# MODELING THE MODELER:
# A NORMATIVE THEORY OF EXPERIMENTAL DESIGN

**Evan Piermont**
Royal Holloway, University of London

joint with

**Fernando Payró**
UAB, BSE & CODE

Antai College of Economics and Management, Jiao Tong University
April 2025

- Decision theoretic analysis of experiments: analyst's preference over experiments
- The value of an exp is determined by what it allows to identify
- In principle any kind of experiment, but adapt the analysis to ***revealed preference***
    - Laboratory economics experiments
    - Any agent who can probe at the preference of other agents

- We propose three normative principles for experimental design
  - minimal rationality properties, independent of specific motivations
  - We will specifically think about revealed preference experiments
- We show that they imply a particular representation
  - Relates a experiment to the expected value of identification
  - Unifies many distinct models of experimentation
  - Axiomatic characterization for Bayesian Experimental Design
  - Test for analyst to make sure they do not have an “agenda”

- A space of parameters $\mathcal{U}$
- Experiment $A$ has possible observable outcomes $\{P_1, \ldots, P_n\}$
- Observing $P$ ***identifies*** a set of parameters:
    - $W_{A,P} \subseteq \mathcal{U}$ consistent with observation
- An "ideal" experiment should induce a partition over $\mathcal{U}$
    - Not always possible (some parameters might not uniquely determine an observation)

Ranking over experiments should reflect the value of potential identification

# Normative Principles

**Structural Invariance**: Two experiments that identify the sets of parameters are equally valued

**Information Monotonicity**: Experiments that induce sharper identification are (weakly) better

**Identification Separability**: The value of identifying a set of parameters should *not* depend on counterfactuals

# Expected Identification Value

These principles characterize ranking according to *expected identification value*

- Exists some $\tau$: for $W \subseteq \mathcal{U}$, $\tau(W)$ is the value of identifying $W$
- Experiments are valued according to:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

- where $\mu$ is the (exogenous) prior probability

# Special Case: Entropy

$$\tau(W) = -\log(\mu(W))$$

- Value of experiment is expected reduction in entropy

# Special Case: Hypothesis Testing

$$\tau(W) = \begin{cases} 1 & \text{if } W \subseteq W^* \text{ or } W^* \subseteq W^c \\ 0 & \text{otherwise}\,. \end{cases}$$

- Hypothesis: the parameter lies in $W^*$
- Value of exp is the probability the hypothesis can be accepted or rejected

# Special Case: Actions

$$\tau(W) = \max_{a \in \mathbb{A}} \int_W \xi(a, u)\, d\mu.$$

- The analyst will take action $a \in \mathbb{A}$
- Utility of outcome depends on the parameter: $\xi(a, u)$
- Value of exp is expected value of conditionally optimal action

# Related literature

Decision Theory

- Dekel, Lipman and Rustichini (2001) “Representing Preferences with a Unique Subjective State Space”
- Gilboa and Lerher (1991) “The value of information-An axiomatic approach”
- Ergin and Sarver (2015) “Hidden actions and preferences for timing of resolution of uncertainty”

Statistics

- Lindley (1972) “Bayesian statistics: A review”
- Chaloner and Verdinelli (1995) “Bayesian experimental design: A review”

# ❦ Model ❦

An analyst (she) wishes to infer a subject's (he) utility function over $Z$:

- Revealed Preference: she should offer a menu $A \subseteq Z$ and observe the subject's choice
- Different menus offer different "inference" opportunities
- Ranking over menus will depend on the goals for the analyst

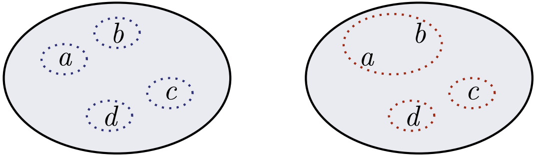
# Experimental Environment

- $Z$ set of alternatives
- $\mathcal{U} \subseteq \{u : Z \to \mathbb{R}\}$ set of utility functions over $Z$
- $\Omega$ algebra of measurable sets of $\mathcal{U}$
- $\mu$ prior over $(\mathcal{U}, \Omega)$

The tuple $(Z, \mathcal{U}, \Omega, \mu)$ constitutes a theory for a Bayesian experimenter

An **experiment** $e = (A, \mathscr{P})$ is a pair:

- $A \subseteq Z$ is finite decision problem
- $\mathscr{P}$ is a partition of $A$




- Represents observability constraints
- Allows for dynamic experiments, non-lab settings, etc

