Rental Harmony With an Understanding of Inequality

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1 Introduction and Related Works

In recent years, the field of Economics and Computation has begun to investigate standard economic models by challenging conventional assumptions. This paper focuses on a model that deviates from the traditional zero-sum transfer of utility assumption, reimagining money as a divisible good with marginal values. By doing so, this approach incorporates a multiplicative weight to the prices in utility, letting us think about the classic fair division problem “Rental Harmony” in a novel manner.

The significance of this research is underscored by the escalating issue of income inequality in the United States and globally. In numerous roommate situations, individuals often come from diverse wealth backgrounds and have disparate income levels. Additionally, rental rates are soaring domestically. By applying this relatively novel economic model to the Rental Harmony problem, we aim to address the inherent inequities present in classical solutions.

The Rental Harmony problem is a well studied problem, both from an ordinal [2] and an cardinal perspective [6] [1]. Recent work has also begun to consider more intuitively fair envy-free solutions like min-max [3]. This work seeks to expand upon these results by combining these problem statements with a more complicated model of utility, that being the welfare weight utility model. This model was initially introduced in Weitzman's seminal paper in 1977 [8]. Examples of its expansion can be found in Dworczak, Kominers, Akbarpour '21 paper [4] in which they explored standard 2 sided buyer seller markets, as well as in Saez, Stantcheva, '16 paper [5] which looked at optimal tax theory.

2 Model

Our model consists of the following setting. We have a set of agents N indexed by the variable i who need to be assigned to a set of rooms M indexed by the variable j. Each agent has a non-negative valuation for each room, v_{ij}, such that $\sum_r v_{ij} = R$, where R is the total rent. Each agent also has a "welfare weight", w_i, which is the marginal value for a dollar. As mentioned before, this can be thought of as each agent's marginal utility for a dollar. Given this setting, a solution consists of a bijection between agents and rooms σ and a price vector $\vec{p}$ where $\sigma(i)$ is the room assigned to agent i and p_j is the price paid for room r, with $\sum_r p_j = R$ and $p_j \geq 0$. Given a particular solution, each agent has utility $v_{i\sigma(i)} - w_i p_{\sigma(i)}$.

Under our model, we hope to find solutions that satisfy non-negative and envy-free. These definitions are formalized below.

Definition 2.1. *A solution* $(\sigma, \vec{p})$ *is non-negative if and only if* $\forall r \in M : p_r \geq 0$

Definition 2.2. *A solution* $(\sigma, \vec{p})$ *is envy-free if and only if* $\forall i, k \in N : v_{i\sigma(i)} - w_i p_{\sigma(i)} \geq v_{k\sigma(k)} - w_i p_{\sigma(k)}$

3 Classification

We now begin our theoretical analysis of this model, and look at classifying when a non-negative (NN) and envy-free (EF) solution exists and when it does not exist within our model. Before discussing the results, we define a notion that will be critical in our analysis of providing a classification, the idea of the *required set*.

Definition 3.1. *For each agent* i *we define a required set* $\mathcal{R}_i$ *as the set of rooms that, if allocated to agent* i*, there exists some price vector* $\vec{p}$ *such that agent* i *will not envy another other agent.*

In other words, in an EF and NN solution, agent i must be assigned a room from $\mathcal{R}_i$. To help intuition, we provide an example. Let $R = 20$, $v_1 = [10, 5, 3, 2]$, and $w_1 = \frac{1}{4}$. Notice that if we allocate room 4 to this agent, we must find prices that satisfy the following inequality: $2 - \frac{p_4}{4} \geq 10 - \frac{p_1}{4}$. Note that since we have a non-negativity constraint we also have $p_4 + p_1 \leq 20$. Combining these two inequalities, we find that $p_4 + 32 \leq p_1$ and $p_1 \leq 20 - p_4$, which are obviously satisfiable. Thus, no matter how we determine prices, if we allocate room 4 to this agent, we can never be envy-free. One can then verify that for this agent, their required set is $= \{10, 5\}$.

With this definition in place, we now can provide a negative classification, or if one takes the negation, a necessary condition for a NN and EF solution to exist.

