14.5 Gradients of Functions of Three Variables

Let $f(x, y, z)$ be a function of three variables x, y, and z. Recall that in order to "graph" $f(x, y, z)$ we would require us to have a four-dimensional coordinate system which unfortunately we can not visualize. Instead, we graph the *level surfaces* of $f(x, y, z)$ which are obtained by setting $f(x, y, z)$ equal to some constant k. In particular, if $f(x, y, z) = x^2 + y^2 + z^2$, then the level surface of f for a given constant k is $x^2 + y^2 + z^2 = k$, that is, the sphere of radius $\sqrt{k}$ centered at the origin.

Analogous to the functions of two variables, we can define partial derivatives of $f(x, y, z)$ at each point (a, b, c),

$$f_x(a, b, c)\,, \quad f_y(a, b, c)\,, \quad f_z(a, b, c)\,,$$

by fixing two of the variables and differentiating with respect to the third. For example,

$$\left.\frac{\partial f}{\partial z}\right|_{(a,b,c)} = f_z(a, b, c) = \lim_{h \to 0} \frac{f(a, b, c + h) - f(a, b, c)}{h} = \left.\frac{d}{dz}\right|_{z=c} \Big(f(a, b, z) \Big)\,.$$

Example 17. *Find all the partial derivatives of the function $f(x, y, z) = x^2y + yz$ at the point $(1, 2, 3)$.*

Solution :

In addition to the partial derivatives, we can also define the *directional derivative* of $f(x,y,z)$ at the point (a,b,c) in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ as follows:

$$f_{\vec{u}}(a,b,c) \; = \; \lim_{h\to 0} \frac{f(a+hu_1\,,\, b+hu_2\,,\, c+hu_3) - f(a,b,c)}{h}$$

If the *gradient vector* of $f(x,y,z)$ at the point (a,b,c) is defined as

$$\nabla f(a,b,c) \; = \; f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}\,,$$

we can rewrite the directional derivative of f at (a,b,c) in the direction of $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ as

$$f_{\vec{u}}(a,b,c) \; = \; \nabla f(a,b,c) \bullet \vec{u} \; = \; f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3\,.$$

As before we have the following theorem:

Theorem: Let $f(x,y,z)$ be a differentiable function at (a,b,c) with $\nabla f(a,b,c) \neq \vec{0}$. Then:

(i) $\nabla f(a,b,c)$ points in the *direction of the largest (maximum) rate of increase* of f at (a,b,c).

(ii) The *maximum rate of change* of f at (a,b,c) is $||\nabla f(a,b,c)||$.

(iii) $\nabla f(a,b,c)$ is perpendicular to the contour line of $f(x,y,z)$ at the point (a,b,c).

Example 18. *Suppose that the function $C(x,y,z) = x^2 + y^4 + x^2z^2$ gives concentration of salt, in gr/gal, at any point (x,y,z) of a rectangular tank of water occupying the region*

$$-2 \le x \le 2\,, \qquad -2 \le y \le 2\,, \qquad 0 \le z \le 2\,,$$

(all measurements in meters). Suppose you are at the point $(-1,1,1)$.

(a) In what direction should you move if you want the concentration to increase the fastest?

(b) If you move from $(-1,1,1)$ toward the origin $(0,0,0)$, how fast is the concentration changing?

Solution :

Example 19. *Find the equation of the tangent plane to the ellipsoid* $x^2 + 2y^2 + z^2 = 15$ *at* $(2, 1, 3)$.

Solution :

14.6 Chain Rule

Recall that the Chain Rule for functions of a *single* variable gives the rule for differentiating a *composite function*: if $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, the y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite functions.

Case 1: We start by considering a special case where $z = f(x, y)$ and each of the variables x and y is a function of another variable t, that is, $x = g(t)$ and $y = h(t)$. But this implies that z is indirectly a function of ONE variable t, $z = f\big(g(t), h(t)\big)$, and the Chain Rule gives us the formula for differentiating z as a function of t.

