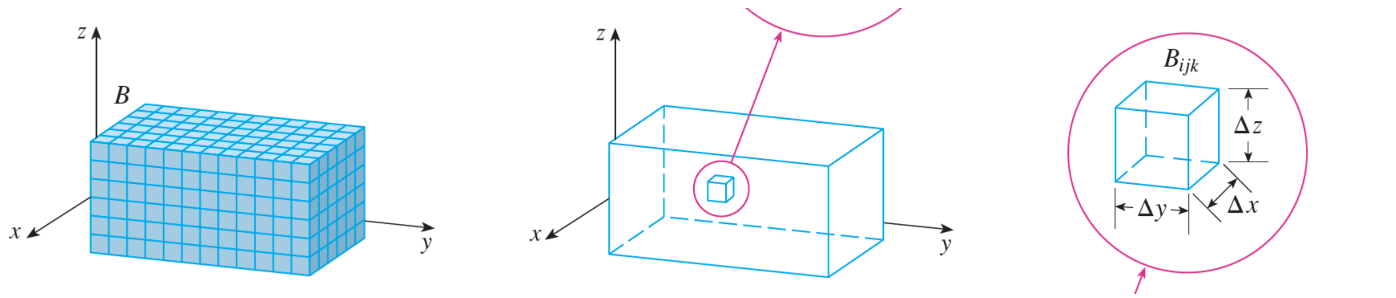
16.3 Triple Integrals

Let $\mathcal{B}$ be a *box* in $3d$-space, that is,

$$\mathcal{B} \;=\; \big\{(x,y,z):\;\; a \le x \le b,\quad c \le y \le d,\quad p \le z \le q\big\}\,. \tag{16.7}$$



Then $\mathcal{B}$ can be subdivided into $\ell\,m\,n$ subsolids, $\mathcal{B}_{ijk}$, where $\quad 1 \le i \le n, \quad 1 \le j \le m, \quad$ and $\quad 1 \le k \le \ell$, that is,

$$\left.\begin{array}{lcll} a \le x \le b & \longrightarrow & \Delta x = \dfrac{b-a}{n} & (n \text{ pieces of equal width } \Delta x) \\[2ex] c \le y \le d & \longrightarrow & \Delta y = \dfrac{d-x}{m} & (m \text{ pieces of equal width } \Delta y) \\[2ex] p \le z \le q & \longrightarrow & \Delta z = \dfrac{q-p}{\ell} & (\ell \text{ pieces of equal width } \Delta z) \end{array}\right\} \quad \text{(subdivision)} \tag{16.8}$$

The Triple Riemann Sum and the Triple Integral

Let $f(x,y,z)$ be a continuous function over a *box* $\mathcal{B}$ as in (16.7).

Furthermore, subdivide $\mathcal{B}$ into ℓmn subsolids $\mathcal{B}_{ijk}$ as in (16.8), each with the volume $\quad \Delta V \;=\; \Delta x\,\Delta y\,\Delta z$.

Then the **triple Riemann sum** is given by

$$\text{Riemann sum} \;=\; \sum_{k=1}^{\ell}\sum_{j=1}^{m}\sum_{i=1}^{n} f\big(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\big)\,\Delta x\,\Delta y\,\Delta z\,,$$

where $\big(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\big)$ is an arbitrary point in the subsolid $\mathcal{B}_{ijk}$.

The **triple integral of $f(x,y,z)$ over the box $\mathcal{B}$** is defined as

$$\begin{aligned} \int_{\mathcal{B}} f(x,y,z)\,dV \;&:=\; \lim_{\ell,\,m,\,n\,\to\,\infty} \sum_{k=1}^{\ell}\sum_{j=1}^{m}\sum_{i=1}^{n} f\big(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\big)\,\Delta x\,\Delta y\,\Delta z \\ &=\; \lim_{\Delta V \to 0} \sum_{\text{all subsolids}} f\big(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}\big)\,\Delta V\,, \end{aligned}$$

if the limit exists.