Theorem 3.2. *If* $\exists S \subseteq M$*, such that* $|\bigcup_{i \in S} \mathcal{R}_i| < |S|$*, then no NN and EF solution exists.*

This is true by the definition of required sets. Since there is some room in S that is not in any agent's required set, it is impossible to give every agent a room in their required set since σ is a bijection. Therefore under this condition is is impossible for an EF solution to exist.

On the other hand, there are situations in which a NN and EF solution does exist. To prove this, we build upon the work in [7], which looks at dividing up a divisible good, and an indivisible good. More precisely, he proves that if agents preferences over the divisible, when given a specific indivisible good, are continuous and monotonic, and for all indivisible items there is some distribution such that the good is the preferred good then there exists an EF solution. Applying our setting to this general framework, we can see that the room prices correspond to the divisible good, and the individual rooms are the indivisible goods. Within this framework, the continuous property and monotonic property are obviously satisfied, as long as one considers the good to be the amount of rent not paid. This then only leaves the third property, which directly relates to our definition of a required set. That being

Corollary 3.3. *If for all $i \in N$, $|\mathcal{R}_i| = |M|$, then we have a NN and EF solution.*

While this does count as a classification, we seek to better contextualize this condition within our setting. To that end we provide this more specific condition which is equivalent to the condition given above. That being:

Theorem 3.4. *If for all $i \in N$, $w_i \geq 1 - (v_{il^*} \cdot n)/R$, where v_{il^*} represents the lowest valued good by the agent, then there exists a NN and EF solution.*

To prove this, we first make an important observation. That being that if can find a price vector $\vec{p}$ such that the lowest valued good is weakly preferred by agent i, then we can find a price vector that does so for all other goods. To see this, notice that agent i's value for every other good is at least v_{il^*}, thus we can obviously find a price vector that will make any other good the preferred good. This is the definition for inclusion in an agent's required set, so we have proven that the existence of some $\vec{p}$ where an agent weakly prefers its lowest valued good is equivalent to $|\mathcal{R}_i| = |M|$.

With this in mind, we only need to find a sufficient condition for having an agent weakly prefer their bottom valued good. To this, we must have the following inequalities satisfied:

$$
\begin{align}
v_{i1} - p_1 w_i &\leq v_{il^*} \tag{1}\\
v_{i2} - p_1 w_i &\leq v_{il^*} \tag{2}\\
v_{i3} - p_2 w_i &\leq v_{il^*} \tag{3}\\
&\dots \tag{4}
\end{align}
$$

Combining these inequalities we get

$$
\begin{align}
\sum_{j \in M \setminus l^*} v_{ij} - R^* w_i &\leq v_{il^*}{}^*(n-1) \tag{5}\\
R - v_z - R^* w_i &\leq v_{il^*}(n-1) \tag{6}\\
R(1 - w_i) &\leq v_{il^*}(n) \tag{7}\\
w_i &\geq 1 - \frac{v_{il^*} \cdot n}{R} \tag{8}
\end{align}
$$

One can also reverse the logic to show that these conditions are equivalent. □

4 Computation

As noted in the introduction, we can adapt previous work on rental harmony within this setting. More specifically, we modify the algorithm described in [6] to fit within our setting. At a high level, we can treat the welfare weights into scalars on the valuation profile, then optimize over that.

Theorem 4.1. *Given a welfare weighted rent division problem V, a NN and EF solution $(\sigma, \vec{p})$ can be computed in polynomial time, when such a solution exists.*

Before providing our algorithm, we first provide two proofs of lemmas that will be critical in proving the correctness of our algorithm. As note above these proofs are adapted from the proofs provided in [6]. Our first lemma requires the following definition.

Definition 4.2. *An allocation σ is welfare-weighted maximizing if* $\sigma \in \arg\max_\sigma \sum_{i \in N} \frac{v_{i\sigma(i)}}{w_i}$

We note that we will also define this be "optimal" within the context of the problem. With this defined, we can now turn to our first lemma.