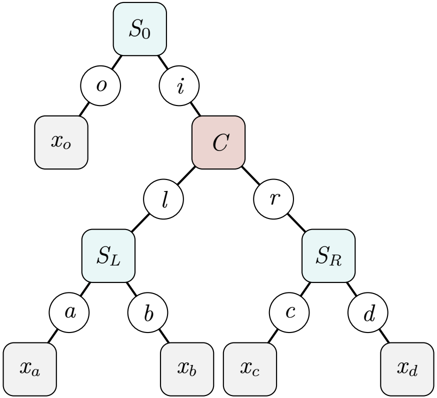
# Example: Dynamic Games

Consider a dynamic game:

- Computer randomizes 50-50
- Subject's strategies:

$$A = \left\{ \begin{array}{l} (i,a,c),(i,b,c),(i,a,d),(i,b,d), \\ (o,a,c),(o,a,d),(o,b,c),(o,b,d) \end{array} \right\}$$

- There are observable paths: $(o)$, $(i, a)$, $(i, b)$, $(i, c)$, and $(i, d)$

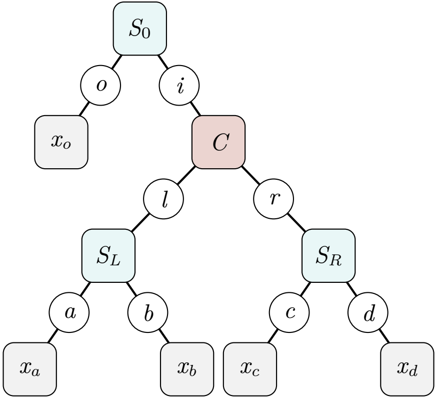


# Example: Dynamic Games

Now consider the following partitions of $A$

$$\mathscr{P}_L = \left\{ \begin{array}{c} \left\{ \begin{array}{c} (o,a,c),(o,b,c), \\ (o,a,d),(o,b,d) \end{array} \right\}, \\ \{\ (i,a,c),(i,a,d)\ \}, \\ \{\ (i,b,c),(i,b,d)\ \} \end{array} \right\}$$

$$\mathscr{P}_R = \left\{ \begin{array}{c} \left\{ \begin{array}{c} (o,a,c),(o,b,c), \\ (o,a,d),(o,b,d) \end{array} \right\}, \\ \{\ (i,a,c),(i,b,c)\ \}, \\ \{\ (i,a,d),(i,b,d)\ \} \end{array} \right\}$$



Given an experiment, $(A, \mathscr{P})$, define the ***identified set***:

$$W_{A,P} = \{u \in \mathcal{U} \mid P \cap \operatorname{argmax}_{x \in A} u(x) \neq \varnothing\}$$

- Observing $P \in \mathscr{P}$ identifies that the subject's utility is in $W_{A,P}$
- We require for an experiment $(A, \mathscr{P})$ that for any $P, Q \in \mathscr{P}$

**(1)** $W_{A,P} \in \Omega$ — measurability

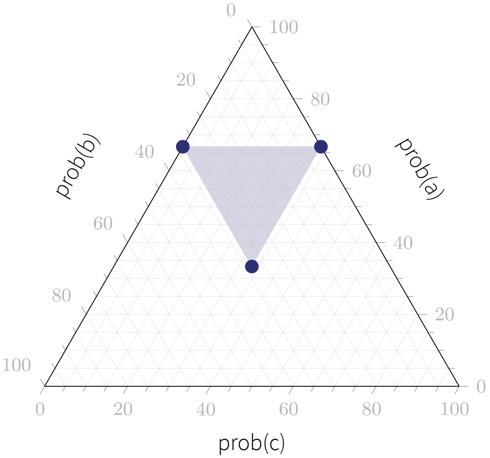
**(2)** $\mu(W_{A,P} \cap W_{A,Q}) = 0$ — zero $\mu$-prob of ties