Fubini's Theorem for Triple Integrals

If $f(x,y,z)$ is continuous on the box $\mathcal{B} = \{(x,y,z) : \ a \le x \le b, \quad c \le y \le d, \quad p \le z \le q\}$, then

$$\begin{aligned}
\int_{\mathcal{B}} f(x,y,z)\,dV &= \int_p^q \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_p^q \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz \\
&= \int_c^d \int_p^q \int_a^b f(x,y,z)\,dx\,dz\,dy = \int_c^d \int_a^b \int_p^q f(x,y,z)\,dz\,dx\,dy \\
&= \int_a^b \int_p^q \int_c^d f(x,y,z)\,dy\,dz\,dx = \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx
\end{aligned}$$

Triple Integral as VOLUME

Suppose $f(x,y,z) = 1 > 0$ for all the points (x,y,z) in the region $\mathcal{B}$. Then $1 \cdot \Delta V = \Delta V$, and therefore

$$\boxed{\text{Volume } (\mathcal{B}) = \int_{\mathcal{B}} 1\,dV = \int_{\mathcal{B}} dV}\ .$$

Triple Integral as Average Value

$$\boxed{\text{Average Value of } f(x,y,z) \text{ on the region } \mathcal{B} = \frac{1}{\text{Volume of } \mathcal{B}} \cdot \int_{\mathcal{B}} f(x,y,z)\,dV = \frac{\int_{\mathcal{B}} f(x,y,z)\,dV}{\int_{\mathcal{B}} dV}}\ .$$

Example 6. *Evaluate* $\int_{\mathcal{B}} f(x,y,z)\,dV$*, where*

$$f(x,y,z) \;=\; xyz^2 \qquad \text{and} \qquad \mathcal{B} = \big\{(x,y,z): \;\; 0 \le x \le 1, \quad -1 \le y \le 2, \quad 0 \le z \le 3\big\}\,.$$

Solution:

Example 7. *Let* $f(x, y, z) = x + y$*. Find* $\int_W f(x, y, z)\, dV$*, where* W *is the solid bounded by the xy-plane, yz-plane, xz-plane, and the plane* $\frac{x}{3} + \frac{y}{2} + \frac{z}{6} = 1$*.*

Solution:

Example 8. *Sketch the solid of integration W corresponding to the iterated integral*

$$\int_W f(x,y,z)\,dV \;=\; \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz\,.$$

Solution:

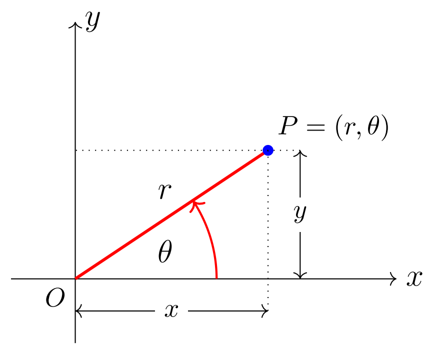
16.4 Double Integrals in Polar Coordinates

Review of Polar Coordinates (Calculus II)

In **polar coordinates** every point P in the xy-plane is described by two coordinates

$$P = (r\,,\,\theta)\,,$$

where *r is the distance of P from the origin O* (called the *pole*), *θ is the angle, in radians, between OP and the positive x-axis* (called the polar axis) *measured counter-clockwise*.



Clearly if we think of the point P in the Cartesian coordinates, that is, $P = (x, y)$, then it is easy to obtain the following relations

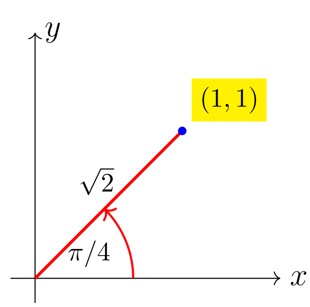
$$x = r\cos(\theta)\,, \qquad y = r\sin(\theta)\,, \qquad\qquad r^2 = x^2 + y^2\,, \qquad \tan(\theta) = \frac{y}{x}\,.$$

Example 9. *Convert polar coordinates in the exercises below to Cartesian coordinates. Give the exact answers.*

(a) $(r\,,\,\theta) = \left(\sqrt{2}\,,\,\frac{\pi}{4}\right)$

Solution:

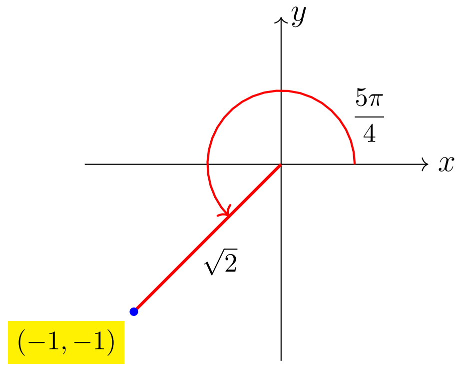
$$x = \sqrt{2}\cos\left(\frac{\pi}{4}\right) = 1\,, \qquad y = \sqrt{2}\sin\left(\frac{\pi}{4}\right) = 1$$



(b) $(r\,,\,\theta) \;=\; \left(\sqrt{2}\,,\,\frac{5\pi}{4}\right)$

Solution:

$$x \;=\; \sqrt{2}\cos\left(\frac{5\pi}{4}\right) \;=\; -1\,, \qquad y \;=\; \sqrt{2}\sin\left(\frac{5\pi}{4}\right) \;=\; -1$$



Conversion of Cartesian Coordinates to Polar Coordinates

Recall from before that given the point $P = (x, y) = (r, \theta)$, where $x \neq 0$, we have

$$\tan(\theta) \;=\; \frac{y}{x}\,, \qquad r^2 \;=\; x^2 + y^2\,,$$

and so

$$\theta \;=\; \arctan\left(\frac{y}{x}\right)\,, \qquad r \;=\; \sqrt{x^2 + y^2}\,.$$

Note that $\theta = \arctan\left(\frac{y}{x}\right)$ is **not** sufficient to tell in which quadrant θ should be. For example, if $P = (x, y) = (1, 1)$, then

$$\arctan\left(\frac{1}{1}\right) \;=\; \arctan(1) \;=\; \frac{\pi}{4}\,,$$

as we need the point P to be in the first quadrant. On the other hand, if $Q = (x, y) = (-1, -1)$, then

$$\theta \;=\; \arctan\left(\frac{-1}{-1}\right) \;=\; \arctan(1) \;=\; \frac{5\pi}{4}\,,$$

as we need the point Q to be in the third quadrant.

Finally note that for a given (x, y) there are many choices of θ. For example, a point $P = (0, 2)$ in the rectangular coordinates can be represented in polar coordinates in many ways

$$P \;=\; \left(2\,,\,\frac{\pi}{2}\right)\,, \quad \text{or} \quad P \;=\; \left(2\,,\,\frac{-3\pi}{2}\right)\,, \quad \text{or} \quad P \;=\; \left(2\,,\,\frac{5\pi}{2}\right)\,, \quad \text{or} \quad \ldots$$

In fact, if $P = (r, \theta)$, then $P = (r, \theta + 2n\pi)$ where n in *any* integer.

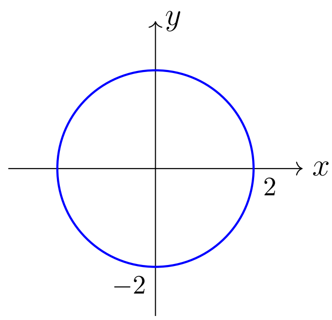
Curves in Polar Coordinates

Example 10. *Describe the curves given by the equations below.*

(a) $r = 2$

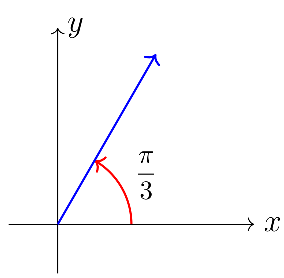
Solution: *This is just a circle of radius 2 centered at the origin since*

$$r = 2 \implies \sqrt{x^2 + y^2} = 2 \implies x^2 + y^2 = 2^2 .$$



(b) $\theta = \frac{\pi}{3}$

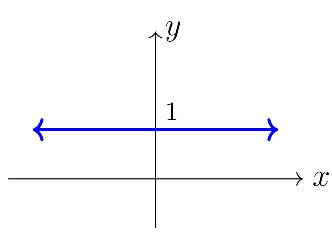
Solution: *This is a half-line with slope* $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ *which extends from the origin.*



(c) $r = \dfrac{1}{\sin(\theta)}$

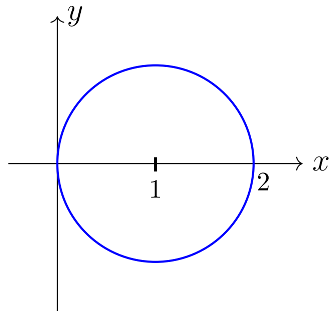
Solution: *This is just a horizontal line* $y = 1$ *since*

$$r = \frac{1}{\sin(\theta)} \implies r\sin(\theta) = 1 \implies y = 1 .$$



(d) $r = 2\cos(\theta)$

Solution: *This is a circle of radius* 1 *centered at* $(1, 0)$.