Lemma 4.3. *If $(\sigma, \vec{p})$ is an envy-free solution, then the assignment σ is an welfare-weighted maximizing allocation.*

Proof: Let $(\sigma, \vec{p})$ be an be an envy-free allocation. From the definition of envy-free, we have for each agent and room assignment π:

$$v_{i\sigma(i)} - w_i p_{\sigma(i)} \geq v_{i\pi(i)} - w_i p_{\pi(i)} \tag{9}$$

We then divide by the welfare weights, combine the inequalities, and cancel like terms to arrive at:

$$\begin{aligned} v_{i\sigma(i)} - w_i p_{\sigma(i)} &\geq v_{i\sigma(i)} - w_i p_{\sigma(i)} && \text{(for every } i \in N) \\ \frac{v_{i\sigma(i)}}{w_i} - p_{\sigma(i)} &\geq \frac{v_{i\pi(i)}}{w_i} - p_{\pi(i)} && \text{(for every } i \in N) \\ \sum_{i \in N} \frac{v_{i\sigma(i)}}{w_i} - R &\geq \sum_{i \in N} \frac{v_{i\pi(i)}}{w_i} - R && (10) \\ \sum_{i \in N} \frac{v_{i\sigma(i)}}{w_i} &\geq \sum_{i \in N} \frac{v_{i\pi(i)}}{w_i} && (11) \end{aligned}$$

Which directly implies welfare-weighted maximizing.

Lemma 4.4. *If $(\sigma, \vec{p})$ is an envy-free allocation, and σ' is a welfare-weighted maximizing assignment, then $(\sigma', \vec{p})$ is also an envy-free solution.*

Proof: Let $(\sigma, \vec{p})$ be an envy-free solution and let σ' be some other welfare-weighted maximizing assignment. From lemma 5.3, we know that σ must also be another assignment that maximizes the weighted welfare. As such we have:

$$\sum_{i \in N} \frac{v_{i\sigma(i)}}{w_i} = \sum_{i \in N} \frac{v_{i\sigma'(i)}}{w_i}$$

Thus, since the total rent also sums to R we have that:

$$\sum_{i \in N} (\frac{v_{i\sigma(i)}}{w_i} - p_{\sigma(i)}) = \sum_{i \in N} (\frac{v_{i\sigma'(i)}}{w_i} - p_{\sigma'(i)}) \tag{12}$$

We also note that $(\sigma . \vec{p})$ is envy-free and thus for all agents i

$$v_{i\sigma(i)} - w_i p_{\sigma(i)} \geq v_{i\sigma'(i)} - w_i p_{\sigma'(i)} \tag{13}$$

Which per above is equivalent to:

$$\frac{v_{i\sigma(i)}}{w_i} - p_{\sigma(i)} \geq \frac{v_{i\sigma'(i)}}{w_i} - p_{\sigma'(i)}$$

We then conclude by noting that due to (3), these inequalities all must be equalities giving us the final inequality, with π being some other assignment:

$$v_{i\sigma(i)} - w_i p_{\sigma(i)} = v_{i\sigma'(i)} - w_i p_{\sigma'(i)} \geq v_{i\pi(i)} - w_i p_{\pi(i)} \tag{14}$$

It then follows directly from this lemma that if an envy-free solution exists $(\sigma, \vec{p})$, then for any arbitrary welfare-weighted maximum allocation σ', we can solve a linear program and find an envy-free solution. Thus we can treat the problems separately. Finding a maximum weighted bipartite matching can be solved in a number of ways, including the Hungarian algorithm which is polynomial in its run time. Additionally, this system of linear inequalities can be solved efficiently. Thus we have proven the theorem.

5 Performance and Comparison to Standard Quasi-linear Utilities

We now turn to looking at the performance of our algorithm and model, from two different perspectives. That being how the utility guarantees compare with respect to individual agents in the, and how the overall system compares. We start with looking at the performance of the system overall.

5.1 Performance of the system

However, before such analysis can begin, we must first ask the question what is appropriate benchmark to use?

