March 23, 2024 **Quiz #4** MTH 243 (Perović)

Max Score: **10 pts** **Name:** Evan O'Neill

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**GRADING:**

- This quiz takes place of our *regular* weekly quiz.
- You will have unlimited time to complete it, though you should *be able to answer* ALL *these questions on the exam in about 15 minutes* and without any resources like calculator or textbook.

**INSTRUCTIONS:** Please read the following instructions **carefully**.

- You are allowed to use *ONLY* our class notes and our textbook that is accessible through WileyPLUS.
- You should work on completing this quiz on *your own*, that is, you should *NOT* work with fellow students, roommates, friends, tutors, online chatting buddies, etc.
- You are NOT allowed to look for answers to these specific question on any of the online forums/platforms or apps!!!
- It is ok to use a calculator to *check* your work, BUT you should be able to these problems on the exam without a calculator!
- **Neatness and organization of your answers/submission matter!**
  - Your answers should be submitted as a *single pdf file*.
  - The uploaded pdf file should be titled `YOURLASTNAME-mth243-quiz4.pdf`.
  - Your answers should be *legible and with no scribbles*.
  - Your answers should be written on *the quiz itself* **OR** if writing on a separate paper, then each new problem should start on a separate page in your pdf submission.

  *Failing to follow submission instructions will result in your final score being reduced.*
- **All work must be shown for full credit!**

**DEADLINE:** **11:59pm on Sat., Mar. 23, 2024, via the Assignments tab on Brightspace.**

$$z_x = 7(x^2+x-y)^6(2x+1)$$
$$z_y = 7(x^2+x-y)^6 \cdot -1$$

1. Compute the following quantities:

(a) $z_x(x,y)$ and $z_y(x,y)$, where $z(x,y) = (x^2+x-y)^7$.

$z_x(x,y) = 7(x^2+x-y)^6 \cdot (2x+1)$

$z_y(x,y) = -7(x^2+x-y)^6$

(b) $\dfrac{\partial}{\partial x}\left[\dfrac{1}{a}e^{-x^2/a^2}\right] = \dfrac{1}{a}\dfrac{\partial}{\partial x}\left[e^{-\frac{x^2}{a^2}}\right] = \dfrac{1}{a}e^{u}\dfrac{\partial}{\partial x}\left(\dfrac{-x^2}{a^2}\right)$

$$\frac{1}{a}e^{\frac{-x^2}{a^2}} \cdot \left(-\frac{2x}{a^2}\right)$$

$\dfrac{\partial}{\partial x}\left[\dfrac{1}{a}e^{-x^2/a^2}\right] = \dfrac{-2xe^{\frac{-x^2}{a^2}}}{a^3}$

(c) $f_x(x,y)$ and $f_y(x,y)$, where $f(x,y) = e^{xy} \cdot \ln(y)$.

$$e^{xy} \cdot \frac{\partial}{\partial x}(xy) = \frac{\partial}{\partial x}(x) \cdot y = y$$

$f_x(x,y) = \cancel{e^{xy} \cdot \frac{\partial}{\partial x}[\ln(y)]} + \frac{\partial}{\partial x}[e^{xy}] \cdot \ln(y) = e^{xy}y\ln(y)$

$f_y(x,y) = e^{xy} \cdot \frac{\partial}{\partial y}[\ln(y)] + \frac{\partial}{\partial y}[e^{xy}] \cdot \ln(y)$ $\left(\frac{\partial}{\partial y}[y \cdot x] \cdot e^{xy}\right)$

$$\frac{e^{xy}}{y} + xe^{xy}\ln(y)$$

$f_x(x,y) = \ln(y) \cdot ye^{xy}$

$f_y(x,y) = \dfrac{e^{xy}}{y} + xe^{xy}\ln(y)$

(d) $f_t(t,x)$ and $f_x(t,x)$, where $f(t,x) = \cos^2(t+x) + e^{e^{\sin(t+x)}}$.

