MTH 243: Multivariable Calculus

Exam #1 - Study Guide

Disclaimer: This list is meant to be an *approximate guide*; please note that you are responsible for all the material covered from Sections §12.1 – §12.5, §13.1 – §13.4, §15.1 – §15.3.

Time and Location: **Thursday, February 29, 9:30am, regular classroom.**

General Rules:

- Bring your URI ID.
- Bring several sharp pencils and an eraser. **NO calculators allowed.**
- Start by reviewing past quizzes. Solutions are posted on Brightspace. Don't just read the solutions, work them out again from scratch and check your work! Modify the problems. See if you can still solve them.
- Do the same with problems that were discussed in class. Now repeat the process with the suggested homework problems.
- The list below contains key points from each chapter. It does not mean that I will ask you about every single topic, but you are still responsible for knowing them. Additional, and **more challenging**, practice problems are included at the end so you can test your understanding.

1. **§12.1:** Functions of Two Variables
   - Understand the notions of *domain* and *range* for functions of two and three variables (lecture notes).
   - Four ways of specifying a function: *verbal*, *algebraic*, *tabular*, and *graphical*.
   - Given the xyz-coordinate system, be able to plot given points (see lecture notes for specifics).
   - Know about the xy-, xz-, and yz-planes and planes parallel to them..
   - Given two points be able to compute the *distance* between them, as well as the *midpoint* of the line segment connecting the two points (see lecture notes and quiz #1).
   - Understand the difference between a *sphere* and a *ball* and their corresponding equations.
2. **§12.2:** Graphs and Surfaces
   - Be able to *graph* and/or *recognize* x- and y-cross sections of $f(x, y)$. The curves that you are responsible for knowing how to graph include parabolas, circles, and ellipses. While I will not ask you to graph hyperbolas by hand, you should be able to recognize an equation of a hyperbola. If you haven't done it by now, you can refresh your memory on this topic by reading "Review of Conic Sections" posted on Brightspace.
   - What are *cylinders*? How can you recognize graphs of cylinders? (see *class notes* corresponding to Sections 12.2 & 12.3)
   - What are *surfaces of revolution*? How can you recognize graphs of surfaces of revolution? (again, see *class notes* corresponding to Sections 12.2 & 12.3)
   - Practice matching functions with given graphs – see #5, #6, #28, #29.
3. **§13.3:** Contour Diagrams
   - What is a *contour diagram* for a function $f(x, y)$?
   - Be able to *find* contour curves of $f(x, y)$ and be able to *sketch* them.
   - Practice matching functions with contour diagrams – see #1–#4, #11, #13, #31.

4. **§12.4:** Linear Functions
    - Be able to determine whether a given function is linear or not.
    - Find an equation of a plane with given x- and y- slopes and a point.
    - Be able to graph a plane given its equation.
    - Be able to determine whether a given table or a contour diagram could represent a linear function.
5. **§12.5:** Functions of Three Variables
    - What is a *level set*, or *level curve*, of a function of three variables $f(x, y, z)$?
    - Be able to identify surfaces from page 734 (also last page in class notes from Section 12.3).
6. **§13.1:** Displacement Vectors
    - *Addition*, *subtraction*, and *scalar multiplication* of vectors, together with the geometric interpretations of these operations.
    - Displacement vector between two points.
    - Be able to determine when two vectors are *parallel*.
    - Be able to compute *magnitude* of a vector and determine whether a vector is a *unit* vector or not.
    - Be able to give an example of a vector that has specified *magnitude* and pointing in the (opposite) direction of a given vector.
    - Be able to resolve a vector into $\vec{i}$, $\vec{j}$, and $\vec{k}$ components.
7. **§13.2:** Vectors in General
    - Know the difference between velocity and speed.
    - Know the properties of vector addition and scalar multiplication (see pg. 729).
    - Study problems #6 – #10 on page 732.
8. **§13.3:** The Dot Product
    - *Algebraic* and *geometric* definitions of the dot product.
    - Basic properties of the dot product.
    - The *geometric meaning* of the dot product in terms of lengths and angles – in particular the formula

    $$\vec{a} \bullet \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta)\,.$$

    - Resolving vectors into components. See Application 2 from class notes for Section 12.3.
    - Be able to find an equation of a plane containing the point (x_0, y_0, z_0) and has a normal vector $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$.
    - Be able to determine whether two planes are parallel.
9. **§13.4:** The Cross Product
    - *Algebraic* and *geometric* definitions of the cross product (see pg.745).
    - Basic properties of the cross product.
    - The *geometric meaning* of the cross product in terms of lengths and angles – in particular the formula

    $$||\vec{a} \times \vec{b}|| = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \sin(\theta)\,,$$

    and direction given by the right-hand rule.
    - Be able to find an equation of a plane given three points.
    - Be able to find the area of a parallelogram/triangle spanned by two vectors (see problems 29,30 on page 749).