Scholars familiar with the envy-free rent division will recall that, in the standard quasi-linear utility model, an envy-free allocation produces what is essentially a 1st welfare theorem. That is, that the allocation produced by any envy-free allocation algorithm in the standard quasi-linear utility model will induce what is a welfare maximizing allocation. Formally this can be written as

$$\sigma \in \arg\max_{\sigma} \sum_{i \in N} v_{i\sigma(i)} \tag{15}$$

Note that this a departure from what our allocation must satisfy as discussed in the previous section. As any envy-free allocation in our welfare weights utility model must satisfy (1).

Our reasoning for choosing the welfare-maximizing allocation as our benchmark is two fold. For one, we feel that it captures the loss in utility that is caused by the inequality, as if all the welfare weights are equivalent, then our welfare-weighted maximizing allocation then becomes just a welfare maximizing allocation. Thus, using this as our benchmark, we can begin to understand how inequality can affect the outcomes in fair division. The second is that it can also be interpreted as the cost of enforcing the envy-free constraint. As if we simply cared about creating the most "efficient" outcome we would simply take the welfare maximizing allocation, which we would likely not be able to satisfy envy-freeness (given a large enough variance in the welfare weights).

With this as our benchmark we now turn to measuring the performance of an envy-free solution within the welfare weights model. We note that these bounds are not of the form we wish, as they are only with respect to a single agents welfare weight. These bounds were produced with the idea of understanding the full structure of the problem, which is a structure we have yet to crack. However, we do feel that these bounds do provide insight to the performance benchmark and as such we include them.

Theorem 5.1. *The welfare-weighted allocation, with respect to a single agents increase in welfare weight, still maintains at least a $\frac{1}{3}$ approximation of the total welfare we could achieve without inequality or envy-freeness.*

First, we formalize our benchmark, specifically we want a lower bound on the following:

$$\frac{\sum_{i \in N} v_{\sigma(i)}}{\sum_{i \in N} v_{i\pi(i)}} \tag{16}$$

where σ is some welfare-weighted maximizing allocation, and π is a welfare maximizing allocation. Note that because we are looking at only a single divergence, all but one agents welfare weight, say agent s, will be equal to 1. We then lower bound the performance of σ, by the performance of the optimal matching without agent s. We call this matching σ'. Note that we are not removing any rooms from the allocation, only agent s. Notice that this provides a lower bound, as every other agent has a welfare weight of 1. This then gives us

$$\frac{\sum_{i \in N} v_{\sigma(i)}}{\sum_{i \in N} v_{i\pi(i)}} \geq \frac{\sum_{i \in N \setminus s} v_{\sigma'(i)}}{\sum_{i \in N} v_{i\pi(i)}} \tag{17}$$

We then claim that this lower bound guarantees us a 1/3-approximation ratio. Which we show through the following sufficient statements.

$$3 \cdot \sum_{i \in N \setminus s} v_{\sigma'(i)} \geq \sum_{i \in N} v_{i\pi(i)} \tag{18}$$

$$2 \cdot \sum_{i \in N \setminus s} v_{\sigma'(i)} \geq v_{s\pi(s)} \tag{19}$$

Where (19) follows from σ' being optimal for every agent other than s. We then prove the second sufficient condition, by using the following lemma, with the proof of it provided in the appendix.

Lemma 5.2. *Given a subset of agents $S \subseteq N$ whose value profiles v_i satisfy $\sum_j v_{ij} = R$ for all $i \in S$, it will always be possible to find an allocation with only those agents with value equal to $|S| \cdot \frac{R}{n}$.*

The minimum value for this ends up being when $n = 2$ and $S = 1$, with the highest value for a single agent being R. Plugging in to the inequality gives us:

$$2\frac{R}{2} \geq R \tag{20}$$

$$R \geq R \tag{21}$$

Which completes the proof of Theorem 5.1 We also note, that in the limit as of n, this approximation ratio approaches $\frac{1}{2}$. □

To conclude this section, we provide a quick example to show that this lower-bounding scheme is tight, and a brief discussion on it. Let the value profile of agent 1 be $v_1 = [R, 0]$ and the value profile of agent 2 be $v_2 = [\frac{R}{2}, \frac{R}{2}]$. Then the best we can hope for with this lower-bounding scheme is a 1/3-approximation. Additionally, we note that this lower-bounding scheme does not take into account the welfare weights of the individual agent, and thus a closer analysis based on the welfare-weights may achieve a better bound. However, in the limit of the welfare weight, it should converge to our given bound.

