Chapter 16

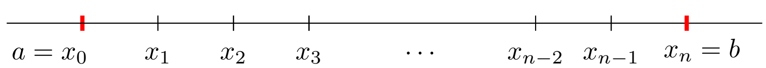
Integrating Functions of Several Variables

16.0 Review of Riemann Sums for Single Variable Functions

Let $f(x)$ be a *continuous* function defined on a closed interval $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Divide $[a,b]$ into n equal subintervals of width

$$\Delta x = \frac{b-a}{n},$$

which are disjoint except the endpoints.



For $1 \leq i \leq n$, the i^{th} subinterval is given by

$$I_i = \left[x_{i-1}, x_i\right] = \left[a + (i-1)\Delta x, a + i\Delta x\right].$$

The **left-hand Riemann sum** of f on $[a,b]$ is defined as

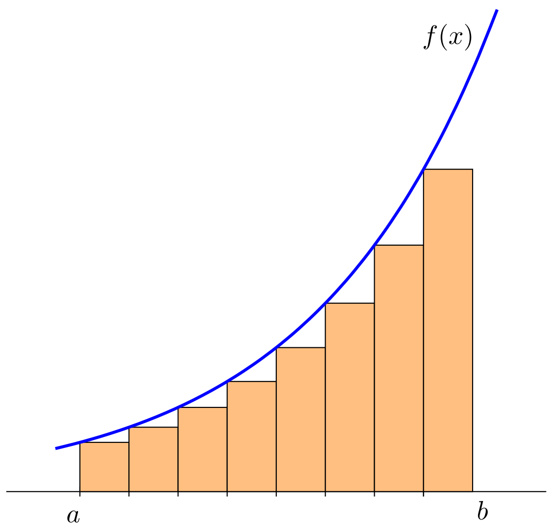
$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-2})\Delta x + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

or graphically as in Figure 16.1a.

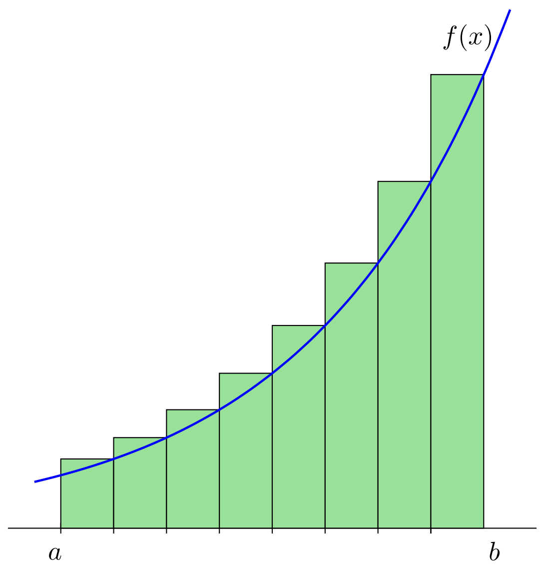
Similarly, the **right-hand Riemann sum** of f on $[a,b]$ is defined as

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x,$$

or graphically as in Figure 16.1b.



(a) Left-hand Riemann Sum



(b) Right-hand Riemann Sum

Figure 16.1: Riemann Sums of $f(x)$ on the interval $[a, b]$

Riemann Sums and Definite Integral $\int_a^b f(x)\,dx$

Suppose f is *continuous* for $a \leq x \leq b$. The **definite integral** of f from a to b, written

$$\int_a^b f(x)\,dx\,,$$

is the limit of the left-hand or right-hand sums with n subdivisions of $a \leq x \leq b$ as n gets arbitrarily large. In other words,

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\Big(\text{Left-hand Reimann sum}\Big) \;=\; \lim_{n\to\infty}\left(\sum_{i=0}^{n-1} f(x_i)\Delta x\right)$$

and

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\Big(\text{Right-hand Reimann sum}\Big) \;=\; \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right).$$

More generally, we have

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right), \qquad x_i^* \in \big[\,x_{i-1}, x_i\,\big]\,.$$

Here f is called the **integrand**, and a and b are called the **limits of integration**.