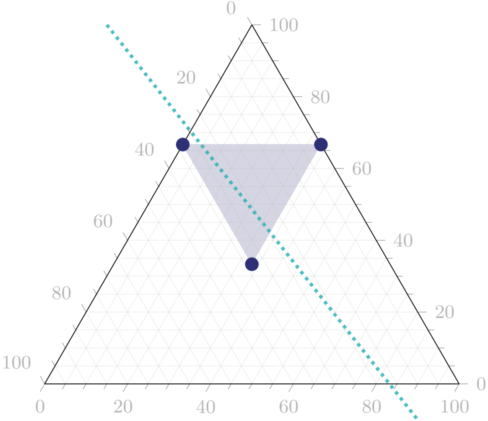
# Example: EU Preferences

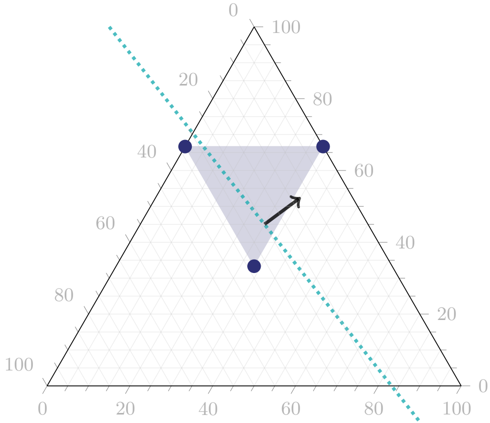
We can example identifying EU preferences as an example:

- $Z$ is lotteries over $\{a, b, c\}$
- $\mathcal{U}$ is affine functions

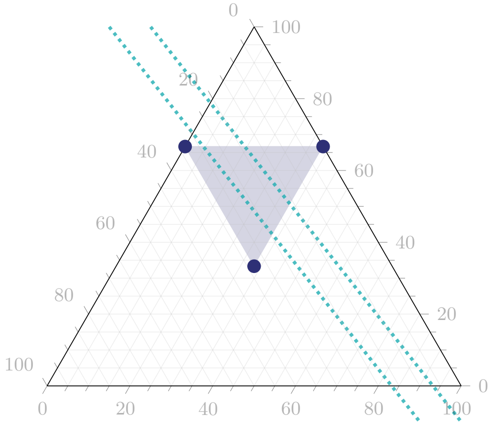
$$\{\tfrac{2}{3}a + \tfrac{1}{3}b, \tfrac{2}{3}a + \tfrac{1}{3}c, \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c\}$$



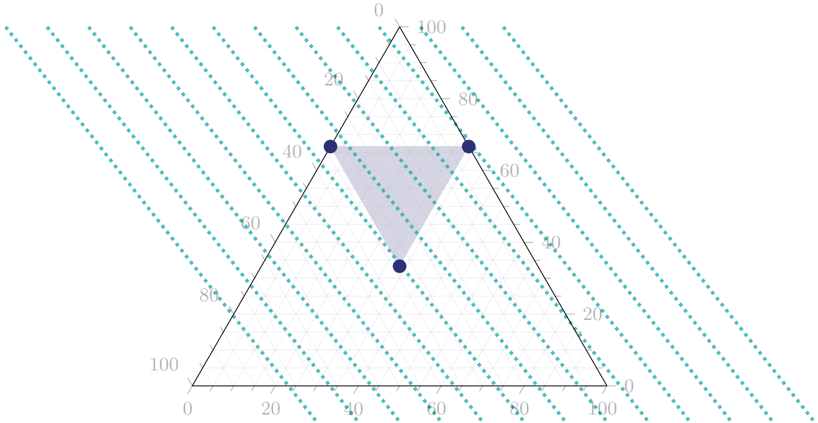

0
20
40
60
80
100
100
80
60
40
20
0
0
20
40
60
80
100

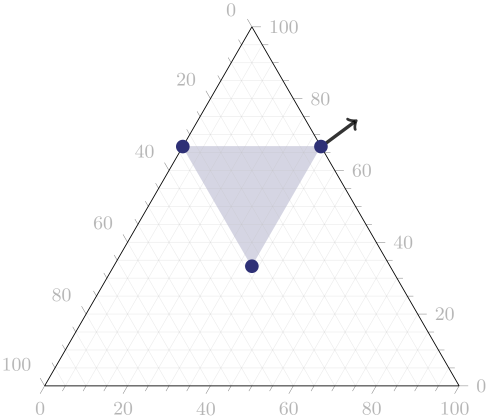


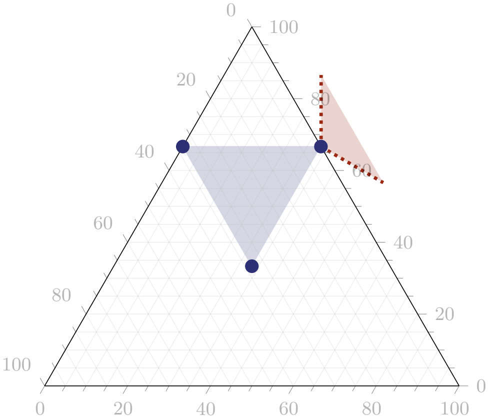
0
20
40
60
80
100
100
80
60
40
20
0
0
20
40
60
80
100