To see this, remember that $\cos(\theta) = \dfrac{x}{r}$, *so*

$$r = 2\cos(\theta) = \frac{2x}{r} \quad\Longrightarrow\quad r^2 = 2x\,.$$

Combining this relation with the fact that $r^2 = x^2 + y^2$ *we obtain*

$$x^2 + y^2 = 2x \qquad\Longleftrightarrow\qquad x^2 - 2x + y^2 = 0\,.$$

Completing the square gives

$$\begin{aligned} x^2 + 2(-1)x + y^2 &= 0 \\ x^2 + 2(-1)x + (-1)^2 - (-1)^2 + y^2 &= 0 \\ (x-1)^2 - 1 + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

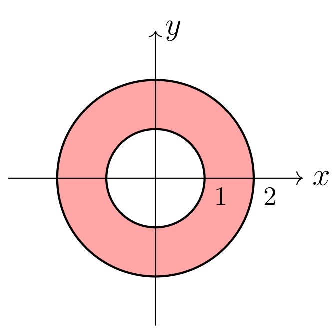
which indeed is the equation of a circle of radius 1 *with center at* $(1, 0)$.

Regions in Polar Coordinates

Example 11. *Sketch the regions given by the relations below.*

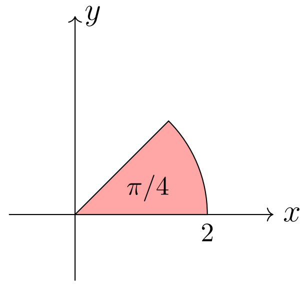
(a) $1 \le r \le 2$

Solution: *Annulus (ring) of inner radius* 1 *and outer radius* 2.



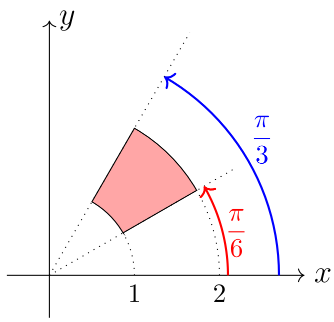
(b) $0 \leq r \leq 2 \quad and \quad 0 \leq \theta \leq \frac{\pi}{4}$

Solution:



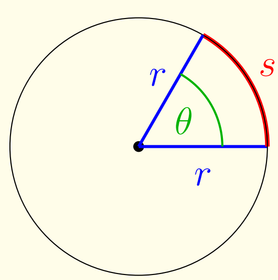
(c) $1 \leq r \leq 2 \quad and \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

Solution: "Curvy rectangle"



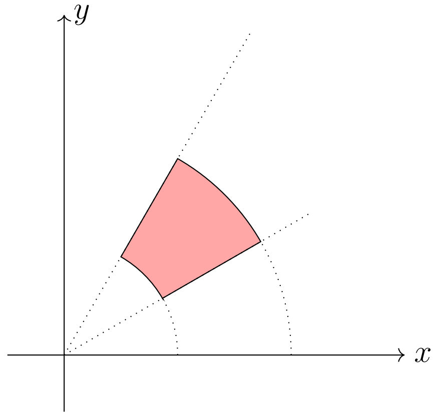
Definition of "RADIAN"

The **"radian" measure** of a central angle of a circle, θ, is defined as the **ratio** of the length of the arc the angle subtends, s, and the radius of the circle, r.

