MTH 243: Exam #2 – Logistics

Exam Structure:

- Exam is scheduled to be held **on campus in person:**
  - **Time:** Starts **at 9:30am on Thursday, April 4**
  - **Location:** **CBLS 152 (our regular classroom)**.
- Please arrive a few minutes early so everyone can be settled in time to start the exam.
- If you need alternative accommodations due to valid reason, please send me an email right away!
- This exam will closely follow the setup from Quizzes, but with a **lot less time allocated** for it. If you found yourself spending more time on the quizzes than recommended, then you are very likely going to run out of time on this in-class exam.
  - **Type I Questions:** Multiple choice, true/false, and matching.
  - **Type II Questions:** Show your work!

General Exam Policies:

- You must bring your URI Photo ID with you to each exam, and you must show it to a proctor as you hand in your exam.
- In general, proctors will not answer any questions regarding the content of the exam. In particular, you are expected to know and remember all relevant definitions and terminology.
- *During the exam, you may not leave the room for any reason!!!* Please remember to use the bathroom before the exam. By the exam time, anyone needing alternate accommodations should have notified me.
- No calculators or notes of any kind may be used on the exam.
- No cell phones, MP3 players, or any electronic devices of any kind may be used or even accessible to you at any time during the exam. Any student found with any electronic device for any reason during the exam will be considered to be cheating.

General Suggestions:

- Bring several sharp pencils and an eraser or if you wish you may use the pen (not recommended).
- *Start by reviewing quizzes and suggested practice problems.* Solutions were posted on Brightspace and/or discussed during office hours (per students' requests). Do not just read the solutions, work them out again from scratch and check your work! Modify the problems. See if you can still solve them.
- *Do the same with problems that were discussed in lecture notes. Now repeat the process with the suggested practice problems.*

Disclaimer: This list is meant to be an *approximate guide*; please note that you are responsible for all the material covered from Sections 14.1–14.7, 15.1, and 16.1 16.2 is NO longer part of the second exam!!!

- The list below contains key points from each section. It does not mean that I will ask you about every single topic, but you are still responsible for knowing it. Additional practice problems are included at the end so you can test your understanding.

1. **§14.1:** The Partial Derivative
   - The limit definition of the *partial derivative* of a function $f(x, y)$ with respect to one of its variables.
   - *Geometric interpretation* of partial derivatives $f_x(x, y)$ and $f_y(x, y)$ (see class notes).
   - Be able to estimate partial derivatives f_x and f_y at a point given the contour plot of $f(x, y)$ or table of values.

2. **§14.2:** Computing Partial Derivatives Algebraically
   - Know how to compute partial derivatives algebraically. *If you do not know how to do this, you will not be able to answer majority of the questions below!!!*

3. **§14.3:** Local Linearity and the Differential
   - Equation of the tangent plane to $f(x, y)$ near the point $(a, b) \Longrightarrow$ *local linearization* of f near $(x, y) = (a, b)$.
   - Using the local linearization of $f(x, y)$ to approximate value of f at a nearby point (Example 2 (pg.801); Problem #37 (pg.807)).
   - Know the notion of the *differential of the function.*

4. **§14.4 + §14.5:** Gradients and Directional Derivatives
   - Be able to compute the *directional derivative* of $f(x, y, z)$ in the direction of a vector $\vec{v}$. Recall that before computing the desired directional derivative, you must first make $\vec{v}$ the unit vector pointing in the same direction as $\vec{v}$.
   - At a given point, in what direction the directional derivative of a function f is greatest? What is the value of directional derivative in this direction? What does it mean in terms of f increasing/decreasing?
   - Be able to find the direction of greatest rate of change of a function at a given point.
   - Ggiven a contour plot of $f(x, y)$ or table of values, be able to estimate f_x and f_y of a function at various points.
   - Be able to find an *equation of the tangent plane to a surface at a point.*
   - Know the *geometric properties* of the gradient vector (see pages 812 & 819).
   - Application problems involving rates of change (#70(b,c) on page 826; #68 on page 826; class notes and review problems)

5. **§14.6:** The Chain Rule
   - Use the Chain Rule to find derivatives of composite functions.
   - Be able to draw appropriate diagrams for differentiating functions of many variables (see page 828 and class notes).

