Part 3 Essay Excerpt

EXCERPT BEGINS ... We now take a closer look at the possible cloud support mechanisms already alluded to. Here we assume a globally bound molecular cloud and derive a more complete version of the Virial Theorem. The theory of un-bound clouds is discussed in the following Chapter.

The generalised hydrostatic equation for a fluid which has an ambient magnetic field permeating it is:

$$\rho\frac{Du}{Dt} = -\vec{\nabla}P - \rho\vec{\nabla}\Phi_g + (\vec{J}\times\vec{B}) \tag{1}$$

where Du/Dt is the 'convective'/'Lagrangian' time-derivative, defined by:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla}) \tag{2}$$

Of course from Maxwell's equations we know that:

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \tag{3}$$

For hereon we assume that the electric field changes slowly and we ignore this term in the equation so that we can re-cast Equation (1) as:

$$\rho\frac{Du}{Dt} = -\vec{\nabla}P - \rho\vec{\nabla}\Phi_g + \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}\times\vec{B}) \tag{4}$$

which expands to:

$$\rho\frac{Du}{Dt} = -\vec{\nabla}P - \rho\vec{\nabla}\Phi_g + \frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B} + \frac{1}{2\mu_0}(\vec{\nabla}|\vec{B}|^2) \tag{5}$$

We can simply see what each term in this expression corresponds to: terms 1 and 2 on the RHS are the normal thermal pressure and gravity terms; term 4 is a magnetic pressure term; while term 3 is an extra tension due to curved magnetic field lines. To push forward to derive the Virial Theorem we now take the scalar product of Equation (5) with the position vector $\vec{r}$ and integrate this over the volume, V:

$$\begin{aligned}\int_V \rho\tfrac{\partial u}{\partial t}\cdot\vec{r}dV + \int_V \rho(\vec{u}\cdot\vec{\nabla})\vec{u}\cdot\vec{r}dV = -\int_V \vec{\nabla}P\cdot\vec{r}dV - \int_V \rho\vec{\nabla}\Phi_g\cdot\vec{r}dV \\ +\tfrac{1}{\mu_0}\int_V(\vec{B}\cdot\vec{\nabla})\vec{B}\cdot\vec{r}dV + \tfrac{1}{2\mu_0}\int_V(\vec{\nabla}|\vec{B}|^2)\cdot\vec{r}dV\end{aligned} \tag{6}$$

This integration is normally performed in components. Thus the LHS of Equation (6) becomes:

$$\int_V \rho x_i\frac{\partial u_i}{\partial t}dV + \int_V \rho x_i u_j\partial_j u_j dV \tag{7}$$

Now by noting the continuity equation which (in components) is $\partial_j(\rho u_j) = -\frac{\partial\rho}{\partial t}$ and also that $\rho dV = dm$ the LHS becomes:

$$LHS = \frac{1}{2}\ddot{I} - 2T \tag{8}$$

where the moment of inertia, I, and the kinetic energy, T, are defined by the usual expressions:

$$I = \int_M x_i x_j dm \tag{9}$$

$$T = \frac{1}{2}\int_M u_i u_j dm \tag{10}$$

The first term on the RHS can be worked out by integration by parts

$$-\int x_i \partial_i P dV = -\int \partial_i (x_i P) dV + \int (\partial_i x_i) P dV \tag{11}$$

and adopting $P = (5/3 - 1)u$ we get:

$$\Rightarrow -\int x_i \partial_i P dV = -\int P \vec{r} \cdot d\vec{S} + 2U \tag{12}$$

The second RHS term is:

$$-\int_V \rho \vec{\nabla} \Phi_g \cdot \vec{r} dV = -\int \vec{r} \cdot \vec{g} dm \tag{13}$$

But we can write this as:

$$\int \vec{r} \cdot \vec{g} dm = -\int\int \frac{G(\vec{r} - \vec{r'}) \cdot \vec{r}}{|\vec{r} - \vec{r'}|^3} dm dm' = -\int\int \frac{G(\vec{r'} - \vec{r}) \cdot \vec{r'}}{|\vec{r} - \vec{r'}|^3} dm dm' \tag{14}$$

Adding these two ways together and dividing by two we get:

$$\int \vec{r} \cdot \vec{g} dm = -\frac{1}{2}\int\int G \frac{dm dm'}{|\vec{r} - \vec{r'}|} = \Omega \tag{15}$$

We next evaluate the third and fouth RHS terms noting the definition that:

$$M_B = \frac{1}{2\mu_0}\int_V |\vec{B}|^2 dV \tag{16}$$

The third term is given by:

$$\frac{1}{\mu_0}\int x_i B_j \partial_j B_i dV = \frac{1}{\mu_0}\int \partial_j (x_i B_j B_i) dV - \frac{1}{\mu_0}\int B_i \partial_j (x_i B_j) dV \tag{17}$$

$$\Rightarrow \frac{1}{\mu_0}\int x_i B_j \partial_j B_i dV = \frac{1}{\mu_0}\int \partial_j (x_i B_j B_i) dV - \frac{1}{\mu_0}\int B_i B_i dV \tag{18}$$

$$\Rightarrow \frac{1}{\mu_0}\int (\vec{r} \cdot \vec{B}) \vec{B} \cdot \vec{n} dV = \frac{1}{\mu_0}\int \partial_j (x_i B_j B_i) dV - 2M_B \tag{19}$$

The fourth term is:

$$\frac{1}{2\mu_0}\int x_i \partial_i (B_j B_j) dV = \frac{1}{2\mu_0}\int \partial_i (x_i B_j B_j) dV + \frac{1}{2\mu_0}\int (\partial_i x_i) B_j B_j dV \tag{20}$$

$$\Rightarrow \frac{1}{2\mu_0}\int x_i \partial_i (B_j B_j) dV = \frac{1}{2\mu_0}\int B^2 \vec{r} \cdot \vec{n} dV + 3M_B \tag{21}$$

Combining all our results we can now state the Virial Theorem.

$$\frac{1}{2}\ddot{I} = 2T + 2U + \Omega + M_B - \int_V \left(P + \frac{1}{\mu_0} B^2\right) \vec{r} \cdot \vec{n} dV + \frac{1}{\mu_0}\int_V (\vec{r} \cdot \vec{B}) \vec{B} \cdot \vec{n} dV \tag{22}$$

It is usual also to ignore the two last surface integrals on the RHS as these involve effects of external pressure. We can safely ignore these terms for strongly self-gravitating objects. This is definitely justified for dense cores but may not be ok to do on a more global scale across the entire cloud complex. Furthermore if the clouds are stable (i.e. in a force balance) for large times we can ignore the LHS also. With these terms ignored we have our working Virial Theorem:

$$\boxed{2T + 2U + \Omega + M_B = 0} \tag{23}$$

i.e. for stability Ω must be balanced by the three other terms. We now investigate the relative importance of these three support mechanisms ... EXCERPT ENDS