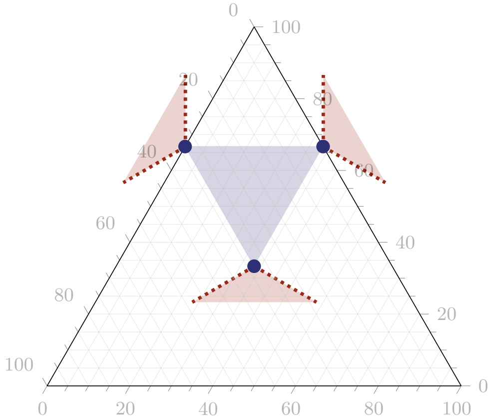
0
20
40
60
80
100


0
20
40
60
80
100
100
80
60
40
20
0
0
20
40
60
80
100

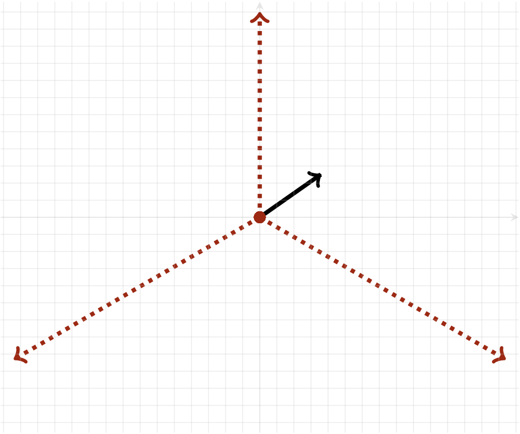

0
20
40
60
80
100
100
80
60
40
20
0
0
20
40
60
80
100



0
20
40
60
80
100
0
20
40
60
80
100
0
20
40
60
80
100


0
100
20
80
40
60
60
40
80
20
100
0
0
20
40
60
80
100

$\mathcal{U} \cong \mathbb{R}^2$



## $\mu$-equivalence

Call $\{W_1, \ldots, W_n\}$ and $\{V_1, \ldots, V_m\}$, families of subsets of $\mathcal{U}$, $\mu$-*equivalent* if for all $W_i$ and $V_j$:

$$\mu(W_i) = \mu(W_i \cap V_j) \text{ for some } j \quad \text{and,}$$
$$\mu(V_j) = \mu(W_i \cap V_j) \text{ for some } i$$

- Such collections identify the same sets of utilities up to a measure zero
- Take $[0, 1]$ with $\lambda$ the Lebesgue measure. The following are $\lambda$-equivalent:
  - $\{[0, \frac{1}{2}), (\frac{1}{2}, 1]\}$; $\{[0, \frac{1}{2}), \{\frac{1}{2}\}, (\frac{1}{2}, 1]\}$, $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$

# Rich Experimental Settings

We say a set of experiments $\mathbb{E}$ is ***rich*** if

**(1)** $(A, \mathscr{P}) \in \mathbb{E} \to (A, \mathscr{Q}) \in \mathbb{E}$ whenever $\mathscr{Q}$ is a coarsening of $\mathscr{P}$

**(2)** For any finite $\Omega$-measurable partition of $\mathcal{U}$, there exists an experiment $(A, \mathscr{P})$ such that $\{W_{A,P}\}_{P \in \mathscr{P}}$ is $\mu$ equivalent

- Any partition can be approximated up to 0 probability events
- For the EU model, the set of all experiments is rich for any "regular" $\mu$

# Primitive

- Our primitive is a ranking $\succcurlyeq$ over the set of all random experiments
- A *random experiment* is a lottery over some (fixed) rich set $\mathbb{E}$

# Normative Principles

"Two experiments that identify the sets of parameters are equally valued"

**(P1) - Structural Invariance**

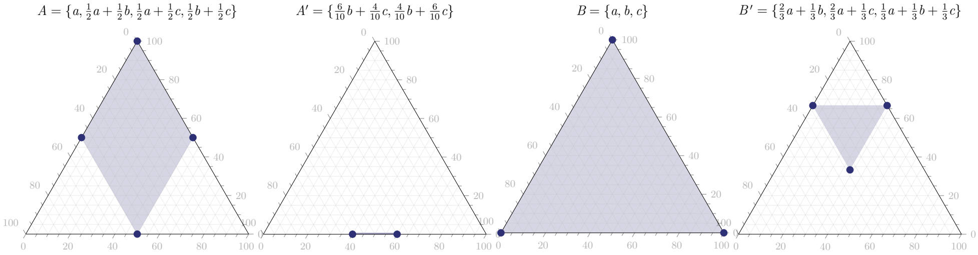
If $\{ W_{A,P} | P \in \mathscr{P} \}$ is $\mu$-equivalent to $\{ W_{B,Q} | Q \in \mathscr{Q} \}$ then $(A, \mathscr{P}) \sim (B, \mathscr{Q})$.

