Virial Theorem #1
Evan Ó Catháin

github.com/evanocathain/Virial_Theorem

The Virial Theorem applies to any stable system of 'particles' bound by conservative forces

$\underline{\nabla} \times \underline{F} = 0 \Rightarrow \underline{F} = -\underline{\nabla} V$, e.g. gravity

particles, fluid elements, stars in a galaxy, galaxies in a cluster, ...

Hydrostatic Equilibrium

General Form:

cf. Euler Equations, Navier-Stokes Equations

$$\rho \frac{D\underline{u}}{Dt} = -\underline{\nabla} P - \rho \underline{\nabla} \Phi + (\underline{J} \times \underline{B})$$

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho (\underline{u} \cdot \underline{\nabla}) \underline{u}$$

If we $\int$ [this equation] $\cdot \hat{r}\, dV$ get the general Virial Theorem

$$\frac{1}{2}\ddot{I} - 2T = 2U + \Omega + M_B$$

$\frac{1}{2}\ddot{I}$: total rotational energy; $2T$: total kinetic energy; $2U$: total thermal (internal) energy; Ω: total grav. binding energy; M_B: total magnetic pressure

This is the general form of the Virial Theorem.
Different simplified versions of this hold in different scenarios.

In a star $\Rightarrow$ will shortly derive this from eqns. of stellar structure
$\Rightarrow$ in a frame where the star is not moving or rotating $\Rightarrow$ LHS $= 0$
& magnetic fields are negligible $\Rightarrow M_B = 0$
$\Rightarrow 2U + \Omega = 0$

In a galaxy (cluster) $\Rightarrow$ stable orbits $\Rightarrow \ddot{I} = 0$ & magnetic fields $\lesssim \mu G \Rightarrow M_B \to 0$
density of gas very small $\Rightarrow U = 0$
$\Rightarrow 2T + \Omega = 0$

In a molecular cloud $\Rightarrow$ conditions such that $\ddot{I} \approx U \approx 0$
$\Rightarrow 2T + \Omega + M_B = 0$
$\Rightarrow$ if $-\Omega > 2T + M_B \Rightarrow$ cloud collapses

If $M_{Cloud} > M_{Jeans}$ star formation happens. Magnetic fields slow the collapse & set the scale for stars to be $\sim 0.1\,M_\odot$. Without them would be $\sim$ few $M_{2\!\!+}$ & no H burning.

Virial Theorem #2
Evan Ó Catháin

NB T, U terms are often conflated, perhaps because thermal energy is itself also a form of kinetic energy

$$T = \text{kinetic energy} = \int dm\, v^2$$ the velocity of the 'particle'

$$U = \text{thermal energy} = \int u\, dV = \frac{3}{2}\int NkT\, dV$$

[ideal gas]

Temperature related to kinetic energy of constituent 'atoms' making up 'particle'

In a star the derivation of the Virial Theorem goes as follows

$$① \quad \frac{dP}{dr} = -\frac{G\rho(r)m(r)}{r^2} \qquad ② \quad \frac{dm}{dr} = 4\pi r^2\rho(r)$$

$$\frac{①}{②} \times 4\pi r^3 \Rightarrow 4\pi r^3 \frac{dP}{dr}\frac{dr}{dm} = -\frac{G\rho(r)m(r)}{r^2}\frac{4\pi r^3}{4\pi r^2\rho(r)}$$

$$\Rightarrow 4\pi r^3\, dP = -\frac{Gm(r)\,dm}{r}$$

$$\Rightarrow 3\int_{\text{centre}}^{\text{surface}} V\, dP = \underbrace{-\int_{\text{centre}}^{\text{surface}} \frac{G\,m(r)\,dm}{r}}_{\text{grav. binding energy} \equiv \Omega}$$

$$\Rightarrow 3\left([VP]_{\underset{V=0}{\text{centre}}}^{\overset{P=0}{\text{surface}}} - \int_{\text{centre}}^{\text{surface}} P\,dV \right) = \Omega$$

$$\Rightarrow 3\int P\,dV + \Omega = 0$$

$$P = P_{gas} + P_{rad} = NkT + \frac{a}{3}T^4$$

if gas pressure dominates, often the case for many MS stars, e.g. the Sun

$$\Rightarrow 2U + \Omega = 0$$

What can we learn from this?

Example 1: Could the sun be powered by gravitational contraction or cooling?

$$\Omega = -\int_0^{M_\odot} \frac{G\,m(r)\,dm}{r}$$

Assume $\rho(r) = \rho_0 = \text{constant}$

$$\Rightarrow \Omega = -\frac{3}{5}\frac{GM_*^2}{R_*}$$

For the Sun $\Rightarrow \Omega \approx 2 \times 10^{41}\,J$

$L_\odot = \dfrac{E}{\text{time}}$ $\Rightarrow$ if powered by contraction $\Rightarrow$ Age of Sun $\approx \dfrac{2\times 10^{41}\,J}{4\times 10^{26}\,J/s} \approx 20\,Myr$

$\Omega = -2U \Rightarrow$ if powered by cooling $\Rightarrow$ Age of Sun $\approx 10\,Myr$

Virial Theorem #3
Evan Ó Catháin

Age of Sun ⇒ when this 'age' was first calculated using the Virial Theorem it was already known to be << age of the Earth and so it was clear that another power source was in action, which turns out to be nuclear burning.

Example 2: How hot are stars?

$$3\int P\,dV = -\Omega$$

$$P = nkT = kT\frac{\rho}{\langle m\rangle}, \quad dm = \rho\, dV \quad \Rightarrow \frac{3k}{\langle m\rangle}\int T\,dm = \int \frac{G\,m(r)\,dm}{r}$$

ave mass of a particle

For a star with radius R_*, integral over r from 0 to R_s
Inside star $r < R_s \Rightarrow \frac{1}{r} > \frac{1}{R_s}$

$$\Rightarrow \frac{3k}{\langle m\rangle}\int T\,dm > \frac{G}{R_s}\int_0^{M_*} m\,dm$$

$$\Rightarrow \frac{3k}{\langle m\rangle}\langle T\rangle M_* > \frac{G}{R_s}\frac{M_*^2}{2} \Rightarrow \langle T\rangle > \frac{G\langle m\rangle M_*}{6kR_s}$$

Sun mostly made of H, $m = 1.67\times10^{-27}$ kg $\Rightarrow \langle T\rangle_\odot \gtrsim 3\times10^6$ K

Example 3: How does the Virial Theorem indicate the existence of dark matter?

In a galaxy, or galaxy cluster: $2T + \Omega = 0$
From spectral line observations can obtain velocities from Doppler shift values
⇒ gives 1-D velocities $\sigma = v_{1D}$. But for spherical symmetry we have $v_{3D}^2 = 3\sigma^2$

$$\Rightarrow 2T = m_* 3\sigma^2 \text{ for each 'particle' of mass } m_*$$

$$\Rightarrow m_*(3\sigma^2) = \underbrace{\frac{3}{5}\frac{GMm_*}{R}}$$

assuming uniform density for simplicity

$$\Rightarrow M = \frac{5\sigma^2 R}{G}$$

For MW, $\sigma \approx 100$ km/s, $R \approx 100$ kpc $\Rightarrow M_{100} \approx 10^{12} M_\odot$
But stellar mass, based on light emitted (recall $L \propto M^{\sim3.5}$ on Main Sequence) is $M_{stellar} \approx 10^{11} M_\odot$ ⇒ Dark Matter