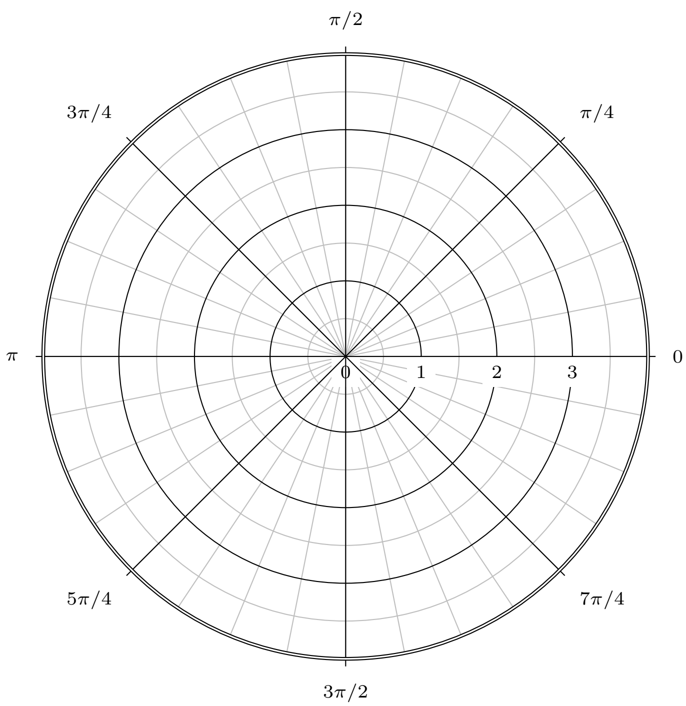


$$\theta\,(\text{in radians}) = \frac{s}{r} \qquad \Longrightarrow \qquad \boxed{s = r \cdot \theta}\,.$$

Area in Polar Coordinates



Construction of Double Integrals in Polar Coordinates

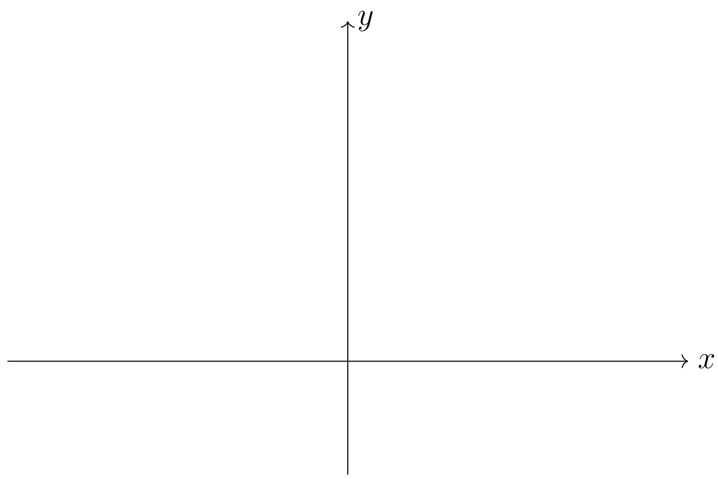


Example 12. *Let $\mathcal{R}$ be the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.*

(a) *Sketch the region $\mathcal{R}$.*

(b) *Evaluate* $\displaystyle\int_{\mathcal{R}} \left(3x + 4y^2\right) dA.$

Solution:



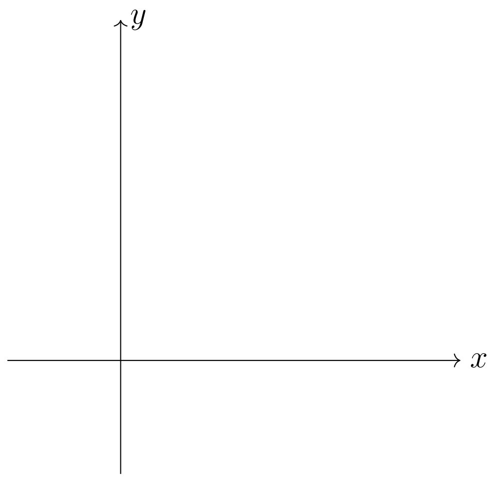
Example 13. *Consider the following* iterated integral $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \big(x\, y\big)\, dxdy$

(a) *Sketch the region of integration.*

(b) *Describe the region of integration in polar coordinates.*

(c) *Set up the original integral as an iterated integral in polar coordinates and evaluate it.*

Solution:



Example 14. *The density of insects in a circular region*

$$100 \le \sqrt{x^2+y^2} \le 200 \tag{16.9}$$

around a circular lake $0 \le \sqrt{x^2+y^2} \le 100$, *where* x *and* y *are in meters, is given by*

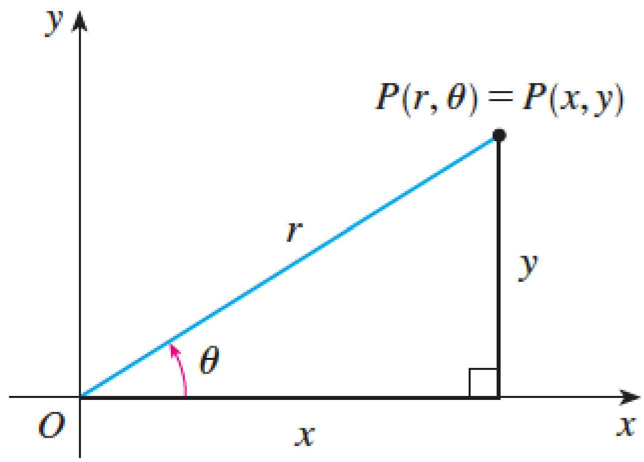
$$d(x,y) = \frac{1000}{\sqrt{x^2+y^2}} \frac{\text{insects}}{m^2}. \tag{16.10}$$

Find the total number of insects in the region.

Solution:

16.5 Cylindrical and Spherical Coordinates

Brief Review of Polar and Rectangular Coordinates



From Polar to Cartesian

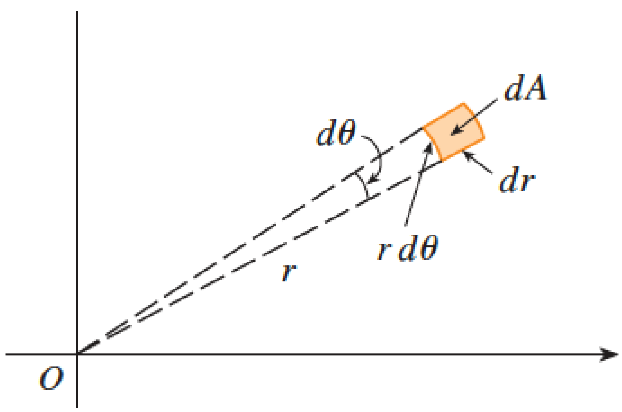
$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

From Cartesian to Polar

$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x}$$



$$dA = r\,dr\,d\theta$$

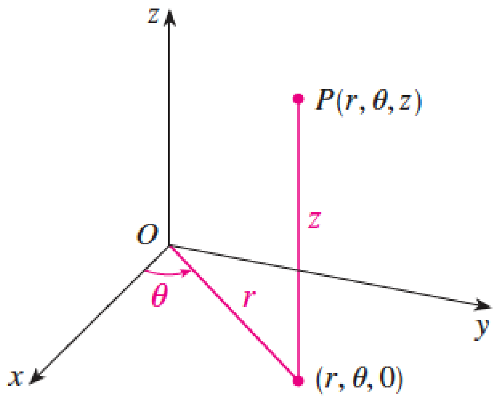
If f is continuous on a polar region of the form

$$\mathcal{D} = \Big\{ (r,\theta) \,|\, \alpha \le \theta \le \beta\,,\ h_1(\theta) \le r \le h_2(\theta) \Big\}$$

then

$$\int_{\mathcal{D}} f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f\big(r\cos(\theta), r\sin(\theta)\big)\,r\,dr\,d\theta\,.$$

Cylindrical Coordinates



Bounds on r, θ, and z

From Cylindrical to Cartesian

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

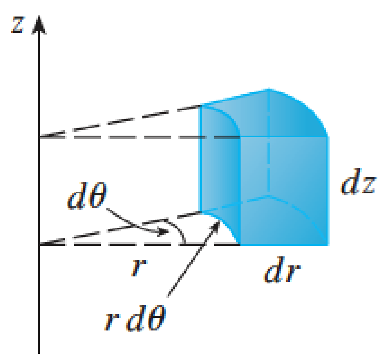
$$z = z$$

From Cartesian to Cylindrical

$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x}$$

$$z = z$$



$$dV = r\,dz\,dr\,d\theta$$

If f is continuous on a region $\mathcal{E}$ of the form

$$\mathcal{E} = \left\{ (r, \theta, z) \,\middle|\, u_1(r,\theta) \le z \le u_2(r,\theta) \;,\; h_1(\theta) \le r \le h_2(\theta) \;,\; \alpha \le \theta \le \beta \right\}$$

then

$$\int_{\mathcal{E}} f(x,y,z)\,dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f\big(r\cos(\theta), r\sin(\theta), z\big)\, r\,dz\,dr\,d\theta\,.$$