- Structural properties of experiments are irrelevant
- Also, 0-probability events are irrelevant

Consider our EU maximizing subject choosing lotteries over $\{a, b, c\}$.

$$\begin{aligned} \text{EXP A}: A \; &= \{a,\;\; \tfrac{1}{2}a + \tfrac{1}{2}b,\;\; \tfrac{1}{2}a + \tfrac{1}{2}c,\;\; \tfrac{1}{2}b + \tfrac{1}{2}c\} \\ A^{'} &= \{\tfrac{6}{10}b + \tfrac{4}{10}c,\;\; \tfrac{4}{10}b + \tfrac{6}{10}c\}. \end{aligned}$$

$$\begin{aligned} \text{EXP B}: B \; &= \{a,\;\; b,\;\; c\} \\ B^{'} &= \{\tfrac{2}{3}a + \tfrac{1}{3}b,\;\; \tfrac{2}{3}a + \tfrac{1}{3}c,\;\; \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c\}. \end{aligned}$$

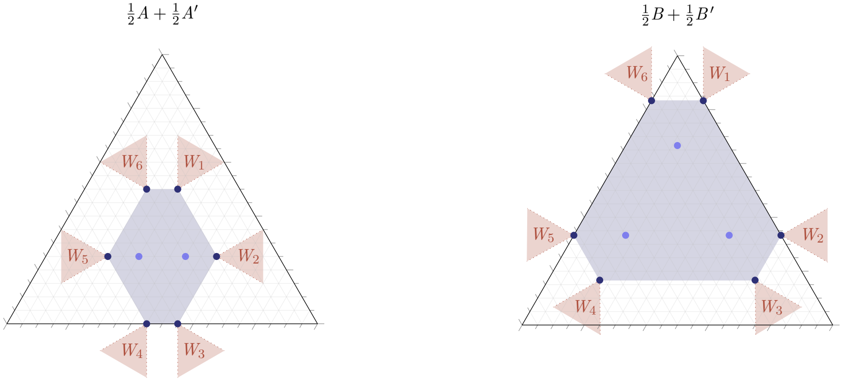


$A = \{a, \frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}c, \frac{1}{2}b + \frac{1}{2}c\}$
$A' = \{\frac{6}{10}b + \frac{4}{10}c, \frac{4}{10}b + \frac{6}{10}c\}$
$B = \{a, b, c\}$
$B' = \{\frac{2}{3}a + \frac{1}{3}b, \frac{2}{3}a + \frac{1}{3}c, \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c\}$

- Linearity states that

$$\left.\begin{array}{l} x \in \arg\max_A u(\cdot) \\ y \in \arg\max_B u(\cdot) \end{array}\right\} \quad \text{if and only if} \quad \alpha x + (1-\alpha) y \in \arg \max_{\alpha A + (1-\alpha) B} u(\cdot)$$

- Therefore, observing $A$ followed by $A'$ is equivalent to observing $\frac{1}{2} A + \frac{1}{2} A'$



$\frac{1}{2}A + \frac{1}{2}A'$
$W_1$
$W_2$
$W_3$
$W_4$
$W_5$
$W_6$
$\frac{1}{2}B + \frac{1}{2}B'$
$W_1$
$W_2$
$W_3$
$W_4$
$W_5$
$W_6$

- Structural invariance reflects the symmetries of the given domain
- With linear utility, the symmetry is ***translation invariance***:

**Structural Invariance for Expected Utility**

$$(A, \{P_1, \ldots P_n\}) \sim (A + B, \{P_1 + B, \ldots P_n + B\})$$

This isn't exactly correct, since $\{P_1 + B, \ldots P_n + B\}$ might have overlaps....

"Experiments that induce sharper identification are (weakly) better"

**(P2) - Information Monotonicity**

If $\mathscr{P}$ refines $\mathscr{Q}$ then $(A, \mathscr{P}) \succcurlyeq (A, \mathscr{Q})$.

