{n,\pi}(p;\eta^*_\pi) - z_\pi(p;\eta^*_\pi)|| = O_p(n^{-1/2})$.

For the first term, Lemma 17 implies that

$$\sup_{\pi\in\Pi}\sqrt{n}||z_\pi(\hat{P}_\pi;\eta) - z_\pi(\hat{P}_\pi;\hat{\eta})|| \leq A_n\sup_{\pi\in\Pi}\sqrt{n}||\hat{P}_\pi - p^*_\pi|| + o_p(1)$$

where $A_n = o_p(1)$ by Assumption 5. Plugging these bounds for U_{1n} and U_{2n} back into (23), we have now shown a version of (22), which completes the proof:

$$\sup_{\pi\in\Pi}\sqrt{n}||\hat{P}_\pi - p^*_\pi|| \leq ||(\nabla_p z_\pi(p^*_\pi))^{-1}||||\Gamma^z_{n,\pi}(p^*_\pi;\eta^*_\pi)|| + (M||\hat{P}_\pi - p^*_\pi|| + o_p(1))||\hat{P}_\pi - p^*_\pi|| + o_p(1) + O_p(1).$$

Under the assumptions of Theorem 2, we can follow the above argument pointwise for each $\pi \in \Pi$ rather than uniformly over π. For the pointwise results, whenever Lemma 16 is used in the above argument, we only need the part that is uniform over $p \in \mathcal{S}$, which requires only the assumptions of Theorem 2.

□

Lemma 20. *Let $\mathcal{F}$ be a class of measurable functions $f : \mathcal{X} \to [-M, +M]$, where $M \in \mathbb{R}$ and $M < \infty$. For some constants $V \geq 1$ and $K \geq 1$, $\sup_Q \log N(\varepsilon, \mathcal{F}, L_2(Q)) \leq \left(\frac{K}{\varepsilon}\right)^V$, for every $0 < \varepsilon < K$. Then, there a finite constant C such that*

$$\mathbb{P}\left(\left|\sup_{f\in\mathcal{F}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(X_i) - \mathbb{E}[f(X_i)]\right| > t\right) \leq Ct^V e^{-2t^2}.$$

Proof. This tail bound is Theorem 2.14.9 of van der Vaart & Wellner (1997) (Theorem 2.14.28 in the second edition). Note to match the conditions of the theorem exactly, we need to rescale f to map to $[0, 1]$, which affects the constant in the tail bound from the original theorem. □

Lemma 21. *Lipschitz composition rules for uniform covering numbers. $\mathcal{F}_1, \ldots \mathcal{F}_K$ are classes of measurable functions from $\mathcal{Z} \to \mathbb{R}$. Let $\psi(\mathcal{F}) = \{\psi(f_1, f_2, f_3, \ldots, f_K) : f_1 \in \mathcal{F}_1, \ldots, f_K \in \mathcal{F}_K\}$ be a class that combines each of these functions, where the map $\psi : \mathbb{R}^k \to \mathbb{R}$ is Lipschitz in that*

$$|\psi(f(z)) - \psi(g(z))|^2 \leq \sum_{i=1}^{n} L_k^2 |f_k(z) - g_i(z)|^2.$$

for every $f, g \in \mathcal{F}_1 \times \ldots \times F_K$ and every $z \in \mathcal{Z}$ and L is positive. Then,

$$\sup_Q N(\varepsilon||L \cdot F||_{Q,2}, \psi(\mathcal{F}), L_2(Q)) \leq \prod_{k=1}^{K} \sup_{R_j} N(\varepsilon||F_k||_{R_j,2}, \mathcal{F}_k, L_2(R_j)),$$

where $L \cdot F(z) = \sum_{k=1}^{K} (L_k^2 F_k^2(z))^{0.5}$.

Proof. This is Lemma A.6 of Chernozhukov et al. (2014). □