MTH 243 - Exam 2 - Spring 2024 - Version A

	Correct	Pts
#1 (a)	TRUE	1
#1 (b)	TRUE	1
#1 (c)	TRUE	1
#1 (d)	FALSE	1
#1 (e)	TRUE	1

Problem #2

Function	Region			
	R1	R2	R3	R4
f(x,y)	zero	pos.	pos.	neg.
g(x,y)	zero	pos.	zero	neg.
h(x,y)	zero	neg.	neg.	zero

#3	C	3
#4	B	3
#5	F	4
#6	D	4
#7	E	5

MTH 243 - Exam 2 - Spring 2024 - Version B

	Correct	Pts
#1 (a)	TRUE	1
#1 (b)	TRUE	1
#1 (c)	FALSE	1
#1 (d)	TRUE	1
#1 (e)	TRUE	1

Problem #2

Function	Region			
	R1	R2	R3	R4
f(x,y)	zero	pos.	zero	neg.
g(x,y)	zero	neg.	neg.	zero
h(x,y)	zero	pos.	pos.	neg.

#3	C	3
#4	B	3
#5	B	4
#6	C	4
#7	F	5

PROBLEM STATEMENT

Find $f_{xx}(x,y)$, where $f(x,y) = x\,y^2 + y\,e^{x^2} + 5.$

SOLUTION

$$f_x(x,y) = \frac{\partial}{\partial x}\left[xy^2 + ye^{x^2} + 5\right]$$

$$= y^2 + y \cdot 2x \cdot e^{x^2} + 0$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(x,y)\right]$$

$$= \frac{\partial}{\partial x}\left[y^2 + y \cdot 2x \cdot e^{x^2}\right]$$

$$= 0 + 2y\left[1 \cdot e^{x^2} + x \cdot 2x \cdot e^{x^2}\right] \quad \text{(product rule)}$$

$$= 2ye^{x^2} + 4x^2ye^{x^2}$$

$$\boxed{= 2ye^{x^2}\left(1 + 2x^2\right)}$$

PROBLEM STATEMENT

Find $f_{xy}(\pi, 1)$, where $f(x, y) = x\,\sin(x\,y^2)$.

SOLUTION

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x \cdot \sin(xy^2)\right]$$

$$= 1 \cdot \sin(xy^2) + x \cdot \cos(xy^2) \cdot y^2$$

$$= \sin(xy^2) + xy^2\cos(xy^2)$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(x,y)\right]$$

$$= \frac{\partial}{\partial y}\left[\sin(xy^2) + xy^2\cos(xy^2)\right]$$

$$= 2xy \cdot \cos(xy^2) + x \cdot \left[2y \cdot \cos(xy^2) + y^2 \cdot \left(-\sin(xy^2)\right) \cdot 2xy\right]$$

$$= 2xy \cdot \cos(xy^2) + 2xy \cdot \cos(xy^2) - 2x^2y^3 \cdot \sin(xy^2)$$

$$= 4xy \cdot \cos(xy^2) - 2x^2y^3\sin(xy^2)$$

$$f_{xy}(\pi,1) = 4 \cdot \pi \cdot 1 \cdot \underbrace{\cos(\pi)}_{=-1} - 2 \cdot \pi^2 \cdot 1 \cdot \underbrace{\sin(\pi)}_{=0}$$

$$\boxed{= -4\pi}$$

PROBLEM STATEMENT

COMPUTE directional derivative of $f(x,y,z)$ at the point $(1,1,1)$ in the direction of vector $\vec{u}$, where

$$f(x,y,z) = x^2y + yz^2 \qquad \text{and} \qquad \vec{u} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

SOLUTION

$f_x(x,y,z) = 2xy$ $\qquad$ $f_x(1,1,1) = 2$

$f_y(x,y,z) = x^2 + z^2$ $\qquad$ $f_y(1,1,1) = 2$

$f_z(x,y,z) = 2yz$ $\qquad$ $f_z(1,1,1) = 2$

$$\nabla f(1,1,1) = \langle 2, 2, 2 \rangle$$

Since $\vec{u}$ is already a unit vector we have that the directional derivative of f in direction $\vec{u}$ and point $(1,1,1)$, $f_{\vec{u}}(1,1,1)$, is given by

$$f_{\vec{u}}(1,1,1) = \nabla f(1,1,1) \cdot \vec{u}$$

$$= \langle 2,2,2 \rangle \cdot \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3} \rangle$$

$$= \frac{4}{3} + \frac{2}{3} + \frac{4}{3}$$

$$\boxed{= \frac{10}{3}}$$

PROBLEM STATEMENT

Find the equation of the tangent plane to the graph of the function

$$f(x, y) = x - \frac{y^2}{2}$$

at the point $(2, 4, -6)$.

SOLUTION

$f_x(x,y) = 1 \qquad f_x(2,4) = 1$

$f_y(x,y) = -y \qquad f_y(2,4) = -4$

The equation of the tangent plane to the graph of $f(x) = x - \frac{y^2}{2}$ at the point $(2,4,-6)$ is given by

$$z = f(2,4) + f_x(2,4)\cdot(x-2) + f_y(2,4)\cdot(y-4)$$

$$\boxed{\begin{gathered} z = -6 + 1\cdot(x-2) - 4(y-4) \\ \text{or} \\ x - 4y - z = -8 \\ \text{or} \\ -x + 4y + z = 8 \end{gathered}}$$

PROBLEM STATEMENT

Find the **relative extrema** of

$$f(x,y) = -\frac{2}{3}x^3 + 4xy - 2y^2 + 1\,.$$

SOLUTION

$f_x(x,y) = -2x^2 + 4y$

$f_y(x,y) = 4x - 4y$

To find critical point(s) set $f_x(x,y) = f_y(x,y) = 0$.