5.2 Utility Guarantees

To conclude our performance analysis, we examine the utility guarantees in the two different models. Specifically, we observe that in the standard quasi-linear utility model, envy-freeness combined with values summing to R ensures that no agent receives negative utility. To demonstrate this, we present a brief proof by contradiction.

Suppose that in some quasi-linear utility envy-free solution $(\sigma, \vec{p})$, agent i receives a negative allocation. Consequently, $v_{i\sigma(i)} \leq p_{\sigma(i)}$. However, given that $\sum_j v_{ij} = \sum_j p_j$, there must exist another allocation with positive utility, which agent i would be envious of, leading to a contradiction. (While we rediscovered this proof during the writing of this paper, we acknowledge that it is certainly not original; however, we lack the proper citation for its initial introduction).

In contrast to the quasi-linear utility model, the welfare-weights model does not exhibit the same guarantee. To illustrate this, consider two agents who value both rooms equally and assign a welfare weight significantly higher than the total rent. In future work, we plan to investigate the utilities of solutions in the quasi-linear model when applied to the welfare-weights model.

6 Implementation

The implementation and code supporting our project are accessible through the public GitHub repository found at the following link: `https://github.com/evanphoward/CS238`. Below is a concise overview of the key components implemented in this paper.

Our primary contribution is an algorithm that identifies an EF and NN solution when one exists and terminates gracefully when no such solution is available. This algorithm initially employs the NetworkX library to discover a room assignment σ that maximizes the value $\sum_i \frac{v_{i\sigma(i)}}{w_i}$, which can be reduced to a maximum weight matching problem. Following this, the assignment is used to formulate a system of linear inequalities, the solution of which yields a price vector $\vec{p}$ that satisfies both non-negativity and envy-freeness. The SciPy library is utilized to solve this system through linear programming. In cases where a solution is unattainable, the linear programming library call fails gracefully, allowing our algorithm to output an error message.

In addition to the primary algorithm, we have implemented a variety of supplementary tools that aided in our theoretical analysis and could prove valuable for future research in this setting. These tools encompass an envy-freeness check for a given setup and solution, an evaluation of the sufficient condition we derived, and functions that plot the existence of EF and NN solutions across diverse welfare weights in 2 and 3 agent cases. See figures 1 - 6 in the appendix for examples of these plots. Although we do lack a full classification for when EF and NN solutions exist, these tools can be used to verify when such a solution does exist in polynomial time.

7 Conclusion & Open Problems

In conclusion, this research presents a novel rent division framework that advances our understanding of the problem and contributes to the field in several significant ways. First, we establish and prove a sufficient condition and a necessary condition for the existence of EF and NN solutions. We then adapt and prove a poly-time algorithm for finding such solutions. We then begin to analyze the loss in welfare from becoming inequality aware, as well as provide an efficient implementation of the algorithm provided.

In future work, we hope to dive into the numerous open problems still exist within this framework, as well as in closely related settings. Potential avenues for this include fully classifying the conditions under which EF and NN solutions exist, and implementing a Minimax algorithm that adheres to our solution constraints. Additionally, expanding the framework to encompass more complex scenarios, such as a roommate setting (where multiple agents share a room) or non-constant welfare weights (where the weights vary based on factors such as price-to-budget constraints), could provide valuable insights and advancements in the field. Finally, we would like to develop better bounds on the loss in welfare when inequality is high, as well as better understand how the utilities compare across these different models.

By addressing these open problems and exploring related settings, we can build on the foundation established in this research to develop a more comprehensive understanding of rent division and its broader implications. Our hope is that these findings inspire further investigation and collaboration, ultimately leading to more efficient and equitable rent allocation solutions.