- Preference respects Blackwell order

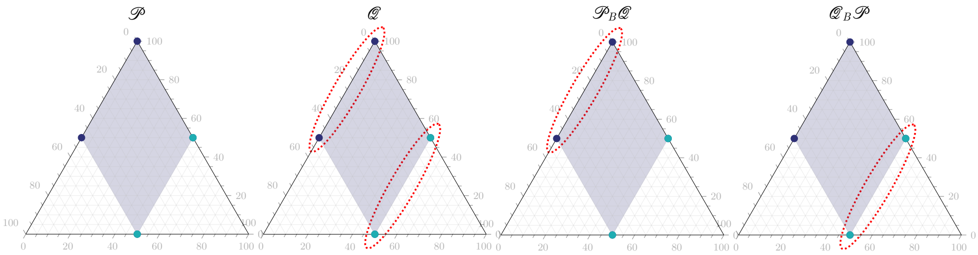
"The value of identification not depend on counterfactuals"

## (P3) - Identification Separability

$$\tfrac{1}{2}(A,\mathscr{P}) + \tfrac{1}{2}(A,\mathscr{Q}) \sim \tfrac{1}{2}(A,\mathscr{P}_B\mathscr{Q}) + \tfrac{1}{2}(A,\mathscr{Q}_B\mathscr{P}).$$

- $\mathscr{P}$ and $\mathscr{Q}$ partitions of $A$ and $B \subseteq A$, then $\mathscr{P}_B\mathscr{Q}$ denotes the partition that coincides with $\mathscr{P}$ over $B$ and with $\mathscr{Q}$ over $A \setminus B$

Consider decision problem $A$ (from before) with the following partitions




- The set $B$ is the two south-east lotteries (in teal)

**Theorem**

Let $\succcurlyeq$ be an expected utility preference, represented by index $F : \mathbb{E} \to \mathbb{R}$.

Then $\succcurlyeq$ satisfies **P1-3** if and only if there exists a $\tau : \Omega \to \mathbb{R}$ such that:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

with $W \subseteq V$ implies

- $\tau(W)\mu(W|V) + \tau(V \setminus W)(1 - \mu(W|V)) \geq \tau(V)$
- $\mu(W) = \mu(V)$ implies $\tau(W) = \tau(V)$

Representation reflects our normative principles:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

- Only dependents on $W_{A,P} \rightarrow$ Structural Invariance
- Additive $\rightarrow$ Identification Separability
- $\tau(W)\mu(W|V) + \tau(V \setminus W)(1\text{-}\mu(W|V)) \geq \tau(V)$ Monotonicity

# ❦ Entropy ❦

- Entropy is a common measure of information
- Entropy of a probability measure is

$$-\sum_{x \in \mathrm{supp}(\mu)} \log(\mu(x))\mu(x)$$

- The experimenter's value for an experiment is the (expected) entropy of the induced identification

$$F(A, \mathscr{P}) = -\sum_{P \in \mathscr{P}} \log(\mu(W_{A,P}))\mu(W_{A,P})$$

We can specialize each of the normative principals to this context

## Structural Invariance for Entropy: Symmetry

Fix $(A, \{P_1, \ldots P_n\})$ and $(B, \{Q_1, \ldots Q_n\})$. Then if for all $i \leq n$,

$$|\mu(W_{B,Q_i}) - \tfrac{1}{n}| \geq |\mu(W_{A,P_i}) - \tfrac{1}{n}|$$

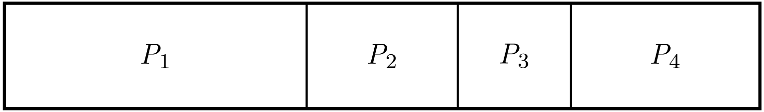
it follows that

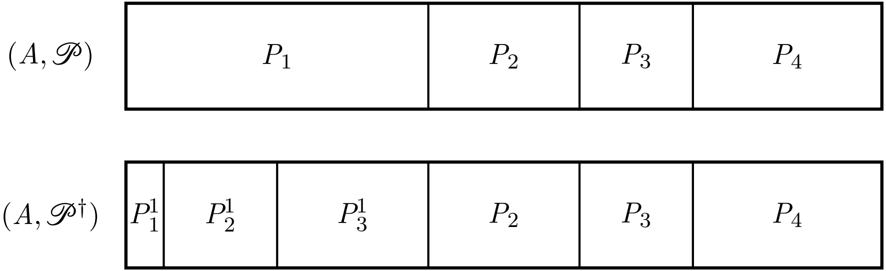
$$(A, \{P_1 \ldots P_n\}) \succcurlyeq (B, \{Q_1 \ldots Q_n\}).$$