$f_y(x,y) = 4x - 4y = 0$

$\Rightarrow x = y$ — $(\bigstar)$

$f_{xx}(x,y) = -4x$

$f_{xy}(x,y) = 4$

$f_{yy}(x,y) = -4$

$f_x(x,y) = -2x^2 + 4y = 0$

now substitute condition $(\bigstar)$

$-2x^2 + 4x = 0$

$-2x(x-2) = 0$

$x = 0$ or $x = 2$

So the critical points are:

$(0,0)$ and $(2,2)$

(a,b) Critical Pt.	$f_{xx}(a,b)$	$f_{yy}(a,b)$	$f_{xy}(a,b)$	$f_{xx}(a,b)\cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$	Classification
$(0,0)$	0	-4	4	$(0)(-4) - (4)^2 = -16 < 0$	saddlepoint
$(2,2)$	-8	-4	4	$(-8)(-4) - (4)^2 = 16 > 0$	local maximum

Since $f_{xx}(2,2)\cdot f_{yy}(2,2) - [f_{xy}(2,2)]^2 > 0$ and $f_{xx}(2,2) < 0$, we have that $(2,2)$ is a $\boxed{\text{local maximum}}$.

PROBLEM STATEMENT

Find a **direction resulting in no change** of $f(x,y) = x^3\, e^{-2y}$ at $x = 1$ and $y = 0$?

Include all supporting computations and briefly justify your answer.

SOLUTION

$f_x(x,y) = 3x^2 e^{-2y}$ $\quad f_x(1,0) = 3\cdot(1)^2 e^0 = 3$

$f_y(x,y) = -2x^3 e^{-2y}$ $\quad f_y(1,0) = -2\cdot(1)^3 e^0 = -2$

$$\nabla f(1,0) = 3\vec{i} - 2\vec{j} = \langle 3,-2\rangle.$$

Direction that will result in no change of $f(x,y)$ at $(1,0)$ is any vector that is perpendicular/orthogonal to $\nabla f(1,0)$.

e.g. $\vec{u} = \langle 2,3\rangle$

since $\nabla f(1,0)\cdot\vec{u} = \langle 3,-2\rangle\cdot\langle 2,3\rangle = 0.$

⊛ Obviously there are infinitely many vectors you could choose, but they will all be non zero scalar multiplies of $\vec{u} = \langle 2,3\rangle$

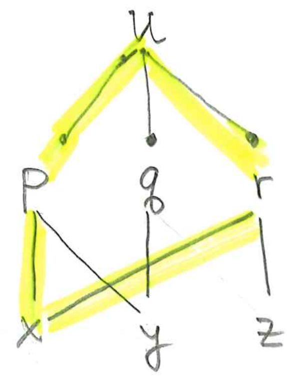
PROBLEM STATEMENT

Find $$\left.\frac{\partial u}{\partial x}\right|_{(x,y,z)=(1,2,5)} = u_x(1,2,5) \qquad \text{where}$$

$$u(p,q,r) = p^2 - q^2 - r\,, \qquad p(x,y,z) = x\,y\,, \qquad q(x,y,z) = y^2\,, \qquad r(x,y,z) = x\,z\,.$$

Include all supporting computations and clearly label your steps.

SOLUTION



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\underbrace{\frac{\partial q}{\partial x}}_{=0} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= 2p\cdot y + (-1)\cdot z$$

$$p(1,2,5) = 1\cdot 2 = 2$$

$$q(1,2,5) = (2)^2 = 4$$

$$r(1,2,5) = 1\cdot 5 = 5$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x,y,z)=(1,2,5)} = 2\cdot p(1,2,5)\cdot 2 - 1\cdot 5$$

$$= 2\cdot 2\cdot 2 - 5$$

$$\boxed{= 3}$$

PROBLEM STATEMENT

Let $z(x, y)$ be the function *implicitly defined* as the solution to

$$x + y + z + \sin(x\,y\,z) \;=\; 3 + \frac{\pi}{2} \qquad (\star)$$

that satisfies $z(1,1) = \frac{\pi}{2}$. **Find** $\frac{\partial z}{\partial x}(1,1) = z_x(1,1)$.

SOLUTION Start by taking partial derivative with respect to "x" of both sides of $(\star)$.

$$\frac{\partial}{\partial x}\left[x+y+z+\sin(xyz)\right] = \frac{\partial}{\partial x}\left[3+\pi/2\right]$$

$$\frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[y] + \frac{\partial}{\partial x}[z] + \frac{\partial}{\partial x}[\sin(xyz)] = 0$$

$$1 + 0 + \frac{\partial z}{\partial x} + \cos(xyz)\cdot\frac{\partial}{\partial x}[xyz] = 0$$

$$1 + \frac{\partial z}{\partial x} + \cos(xyz)\cdot\left(yz + x\cdot y\,\frac{\partial z}{\partial x}\right) = 0$$

$$1 + \underline{\underline{\frac{\partial z}{\partial x}}} + y\cdot z\cdot\cos(xyz) + \underline{\underline{x\cdot y\cdot\cos(xyz)\cdot\frac{\partial z}{\partial x}}} = 0$$

$$\frac{\partial z}{\partial x}\cdot\left(1 + xy\cos(xyz)\right) = -1 - yz\cos(xyz)$$

$$\frac{\partial z}{\partial x} = \frac{-1 - yz\cos(xyz)}{1 + xy\cos(xyz)}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(x,y)=(1,1)} = \frac{-1 - 1\cdot 1\cdot\cos(1\cdot 1\cdot z(1,1))}{1 + 1\cdot 1\cdot\cos(1\cdot 1\cdot z(1,1))} = \frac{-1-\cos(\pi/2)}{1+\cos(\pi/2)} = \boxed{-1}$$