Translation & Implication

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Mechanism design / contract theory is the design of incentive structures.

This requires:

- explaining rules/game/outcomes to agents
- interpreting the messages of the agents

Economists like to model uncertainty via state-spaces

Real communication (contracts, messages, etc) regards state**ments** not states

- This is dependent of language of the decision maker
- Awareness complicates the picture
- But, what if agents speak different languages or are differentially aware?

This paper is a theory of translation

- How agents' language represents their understanding of uncertainty
- How this might be communicated between agents
- When can different agent's views be unified into a universal perspective

Model: Languages

A language is a bounded distributive algebra $\mathcal{L}$ representing sets of statements that can be true or false:

- $\mathtt{T}$ and $\mathtt{F}$ are distinguished elements
- closed under disjunction $\vee$
    - if λ and η are statements in $\mathcal{L}$, then so is $\lambda \vee \eta$
    - interpreted as OR; at least one of λ and η is true
- closed under conjunction $\wedge$
    - if λ and η are statements in $\mathcal{L}$, then so is $\lambda \wedge \eta$
    - interpreted as AND; both λ and η are true

Formally, $\mathcal{L}$ must satisfy the following properties:

- **Commutativity**
  - $\lambda \vee \eta = \eta \vee \lambda$
  - $\lambda \wedge \eta = \eta \wedge \lambda$
- **Associativity**
  - $(\lambda \vee \eta) \vee \mu = \lambda \vee (\eta \vee \mu)$
  - $(\lambda \wedge \eta) \wedge \mu = \lambda \wedge (\eta \wedge \mu)$
- **Absorption**
  - $\lambda \vee (\lambda \wedge \eta) = \lambda$
  - $\lambda \wedge (\lambda \vee \eta) = \lambda$
- **Bounds**
  - $\lambda \vee \mathtt{F} = \lambda$
  - $\lambda \wedge \mathtt{T} = \lambda$
- **Distributivity**
  - $\lambda \wedge (\eta \vee \mu) = (\lambda \wedge \eta) \vee (\lambda \wedge \mu)$
  - $\lambda \vee (\eta \wedge \mu) = (\lambda \vee \eta) \wedge (\lambda \vee \mu)$

The operations $\vee$ and $\wedge$ induce a natural partial order on $\mathcal{L}$

$$\lambda \Rightarrow_{\mathcal{L}} \eta \quad \text{iff} \quad \lambda \wedge \eta = \lambda \qquad (\text{iff} \quad \lambda \vee \eta = \eta)$$

- captures implication: whenever λ is true, η is also true
- $\Rightarrow_{\mathcal{L}}$ is reflexive, antisymmetric, and transitive
- $\mathtt{F} \Rightarrow_{\mathcal{L}} \lambda \Rightarrow_{\mathcal{L}} \mathtt{T}$ for all $\lambda \in \mathcal{L}$

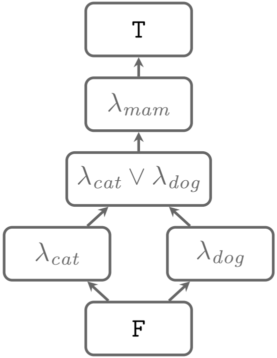
Example

$\mathcal{L}$ is constructed from the primitive statements

- λ_{cat} = "*Martin* is a cat"
- λ_{dog} = "*Martin* is a dog"
- λ_{mam} = "*Martin* is a mammal"

under the axioms / presumption that:

- $(\lambda_{cat} \wedge \lambda_{dog}) \Rightarrow_{\mathcal{L}} \mathtt{F}$
  - It is not possible to be a dog and a cat
- $(\lambda_{cat} \vee \lambda_{dog}) \Rightarrow_{\mathcal{L}} \lambda_{mam}$
  - If Martin is a dog or is a cat, he is a mammal



A truth assignment for $\mathcal{L}$ is a function

$$w : \mathcal{L} \to \{0, 1\}$$

such that for all $\lambda, \eta \in L$:

- $w(\mathtt{T}) = 1$ and $w(\mathtt{F}) = 0$
- $w(\lambda \vee \eta) = \max\{w(\lambda), w(\eta)\}$
- $w(\lambda \wedge \eta) = \min\{w(\lambda), w(\eta)\}$

Truth assignments preserve the implication ordering:

$$
\begin{aligned}
\lambda \Rightarrow_{\mathcal{L}} \eta \qquad & \text{iff} \qquad \lambda = \lambda \wedge \eta \\
& \text{iff} \qquad w(\lambda) = \min\{w(\lambda), w(\eta)\} \\
& \text{iff} \qquad w(\lambda) \leq w(\eta)
\end{aligned}
$$

Let $W(\mathcal{L})$ is the set of truth assignments for $\mathcal{L}$; $W(\mathcal{L})$ is a state-space for $\mathcal{L}$:

- The event corresponding to λ is $\mathbf{E}(\lambda) = \{w \in W \mid w(\lambda) = 1\}$
- Implication is containment: $\lambda \Rightarrow \eta$ if and only if $\mathbf{E}(\lambda) \subseteq \mathbf{E}(\eta)$
- Tautology ($\mathtt{T}$) maps to entire state-space, Contradiction ($\mathtt{F}$) to empty-set
- $\mathbf{E}(\lambda \vee \eta) = \mathbf{E}(\lambda) \cup \mathbf{E}(\eta), \qquad \mathbf{E}(\lambda \wedge \eta) = \mathbf{E}(\lambda) \cap \mathbf{E}(\eta)$

Example

$\lambda_{cat} = 1$	$\lambda_{dog} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$
$\lambda_{mam} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$
$\lambda_{dog} = 0$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$	$\lambda_{mam} = 0$

w_1^i	w_2^i	w_3^i	w_4

Example

$\lambda_{cat} = 1$	$\lambda_{dog} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$
$\lambda_{mam} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$
$\lambda_{dog} = 0$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$	$\lambda_{mam} = 0$
w_1^i	w_2^i	w_3^i	w_4

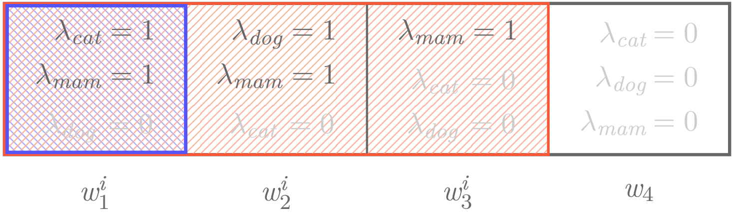
Event $\mathbf{E}(\lambda_{cat})$ = "Martin is a cat"