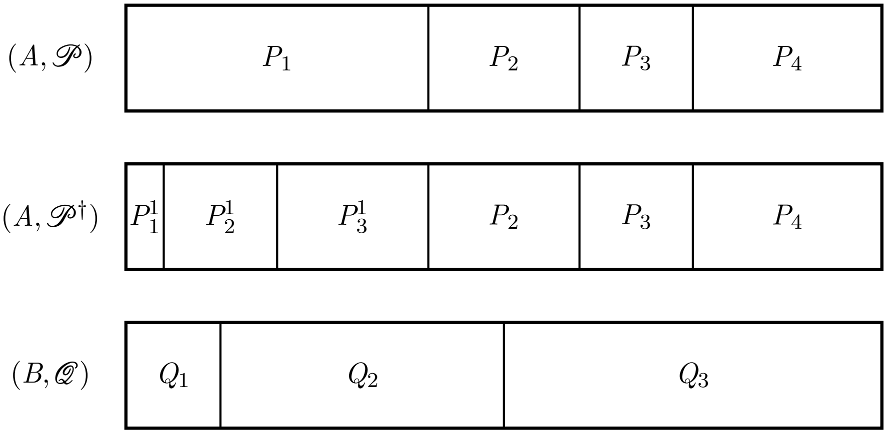
- implies structural invariance:

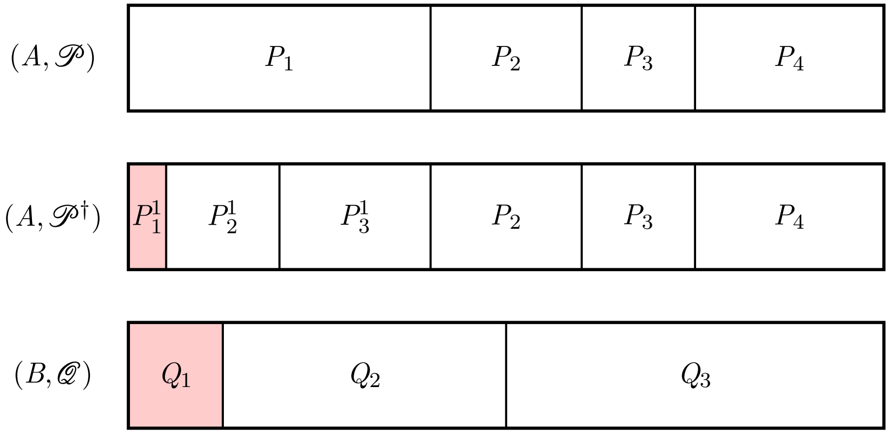
- Fix $(A, \mathscr{P} = \{P_1, \ldots P_n\})$ and let $\mathscr{P}^1 = \{P_1^1, \ldots P_k^1\}$ partition $P_1$.
- Then $\mathscr{P}^\dagger = \{P_1^1, \ldots P_k^1, P_2, \ldots P_n\}$ is also partition of $A$.
    - As if observing $\mathscr{P}$ and then if $P_1$ is realized, further observing $\mathscr{P}^1$
    - $\mathscr{P}$ observed with prob 1, $\mathscr{P}^1$ observed with probability $\mu(W_{A,\mathscr{P}_1})$
- Let $(B, \mathscr{Q} = \{Q_1, \ldots Q_k\})$ with $\mu(W_{B,Q_i}) = \mu(W_{A,P_i^1} \mid W_{A,P_1})$
    - Observing $\mathscr{Q}$ has same 'informational content' as observing $\mathscr{P}^1$ conditional on realization of $P_1$