Theorem: Suppose that $f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\ .$$

Remark: We now provide some intuition behind this theorem. Start by observing that a change of Δt in t produces changes of Δx in x and Δy in y. More specifically,

$$\Delta x = g(t + \Delta t) - g(t) \quad \text{and} \quad \Delta y = h(t + \Delta t) - h(t)\,.$$

These changes in turn produce a change of Δz in z which due to the fact that $z = f(x, y)$ is a differentiable function can be approximated by

$$\Delta z \approx \frac{\partial f}{\partial x}\cdot\Delta x + \frac{\partial f}{\partial y}\cdot\Delta y\,. \tag{14.5}$$

Dividing the both sides of equation (14.5) gives

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial f}{\partial x}\cdot\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\cdot\frac{\Delta y}{\Delta t}\,. \tag{14.6}$$

If $\Delta t \to 0$, then

$$\frac{\Delta x}{\Delta t} = \frac{g(t+\Delta t) - g(t)}{\Delta t} \longrightarrow \frac{dx}{dt} \qquad \text{and} \qquad \frac{\Delta y}{\Delta t} = \frac{h(t+\Delta t) - h(t)}{\Delta t} \longrightarrow \frac{dy}{dt}\,.$$

Thus, by letting $\Delta t \to 0$ we obtain the "infinitesimal" version of equation (14.6)

$$\frac{dz}{dt} \approx \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\ . \tag{14.7}$$

Example 20. *If* $z(x, y) = x^2y + 3xy^4$*, where* $x = \sin(2t)$ *and* $y = \cos(t)$*, find* dz/dt *when* $t = 0$*.*

Solution :

<u>Case 2:</u> For our next case, we consider the the situation where $z = f(x, y)$, but BOTH x and y are now functions of TWO variables u and v, namely, $x = g(u, v)$ and $y = h(u, v)$. Then z is indirectly a function of of two variables u and v and we wish to determine $\partial z/\partial u$ and $\partial x/\partial v$. Recall that in computing $\partial z/\partial u$ we hold v fixed and compute the ordinary derivative of z with respect to u. Therefore, by applying the previous theorem we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

A similar argument holds for $\partial z/\partial v$, which leads us to the following result:

Theorem: Suppose that $f(x, y)$ is a differentiable function of x and y, where $x = g(u, v)$ and $y = h(u, v)$ are both differentiable functions of u and v. Then:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

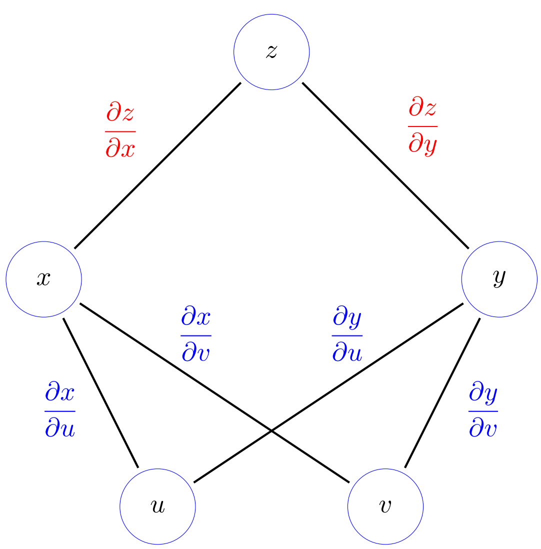
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

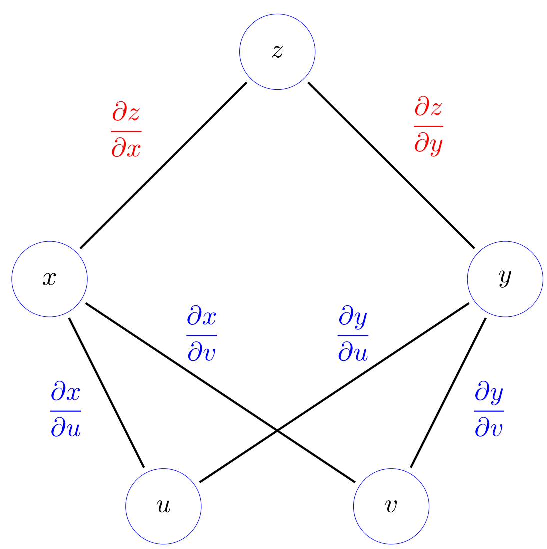
Example 21. *If $z(x, y) = e^x \sin(y)$, where $x(u, v) = u \cdot v^2$ and $y(u, v) = u^2 \cdot v$, then find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.*

Solution :

Question: Now that we have experience with computing with the Chain Rule, is there an easier way to remember it???