10. **§14.1:** The Partial Derivative

   - The limit definition of the *partial derivative* of a function $f(x, y)$ with respect to one of its variables.
   - *Geometric interpretation* of partial derivatives $f_x(x, y)$ and $f_y(x, y)$ (see class notes).
   - Be able to estimate partial derivatives f_x and f_y at a point given the contour plot of $f(x, y)$ or table of values.

11. **§14.2:** Computing Partial Derivatives Algebraically

   - Know how to compute partial derivatives algebraically. *If you do not know how to do this, you will not be able to answer majority of the questions below!!!*

12. **§14.3:** Local Linearity and the Differential

   - Equation of the tangent plane to $f(x, y)$ near the point $(a, b) \Longrightarrow$ *local linearization* of f near $(x, y) = (a, b)$.
   - Using the local linearization of $f(x, y)$ to approximate value of f at a nearby point (Example 2 (pg.801); Problem #37 (pg.807)).
   - Know the notion of the *differential of the function.*

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Additional Practice Problems

1. What is the *domain* of the function
$$f(x, y) \;=\; \frac{xy}{\sqrt{x - y}}\,?$$
Shade the portion of the xy-plane that corresponds to the domain of $f(x, y)$.

2. What is the *domain* of the function
$$g(x, y) \;=\; \frac{x}{x^2 + y^2}\,?$$
Shade the portion of the xy-plane that corresponds to the domain of $g(x, y)$.

3. For what values of t are $\vec{u} = t\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = t\vec{i} + t\vec{j} - 2\vec{k}$ perpendicular? Are there values of t for which $\vec{u}$ and $\vec{v}$ are parallel?

4. Show that if $\vec{u}$ and $\vec{v}$ are two vectors such that
$$\vec{u} \cdot \vec{w} \;=\; \vec{v} \cdot \vec{w}$$
for *every* vector $\vec{w}$, then
$$\vec{u} \;=\; \vec{v}\,.$$

5. Determine which of the following statements is true and justify your answer (you might find useful properties on pg.729 and pg.735):

   (a) For any 3-dimensional vectors $\vec{u}, \vec{v}, \vec{w}$, we have

   $$\left(\vec{u} \bullet \vec{w}\right) \cdot \vec{w} \;=\; \vec{u} \cdot \left(\vec{v} \bullet \vec{w}\right).$$

   (b) For any two vectors $\vec{u}$ and $\vec{v}$:

   $$\left(\vec{u} + \vec{v}\,\right) \bullet \left(\vec{u} - \vec{v}\,\right) \;=\; \left|\left|\vec{u}\,\right|\right|^2 - \left|\left|\vec{v}\,\right|\right|^2.$$

   (c) The triangle in 3-space with vertices $(1,1,0)$, $(0,1,0)$, and $(0,1,1)$ has a right angle.

6. For any two vectors $\vec{v}$ and $\vec{w}$, consider the following function of t:

   $$q(t) \;=\; \left(\vec{v} + t\vec{w}\right) \bullet \left(\vec{v} + t\vec{w}\right).$$

   (a) Explain why $q(t) \geq 0$ for all real t.

   (b) Expand $q(t)$ as a quadratic polynomial in t using the properties on page 735.

   (c) Using the discriminant of the quadratic to show that

   $$\left|\vec{v} \bullet \vec{w}\,\right| \;\leq\; \left|\left|\vec{v}\,\right|\right| \cdot \left|\left|\vec{w}\,\right|\right|.$$

   Hint: For part (c) you will need to use the quadratic formula for finding zeros of $ax^2 + bx + c = 0$ which is given by

   $$x_{1,2} \;=\; \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic equation $ax^2 + bx + c$ and it plays the crucial role in determining whether the equation $ax^2 + bx + c = 0$ has zero, one, or two real distinct solutions.

7. Show that for any two vectors $\vec{v}$ and $\vec{w}$ we have $\vec{v} \times \vec{w} \;=\; -\vec{w} \times \vec{v}$.

8. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three arbitrary vectors. Show that

   $$\vec{a} \bullet \left(\vec{b} \times \vec{c}\,\right) \;=\; \vec{b} \bullet \left(\vec{c} \times \vec{a}\,\right) \;=\; \vec{c} \bullet \left(\vec{a} \times \vec{b}\,\right)$$

9. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three arbitrary vectors. Show that

   $$\vec{a} \times \left(\vec{b} \times \vec{c}\,\right) \;=\; \vec{b} \cdot \left(\vec{a} \bullet \vec{c}\,\right) - \vec{c} \cdot \left(\vec{a} \bullet \vec{b}\,\right).$$

10. Review "Practice Problems" about surfaces of revolution from the class notes for Sections 12.2 and 12.3 that is posted on Brightspace.