Example

w_1^i	w_2^i	w_3^i	w_4
$\lambda_{cat} = 1$	$\lambda_{dog} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$
$\lambda_{mam} = 1$	$\lambda_{mam} = 1$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$
$\lambda_{dog} = 0$	$\lambda_{cat} = 0$	$\lambda_{dog} = 0$	$\lambda_{mam} = 0$

Event $\mathbf{E}(\lambda_{mam})$ = “Martin is a mammal”

Example



$\mathbf{E}(\lambda_{cat}) \subseteq \mathbf{E}(\lambda_{mam})$, cat implies mammal

Model: Translation

There are two agents, 1 and 2, each endowed with a language $\mathcal{L}_1$ and $\mathcal{L}_2$

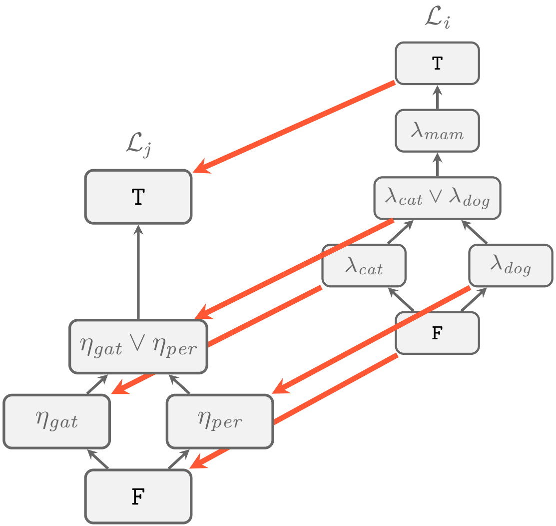
- let $\Rightarrow_i$ denote implication in i's language
- let $\mathtt{T}_i$ and $\mathtt{F}_i$ denote the tautology and contradiction
- A translation operator (from i to j) is a function from i's language to j's

Example

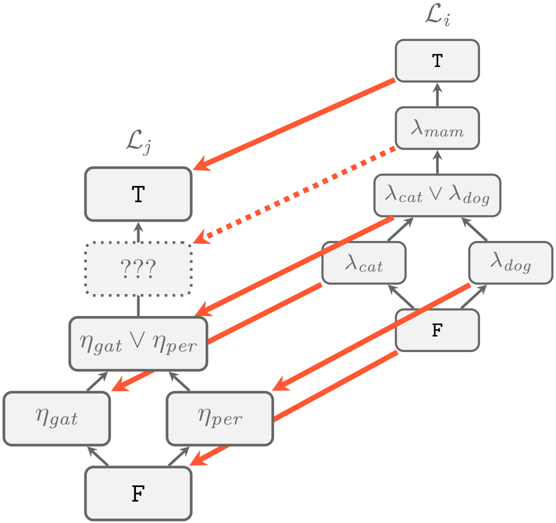
Consider a Spanish speaker, $\mathcal{L}_j$, who is never heard of ‘mammals’

- η_{gat} =“*Martin* es un gato”
- η_{per} =“*Martin* es un perro”

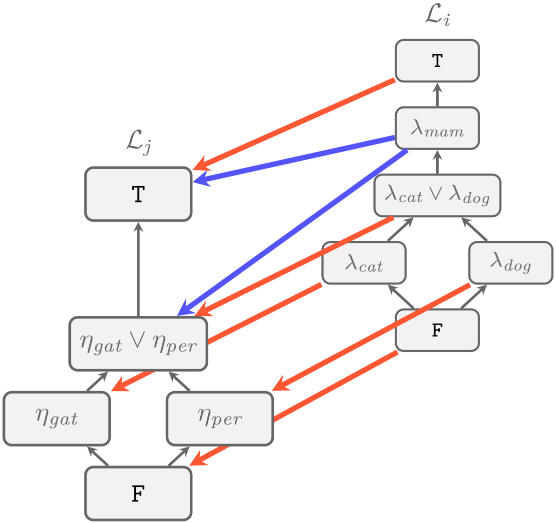
$\eta_{gat} = 1$	$\eta_{per} = 1$	$\eta_{gat} = 0$
$\eta_{per} = 0$	$\eta_{gat} = 0$	$\eta_{per} = 0$
$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_3$



$\mathcal{L}_i$
T
λ_{mam}
$\lambda_{cat} \vee \lambda_{dog}$
λ_{cat}
λ_{dog}
F
$\mathcal{L}_j$
T
$\eta_{gat} \vee \eta_{per}$
η_{gat}
η_{per}
F



$\mathcal{L}_i$
T
λ_{mam}
$\lambda_{cat} \vee \lambda_{dog}$
λ_{cat}
λ_{dog}
F
$\mathcal{L}_j$
T
???
$\eta_{gat} \vee \eta_{per}$
η_{gat}
η_{per}
F



$\mathcal{L}_i$
T
λ_{mam}
$\lambda_{cat} \vee \lambda_{dog}$
λ_{cat}
λ_{dog}
F
$\mathcal{L}_j$
T
$\eta_{gat} \vee \eta_{per}$
η_{gat}
η_{per}
F

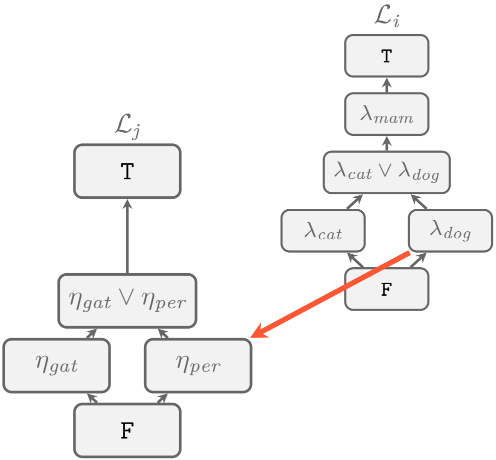
We consider two kinds of translations

- Inner Translation operator $\mathrm{T}^{-}_{i \to j}$
  - From ‘below’
  - Provides a *more specific* approximation
- Outer Translation operator $\mathrm{T}^{+}_{i \to j}$
  - From ‘above’
  - Provides a *more general* approximation

Translation of λ_{dog}

$$\mathsf{T}^{-}_{i\to j}(\lambda_{dog}) = \eta_{per} = \mathsf{T}^{+}_{i\to j}(\lambda_{dog})$$

- No gap between $\mathsf{T}^{-}_{i\to j}$ and $\mathsf{T}^{+}_{i\to j}$
- This is a 'perfect' translation



