**PROBLEM STATEMENT**

Find the Taylor polynomial of degree two for $f(x, y) = \sin(2x) + \cos(y)$ about $(0, 0)$.

**SOLUTION**

The quadratic Taylor polynomial for $f(x, y)$ about $(0, 0)$ is given by

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2}f_{xx}(0,0)(x-0)^2 \\ &\quad + f_{xy}(0,0)(x-0)(y-0) + \frac{1}{2}f_{yy}(0,0)(y-0)^2 \end{aligned}$$

$$\begin{aligned} f_x(x,y) &= 2\cos(2x) & f_x(0,0) &= 2 \\ f_y(x,y) &= -\sin(y) & f_y(0,0) &= 0 \\ f_{xx}(x,y) &= -4\sin(2x) & f_y(0,0) &= 0 \\ f_{xy}(x,y) &= 0 & f_{xy}(0,0) &= 0 \\ f_{yy}(x,y) &= -\cos(y) & f_{yy}(0,0) &= -1 \end{aligned}$$

Since $f(0, 0) = 1$, we obtain

$$\begin{aligned} Q(x,y) &= 1 + 2 \cdot x + 0 \cdot y + \frac{1}{2} \cdot 0 \cdot (x)^2 + 0 \cdot x \cdot y + \frac{1}{2} \cdot (-1) \cdot (y)^2 \\ &= \boxed{1 + 2x - \frac{1}{2}y^2} \end{aligned}$$

**PROBLEM STATEMENT**

If $u(x,t) = e^{at}\sin(bx)$ satisfies the heat equation

$$u_t(x,t) = u_{xx}(x,t)\,, \tag{1}$$

then find the relationship between $a$ and $b$.

**SOLUTION**

$$\begin{aligned} u_t &= a \cdot e^{at}\sin(bx) \\ u_x &= b \cdot e^{at}\cos(bx) \\ u_{xx} &= -b^2 e^{at}\sin(bx) \end{aligned}$$

If $u(x,t) = e^{at}\sin(bx)$ satisfies $u_t = u_{xx}$, then it must be

$$\begin{aligned} a \cdot e^{at}\sin(bx) &= -b^2 \cdot e^{at}\sin(bx) \\ (a+b^2)e^{at}\sin(bx) &= 0 \\ a^2+b^2 &= 0 \\ a &= -b^2 \end{aligned}$$

**PROBLEM STATEMENT**

Suppose function $f(x, y)$ is differentiable and $f_x(2, 1) = -3$, $f_y(2, 1) = 4$, $f(2, 1) = 7$.

(a) Give an equation for the tangent plane to the graph of $f$ at $x = 2$ and $y = 1$.

(b) Give an equation for the tangent line to the contour for $f$ at $x = 2$ and $y = 1$.

**SOLUTION**

(a) An equation for the tangent plane is given by

$$\begin{aligned} z &= f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) \\ z &= 7 + (-3)(x - 2) + 4(y - 1) \\ \Aboxed{z &= -3x + 4y + 9} \end{aligned}$$

(b) We start by recalling that

$$\nabla f(2, 1) \;=\; f_x(2, 1)\vec{i} + f_y(2, 1)\vec{j} \;=\; -3\vec{i} + 4\vec{j}$$

is perpendicular to the contour of $f$ at $x = 2$, $y = 1$. Consequently, the tangent line of $f$ at $x = 2$, $y = 1$ is also perpendicular to $-3\vec{i} + 4\vec{j}$, and so the slope of this tangent line is $\frac{3}{4}$.
Finally, the desired tangent line is given by

$$\begin{aligned} y - 1 &= \frac{3}{4}(x - 2) \\ \Aboxed{y &= \frac{3}{4}x - \frac{1}{2}} \end{aligned}$$

Alternatively, you can use the equation of tangent plane and set $z = f(2, 1) = 7$. One attains

$$\begin{aligned} 7 &= -3x + 4y + 9 \\ -2 &= -3x + 4y \\ y &= \frac{3}{4}x - \frac{1}{2} \end{aligned}$$

**PROBLEM STATEMENT**

The function $f$ gives temperature in °C and $x$ and $y$ are in centimeters, and

$$f_x(2,1) \;=\; -3\,, \qquad f_y(2,1) \;=\; 4\,, \qquad f(2,1) \;=\; 7\,.$$

A bug leaves $(2,1)$ at 3 cm/min so that it cools off as fast as possible.

(a) In which direction does the bug head?

(b) At what rate does it cool off, in °C/min?

**SOLUTION**

(a) The bug is heading in the direction of $-\nabla f(2,1)$. Note that

$$\begin{aligned}\nabla f(2,1) \;&=\; f_x(2,1)\,\vec{i} + f_y(2,1)\,\vec{j}\\ \nabla f(2,1) \;&=\; -3\,\vec{i} + 4\,\vec{j}\end{aligned}$$

Thus the direction of fastest cooling off is

$$-\nabla f(2,1) \;=\; 3\,\vec{i} - 4\,\vec{j}$$

and the rate of cooling off in degrees per centimeter is given by the magnitude of $-\nabla f(2,1)$, namely

$$\|{-\nabla f(2,1)}\| \;=\; \sqrt{3^2 + (-4)^2} \;=\; \boxed{5\,\frac{\text{degrees }^\circ C}{\text{cm}}}.$$

(b) Since the bug is moving 3 cm/min, the rate of change in $^\circ C$ per minute is given by

$$5\,\frac{\text{degrees }^\circ C}{\text{cm}} \cdot 3\frac{\text{cm}}{\text{min}} \;=\; \boxed{15\,\frac{\text{degrees }^\circ C}{\text{min}}}.$$

**PROBLEM STATEMENT**

(a) Find an equation of the tangent plane to the surface $2x^2 - 2xy^2 + az = a$ at the point $(1, 1, 1)$ (treat "$a$" as a constant).

(b) For which value of "$a$" does the tangent plane pass through the origin?

**SOLUTION**

(a) The surface $2x^2 - 2xy^2 + az = a$ is the level surface $f(x, y, z) = 0$ where $f(x, y, z) = 2x^2 - 2xy^2 + az$. Now recall that $\nabla f(1, 1, 1)$ is a normal vector for the tangent plane at $(1, 1, 1)$.

$$\begin{aligned} \nabla f(x, y, z) &= (4x - 2y^2)\,\vec{i} + (-4xy)\,\vec{j} + a\,\vec{k} \\ \nabla f(1, 1, 1) &= 2\,\vec{i} - 4\,\vec{j} + a\,\vec{k} \end{aligned}$$

Thus the equation of the desired tangent plane is

$$\nabla f(1, 1, 1) \cdot \Big( (x - 1)\,\vec{i} + (y - 1)\,\vec{j} + (z - 1)\,\vec{k} \Big) = 0$$

$$2(x - 1) - 4(y - 1) + a(z - 1) = 0$$

$$\boxed{2x - 4y + az = a - 2} \tag{2}$$

(b) Substitute $x = 0$, $y = 0$, $z = 0$ into (2) to obtain

$$2 \cdot 0 - 4 \cdot 0 + a \cdot 0 = a - 2 \qquad \text{so that} \qquad \boxed{\text{a=2}}$$

**PROBLEM STATEMENT**

Let $x, y, z$ be in meters. At the point $(x, y, z)$ in space, the temperature , $H$, in °C, is given by

$$H \; = \; e^{-(x^2+2y^2+3z^2)}\,. \tag{3}$$

(a) A particle at the point $(2, 1, 5)$ starts to move in the direction of increasing $x$. How fast is the temperature changing with respect to distance? Give units.

(b) If the particle in part (a) moves at 10 meters/sec, how fast is the temperature changing with respect to time? Give units.

(c) What is the maximum rate of change of temperature with respect to distance at the point $(2, 1, 5)$?

**SOLUTION**

(a)

$$\left.\frac{\partial H}{\partial x}\right|_{(2,4,5)} \; = \; \left.\left(-2x \cdot e^{-(x^2+y^2+3z^2)}\right)\right|_{(2,1,5)} \; = \; \boxed{-4e^{-81} \;\; {}^\circ C/\text{meter}}.$$

(b) By the Chain Rule, we have

$$\frac{\partial H}{\partial t} \; = \; \frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} \; = \; -4e^{-81} \;\; {}^\circ C/\text{meter} \; \cdot \; 10\frac{\text{meter}}{sec} \; = \; \boxed{-40e^{-81} \;\; {}^\circ C/\text{sec}}.$$

(c) The magnitude of the gradient gives the maximum rate of change, so

$$\nabla H\Big|_{(2,1,5)} \; = \; -e^{-(x^2+2y^2+3z^2)}\left.\left(2x\,\vec{i} + 4j\,\vec{j} + 6z\,\vec{k}\right)\right|_{(2,1,5)} \; = \; -e^{81}\left(4\,\vec{i} + 4\,\vec{j} + 30\,\vec{k}\right).$$

Thus

$$\|\nabla H(2,1,5)\| \; = \; e^{-81}\sqrt{4^2 + 4^2 + 30^2} \; = \; \boxed{\sqrt{932}e^{-81} \;\; \frac{{}^\circ C}{\text{meter}}}.$$

**PROBLEM STATEMENT**

You are standing at the point $(1, 1, 3)$ on the hill whose equation is given by $z = 5y - x^2 - y^2$.

(a) If you choose to climb in the direction of steepest ascent, what is your initial rate of ascent relative to the horizontal distance?

(b) If you decide to go straight northwest, will you be ascending or descending? At what rate?

(c) If you decide to maintain your altitude, in what directions can you go?

**SOLUTION**

(a) The direction of steepest ascent is given by $\nabla f(1, 1)$:

$$\nabla f(1,1) \;=\; -2x\,\vec{i} + (5-2y)\,\vec{j}\,\Big|_{(1,1)} \;=\; \boxed{-2\,\vec{i} + 3\,\vec{j}}.$$

Thus the initial rate of steepest ascent is $\|\nabla f(1,1)\| = \sqrt{(-2)^2 + (3)^2} = \boxed{\sqrt{13}}$ meters ascended for each horizontal meter covered.

(b) In order to go straight northwest, we want to travel along the vector $\vec{v} = -\vec{i} + \vec{j}$. A unit vector that points in the direction of $\vec{v}$ is therefore given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{2}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}.$$

So,

$$f_{\vec{u}} = \nabla f(1,1)\cdot\vec{u} = (-2\,\vec{i} + 3\,\vec{j})\cdot\left(-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \boxed{\frac{5}{\sqrt{2}}}$$

meters ascended for each meter ascended.

(c) We are looking for direction $\vec{u}$ such that $f_{\vec{u}}(1,1) = 0$. Such vector $\vec{u}$ must be perpendicular to $\nabla f(1,1)$, i.e., $\nabla f(1,1)\cdot\vec{u} = 0$. By inspection we attain that two such possible directions are

$$\boxed{\vec{u} = -3\,\vec{i} - 2\,\vec{j} \quad \text{and} \quad \vec{u} = 3\,\vec{i} + 2\,\vec{j}}.$$