16.1 The Double Integral of a Function of Two Variables

We start this chapter by extending the notion of the *integral* to functions of two variables. We have previously seen that the area between the graph of nonnegative single-variable function $f(x)$ on $[a,b]$ and the x-axis is equal to $\int_a^b f(x)\,dx$.

But what about if we have a function of two variables, say $f(x,y)$? We know that the graph of a function of two variables $z=f(x,y)$ is an object in three-dimensional space – two dimensions for the input and one dimension for the output.

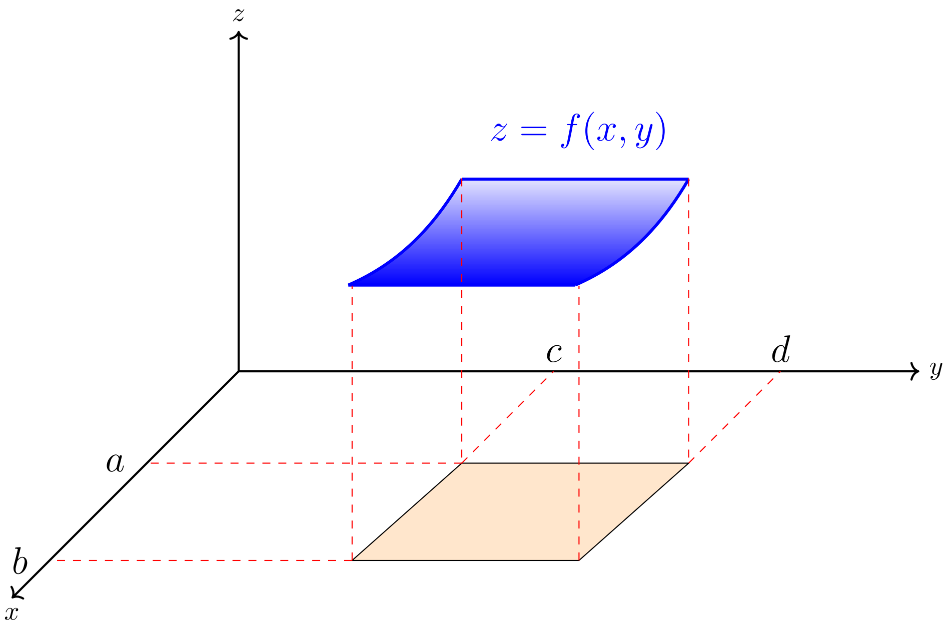


Figure 16.2: Graph of non-negative $f(x,y)$ of the region $\mathcal{R} : \{(x,y) \,:\, a \le x \le b\,,\, c \le y \le d\}$

Double Integral of $f(x,y)$ over a region $\mathcal{R}$

Let $f(x,y)$ be a function of two variables defined over some region $\mathcal{R}$. The **double integral** of the function $f(x,y)$ on the region $\mathcal{R}$ is the *volume* between the graph $z=f(x,y)$ and the xy-plane.

- If the graph of $f(x,y)$ is *below* the xy-plane, then the volume is counted *negatively*.
- If the graph of $f(x,y)$ is *above* the xy-plane, then the volume is counted *positively*.
- The volume V is written as $\quad V = \int\limits_{\mathcal{R}} f(x,y)\,dA\,.$

The quantity "dA" is an infinitesimal element of the area of the region $\mathcal{R}$.

Mathematical Construction of the Double Integral

The double integral over a region $\mathcal{R}$ is constructed in a way that is very similar to the way we learned to construct integrals of functions of one variable in previous courses. For the sake of simplicity, we start by assuming that our region $\mathcal{R}$ is a rectangle as in Figure 16.3.