Work on next page

$$f_t(t,x) = -2\cos(t+x)\sin(t+x) + \left(e^{e^{\sin(t+x)}}\right)\left(e^{\sin(t+x)}\right)\left(\cos(t+x)\right)$$

$$f_x(t,x) = -2\cos(t+x)\sin(t+x) + \left(e^{e^{\sin(t+x)}}\right)\left(e^{\sin(t+x)}\right)\left(\cos(t+x)\right)$$

(e) $g_x(x,y)$ and $g_y(x,y)$, where $g(x,y) = \dfrac{\ln(\sin^2(y)+3)}{x^2y+y^2x}$.

Work on Back

$$g_x(x,y) = \frac{-\ln(\sin^2(y)+3)(2xy+y^2)}{(x^2y+y^2x)^2}$$

$$g_y(x,y) = \frac{\sin(2y)(x^2y+y^2x) - (\ln(\sin^2(y)+3))(x^2+2yx)}{(x^2y+y^2x)^2(\sin^2(y)+3)}$$

(d) $f_t(t,x)$ and $f_x(t,x)$, where $f(t,x) = \cos^2(t+x) + e^{e^{\sin(t+x)}}$.

d) $\quad u = t+x \qquad v = e^{\sin(t+x)} \qquad w = \sin(t+x)$

$$f_t(t,x) = \frac{\partial}{\partial t}[\cos^2(u)] \cdot \frac{\partial}{\partial t}[t+x] + \frac{\partial}{\partial t}[e^v] \cdot \frac{\partial}{\partial t}[e^w] \frac{\partial}{\partial t}[\sin(u)] \cdot \frac{\partial}{\partial t}[t+x]$$

$$f_t(t,x) = -2\cos(t+x)(\sin(t+x)) \cdot 1 + e^{e^{\sin(t+x)}} \cdot e^{\sin(t+x)} \cdot \cos(t+x) \cdot 1$$

$$= -2\cos(t+x)\sin(t+x) + \left(e^{e^{\sin(t+x)}}\right)\left(e^{\sin(t+x)}\right)\left(\cos(t+x)\right)$$

$$f_x(t,x) = \frac{\partial}{\partial x}[\cos^2(u)] \cdot \frac{\partial}{\partial x}[t+x] + \frac{\partial}{\partial x}[e^v] \cdot \frac{\partial}{\partial x}[e^w] \frac{\partial}{\partial x}[\sin(u)] \cdot \frac{\partial}{\partial x}[t+x]$$

$$f_x(t,x) = -2\cos(t+x)(\sin(t+x)) \cdot 1 + e^{e^{\sin(t+x)}} \cdot e^{\sin(t+x)} \cdot \cos(t+x) \cdot 1$$

$$= -2\cos(t+x)\sin(t+x) + \left(e^{e^{\sin(t+x)}}\right)\left(e^{\sin(t+x)}\right)\left(\cos(t+x)\right)$$

(e) $g_x(x,y)$ and $g_y(x,y)$, where $g(x,y) = \dfrac{\ln(\sin^2(y)+3)}{x^2y + y^2x}$.

$$\frac{f'g - fg'}{g^2}$$

e) $$\frac{\frac{\partial}{\partial x}[\ln(\sin^2(y)+3)]\cancel{(x^2y+y^2x)} - (\ln(\sin^2(y)+3))\frac{\partial}{\partial x}[x^2y+y^2x]}{(x^2y+y^2x)^2}$$

$\frac{1}{\sin^2(y)+3} \cdot \frac{\partial}{\partial x}[\cancel{\sin^2(y)+3}] = 0$

$\left(\frac{\partial}{\partial x}[x^2]y + \frac{\partial}{\partial x}[y]x^2\right) + \left(\frac{\partial}{\partial x}[y^2]x + \frac{\partial}{\partial x}[x]y^2\right)$

$2xy + 0 + 0 + y^2$

$2xy + y^2$

$$g_x(x,y) = \frac{-\ln(\sin^2(y)+3)(2xy+y^2)}{(x^2y+y^2x)^2}$$

$$\frac{\frac{\partial}{\partial y}[\ln(\sin^2(y)+3)](x^2y+y^2x) - (\ln(\sin^2(y)+3))\frac{\partial}{\partial y}[x^2y+y^2x]}{(x^2y+y^2x)^2}$$