6. **§14.7:** Second-order Partial Derivatives
   - Be able to compute *second order partial derivatives* for functions of two and three variables.
   - Mixed partials are *usually* equal (Clairaut's Theorem), i.e., $f_{xy}(x, y) = f_{yx}(y, x)$.
   - Be able to find *Taylor polynomials* of degree *one* and *two* for $f(x, y)$ near the point (a, b) (see page 840).
   - Be able to determine if a function is a solution to a given partial differential equation (see class notes).

7. **§15.1:** Critical Points: Local Extrema and Saddle Points
   - Know what *critical points* are and be able to find them.
   - Be able to recognize the critical points from a given contour plot.
   - Be able to classify critical points using the *second derivative test for functions* $f(x, y)$ (see page 860).
8. **§16.1:** The Definite Integral of a Function of Two Variables
   - Know the definition of the *definite integral* of $f(x, y)$ over s rectangular region R.
   - Be able to approximate $\int_R f\, dA$ via Riemann sums.
   - Different interpretations of $\int_R f\, dA$ (see pages 870–872).
9. **§16.2:** Iterated Integrals – **NOT INCLUDED**
   - Know how to evaluate *iterated integrals*. **NOT INCLUDED**
   - Be able to sketch region of integration. **NOT INCLUDED**
   - Be able to rewrite a given iterated integral by reversing order of integration. **NOT INCLUDED**
   - Be able to set up and evaluate iterated integral in order to find $\int_R f\, dA$. **NOT INCLUDED**

Additional Practice Problems

Chapter 14: Differentiation of Multivariable Functions

1. Compute all first and second order partial derivatives for the following functions.
   (a) $f(x, y) = (x + y)^2$
   (b) $f(x, y, z) = x^2y + x^2z - yz^2$
   (c) $h(x, y) = e^{2x+y^2}$
   (d) $g(s, t) = \sin(2s + t)$
   (e) $f(x, y, z) = \sin(x)\sin(y)\sin(z)$
   (f) $h(s, t, u) = se^{tu}$
   (g) $f(x, t) = \sin(4t - 5x)$
   (h) $g(u, v) = e^u \sin(v)$

2. Find the gradient of the following functions.
   (a) **(#30, p.847)** $f(x, y, z) = x^3 + z^3 - xyz$
   (b) **(#34, p.847)** $h(x, y, z) = xe^y + \ln(xz)$

3. Find the directional derivative of the following functions.
   (a) **(#42, p.847)** $f(x, y) = x^3 - y^3$ at $(2, -1)$ in the direction $\vec{i} - \vec{j}$.
   (b) **(#43, p.847)** $f(x, y) = xe^y$ at $(3, 0)$ in the direction of $4\vec{i} - 3\vec{j}$.
   (c) **(#45, p.847)** $f(x, y, z) = 3x^2y^2 + 2yz$ at $(-1, 0, 4)$ in the direction of $\vec{i} - \vec{k}$.

4. **(#18, p.844)** Find the Taylor polynomial of degree two for $f(x, y) = \sin(2x) + \cos(y)$ about $(0, 0)$.

5. **(#47, p.845)** Show that the function $u(x, y) = e^{-x}\sin(y)$ is satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

6. Show that $u(x, y) = e^{x-y}$ is a solution to the partial differential equation $u_x + u_y = 0$.

7. **(solved in class)** Show that $u(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution to the three-dimensional Laplace equation
$$u_{xx} + u_{yy} + u_{zz} = 0\,.$$

8. **(#49, p.845)** If $u(x, t) = e^{at}\sin(bx)$ satisfies the heat equation $u_t(x, t) = u_{xx}(x, t)$, then find the relationship between a and b.

9. **(#100, p.847)** Suppose function $f(x, y)$ is differentiable and $f_x(2, 1) = -3$, $f_y(2, 1) = 4$, $f(2, 1) = 7$.
   (a) Give an equation for the tangent plane to the graph of f at $x = 2$ and $y = 1$.
   (b) Give an equation for the tangent line to the contour for f at $x = 2$ and $y = 1$.

10. **(#105, p.847)** The function f gives temperature in °C and x and y are in centimeters, and $f_x(2, 1) = -3$, $f_y(2, 1) = 4$, $f(2, 1) = 7$. A bug leaves $(2, 1)$ at 3 cm/min so that it cools off as fast as possible.
    (a) In which direction does the bug head?
    (b) At what rate does it cool off, in °C/min?

11. **(#65, p.847)**
    (a) Find an equation of the tangent plane to the surface $2x^2 - 2xy^2 + az = a$ at the point $(1, 1, 1)$ (treat "a" as a constant).
    (b) For which value of "a" does the tangent plane pass through the origin?