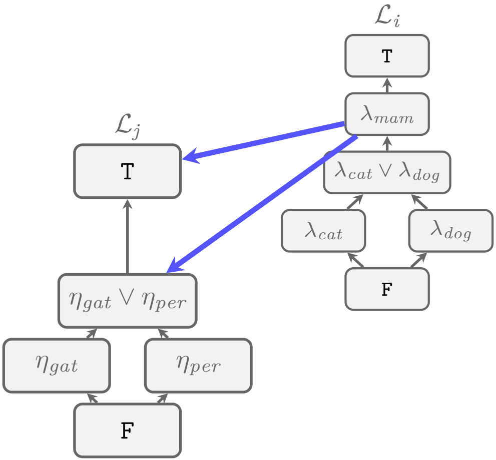
Translation of λ_{mam}

- j has never heard of a mammals,
- no statement captures it exactly
- All cats and dogs are mammals:

$$\mathsf{T}^{-}_{i\to j}(\lambda_{mam}) = (\eta_{gat} \vee \eta_{per})$$

- All mammals are something:

$$\mathsf{T}^{+}_{i\to j}(\lambda_{mam}) = \mathtt{T}_j$$



Consider a set of ‘translation’ operators

$$\mathsf{T} = \langle \mathsf{T}^{-}_{1\to 2}, \mathsf{T}^{+}_{1\to 2}, \mathsf{T}^{-}_{2\to 1}, \mathsf{T}^{+}_{2\to 1} \rangle$$

When does `T` behave like a translation?

- serves as the ‘best approximations’
- preserves logical structure

For a (complete) distributive lattice, 'best approximations' would mean

$$\mathsf{T}^{-}_{i\to j}(\lambda_i) = \bigvee\{\eta_j \in \mathcal{L}_j \mid \eta_j \text{ implies } \lambda_i\}$$

$$\mathsf{T}^{+}_{i\to j}(\lambda_i) = \bigwedge\{\eta_j \in \mathcal{L}_j \mid \lambda_i \text{ implies } \eta_j\}$$

Two issues:

1. What does it mean for η_j to imply λ_i?
2. Infinite joins / meets might not exist

Cross Language Implication

- j is trying to ascertain whether *perro* (dog) implies *mammal*.
  - j could point to various *perros*
  - i affirm that these are all also *mammals*
  - exhibits the implication holds
- now i is trying to ascertain whether *mammal* implies *perro*:
  - i could point to various *mammals*
  - when pointing at a *cat*, j can deny that it is a *mammal*
  - refutes the implication holds

Consider a binary relation $\Rightarrow^\star$ over $\mathcal{L}_i \cup \mathcal{L}_j$

- represents when one statement implies another, *across* languages
- this is, in principle, observable to some outside modeler (as above)
- $\Rightarrow^\star$ must satisfy some consistency properties to maintain the logic of distributive lattices

I1: Within Language Consistency

For all $\lambda_i, \lambda_i' \in \mathcal{L}_i$:

$$\lambda_i \Rightarrow^{\star} \lambda_i' \qquad \text{if and only if} \qquad \lambda_i \Rightarrow_i \lambda_i'.$$

I2: Transitivity

$\Rightarrow^\star$ is transitive

I3: Connective Consistency

Let $\eta_j, \eta_j' \in \mathcal{L}_j$ and $\lambda_i \in \mathcal{L}_i$. Then:

(i) $\lambda_i \Rightarrow^\star \eta_j$ and $\lambda_i \Rightarrow^\star \eta_j'$ implies $\lambda_i \Rightarrow^\star (\eta_j \wedge \eta_j')$

(ii) $\eta_j \Rightarrow^\star \lambda_i$ and $\eta_j' \Rightarrow^\star \lambda_i$ implies $(\eta_j \wedge \eta_j') \Rightarrow^\star \lambda_i$

$*$ if you have a complete lattice, you could ask for arbitrary meets/joins

I4: Principle of Explosion

For all $\lambda_j \in \mathcal{L}_j$,

$$f_i \Rightarrow^\star \lambda_j$$

Given a (complete) lattice, if $\Rightarrow^\star$ satisfies `I1-4`, then

$$\mathsf{T}^{-}_{i\to j}(\lambda_i) = \bigvee\{\eta_j \in \mathcal{L}_j \mid \eta_j \Rightarrow^\star \lambda_i\}$$

$$\mathsf{T}^{+}_{i\to j}(\lambda_i) = \bigwedge\{\eta_j \in \mathcal{L}_j \mid \lambda_i \Rightarrow^\star \eta_j\}$$

is well defined.

Given a (complete) lattice, if $\Rightarrow^\star$ satisfies `I1-4`, then

$$\mathsf{T}^-_{i\to j}(\lambda_i) = \bigvee \{\eta_j \in \mathcal{L}_j \mid \eta_j \Rightarrow^\star \lambda_i\}$$

$$\mathsf{T}^+_{i\to j}(\lambda_i) = \bigwedge \{\eta_j \in \mathcal{L}_j \mid \lambda_i \Rightarrow^\star \eta_j\}$$

is well defined. However,

$$\{\eta_j \in \mathcal{L}_j \mid \lambda_i \Rightarrow^\star \eta_j\}$$

might be empty.

Given a (complete) lattice, if $\Rightarrow^\star$ satisfies `I1-4`, then

$$\mathsf{T}^{-}_{i\to j}(\lambda_i) = \bigvee\{\eta_j \in \mathcal{L}_j \mid \eta_j \Rightarrow^\star \lambda_i\} \in \mathcal{L}_j$$

$$\mathsf{T}^{+}_{i\to j}(\lambda_i) = \bigwedge\{\eta_j \in \mathcal{L}_j \mid \lambda_i \Rightarrow^\star \eta_j\} \in \mathcal{L}_j \cup \{*\}$$

is well defined. However,

$$\{\eta_j \in \mathcal{L}_j \mid \lambda_i \Rightarrow^\star \eta_j\}$$

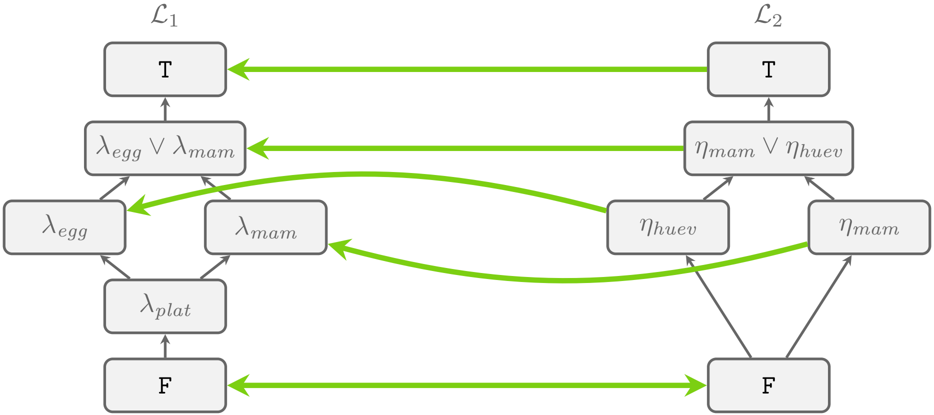
might be empty. Define $\bigwedge \varnothing = *$.