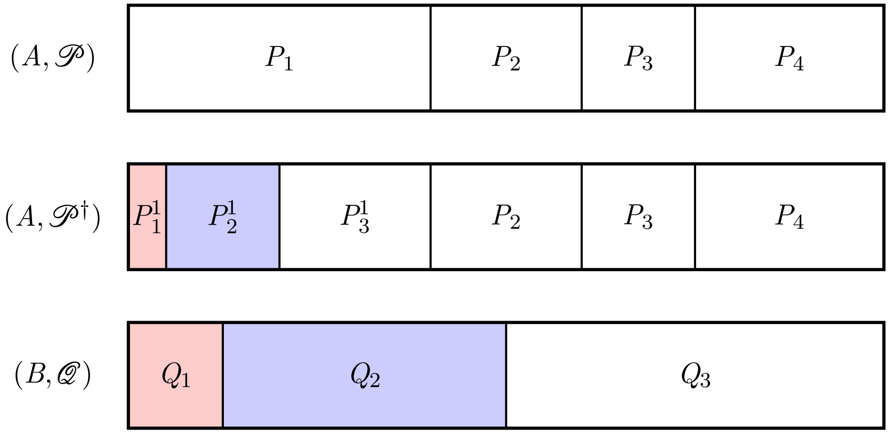
$(A, \mathscr{P})$



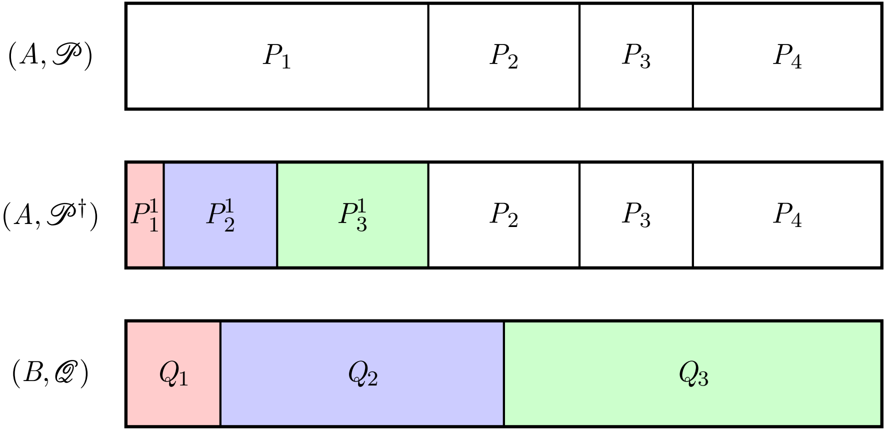

$(A, \mathscr{P})$
$P_1$
$P_2$
$P_3$
$P_4$
$(A, \mathscr{P}^\dagger)$
$P_1^1$
$P_2^1$
$P_3^1$
$P_2$
$P_3$
$P_4$


$(A, \mathscr{P})$
$P_1$
$P_2$
$P_3$
$P_4$
$(A, \mathscr{P}^\dagger)$
$P_1^1$
$P_2^1$
$P_3^1$
$P_2$
$P_3$
$P_4$
$(B, \mathscr{Q})$
$Q_1$
$Q_2$
$Q_3$


$(A, \mathscr{P})$
$P_1$
$P_2$
$P_3$
$P_4$
$(A, \mathscr{P}^\dagger)$
$P_1^1$
$P_2^1$
$P_3^1$
$P_2$
$P_3$
$P_4$
$(B, \mathscr{Q})$
$Q_1$
$Q_2$
$Q_3$



$(A, \mathscr{P})$
$P_1$
$P_2$
$P_3$
$P_4$
$(A, \mathscr{P}^\dagger)$
$P_1^1$
$P_2^1$
$P_3^1$
$P_2$
$P_3$
$P_4$
$(B, \mathscr{Q})$
$Q_1$
$Q_2$
$Q_3$


$(A, \mathscr{P})$
$P_1$
$P_2$
$P_3$
$P_4$
$(A, \mathscr{P}^\dagger)$
$P_1^1$
$P_2^1$
$P_3^1$
$P_2$
$P_3$
$P_4$
$(B, \mathscr{Q})$
$Q_1$
$Q_2$
$Q_3$

| Option 1 | Option 2 |
| --- | --- |
| ◇ Observe $\mathscr{P}^\dagger$ with prob $\alpha$ | ◇ Observe $\mathscr{P}$ with prob $\alpha$ |
| ◇ Observe nothing with prob $(1-\alpha)$ | ◇ Observe $\mathscr{Q}$ with prob $(1-\alpha)$ |

- To be indifferent $\alpha$ must capture the likelihood of reviving extra information
- This is $\alpha = \frac{1}{1+\mu(W_{A,P_1})}$

## Identification Separability for Entropy

Fix $(A, \mathscr{P} = \{P_1, \dots P_n\})$ and $\mathscr{P}^\dagger = \{P_1^1, \dots P_k^1, P_2, \dots P_n\}$.

Set $\alpha = \frac{1}{1+\mu(W_{A,P_1})}$.

Then if $(B, \{Q_1, \dots Q_k\})$ is such that $\mu(W_{B,Q_i}) = \mu(W_{A,P_i^1} \mid W_{A,P_1})$, it follows that

$$\alpha(A, \mathscr{P}^\dagger) + (1-\alpha)(A, \{A\}) \sim \alpha(A, \mathscr{P}) + (1-\alpha)(B, \mathscr{Q})$$

- implies identification separability

**Theorem**

Let $\succcurlyeq$ be an expected utility preference.

Then $\succcurlyeq$ satisfies the entropic versions of `P1-3` if and only if

$$F(A, \mathscr{P}) = -\sum_{P \in \mathscr{P}} \log(\mu(W_{A,P}))\mu(W_{A,P})$$

is an utility index for $\succcurlyeq$

Thank You!