12. **(87, p.847)** Let x, y, z be in meters. At the point (x, y, z) in space, the temperature , H, in °C, is given by

$$H = e^{-(x^2+2y^2+3z^2)}.$$

   (a) A particle at the point $(2, 1, 5)$ starts to move in the direction of increasing x. How fast is the temperature changing with respect to distance? Give units.

   (b) If the particle in part (a) moves at 10 meters/sec, how fast is the temperature changing with respect to time? Give units.

   (c) What is the maximum rate of change of temperature with respect to distance at the point $(2, 1, 5)$?

13. Study problem #21 on page 791 (note that #22 on the same page was solved in class).

14. Study #4 on page 790.

15. Study problems #29 – #34 on page 815.

16. **(#1, p.805)** Find an equation of the plane tangent to the graph of $f(x, y) = ye^{x/y}$ at the point $(1, 1, e)$.

17. **(#7, p.805)** Find an equation of the plane tangent to the surface $x^2y^2 + z - 40 = 0$ at $x = 2$ and $y = 3$.

18. **(#13, p.805)** Find the differential of the function $g(x, t) = x^2 \sin(2t)$ at $(2, \pi/4)$.

19. **(#42, p.847)** Find the local linearization of the function $f(x, y) = x^2y$ at $(3, 1)$.

20. **(solved in class)** Suppose that your weight, w, in pounds, is a function $w = f(c, m)$ of the number of calories, c, you consume daily and the number of minutes, m, you exercise daily.

   (a) What are the units of $\frac{\partial w}{\partial c}(c, m)$?

   (b) What is the practical meaning of the statement $\frac{\partial w}{\partial c}(2100, 20) = 0.007$?

   (c) What are the units of $\frac{\partial w}{\partial m}(c, m)$?

   (d) What is the practical meaning of the statement $\frac{\partial w}{\partial m}(2100, 20) = -0.2$?

   (e) Assume that $w(2100, 20) = 120$, $\frac{\partial w}{\partial m}(2100, 20) = -0.2$. Estimate $w(2100, 25)$.

21. **(#99, p.818)** You are standing at the point $(1, 1, 3)$ on the hill whose equation is given by $z = 5y - x^2 - y^2$.

   (a) If you choose to climb in the direction of steepest ascent, what is your initial rate of ascent relative to the horizontal distance?

   (b) If you decide to go straight northwest, will you be ascending or descending? At what rate?

   (c) If you decide to maintain your altitude, in what directions can you go?

22. **(solved in class)** Suppose that the elevation (in miles) of points on a mountain is given by

$$z \;=\; 2e^{-x^2-y^2} + e^{-(x-2)^2-(y-2)^2},$$

   where x and y are, respectively, the distances east and north of a reference point. If Ingrid finds herself 3 miles east and 1 mile south of the reference point and she is headed southwest, is she going uphill or downhill? In what direction(s) should she head if she wants to stay at the same elevation?

23. **(solved in class)** Suppose that the function $C(x, y, z) = x^2 + y^4 + x^2z^2$ gives concentration of salt, in gr/gal, at any point (x, y, z) of a rectangular tank of water occupying the region

$$-2 \le x \le 2, \qquad -2 \le y \le 2, \qquad 0 \le z \le 2,$$

   (all measurements in meters). Suppose you are at the point $(-1, 1, 1)$.

   (a) In what direction should you move if you want the concentration to increase the fastest?

   (b) If you move from $(-1, 1, 1)$ toward the origin $(0, 0, 0)$, how fast is the concentration changing?

24. **(#40, p.824)**

    (a) Let $f(x, y, z) = x^2 + y^2 - xyz$. Find gradient of f.

    (b) Find the equation of the tangent plane to the surface $f(x, y, z) = 7$ at the point $(2, 3, 1)$.

25. **(#69, p.826)** The temperature of a gas at the point (x, y, z) is given by $G(x, y, z) = x^2 - 5xy + y^2 z$.

    (a) What is the rate of change in the temperature at the points $(1, 2, 3)$ in the direction $\vec{v} = 2\vec{i} + \vec{j} - 4\vec{k}$?

    (b) What is the direction of maximum rate of change of temperature at the point $(1, 2, 3)$?

    (c) What is the maximum rate of change at the point $(1, 2, 3)$?