Example

$\mathcal{L}_1$	$\mathcal{L}_2$
λ_{egg} = "*Tonya* lays eggs"	η_{huev} = "*Tonya* pone huevos"
λ_{mam} = "*Tonya* is a mammal"	η_{mam} = "*Tonya* es un mamífero"
λ_{plat} = "*Tonya* is a platypus"	

- 1 is aware of platypus, so $(\lambda_{egg} \wedge \lambda_{mam}) \Rightarrow_1 \lambda_{plat}$
- 2 is not aware of platypus, so $(\eta_{huev} \wedge \eta_{mam}) \Rightarrow_2 f_2$

The additional components of $\Rightarrow^\star$:



Example

How should we translate λ_{mam} from $1 \to 2$?

- η_{mam} is more specific than λ_{mam}, so $\mathsf{T}^{-}_{1\to 2}(\lambda_{mam}) = \eta_{mam}$ makes sense
- But there is *no* element in $\mathcal{L}_2$ more general
  - Nothing in 2's language allows for platypus
- So, we can set $\mathsf{T}^{+}_{1\to 2}(\lambda_{mam}) = *$
- $*$ represents the inclusion of something the target language is unaware of

Abstract Translation

We can return to our original question: given

$$\mathsf{T} = \langle \mathsf{T}^{-}_{1\to 2}, \mathsf{T}^{+}_{1\to 2}, \mathsf{T}^{-}_{2\to 1}, \mathsf{T}^{+}_{2\to 1} \rangle$$

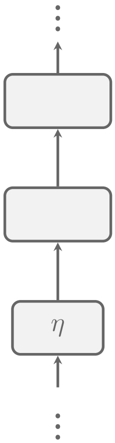
When does T behave like a translation?

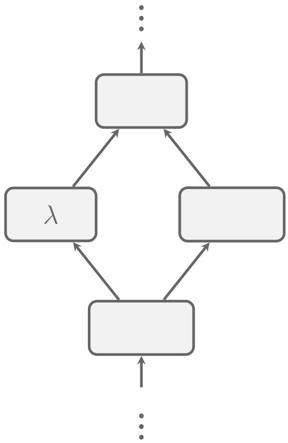
- Each $\mathsf{T}_{i\to j} : \mathcal{L}_i \to \mathcal{L}_j^*$ (where $\mathcal{L}_j^* = \mathcal{L}_j \cup \{*\}$)
- We require two axioms on T

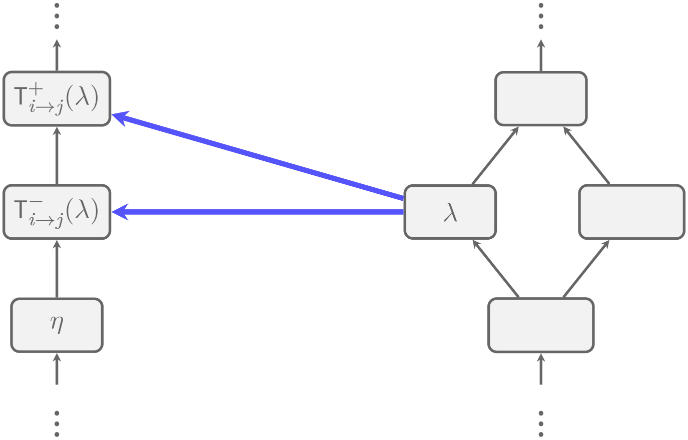
C1: Galois

For all $\lambda \in \mathcal{L}_i^*$ and $\eta \in \mathcal{L}_j^*$:

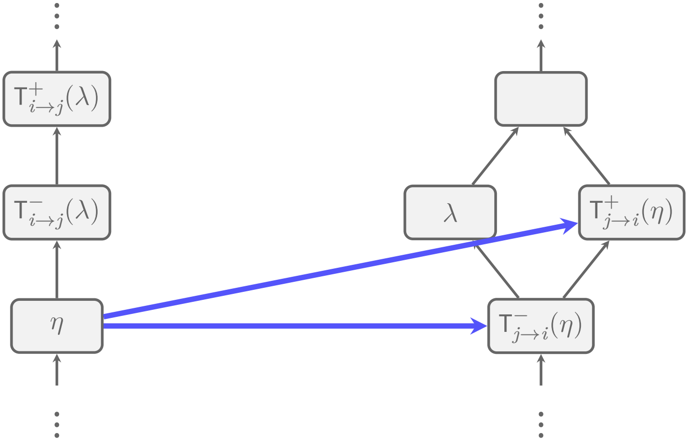
$$\eta \Rightarrow_j \mathsf{T}^{-}_{i\to j}(\lambda) \qquad \text{if and only if} \qquad \mathsf{T}^{+}_{j\to i}(\eta) \Rightarrow_i \lambda.$$


η

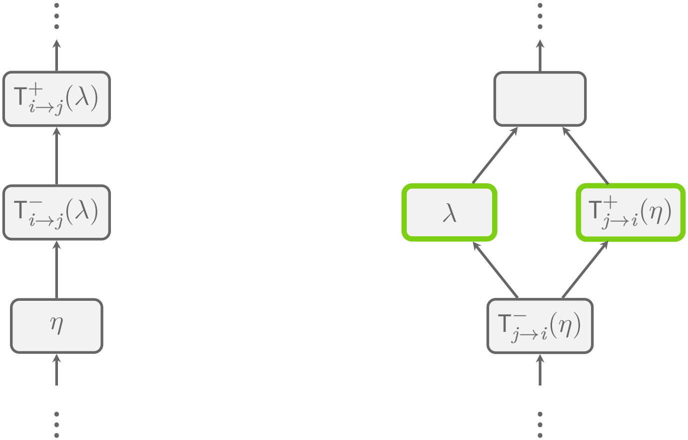

λ



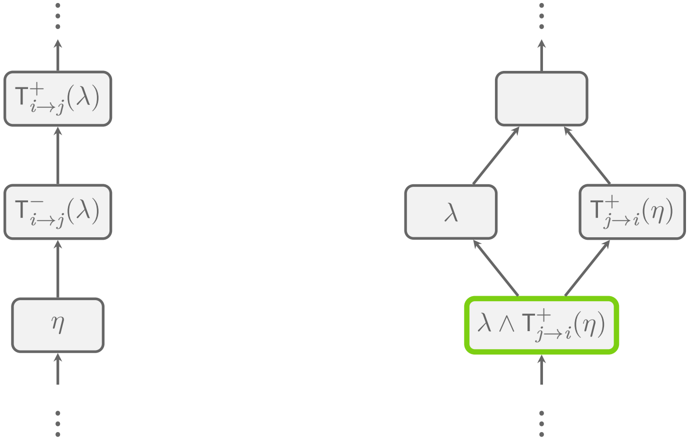
$\eta \Rightarrow_j \mathsf{T}^{-}_{i\to j}(\lambda)$; thus λ is more general than η



If T violates C1 then $\mathsf{T}^+_{j\to i}(\eta) \not\Rightarrow_i \lambda$



So both λ and $T^+_{j\to i}(\eta)$ are more general than η



Then $\lambda \wedge \mathsf{T}^{+}_{j\to i}(\eta)$ is *better* approximation of η from above

C2: Monotone Approximation

For all $\lambda_i \in \mathcal{L}_i^*$ we have $\mathsf{T}^{-}_{i\to j}(\lambda_i) \Rightarrow_j \mathsf{T}^{+}_{i\to j}(\lambda_i)$.