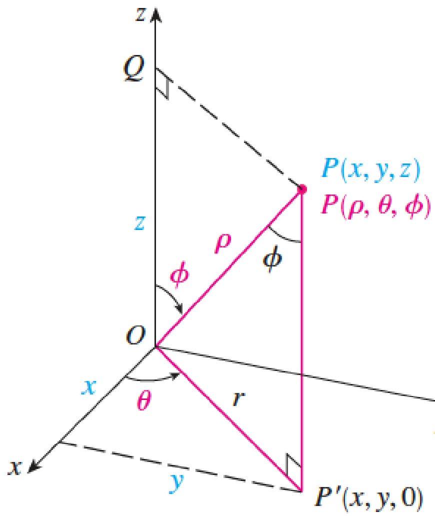
Example 15. *Find* $\int_W f(x,y,z)\,dV$, *where* $f(x,y,z) = \sin(x^2+y^2)$, *and* W *is the solid circular cylinder with height* 4 *units and base of radius* 1 *unit centered on the* z*-axis on the place* $z=-1$.

Solution:

Example 16. *Find* $\int_W f(x, y, z)\, dV$ *in cylindrical coordinates, where* W *is the* inverted circular cone *centered on the* z*-axis with height* 2 *units, "tip" of the cone being located at the origin, and with "base" of radius* 3.

Solution:

Spherical Coordinates



Bounds on ρ, θ, and ϕ

From Spherical to Cartesian

$$x = \rho \sin(\phi)\cos(\theta)$$

$$y = \rho \sin(\phi)\sin(\theta)$$

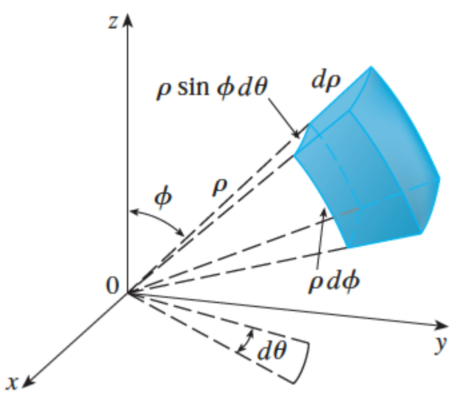
$$z = \rho\cos(\phi)$$

From Cartesian to Spherical

$$\rho^2 = x^2 + y^2 + z^2$$

$$\cos(\phi) = \frac{z}{\rho}$$

$$\cos(\theta) = \frac{x}{\rho\sin(\phi)}$$



$$dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

If $f(x,y,z)$ is continuous on a region $\mathcal{E}$ given by

$$\mathcal{E} = \Big\{ (\rho,\theta,\phi) \,|\, u_1(\rho,\theta) \le \phi \le u_2(\rho,\theta) \;,\; h_1(\rho) \le \theta \le h_2(\rho) \;,\; a \le \rho \le b \Big\}$$

then

$$\int_{\mathcal{E}} f(x,y,z)\,dV = \int_a^b \int_{h_1(\rho)}^{h_2(\rho)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f\Big(\rho\sin(\phi)\cos(\theta)\,,\,\rho\sin(\phi)\sin(\theta)\,,\,\rho\cos(\phi)\Big)\,\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi\,.$$

Example 17. *Evaluate* $\int_W f(x, y, z)\, dV$, *where* $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ *and* W *is the bottom half of the sphere of radius* 5 *centered at the origin.*

Solution:

Example 18. *Give spherical coordinates for the iterated integral over the region bounded between the sphere of radius* 1 *and the sphere of radius* 4 *centered at the origin.*

Solution: