Chapter 15

Optimization: Local and Global Extrema

15.1 Critical Points: Local Extrema and Saddle Points

Definition: Local extrema of a function $f(x, y)$

(a) Function $f(x, y)$ has a **local minimum** at the point (a, b) if for all points (x, y) in some open disc around (a, b) we have

$$f(x, y) \geq f(a, b).$$

(b) Function $f(x, y)$ has a **local maximum** at the point (a, b) if for *all points* (x, y) in some open disc around (a, b) we have

$$f(x, y) \leq f(a, b).$$

Note that if (a, b) is a local extremum of $f(x, y)$, the *both cross-sections* $f(x, b)$ and $f(a, y)$ have a local extremum at a and b, respectively. Thus, the derivatives of $f(x, b)$ and $f(a, y)$ are 0 at $x = a$ and $y = b$, respectively, **or** the derivatives do not exist. Consequently, we have the following result.

Theorem: If $f(x, y)$ has a local extremum at (a, b), then:

(i) $f_x(a, b) = 0$ *or* $f_x(a, b)$ does not exist

AND

(ii) $f_y(a, b) = 0$ *or* $f_y(a, b)$ does not exist.

This leads us to introduce the following definition.

Definition: Critical (Stationary) Point

Point (a, b) is a **critical (stationary) point** of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$ (that is, $\nabla f(a, b) = \vec{0}$) *or* at least one the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ does not exist.

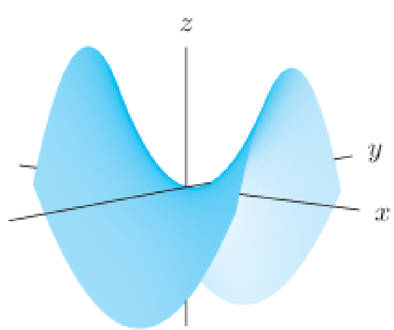
Question: Let $z = f(x, y)$ and assume (a, b) is a critical point of $f(x, y)$. Does (a, b) have to be a local extremum?

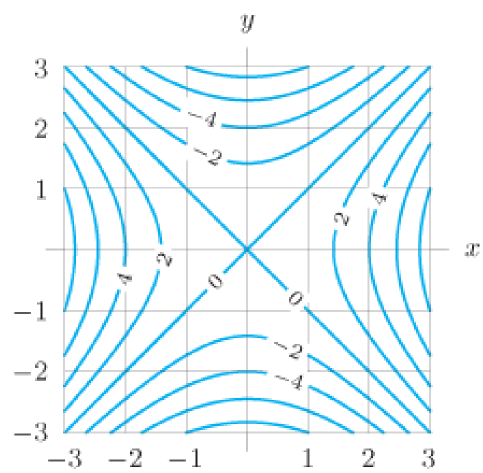
Answer: **NO.** Before we come up with an appropriate example to justify our answer, note that the equivalent of this statement for functions of one variable is also not true (e.g., function $g(x) = x^3$ has a critical point $x = 0$, but it is neither a local minimum nor a local maximum).

Example 1. *Consider the following function $z = f(x, y) = x^2 - y^2$.*

(a) *Determine the critical points of $f(x, y) = x^2 - y^2$.*

(b) *Classify the critical points of $f(x, y) = x^2 - y^2$ by looking at its contour plot.*





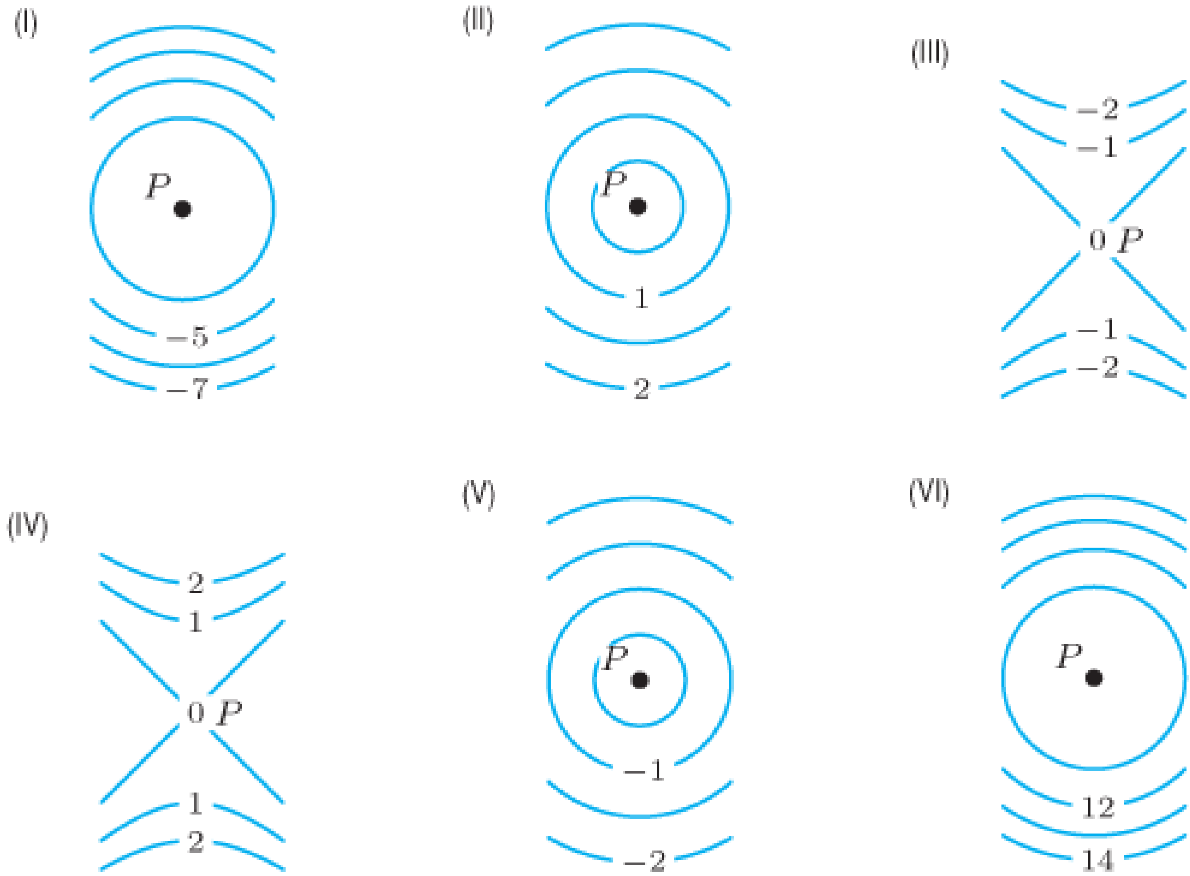
Definition: Saddle Point

Point (a, b) is a **saddle point** of $f(x, y)$ if (a, b) is a *critical point* of $f(x, y)$ **and** in every open disk around (a, b) there exist points (x_1, y_1) and (x_2, y_2) such that

$$f(x_1, y_1) < f(a, b) \qquad \textbf{and} \qquad f(x_2, y_2) > f(a, b)\,.$$

Remark: **IF** $f(x, y)$ has a critical point at (a, b), **THEN** (a, b) is a local maximum, a local minimum, or a saddle point.

Example 2. *Figure below shows level curves of six functions around a critical point P. Does each function have a local maximum, a local minimum, or a saddle point at P?*



Remark: *Note that the contour plot corresponding to a saddle point can look different than the examples (III) and (IV) above. Use Mathematica here to investigate function $f(x, y) = x^4 + y^3$.*

Now a natural question to ask is can we determine algebraically (that is, without an aid of a contour plot) if a critical point is a local minimum, a local maximum, or a saddle point. The answer is yes, and the key is *discriminant of $f(x, y)$* and the *Second Derivative Test of Local Extrema*.

Definition: Let $f(x, y)$ be a function of two variables with continuous second order partial derivatives. Then the **discriminant of $f(x, y)$,** denoted by $D(x, y)$, is given by

$$D(x, y) := f_{xx}(x, y) \cdot f_{yy}(x, y) - \Big[f_{xy}(x, y)\Big]^2$$

Theorem: **Second Derivative Test for Functions of Two Variables**

Suppose (a, b) is a point where $f_x(a, b) = f_y(a, b) = 0$ and consider

$$D(a, b) \,=\, f_{xx}(a, b) \cdot f_{yy}(a, b) \,-\, \Big[f_{xy}(a, b)\Big]^2 .$$

(i) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a local minimum at (a, b).

(ii) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a local maximum at (a, b).

(iii) If $D(a, b) < 0$, then $f(x, y)$ has a saddle point at (a, b).

(iv) If $D(a, b) = 0$, then anything can happen at (a, b).

Example 3. Find *all critical points of function*

$$f(x, y) \,=\, x^3 - 3x + y^3 - 3y\,. \tag{15.1}$$

and **classify** *them as a local minimum, a local maximum, or a saddle point.*