Answer: For illustration purposes, we will assume that $z = z(x, y)$ and $x = g(u, v)$ and $y = h(u, v)$. Now consider a diagram consisting of three levels – main function z, variables x and y, and finally, variables u and v.



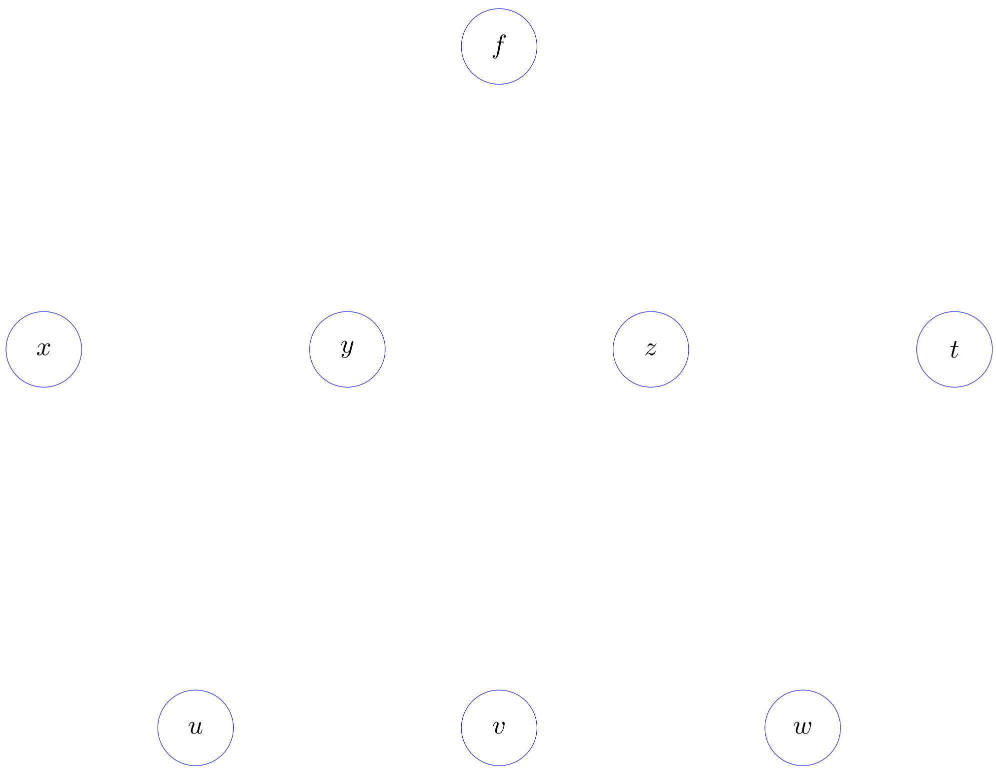


Example 22. *(not in our textbook!!) For a function $f(x, y, z, t)$, where*

$$\begin{aligned} x &= x(u, v, w)\,, \\ y &= y(u, v)\,, \\ z &= z(u, w)\,, \\ t &= t(u)\,, \end{aligned}$$

find the general expressions for $\dfrac{\partial f}{\partial u}$ *and* $\dfrac{\partial f}{\partial w}$.

Solution :



Example 23. *Use the Chain Rule to find* $\dfrac{\partial w}{\partial \rho}$ *and* $\dfrac{\partial w}{\partial \theta}$ *, given that*

$$w(x, y, z) \;=\; x^2 + y^2 + z^2 \tag{14.8}$$

and

$$x \;=\; \rho \cdot \sin(\phi) \cdot \cos(\theta)\,; \qquad y \;=\; \rho \cdot \sin(\phi) \cdot \sin(\theta)\,; \qquad z \;=\; \rho \cdot \cos(\theta)\,. \tag{14.9}$$

Solution :

14.7 Second Order Partial Derivatives

Given $z = f(x, y)$ each partial derivative

$$\frac{\partial z}{\partial x} = f_x(x, y)\,, \qquad \frac{\partial z}{\partial y} = f_y(x, y)\,,$$

is again a function of (x, y). As such, one can also take their partial derivatives. *Partial derivatives of partial derivatives* are called **second-order partial derivatives**. We denote them as follows:

$$\big(f_x\big)_x = f_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\Big[f_x(x, y)\Big]$$

$$\big(f_x\big)_y = f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\Big[f_x(x, y)\Big]$$

$$\big(f_y\big)_x = f_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\Big[f_y(x, y)\Big]$$

$$\big(f_y\big)_y = f_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\Big[f_y(x, y)\Big]$$

Terminology: For a function $z = f(x, y)$, the second-order partial derivatives f_{xy} and f_{yx} are called **mixed partials**.