$u = \sin^2(y)+3$

$\frac{\partial}{\partial y}[\ln(u)] \cdot \frac{\partial}{\partial y}[\sin^2(y)+3]$

$\frac{1}{\sin^2(y)+3} \cdot \sin(2y)$

$\frac{\sin(2y)}{\sin^2(y)+3}$

$\left(\frac{\partial}{\partial y}[x^2]y + \frac{\partial}{\partial y}[y]x^2\right) + \left(\frac{\partial}{\partial y}[y^2]x + \frac{\partial}{\partial y}[x]y^2\right)$

$0 + x^2 + 2yx + 0$

$x^2 + 2yx$

$$g_y(x,y) = \frac{\sin(2y)(x^2y+y^2x) - (\ln(\sin^2(y)+3))(x^2+2yx)}{(x^2y+y^2x)^2(\sin^2(y)+3)}$$

2. Let $(x, y)$ be the function given by $f(x,y) = \ln\left(x^2 + xy\right)$.

(a) Find the *gradient vector* $\nabla f$ at the point $(4, 1)$.

$f_x(x,y) = \frac{\partial}{\partial x}(\ln(x^2+xy)) \cdot \frac{\partial}{\partial x}[x^2+xy] = \frac{2x+y}{x^2+xy}$

$f_x(4,1) = \frac{2(4)+1}{(4)^2+(4)(1)} = \frac{9}{20}$

$f_y(x,y) \frac{\partial}{\partial y}[\ln(x^2+xy)] \cdot \frac{\partial}{\partial y}[x^2+xy] = \frac{x}{x^2+xy}$

$f_y(4,1) = \frac{(4)}{(4)^2+(4)(1)} = \frac{4}{20} = \frac{1}{5}$

$\boxed{\nabla f(4,1) = \left\langle \frac{9}{20}, \frac{1}{5} \right\rangle}$

(b) Find the *directional derivative* of $f(x, y)$ at the point $(4, 1)$ in the direction of the vector $\vec{v} = 4\vec{i} - 3\vec{j}$.

$\|\vec{v}\| = \sqrt{4^2+(-3)^2} = \sqrt{25} = 5$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$

$f_v(4,1) = (\nabla f(4,1)) \cdot \vec{u} = \left(\frac{9}{20}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{5}\right)\left(-\frac{3}{5}\right) = \frac{36}{100} + -\frac{3}{25}$

$= \frac{9}{25} - \frac{3}{25}$

$\boxed{= \frac{6}{25}}$

(c) Find a *local linearization* of $f(x, y)$ at the point $(4, 1)$.

$L(x,y) = f(4,1) + f_x(4,1)(x-4) + f_y(4,1)(y-1)$

$L(x,y) = \ln(20) + \left(\frac{9}{20}\right)(x-4) + \left(\frac{1}{5}\right)(y-1)$

$L(x,y) = \ln(20) + \frac{9}{20}x - \frac{9}{5} + \frac{1}{5}y - \frac{1}{5}$

$\boxed{L(x,y) = \ln(20) + \frac{9}{20}x + \frac{1}{5}y + 2}$

(d) Find the *maximum rate of change* of $f(x, y)$ at the point $(4, 1)$.

Max Rate of Change $= \|\nabla f(4,1)\|$

$= \sqrt{\left(\frac{9}{20}\right)^2 + \left(\frac{1}{5}\right)^2} = \sqrt{\frac{81}{400} + \frac{1}{25}} = \sqrt{\frac{97}{400}}$

Max Rate of Change $= \boxed{\frac{\sqrt{97}}{20}}$

(e) Along which direction at the point $(4, 1)$ will varying $x$ and $y$ result in zero change?

Perpendicular to the gradient vector (where $f(x,y)$ is constant)

any scalar multiple of $\boxed{\left\langle -\frac{1}{5}, \frac{9}{20} \right\rangle}$

**END**