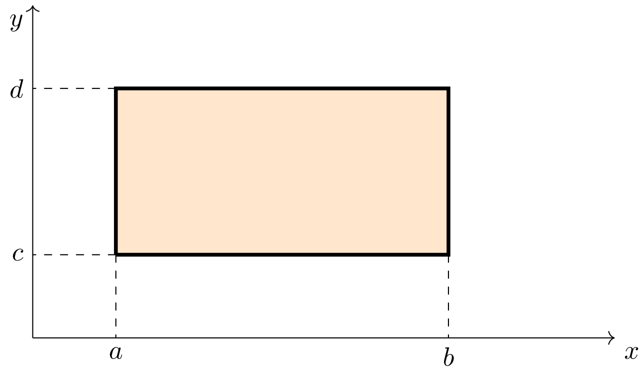


Figure 16.3: $\mathcal{R} = \{(x, y) : a \le x \le b, c \le y \le d\}$

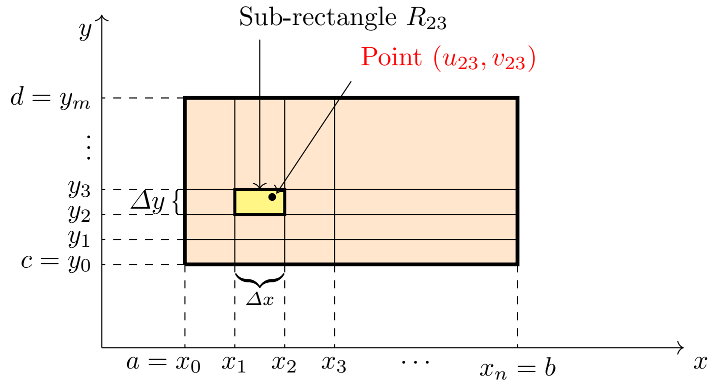


Figure 16.4: R_{ij} = the ij^{th} subrectangle of $\mathcal{R}$

The idea is to first subdivide region $\mathcal{R}$ into smaller rectangles. One way to accomplish this is as follows:

- Subdivide the interval $[a, b]$ into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$.
- Subdivide the interval $[c, d]$ into m equal subintervals of width $\Delta y = \dfrac{d-c}{m}$.
- Then the region $\mathcal{R}$ can be subdivided into $(m \cdot n)$ subrectangles R_{ij} (see Figure 16.4). The area of each subrectangle R_{ij} is given by $\Delta A = \Delta x \cdot \Delta y$.
- In each subrectangle R_{ij} pick an *arbitrary point* (u_{ij}, v_{ij}) – not necessarily one of the corner points of R_{ij}, though that is certainly allowed.
- Construct boxes (or more generally subsolids) which have the small subrectangles R_{ij} as their base and $f(u_{ij}, v_{ij})$ as their height (see Figure 16.5).
- The sum of the volumes of all the boxes (subsolids) approximates V, the double integral of f over the region $\mathcal{R}$.

  Remark: For the sake of simplicity, in Figure 16.5 we only show one box (subsolid). In reality, we have all the boxes put together and added to approximate the volume V.