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A Proof of Missing Lemma

Here we provide a proof of lemma 6.2

Lemma A.1. *Given a subset of agents $S \subseteq N$ who's value profiles v_i satisfy $\sum_j v_{ij} = R$, it will always be possible to find an allocation with only those agents with value equal to $|S| \cdot \frac{R}{n}$.*

To prove this, we provide a simple greedy algorithm that guaranties the performance. Our algorithm works in the following way

1. Match the item and agent with the highest v_{ij}.

2. Repeat this process until all agents have been matched.

To see that this produces an $|S| \cdot \frac{R}{n}$, we give a simple proof by induction, on the the fact that if we have matched s agent's, then we are guarantied atleast $s \cdot \frac{R}{n}$. For the base case, this is obvious as for every agent their must be atleast one item they value at least $\frac{R}{n}$, per the pigeon hole principle. Then, for the induction, notice that there must always be some agent, who has a valuation on an unmatched item that will bring us above our required total. To see this, notice that since they did not match to any item previously, they must have enough value in other items as otherwise they would have been picked already.

B Figures

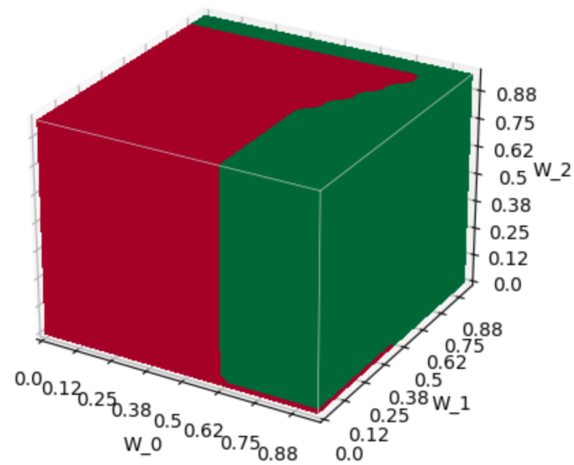

Figure 1:

	V_0	V_1	V_2
Agent 0	7.89	1.02	1.09
Agent 1	7.42	2.33	0.25
Agent 2	4.79	3.93	1.28

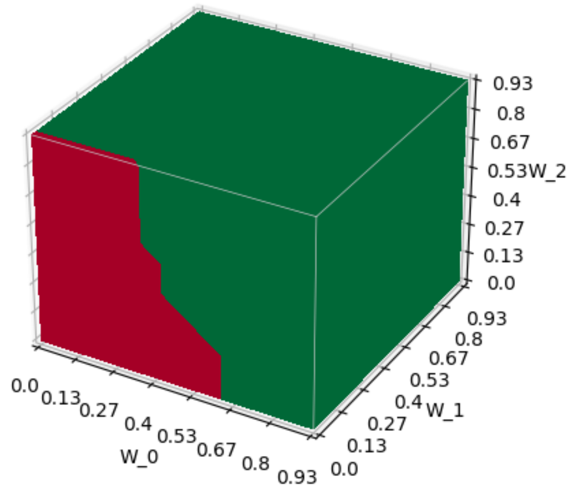

Figure 2:

	V_0	V_1	V_2
Agent 0	6.21	2.69	1.10
Agent 1	5.39	0.22	4.39
Agent 2	3.73	3.51	2.76

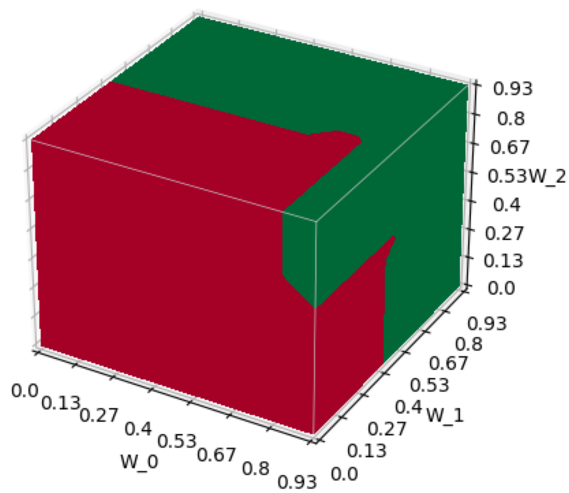

Figure 3:

	V_0	V_1	V_2
Agent 0	8.36	1.33	0.31
Agent 1	6.99	1.09	1.92
Agent 2	7.57	2.33	0.10

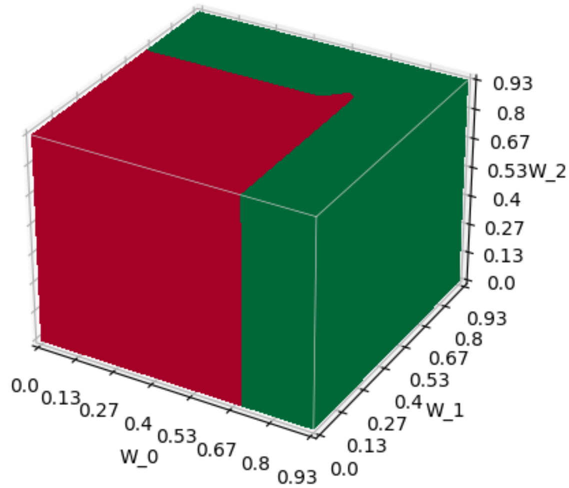

Figure 4:

	V_0	V_1	V_2
Agent 0	1.67	0.82	7.50
Agent 1	0.66	1.32	8.02
Agent 2	4.82	3.71	1.47

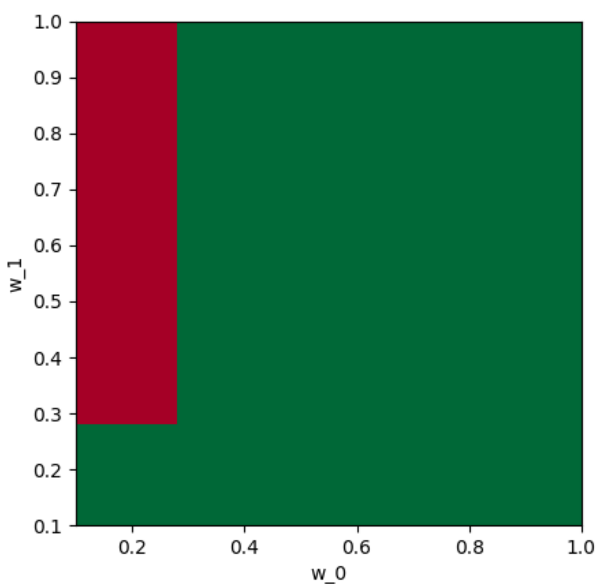

Figure 5:

	V_0	V_1
Agent 0	0.53	9.47
Agent 1	3.94	6.06

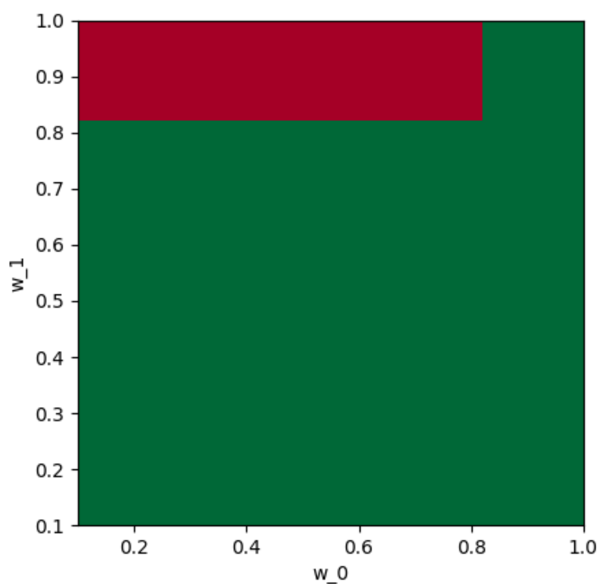

Figure 6:

	V_0	V_1
Agent 0	6.42	3.58
Agent 1	9.41	0.59