Example 24. *Let* $f(x,y) = x^3 + x^2y^2 - y^4$. *Find* all *second-order partial derivatives of* $f(x,y)$. **Solution :**

Clairaut's Theorem: (XVIII century) – (name of this theorem is not in our textbook)

Let $f(x,y)$ be a function with domain $\mathcal{D}$. If all partial derivatives of $f(x,y)$ up to the second order are continuous at an interior point (a,b) of the domain, then

$$f_{xy}(a,b) \;=\; f_{yx}(a,b)\,.$$

Key Point: Under some mild conditions on $f(x,y)$, the *mixed partial derivatives of* f *are equal.*

As is the case with second derivatives of a *single variable* function, there are many uses for the second-order partial derivatives:

- Second Derivative Test for Local Extrema (Ch. 15)
- Taylor approximations of order two
- ...

Recall that so far we have seen **Taylor Polynomial of Degree ONE** that approximates $f(x,y)$ for all (x,y) *near* (a,b):

$$f(x,y) \approx L(x,y) \;=\; f(a,b) + f_x(a,b)\cdot(x-a) \;+\; f_y(a,b)\cdot(y-b)\,.$$

Second Degree Taylor Polynomial for $f(x, y)$ near (a, b)

Let $f(x, y)$ be such that all second-order partial derivatives are continuous at the point (a, b). Then the **Taylor polynomial of degree TWO** of $f(x, y)$ near (a, b) is given by

$$Q(x, y) = f(a, b) + f_x(a, b) \cdot (x - a) + f_y(a, b) \cdot (y - b) + \dots$$

$$\dots + \frac{1}{2} f_{xx}(a, b) \cdot (x - a)^2 + f_{xy}(a, b) \cdot (x - a) \cdot (y - b) + \frac{1}{2} f_{yy}(a, b) \cdot (y - b)^2 .$$

Remark: This idea can be quite easily extended to functions of three, four, ... variables, but it requires using tools from linear algebra/matrix theory.

Example 25. *Find the quadratic Taylor polynomial about $(0, 0)$ for the function*

$$f(x, y) = (y - 1)(x + 1)^2 .$$

Solution : *We start by computing appropriate partial derivatives:*

$f_x(x, y) =$ $\qquad$ $f_x(0, 0) =$

$f_{xx}(x, y) =$ $\qquad$ $f_{xx}(0, 0) =$

$f_{xy}(x, y) =$ $\qquad$ $f_{xy}(0, 0) =$

$f_y(x, y) =$ $\qquad$ $f_y(0, 0) =$

$f_{yy}(x, y) =$ $\qquad$ $f_{yy}(0, 0) =$

$f_{yx}(x, y) =$ $\qquad$ $f_{yx}(0, 0) =$

Many physical systems are readily described by *partial differential equations* in which derivatives of an unknown function are related to one another and/or to the function itself. For example, the **Laplace Equation**

$$\frac{\delta^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

can be used to model equilibrium temperature distribution in *planar* region or shape of a soap film suspended on a wire frame.

Example 26. *Show that* $u(x, y) = \ln(x^2 + y^2)$ *is* a *solution to the Laplace's equation.*

Example 27. *Verify that* $f(t, x) = \cos^2(t + x) + e^{e^{\sin(t+x)}}$ *is a solution of the* transport equation $f_t(t, x) = f_x(t, x)$ *(this equation models the concentration of a substance flowing in a fluid at a constant rate).*

Example 28. *Show that* $f(t, x) = \sin(x - t) + \sin(x + t)$ *satisfies the* wave equation $f_{tt}(t, x) = f_{xx}(t, x)$ *(this equation governs the motion of light or sound).*

Example 29. *Show that* $f(t, x) = \dfrac{1}{\sqrt{t}}\, e^{-x^2/(4t)}$ *satisfies the* heat equation $f_t(t, x) = f_{xx}(t, x)$ *(this equation describes diffusion of heat or spread of an epidemic).*