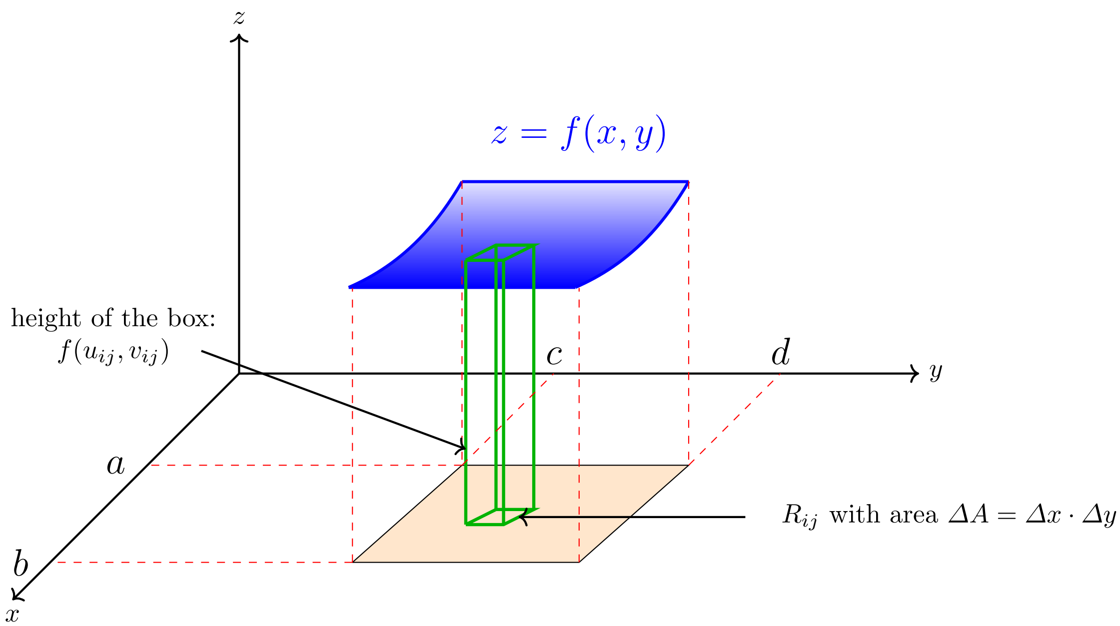


Figure 16.5: Single box (subsolid) with volume $(height) \cdot (base) = f(u_{ij}, v_{ij}) \cdot \Delta A$.

The key is to realize that in the limit in which the subrectangles R_{ij} become of infinitely small sides, the region $\mathcal{R}$ can be covered exactly by using these tiny subrectangels. Therefore, one can calculate the volume V exactly by adding the volume of all the boxes in the limit. In fact, this is how the double integral of f is constructed mathematically.

Definition: Double Integral

Let $f(x, y)$ be a *continuous* function on a rectangular region $\mathcal{R}$, where

$$\mathcal{R} \,=\, \big\{(x, y) \,:\, a \le x \le b \,,\, c \le y \le d\,\big\}\,.$$

If (u_{ij}, v_{ij}) is *any* point in the ij^{th} sub-rectangle, we define the **definite integral** of f over $\mathcal{R}$

$$\int_{\mathcal{R}} f\,dA \;\stackrel{\text{def}}{=}\; \lim_{\Delta A \to 0} \sum_j \left(\sum_i f(u_{ij}, v_{ij}) \Delta A \right) \;=\; \lim_{\Delta x, \Delta y \to 0} \sum_j \left(\sum_i f(u_{ij}, v_{ij}) \Delta x \Delta y \right).$$

Such an integral is called a **double integral**.

Here, Δx and Δy are the lengths of the sides of smaller subrectangles and ΔA is the area of these subrectangles $\Delta A = \Delta x \cdot \Delta y$.

Example 1. *Values of $f(x,y)$ are given in the table below*

y \ x	1	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	4

Let $\mathcal{R}$ be the rectangle defined by $\mathcal{R} = \{(x,y) : 1 \leq x \leq 1.2\,,\, 2 \leq y \leq 2.4\}$. Find Riemann sums which are reasonable over and underestimates for $\int_{\mathcal{R}} f(x,y)dA$ with $\Delta x = 0.1$ and $\Delta y = 0.2$.

