Chapter 17

Parametrization and Vector Fields

17.1 Parametrized Curves

Our experience with describing plane curves so far primarily consists of one of the following:

- expressing "y" as a function of "x", that is, $y = f(x)$,
- expressing "x" as a function of "y", that is, $x = g(y)$,
- giving a relation between "y" and "x" that defines "y" *implicitly* as a function of "x", that is, $f(x, y) = 0$.

However, some curves are not readily expressed using either of these methods, e.g., see Figure 17.1

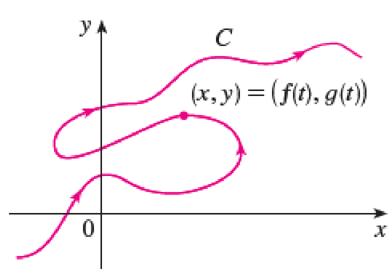


Figure 17.1: Generic Planar Curve

Imagine that a particle moves along the curve C as shown in Figure 17.1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. *Such a pair of equations is often a a new and more convenient way of describing a curve and gives rise to the following definition.*

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t) \tag{17.1}$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = \big(f(t), g(t)\big)$ varies and traces out a curve C, which we call a **parametric curve**. *The parameter t does not necessarily represent time and, in fact, we could use any letter other than t for the parameter (another typical choice is θ which could denote an angle).* But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = \big(f(t), g(t)\big)$ as the position of a particle at time t.

Example 1. **Sketch** *and* **identify** *the curve defined by the parametric equations*

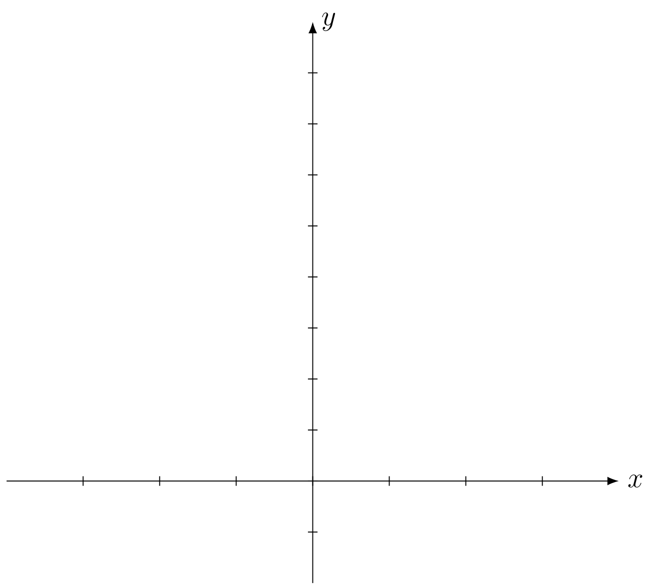
$$x = t + 1 \qquad y = t^2 - 2t\,, \tag{17.2}$$

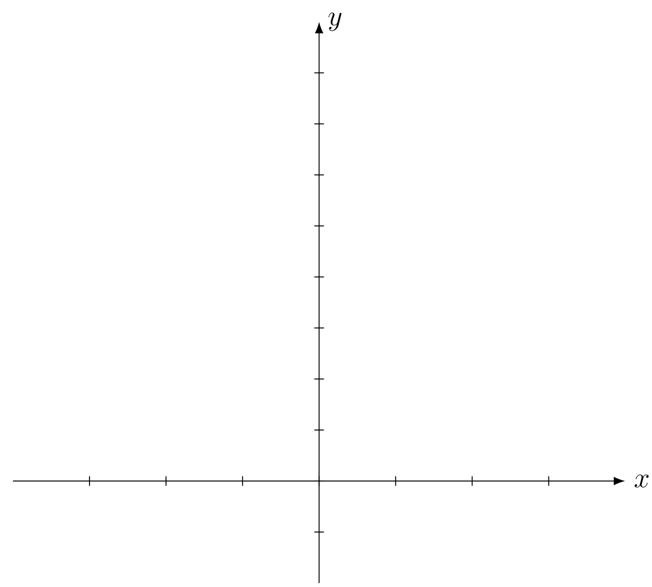
where:

(a) $-\infty < t < +\infty$ (b) $1 \leq t \leq 3$

Solution: *We start by selecting several values of t and finding the corresponding x and y values.*

t	x	y
-2		
-1		
0		
1		
2		
3		
4		





Example 2. **Sketch** *and* **identify** *the curve defined by the parametric equations*

$$x = \cos(\theta) \qquad y = \sin(\theta)\,, \tag{17.3}$$

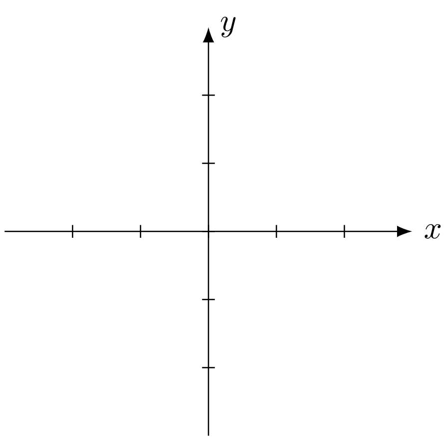
where:

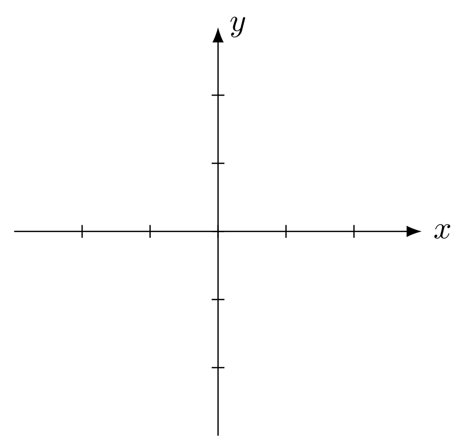
(a) $0 \leq \theta < 2\pi$

(b) $\dfrac{\pi}{2} \leq \theta \leq \dfrac{3\pi}{2}$

Solution: *We start by picking some values for θ and find the corresponding values of x and y.*

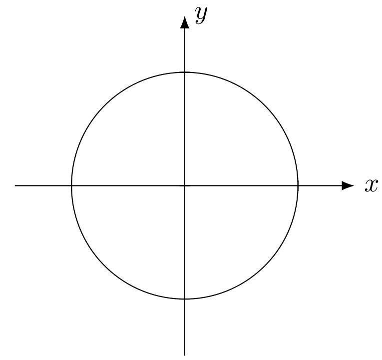
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0



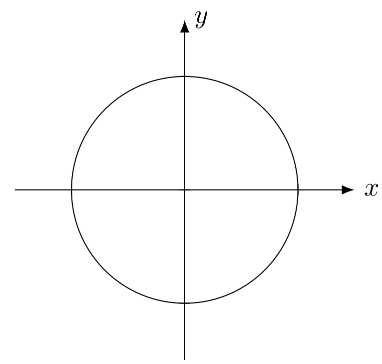


Example 3. Sketch *and* **identify** *the curve defined by the parametric equations given below:*

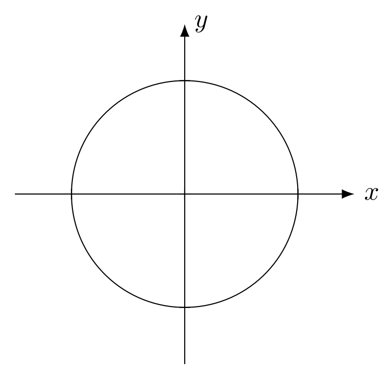
(a) $x = \cos(t) \qquad y = \sin(t), \qquad 0 \le t \le 2\pi$



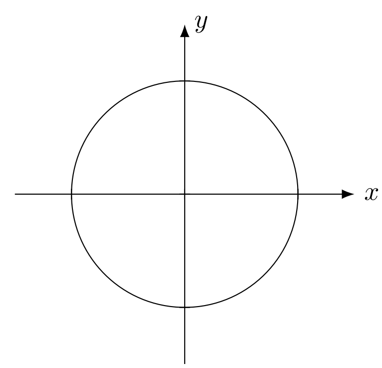
(b) $x = \sin(t) \qquad y = \cos(t), \qquad 0 \le t \le 2\pi$



(c) $x = \sin(2t) \qquad y = \cos(2t), \qquad 0 \le t \le \pi$



(d) $x = \sin(2t) \qquad y = \cos(2t), \qquad \frac{5\pi}{2} \le t \le \frac{7\pi}{2}$



Parametric Equations in Three Dimensions

A **parametric curve** in the xyz-space is a curve described by parametric equations

$$x = f(t)\,, \qquad y = g(t)\,, \qquad z = h(t)\,,$$

where the **parameter** t changes in some interval $\mathcal{I} = [\,a\,,b\,]$.

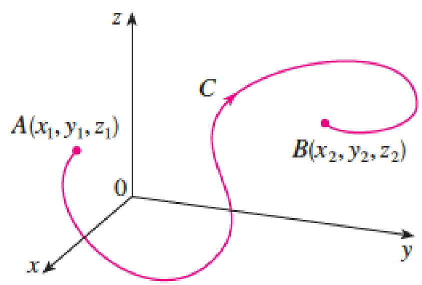


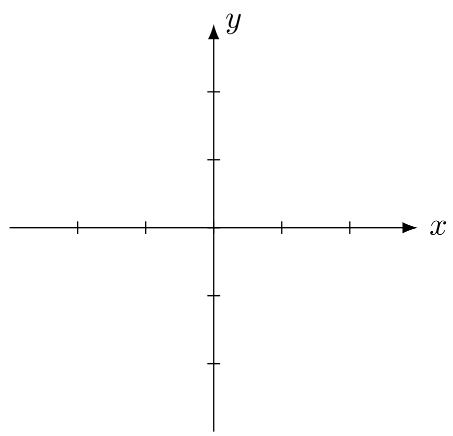
Figure 17.2: Generic Curve in the xyz-space

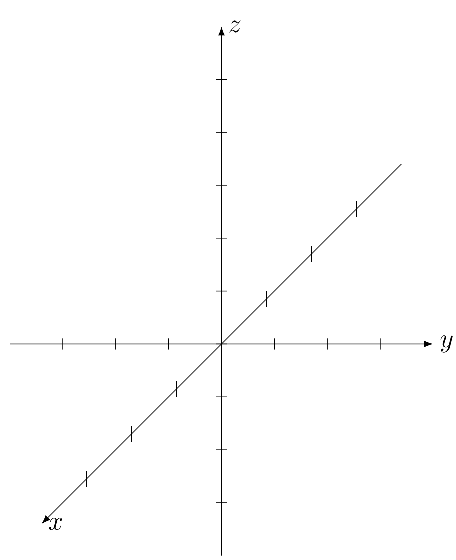
As in the case of planar curves, for each value of parameter t there is a corresponding point $(x, y, z) = \big(f(t), g(t), h(t)\big)$. As t changes, the point moves along C describing a motion along the path C. The path C has

- **initial point** $\big(x_1, y_1, z_1\big) = \big(f(a), g(a), h(a)\big)$,
- **terminal point** $\big(x_2, y_2, z_2\big) = \big(f(b), g(b), h(b)\big)$.

Example 4. **Sketch** *and* **identify** *the curve defined by the parametric equations given below:*

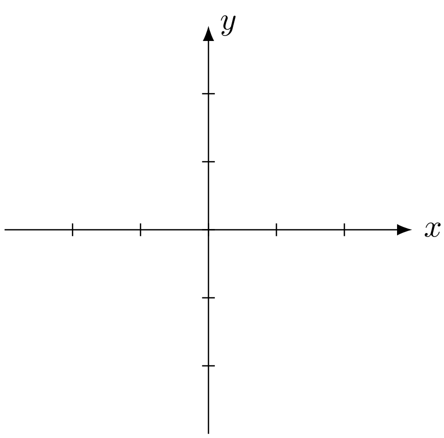
$$x = 2\cos(t)\,, \qquad y = 2\sin(t)\,, \qquad z = t\,, \qquad t \geq 0\,.$$

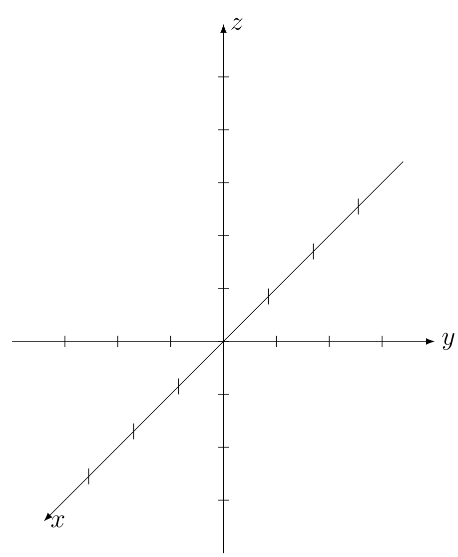




Example 5. Sketch *and* **identify** *the curve defined by the parametric equations given below:*

$$x = t \cdot \cos(t)\,, \qquad y = t \cdot \sin(t)\,, \qquad z = t\,, \qquad t \geq 0\,.$$





Parametrization in Vector Form

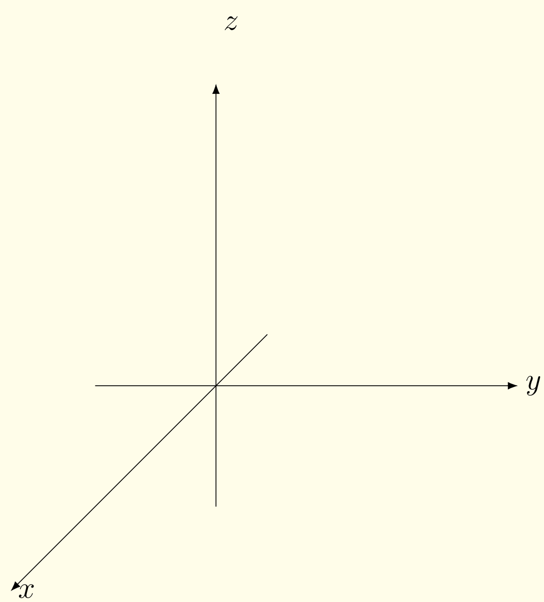
Let a parametrized curve

$$x = f(t)\,, \qquad y = g(t)\,, \qquad z = h(t)\,, \qquad t \in \mathcal{I} = [a, b]$$

be given. We can write the parameterization as:

$$\vec{r}\,(t) \;=\; f(t)\,\vec{i} \;+\; g(t)\,\vec{j} \;+\; h(t)\,\vec{k}\;.$$

$\vec{r}\,(t)$ is called the **position vector**. The position vector traces the curve as t changes.



Parametric Equation of a Line in Vector Form

Let L be the **line passing through a point** (x_0, y_0, z_0) **and parallel to** $\vec{w} = w_1\,\vec{i} + w_2\,\vec{j} + w_3\,\vec{k}$. Then

$$L: \quad x(t) = x_0 + t \cdot w_1\,, \quad y(t) = y_0 + t \cdot w_2\,, \quad z(t) = z_0 + t \cdot w_3\,, \quad -\infty < t < \infty\,.$$

In the **vector form** the same line can be written as

$$L: \quad \vec{r}\,(t) \;=\; \vec{r_0}(t) \;+\; t \cdot \vec{w}\,, \qquad \vec{r_0}(t) \;=\; x_0\,\vec{i} + y_0\,\vec{j} + z_0\,\vec{k}\,, \qquad -\infty < t < \infty\,.$$

Example 6. *Consider the following two points* $P_0 = (2, -1, 3)$ *and* $P_1 = (-1, 5, 4)$*. Find a parametric representation of:*

(a) *The* line*,* L*, through the points* P_0 *and* P_1*.*

(b) *The* line segment*,* S*, from* P_0 *to* P_1*.*

Solution:

Example 7. *Find a parametrization of the line perpendicular to the plane* $z = 2x - 3y + 7$ *and through the point* $(1, 1, 6)$.

Solution:

Example 8. *Determine whether the following line is parallel to the plane $2x - 3y + 5z = 5$:*

$$x = 5 + 7t\,, \qquad y = 4 + 3t\,, \qquad z = -3 - 2t\,.$$

Solution:

Example 9. *Make sure to study Examples 9, 10, and 11 from Section 17.1 in our textbook (accessible through WileyPLUS).*