26. Use the Chain Rule to find $\dfrac{dz}{dt}$.

    (a) $z = x^2 y + xy^2, \quad x = 2 + t^4, \quad y = 1 - t^3$

    (b) $z = x \ln(x + 2y), \quad x = \sin(t), \quad y = \cos(t)$

27. Use the Chain Rule to find $\dfrac{dw}{dt}$ where $w = xy + yz^2, \quad x = e^t, \quad y = e^t \sin(t), \quad z = e^t \cos(t)$.

28. Use the Chain Rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

    (a) $z = x^2 + xy + y^2, \quad x = s + t, \quad y = st$

    (b) $z = e^r \cos(\theta), \quad r = st, \quad \theta = \sqrt{s^2 + t^2}$.

29. If $z = f(x, y)$, where $x = g(t)$, $y = h(t)$, $g(3) = 2$, $g'(3) = 5$, $h(3) = 7$, $h'(3) = -4$, $f_x(2, 7) = 6$, and $f_y(2, 7) = -8$, find dz/dt when $t = 3$.

30. Use a diagram to write out the Chain Rule for the functions given below. Assume all functions are differentiable.

    (a) $u = f(x, y)$, where $x = x(r, s, t)$ and $y = y(r, s, t)$.

    (b) $w = f(x, y, z)$, where $x = x(t, u)$, $y = y(t, u)$, and $z = z(t)$ (no, it is not a typo for $z = z(t)$).

31. Use the Chain Rule to find $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$, when $s = 1$, $t = 0$, and

$$w = x^2 + y^2 + z^2, \qquad x = st, \qquad y = s\cos(t), \qquad z = s\sin(t).$$

32. Use the Chain Rule to find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial s}$, and $\dfrac{\partial z}{\partial t}$, when $r = 1$, $s = 2$, $t = 0$, and

$$z = \frac{x}{y}, \qquad x = re^{st}, \qquad y = rse^t.$$

Section 15.1: Critical Points and Second Derivative Test

33. Find and classify all critical points of $f(x, y) = x^3 + 3xy^2 + 3y^2 - 15x + 2$.

Sections 16.1 – 16.2: Multiple Integrals

34. **CHAPTER 16: From our textbook**

    (a) **Section 16.1:** **Study suggested practice problems 1, 9, 11, 13, 21, 25, 29 on page 896/897.** All listed problems from §16.1 have answers/solutions accessible through WileyPlus.

STILL INCLUDED on Exam 2

Everything below is NOT Included

Section 16.2 is NOT covered in our Second Exam

35. Evaluate the following iterated integrals.

    (a) $\displaystyle\int_{-2}^{2}\int_{0}^{4} (4x^3 + 3xy^2)\, dydx$

    (b) $\displaystyle\int_{0}^{2}\int_{x^2}^{4-x^2} 2x\, dydx$

    (c) $\displaystyle\int_{1}^{2}\int_{0}^{x^2} \frac{1}{x+y}\, dydx$

    (d) $\displaystyle\int_{0}^{1}\int_{0}^{x^3} e^{y/x}\, dydx$

    (e) $\displaystyle\int_{0}^{3/2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} ydxdy$ (this is #36 from below)

36. Find the average height of the paraboloid $z = x^2 + y^2$ over the rectangle

$$R = \Big\{ (x, y) : 0 \leq x \leq 2 \,,\ 0 \leq y \leq 1 \Big\}.$$

37. Consider the following iterated integral $\displaystyle\int_{0}^{3/2}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} ydxdy$.

    (a) Sketch the region of integration.

    (b) Evaluate the given integral.

38. Consider the following iterated integral $\displaystyle\int_{0}^{2}\int_{y/2}^{1} e^{x^2}\, dxdy$.

    (a) Sketch the region of integration.

    (b) Evaluate the given integral by first reversing the order of integration.

39. Consider the following iterated integral $\displaystyle\int_{0}^{1}\int_{y^2}^{1} y\sin(x^2)\, dxdy$.

(a) Sketch the region of integration.

(b) Evaluate the given integral by first reversing the order of integration.

40. **CHAPTER 16: From our textbook**

(a) **Section 16.1:** **Study suggested practice problems 1, 9, 11, 13, 21, 25, 29 on page 896/897.**
All listed problems from §16.1 have answers/solutions accessible through WileyPlus.

(b) **Section 16.2:** **Study suggested practice problems 5, 7, 11, on page 905.**

(c) **Section 16.2:** **Study suggested practice problems 1, 3, 13 on page 905.**