Double Integral as Volume

Assuming that $f(x, y)$ is positive for all $(x, y) \in \mathcal{R}$, then

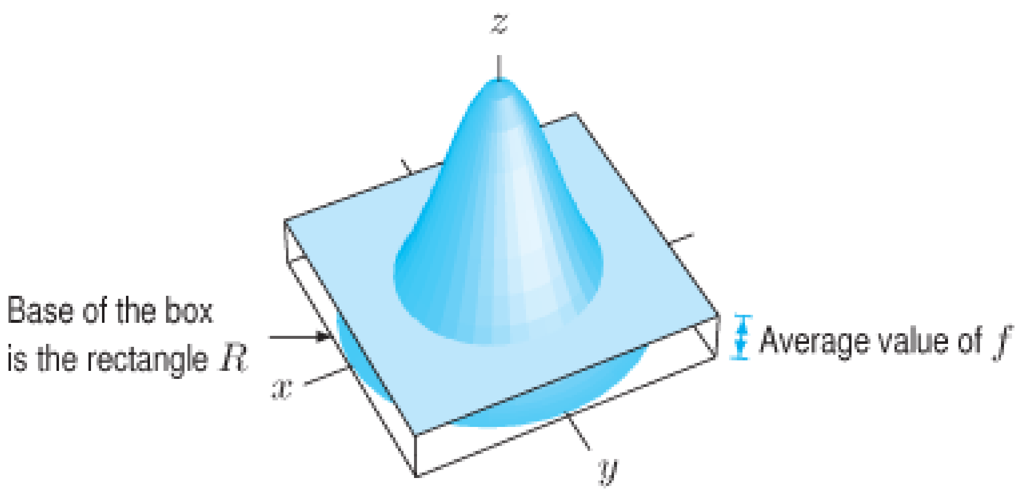
$$\boxed{\text{Volume under graph of } f \text{ above } \mathcal{R} \;=\; \int_{\mathcal{R}} f\, dA}\,.$$

Double Integral as Area

Suppose $f(x, y) = 1 > 0$ for all the points (x, y) in the region $\mathcal{R}$. Then $1 \cdot \Delta A = \Delta A$, and therefore

$$\boxed{\text{Area } (\mathcal{R}) \;=\; \int_{\mathcal{R}} 1\, dA \;=\; \int_{\mathcal{R}} dA}\,.$$

Double Integral as Average Value



$$\boxed{\text{Average Value of } f \text{ on the region } \mathcal{R} \;=\; \frac{1}{\text{Area of } \mathcal{R}} \cdot \int_{\mathcal{R}} f\, dA}\,.$$

16.2 Iterated Integrals

In the previous section we have seen how the double integral is defined mathematically and how in theory double integrals can be computed. However, this is *not* the way *we (in this course)* will compute integrals in most cases. The main objective of this section is to learn how such integrals can be computed in practice.

Iterated Integrals over Rectangular Regions

We start by assuming that $f(x,y) \geq 0$ over the rectangular region $\mathcal{R} = \{(x,y) : a \leq x \leq b\,,\ c \leq y \leq d\}$. In this case, the volume V under the graph of $f(x,y)$ over the region $\mathcal{R}$ was exactly $\int_{\mathcal{R}} f\,dA$. Recall from the previous section (see Figure 16.5) that the volume V can be computed as a sum of volumes of all subsolides (boxes) that we fit under the graph of $f(x,y)$. However, the same volume V can also be computed if we subdivide it into *slabs/slices* instead of tiny boxes (see Figure 16.6)

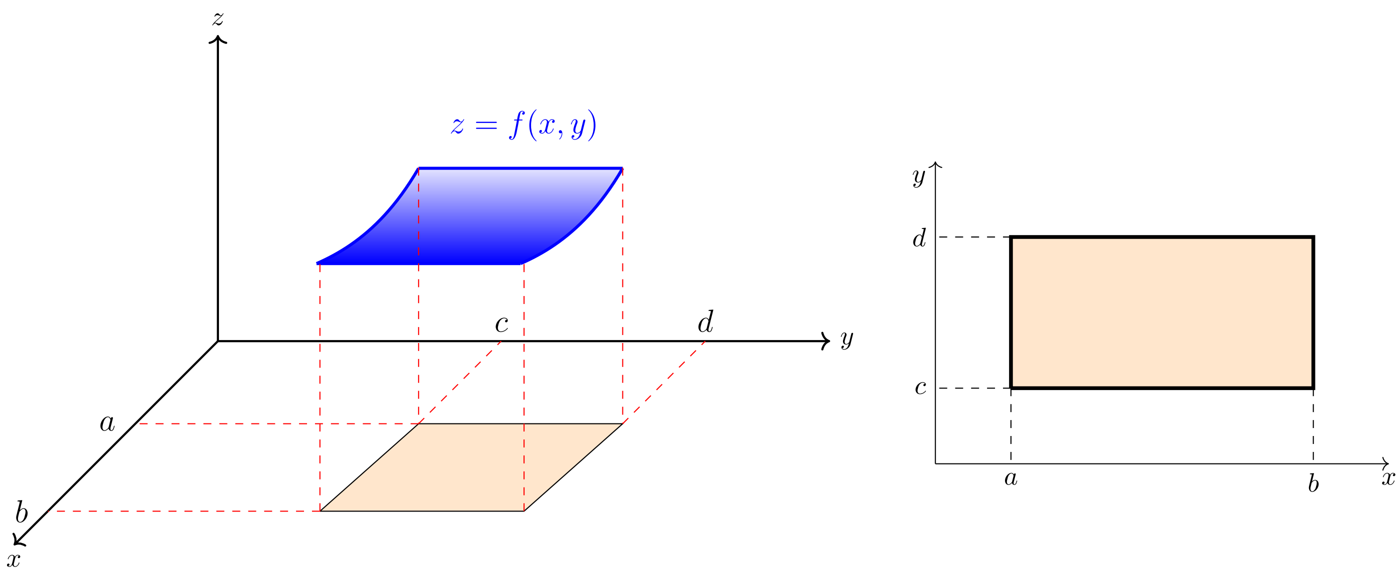


Figure 16.6: $V = \int_{\mathcal{R}} f\,dA$ as the sum of volumes of x-slices

This is how we accomplish the task of approximating $V = \int_{\mathcal{R}} f\,dA$ using different slices.

(i) **Pick an** x in the interval $[a,b]$ and **keep it fixed**. Then *the area under the corresponding cross-section of $f(x,y)$ with x-fixed, $A(x)$,* is given by

$$A(x) = \int_c^d f(x,y)\,dy\,. \tag{16.1}$$

Note that the integral in (16.1) has x-fixed. In other words, this scenario is as if y was the only variable in the integral (16.1) $\longrightarrow$ we know how to do this!

(ii) The volume corresponding to the slice of width Δx and with cross-section $A(x)$ is given by

$$\text{Volume of the } x\text{-slice} = (\text{Area}) \cdot (\text{Width}) = A(x) \cdot \Delta x = \left(\int_c^d f(x,y)\,dy\right) \cdot \Delta x\,. \tag{16.2}$$

(iii) The total volume can then be *approximated* by the sum of volumes of individual slabs, that is,

$$\int_{\mathcal{R}} f\, dA \;=\; \text{Total Volume} \;\approx\; \sum_{\text{all } x\text{-slices}} \left[\left(\int_c^d f(x,y)\, dy \right) \cdot \Delta x \right].$$

(iv) Finally, by letting the width of each x-slice become infinitely small ($\Delta \to 0$) we obtain the exact total volume $V = \int_{\mathcal{R}} f\, dA$ is given by

$$\int_{\mathcal{R}} f\, dA = \text{Total Volume} = \lim_{\Delta x \to 0} \sum_{\text{all } x\text{-slices}} \left[\left(\int_c^d f(x,y)\, dy \right) \cdot \Delta x \right] = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx. \tag{16.3}$$

Definition: The integral $\displaystyle\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$ is called an **iterated integral**.

Here are some conventions regarding iterated integrals to keep in mind:

- Given the iterated integral $\displaystyle\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$, sometimes the integral $\displaystyle\int_c^d f(x,y)\, dy$ is referred to as the **inner/inside integral**.
- Very often we will omit the parenthesis and write
$$\int_a^b \int_c^d f(x,y)\, dy\, dx\,.$$
- If we write
$$\int_a^b \int_c^d f(x,y)\, dy\, dx\,,$$
then we first calculate the y integral at x fixed, and then the x integral.
- If we write
$$\int_c^d \int_a^b f(x,y)\, dx\, dy\,,$$
then we first calculate the x integral at y fixed, and then the y integral.

Example 2. *Compute the volume under the graph of* $f(x, y) = x^2y$ *and the xy-plane over the region*

$$\mathcal{R} = \{(x, y) : 0 \leq x \leq 1,\, 0 \leq y \leq 2\}.$$

Solution:

In the previous example we have observed that

$$\int_0^1 \int_0^2 (x^2 y)\, dy\, dx \;=\; \int_0^2 \int_0^1 (x^2 y)\, dx\, dy\,.$$

This is not a coincidence, in fact, for *sufficiently smooth functions*, *the order of integration does not matter*, that is, one gets the same answer is one integrates in y first or in x first. The following result makes this notion a bit more precise.

Fubini's Theorem

If $f(x,y)$ is continuous on the rectangle $\mathcal{R} = \{(x,y) : a \le x \le b,\, c \le y \le d\}$, then

$$\int_{\mathcal{R}} f(x,y)\, dA \;=\; \int_a^b \int_c^d f(x,y)\, dy\, dx \;=\; \int_c^d \int_a^b f(x,y)\, dx\, dy\,.$$

Iterated Integrals over General Regions

The technique of iterated integrals also works for double integrals over a general region $\mathcal{D}$, but we have to be more careful with the limits of integration (see Figure 16.7)

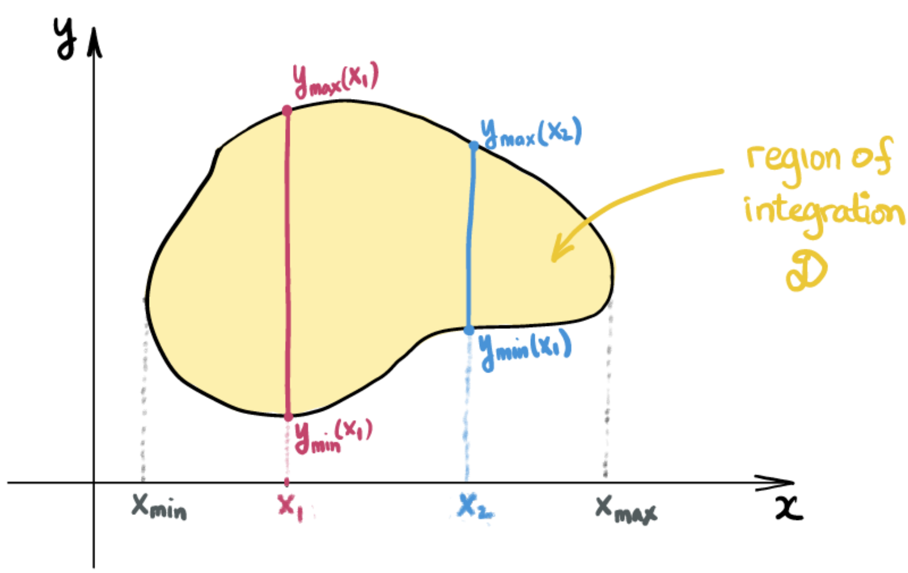


Figure 16.7: General region of integration $\mathcal{D}$: Bounds on y vary with respect to x

The sizes of the interval over which one calculates the y-interval at fixed x *depends* on the value of x. In other words, $y_{\min} = y_{\min}(x)$. Thus one has to write:

$$\int_{\mathcal{D}} f(x,y)\, dA \;=\; \int_{x_{\min}}^{x_{\max}} \left[\int_{y_{\min}(x)}^{y_{\max}(x)} f(x,y)\, dy \right] dx\,.$$

Of course, depending on region $\mathcal{D}$, if one is to calculate the integral to calculate the integral over x at y fixed first, and then the integral over y, then the opposite situation occurs: it is $x_{\min}$ and $x_{\max}$ that depend on y as shown in Figure 16.8.