- The inner translation should be more specific than the outer translation

Theorem

The following are equivalent:

(1) $\mathtt{T}$ satisfies $\mathtt{C1}$ and $\mathtt{C2}$

(2) There exists a (unique) $\Rightarrow^\star$ satisfying $\mathtt{I1}$—$\mathtt{I4}$ such that[1]

$$\eta_j \Rightarrow^\star \lambda_i \quad \text{if and only if} \quad \eta_j \Rightarrow_j \mathsf{T}^-_{i\to j}(\lambda_i), \qquad \text{and} \qquad (\star^-)$$

$$\lambda_i \Rightarrow^\star \eta_j \quad \text{if and only if} \quad \mathsf{T}^+_{i\to j}(\lambda_i) \Rightarrow_j \eta_j \qquad (\star^+)$$

[1] Where $\lambda \Rightarrow_i *$ and $* \not\Rightarrow_i \lambda$ for all $\lambda \in \mathcal{L}_i$

`C1` is abstract in nature, we can characterize it via ‘structural’ properties

T1. $\mathtt{T}^{-}$ and $\mathtt{T}^{+}$ *preserve contradiction*:

$$\mathsf{T}^{-}_{i\to j}(f_i) = \mathsf{T}^{+}_{i\to j}(f_i) = f_j$$

T2. $\mathtt{T}^{-}$ and $\mathtt{T}^{+}$ *preserves implication*:

$$\lambda \Rightarrow_i \lambda' \text{ implies } \begin{cases} \mathsf{T}^{-}_{i\to j}(\lambda) \Rightarrow_j \mathsf{T}^{-}_{i\to j}(\lambda') \text{ and} \\ \mathsf{T}^{+}_{i\to j}(\lambda) \Rightarrow_j \mathsf{T}^{+}_{i\to j}(\lambda') \end{cases}$$

T3. $\mathsf{T}^{-}_{i\to j}$ *preserves conjunction*:

$$\mathsf{T}^{-}_{i\to j}(\lambda \wedge \lambda') = \mathsf{T}^{-}_{i\to j}(\lambda) \wedge \mathsf{T}^{-}_{i\to j}(\lambda')$$

T4. $\mathsf{T}^{+}_{i\to j}$ *preserves disjunction*:

$$\mathsf{T}^{+}_{i\to j}(\lambda \vee \lambda') = \mathsf{T}^{+}_{i\to j}(\lambda) \vee \mathsf{T}^{+}_{i\to j}(\lambda')$$

- $W(\mathcal{L}_i)$ and $W(\mathcal{L}_j)$ act as ‘local’ state-spaces
- If $W^\star$ is some ‘global’ state-space that nests both:
  - Then translation operators appear as inner and outer approximation
  - As in measure theory, etc

Example

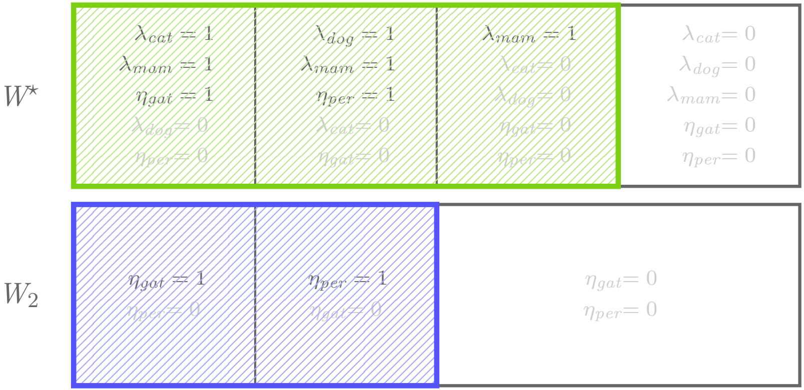
$W^\star$	$\lambda_{cat} = 1$ $\lambda_{mam} = 1$ $\eta_{gat} = 1$ $\lambda_{dog} = 0$ $\eta_{per} = 0$	$\lambda_{dog} = 1$ $\lambda_{mam} = 1$ $\eta_{per} = 1$ $\lambda_{cat} = 0$ $\eta_{gat} = 0$	$\lambda_{mam} = 1$ $\lambda_{cat} = 0$ $\lambda_{dog} = 0$ $\eta_{gat} = 0$ $\eta_{per} = 0$	$\lambda_{cat} = 0$ $\lambda_{dog} = 0$ $\lambda_{mam} = 0$ $\eta_{gat} = 0$ $\eta_{per} = 0$

W_2	$\eta_{gat} = 1$ $\eta_{per} = 0$	$\eta_{per} = 1$ $\eta_{gat} = 0$	$\eta_{gat} = 0$ $\eta_{per} = 0$

Example

$W^\star$	$\lambda_{cat} = 1$ $\lambda_{mam} = 1$ $\eta_{gat} = 1$ $\lambda_{dog} = 0$ $\eta_{per} = 0$	$\lambda_{dog} = 1$ $\lambda_{mam} = 1$ $\eta_{per} = 1$ $\lambda_{cat} = 0$ $\eta_{gat} = 0$	$\lambda_{mam} = 1$ $\lambda_{cat} = 0$ $\lambda_{dog} = 0$ $\eta_{gat} = 0$ $\eta_{per} = 0$	$\lambda_{cat} = 0$ $\lambda_{dog} = 0$ $\lambda_{mam} = 0$ $\eta_{gat} = 0$ $\eta_{per} = 0$
W_2	$\eta_{gat} = 1$ $\eta_{per} = 0$	$\eta_{per} = 1$ $\eta_{gat} = 0$	$\eta_{gat} = 0$ $\eta_{per} = 0$	