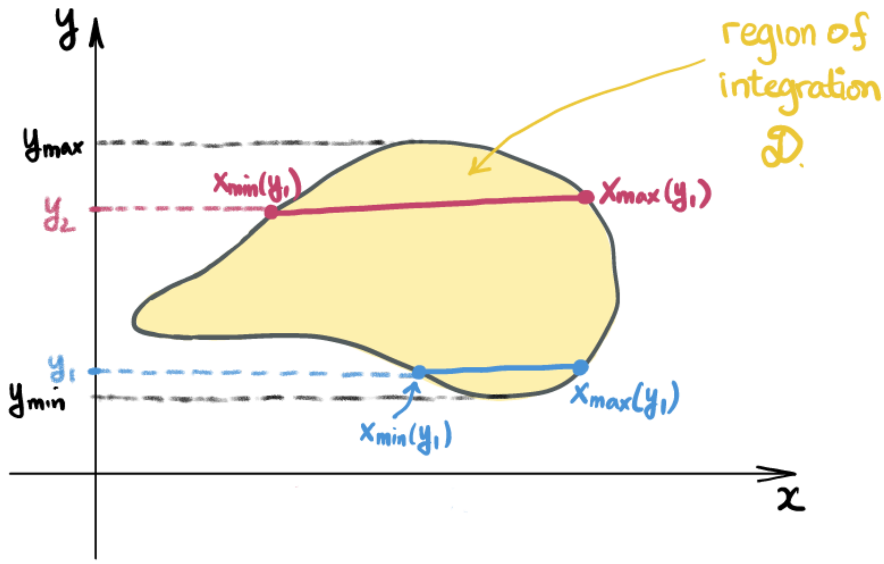


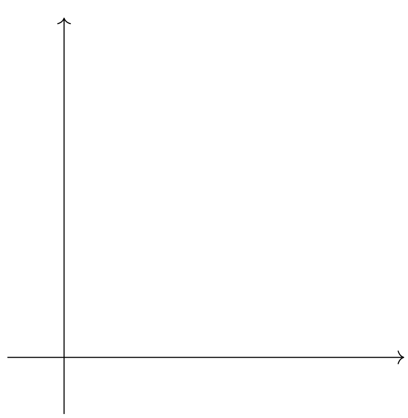
Figure 16.8: General region of integration $\mathcal{D}$: Bounds on x vary with respect to y

In this case, we would have

$$\int_{\mathcal{D}} f(x,y)\,dA \;=\; \int_{y_{\min}}^{y_{\max}} \left[\int_{x_{\min}(y)}^{x_{\max}(y)} f(x,y)\,dx\right] dy\,.$$

Example 3. *Let $\mathcal{D}$ be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Find*

$$\int_{\mathcal{D}} (x+2y)\,dA\,. \tag{16.4}$$

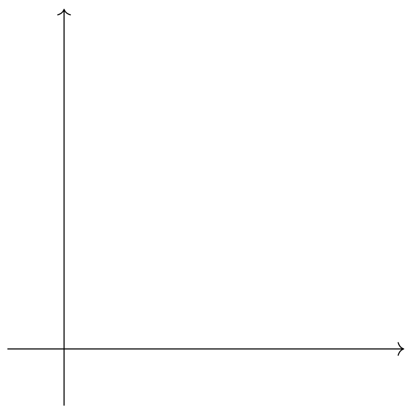


Example 4. *Consider the iterated integral*

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy\,. \tag{16.5}$$

(a) *Sketch the region $\mathcal{R}$ of integration for the corresponding integral* (16.5).

(b) *Evaluate the integral in* (16.5).



Example 5. *Consider a metal plate $\mathcal{R}$ bounded by $y = x$ and $y = x^2$ with density of mass*

$$\delta(x, y) \;=\; 1 + xy\,\frac{kg}{m^2}\,, \tag{16.6}$$

where x and y are measured in meters. Find the total mass of the plate.