Example


$W^\star$
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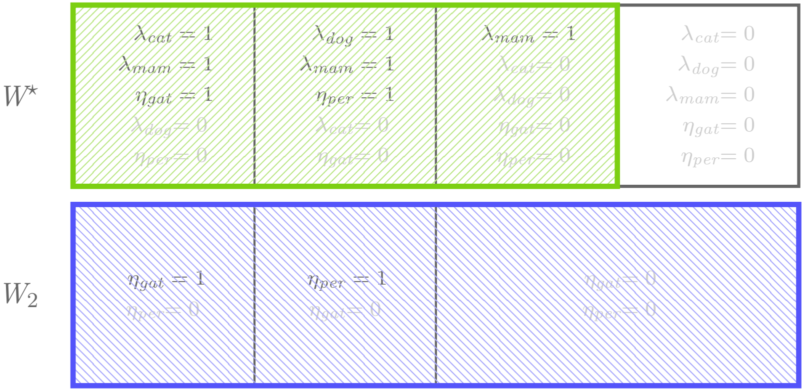
$\eta_{gat} = 0$
$\eta_{per} = 0$
W_2
$\eta_{gat} = 1$
$\eta_{per} = 0$
$\eta_{per} = 1$
$\eta_{gat} = 0$
$\eta_{gat} = 0$
$\eta_{per} = 0$

Example



$W^\star$
$\lambda_{cat} = 1$
$\lambda_{mam} = 1$
$\eta_{gat} = 1$
$\lambda_{dog} = 0$
$\eta_{per} = 0$
$\lambda_{dog} = 1$
$\lambda_{mam} = 1$
$\eta_{per} = 1$
$\lambda_{cat} = 0$
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$\lambda_{cat} = 0$
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$\eta_{gat} = 0$
$\eta_{per} = 0$
$\lambda_{cat} = 0$
$\lambda_{dog} = 0$
$\lambda_{mam} = 0$
$\eta_{gat} = 0$
$\eta_{per} = 0$
W_2
$\eta_{gat} = 1$
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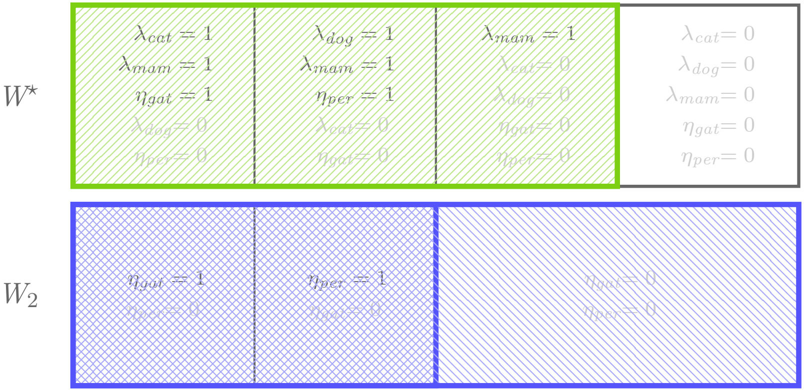
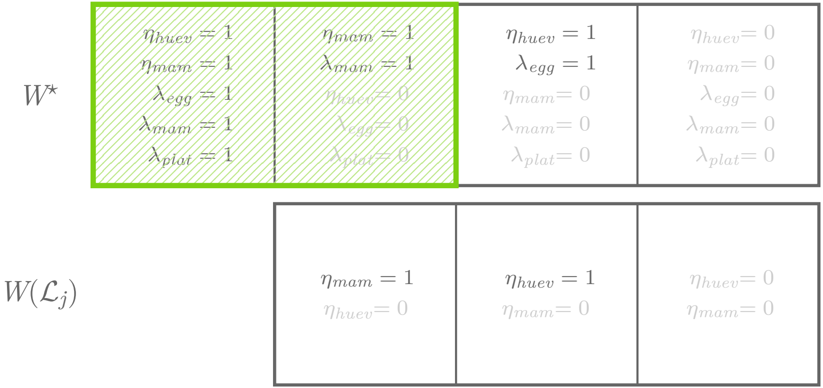
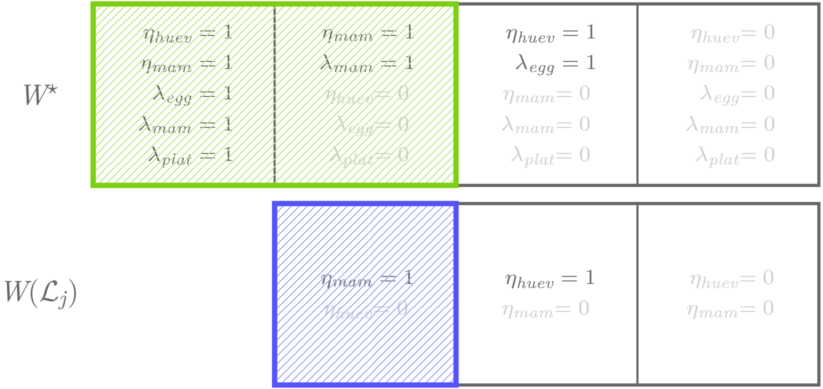
Example

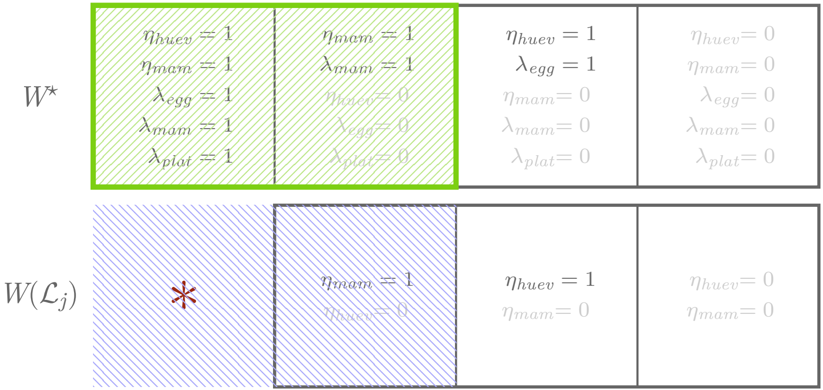

$W^\star$
$\lambda_{cat} = 1$
$\lambda_{mam} = 1$
$\eta_{gat} = 1$
$\lambda_{dog} = 0$
$\eta_{per} = 0$
$\lambda_{dog} = 1$
$\lambda_{mam} = 1$
$\eta_{per} = 1$
$\lambda_{cat} = 0$
$\eta_{gat} = 0$
$\lambda_{mam} = 1$
$\lambda_{cat} = 0$
$\lambda_{dog} = 0$
$\eta_{gat} = 0$
$\eta_{per} = 0$
$\lambda_{cat} = 0$
$\lambda_{dog} = 0$
$\lambda_{mam} = 0$
$\eta_{gat} = 0$
$\eta_{per} = 0$
W_2
$\eta_{gat} = 1$
$\eta_{per} = 0$
$\eta_{per} = 1$
$\eta_{gat} = 0$
$\eta_{gat} = 0$
$\eta_{per} = 0$

$W^\star$	$\eta_{huev} = 1$ $\eta_{mam} = 1$ $\lambda_{egg} = 1$ $\lambda_{mam} = 1$ $\lambda_{plat} = 1$	$\eta_{mam} = 1$ $\lambda_{mam} = 1$ $\eta_{huev} = 0$ $\lambda_{egg} = 0$ $\lambda_{plat} = 0$	$\eta_{huev} = 1$ $\lambda_{egg} = 1$ $\eta_{mam} = 0$ $\lambda_{mam} = 0$ $\lambda_{plat} = 0$	$\eta_{huev} = 0$ $\eta_{mam} = 0$ $\lambda_{egg} = 0$ $\lambda_{mam} = 0$ $\lambda_{plat} = 0$
$W(\mathcal{L}_j)$		$\eta_{mam} = 1$ $\eta_{huev} = 0$	$\eta_{huev} = 1$ $\eta_{mam} = 0$	$\eta_{huev} = 0$ $\eta_{mam} = 0$

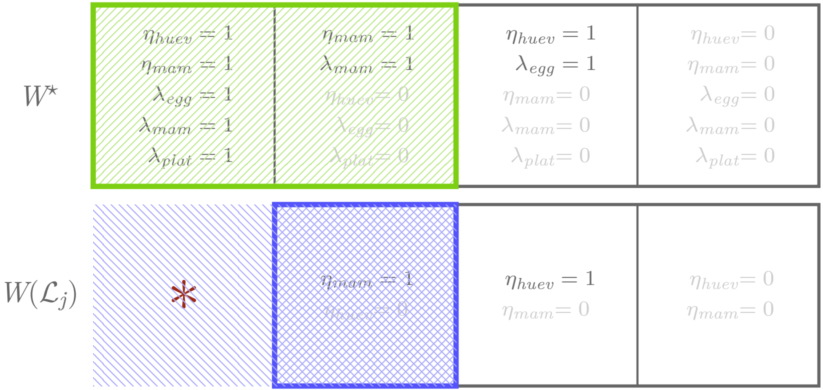

$W^\star$
$\eta_{huev} = 1$
$\eta_{mam} = 1$
$\lambda_{egg} = 1$
$\lambda_{mam} = 1$
$\lambda_{plat} = 1$
$\eta_{mam} = 1$
$\lambda_{mam} = 1$
$\eta_{huev} = 0$
$\lambda_{egg} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 1$
$\lambda_{egg} = 1$
$\eta_{mam} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$
$\lambda_{egg} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$W(\mathcal{L}_j)$
$\eta_{mam} = 1$
$\eta_{huev} = 0$
$\eta_{huev} = 1$
$\eta_{mam} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$


$W^\star$
$\eta_{huev} = 1$, $\eta_{mam} = 1$, $\lambda_{egg} = 1$, $\lambda_{mam} = 1$, $\lambda_{plat} = 1$
$\eta_{mam} = 1$, $\lambda_{mam} = 1$, $\eta_{huev} = 0$, $\lambda_{egg} = 0$, $\lambda_{plat} = 0$
$\eta_{huev} = 1$, $\lambda_{egg} = 1$, $\eta_{mam} = 0$, $\lambda_{mam} = 0$, $\lambda_{plat} = 0$
$\eta_{huev} = 0$, $\eta_{mam} = 0$, $\lambda_{egg} = 0$, $\lambda_{mam} = 0$, $\lambda_{plat} = 0$
$W(\mathcal{L}_j)$
$\eta_{mam} = 1$, $\eta_{huev} = 0$
$\eta_{huev} = 1$, $\eta_{mam} = 0$
$\eta_{huev} = 0$, $\eta_{mam} = 0$

$W^\star$	$\eta_{huev}=1$ $\eta_{mam}=1$ $\lambda_{egg}=1$ $\lambda_{mam}=1$ $\lambda_{plat}=1$	$\eta_{mam}=1$ $\lambda_{mam}=1$ $\eta_{huev}=0$ $\lambda_{egg}=0$ $\lambda_{plat}=0$	$\eta_{huev}=1$ $\lambda_{egg}=1$ $\eta_{mam}=0$ $\lambda_{mam}=0$ $\lambda_{plat}=0$	$\eta_{huev}=0$ $\eta_{mam}=0$ $\lambda_{egg}=0$ $\lambda_{mam}=0$ $\lambda_{plat}=0$
$W(\mathcal{L}_j)$	*	$\eta_{mam}=1$ $\eta_{huev}=0$	$\eta_{huev}=1$ $\eta_{mam}=0$	$\eta_{huev}=0$ $\eta_{mam}=0$



$W^\star$
$\eta_{huev} = 1$
$\eta_{mam} = 1$
$\lambda_{egg} = 1$
$\lambda_{mam} = 1$
$\lambda_{plat} = 1$
$\eta_{mam} = 1$
$\lambda_{mam} = 1$
$\eta_{huev} = 0$
$\lambda_{egg} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 1$
$\lambda_{egg} = 1$
$\eta_{mam} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$
$\lambda_{egg} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$W(\mathcal{L}_j)$
$\eta_{mam} = 1$
$\eta_{huev} = 0$
$\eta_{huev} = 1$
$\eta_{mam} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$

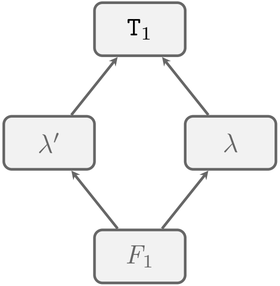

$W^\star$
$\eta_{huev} = 1$
$\eta_{mam} = 1$
$\lambda_{egg} = 1$
$\lambda_{mam} = 1$
$\lambda_{plat} = 1$
$\eta_{mam} = 1$
$\lambda_{mam} = 1$
$\eta_{huev} = 0$
$\lambda_{egg} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 1$
$\lambda_{egg} = 1$
$\eta_{mam} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$
$\lambda_{egg} = 0$
$\lambda_{mam} = 0$
$\lambda_{plat} = 0$
$W(\mathcal{L}_j)$
$\eta_{mam} = 1$
$\eta_{huev} = 0$
$\eta_{huev} = 1$
$\eta_{mam} = 0$
$\eta_{huev} = 0$
$\eta_{mam} = 0$

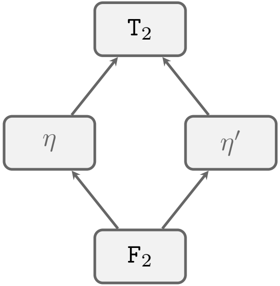
Does such a state-space $W^\star$ always exist?

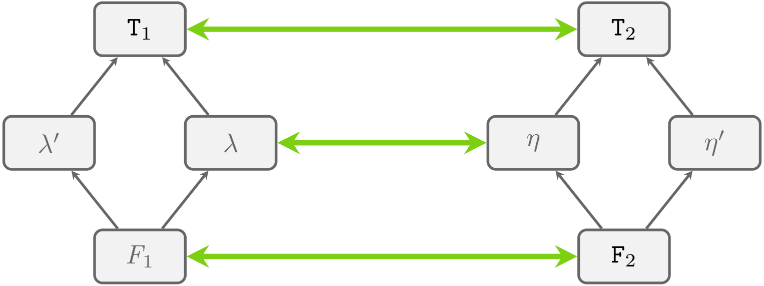
Example

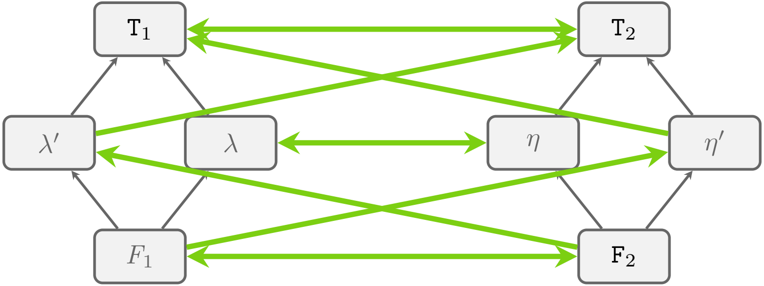
$\mathcal{L}_1$	$\mathcal{L}_2$
$\lambda =$ "God exists"	$\eta =$ "Dios es bueno"
$\lambda' =$ "God does not exist"	$\eta' =$ "Dios es malvado"

- 1 cannot conceive of an evil God, but is unsure a benevolent God exists
  - $\mathtt{T}_1 \Rightarrow_1 (\lambda \vee \lambda'), (\lambda \wedge \lambda') \Rightarrow_1 \mathtt{F}_1$
- 2 defines God as what is in the universe, but is unsure its moral character
  - $\mathtt{T}_2 \Rightarrow_2 (\eta \vee \eta'), (\eta \wedge \eta') \Rightarrow_2 \mathtt{F}_2$


$\mathtt{T}_1$
λ'
λ
F_1


$\mathtt{T}_2$
η
η'
$\mathtt{F}_2$


$\mathtt{T}_1$
$\mathtt{T}_2$
λ'
λ
η
η'
F_1
$\mathtt{F}_2$


$\mathtt{T}_1$
$\mathtt{T}_2$
λ'
λ
η
η'
F_1
$\mathtt{F}_2$

Example

$\mathcal{L}_i$

$\lambda =$ "God exists"

$\lambda' =$ "God does not exist"

$$\mathsf{T}^{-}_{1\to 2}(f_i) = f_j$$
$$\mathsf{T}^{-}_{1\to 2}(\lambda) = \eta$$
$$\mathsf{T}^{-}_{1\to 2}(\lambda') = f_j$$
$$\mathsf{T}^{-}_{1\to 2}(t_i) = t_j$$

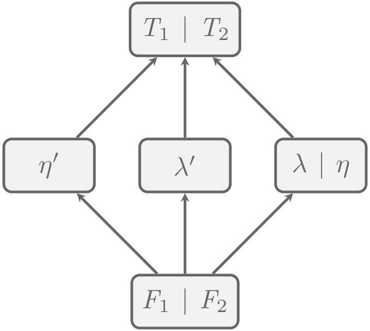
$\mathcal{L}_j$

$\eta =$ "Dios es bueno"

$\eta' =$ "Dios es malvado"

$$\mathsf{T}^{+}_{1\to 2}(f_i) = f_j$$
$$\mathsf{T}^{+}_{1\to 2}(\lambda) = \eta$$
$$\mathsf{T}^{+}_{1\to 2}(\lambda') = t_j$$
$$\mathsf{T}^{+}_{1\to 2}(t_i) = t_j$$

This satisfies our axioms but admits no state-space representation. The induced graph, i.e., $(\mathcal{L}_i \cup \mathcal{L}_j) \,/ \Rightarrow^{\star}$:



is not distributive.

Open question:

what additional axioms does T (equivalently, $\Rightarrow^\star$) need to satisfy to ensure existence of state-space representation?

We can answer this for Boolean languages:

- Assume $\mathcal{L}_i$ and $\mathcal{L}_j$ are Boolean algebras
- Closed under *negation*: if $\lambda \in \mathcal{L}$ then $\neg\lambda \in \mathcal{L}$:
  - $\lambda \wedge \neg\lambda = \mathtt{F}$,
  - $\lambda \vee \neg\lambda = \mathtt{T}$

C3: Duality

For $\lambda_i \in \mathcal{L}_i$ and $\eta_j \in \mathcal{L}_j$ such that $\mathsf{T}^{+}_{i\to j}(\lambda_i) \neq *$:

$$\mathsf{T}^{-}_{i\to j}(\neg\lambda_i) = \neg\mathsf{T}^{+}_{i\to j}(\lambda_i) \wedge \mathsf{T}^{-}_{i\to j}(\mathtt{T}_i).$$

If agents share a view of truth: $\mathsf{T}^{-}_{i\to j}(\mathtt{T}_i) = \mathtt{T}_j = \mathsf{T}^{+}_{i\to j}(\mathtt{T}_i)$ then we can simplify:

$$\mathsf{T}^{-}_{i\to j}(\neg\lambda_i) = \neg\mathsf{T}^{+}_{i\to j}(\lambda_i),$$

for all statements.

I5: Negation Consistency

For $\lambda_i \in \mathcal{L}_i$ and $\eta_j \in \mathcal{L}_j$ such that $\lambda_i \rightarrow^\star \mathtt{T}_j$:

$$\eta_j \Rightarrow^\star \neg\lambda_i \qquad \text{implies} \qquad \lambda_i \Rightarrow^\star \neg\eta_j.$$

Theorem

The following are equivalent:

(1) $\mathtt{T}$ satisfies $\mathtt{C1-3}$

(2) There exists a (unique) $\Rightarrow^\star$ satisfying $\mathtt{I1-5}$ that defines it (via $\star^-$ and $\star^+$).

(3) There exists a joint state-space representation of $\mathcal{L}_i$ and $\mathcal{L}_j$:

State-space is 'locally' Boolean

- $\wedge$ and $\vee$ map to $\cap$ and $\cup$, respectively
- $\mathtt{F}$ maps to $\varnothing$
- $\neg$ maps to *relative* complementation w.r.t. $\mathtt{T}$

Thank You