Three Dimensional Skewed Brownian Motion

Evan Prentice - 50109870

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Abstract - **Brownian motion is traditionally defined as the random motion and interaction of particles within a medium. This definition can then be more generally defined as the movement of any physical phenomena where dynamic the data points experience random fluctuations. Being able to mathematically model this phenomenon clearly has direct applications for predicting the behavior of microscopic chemical processes of many fast moving particles, like diffusion or chemical kinetics, but, it also useful for describing other independent stochastic processes, like stock price movement or traffic flow models. That being said, Brownian motion is primarily a desirable method for simulating processes like the ones listed above due to the normally distributed probability density function (PDF) that it produces as a result. Being able to interpret the results of a simulation as a basic Gaussian-distribution allows the models to be easily estimated with great precision. However, most real world phenomena do not behave according to a perfectly symmetrical normal distribution and instead have asymmetric PDFs that are skewed in a certain direction due to unknowns present in their data. Thus, the purpose of this paper is to analyze how a skewed Gaussian distribution can be generated from three dimensional Brownian motion. In particular, the type of Brownian motion that this paper will focus on is called random walks.**

I. Introduction

Random walks are a variation of Brownian motion, in that they still involve the random movement of discrete particles, but with a few different underlying assumptions. First, it is assumed that any present particle movements happen independently of past and future movements. Second, movements are of a discrete fixed length and happen in at discrete intervals of time. Third, the system of particles do not interact with one another and therefore each particle is independent of the movement and position of the other particles.

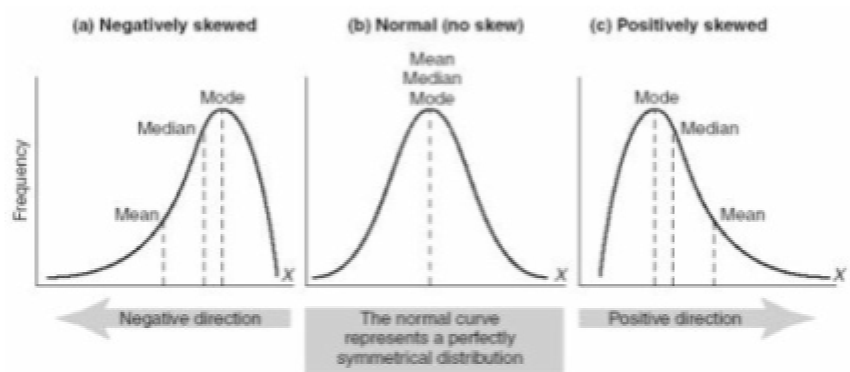

Figure 1

Using the above three assumptions, the random walks of a large amount of particles will generally result in a Gaussian distribution, with respect to the PDF of distance between the particles and origin at a certain time. In other words, the probability distribution for the total distance 'walked' in a random walk will tend towards a normal distribution. This happens because of the Central Limit Theorem, which states "the arithmetic mean of a large enough number of independent random variables, with a finite expected value and variance, will be approximately normally distributed." Thus, if a large enough number of particles are used in the random walk, the resulting PDF has a shape similar to a normal distribution as seen in (b) of Figure 1. The PDF is is symmetric about the mean, and has a mean value that is similar, if not equal to, the mode value. When the mean = mode the PDF can be described as having zero skew, and is therefore has a Gaussian distribution.

In contrast, a skewed Gaussian distribution has a PDF similar to the PDFs in parts (a) and (c) in Figure 1. Skewed Gaussian distributions can be identified by their asymmetrical appearance. The asymmetry or skew is a result of the mean value being significantly larger (positive skew) or smaller (negative skew) than the mode's value. Typically, this only happens when there are outlier data points that have extreme values relative to all the other data points making up the PDF.

For example, in regards to the national wealth of the United States, it is estimated that the extremely wealthy (top 1%) have a total net worth greater than the total net worth of the bottom 90% of the population. This would mean the financial data of the top 1% would have extremely large values compared to the vast majority of the data points, resulting in a positive skew. Even though just 1% of the data would be considered outlier data, there is enough disproportion in the amount of wealth of that 1%, that the mean value is pulled to be larger than the bulk of the population density (mode).

Thus, in order to generate a skewed Gaussian distribution from a random walk, the standard random walk process must be altered so that outlier data points have more extreme values in comparison to the rest of the data values. But first, what is the simple random walk process?

II. Procedure for 3D Random Walk Resulting in a Gaussian Distribution

In a random walk, the path of a single particle in three dimensional space is randomly dictated by the probability of movement in any particular direction. For 3D space, the possible movement directions are just the positive and negative unit vectors of each axis: $+\hat{x},\ -\hat{x},\ +\hat{y},\ -\hat{y},\ +\hat{z},\ -\hat{z}$. Since, this is an unbiased random walk, the particle has equal probabilities to move in any one of the six possible directions and therefore, $P(+\Delta\hat{x}) = P(-\Delta\hat{x}) = P(+\Delta\hat{y}) = P(-\Delta\hat{y}) = P(+\Delta\hat{z}) = P(-\Delta\hat{z}) = 1/6$. To properly simulate this, each of the six directions are assigned a range of values, each having a width of 1/6 so that the sample space can be thought of as a number line from 0 to 1, representing the normalized probabilities of each possible movement direction, as seen in Figure 3. For example, $P(+\Delta\hat{x})$ has the range [0,⅙), $P(-\Delta\hat{x})$ has the range [⅙ , 2/6), $P(+\Delta\hat{y})$ has the range [2/6, 3/6), etc.

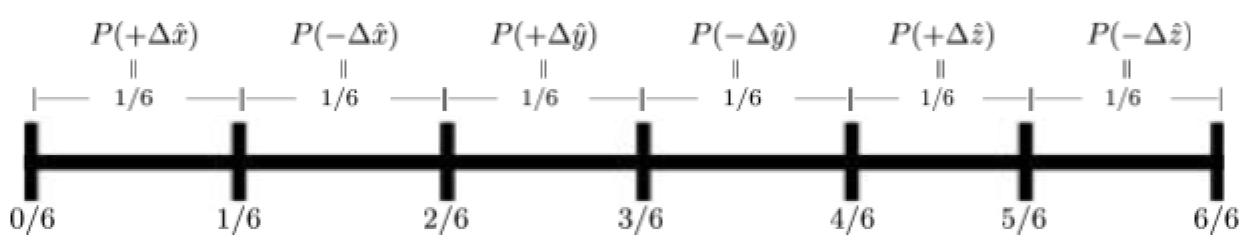

Figure 3

At each discrete interval of time (Δt) a uniform random number in the range of [0,1] will be generated. This random number is what determines what direction the particle moves in each step, depending on which of the six ranges the random number falls in. For example, at step $t = \Delta t$ if the random number has the value of 0.45, the corresponding range is $P(+\Delta\hat{y})$, so the particle will move a discrete distance in the positive y-direction for that step. Repeating this process many times results in a plot similar to Figure 4. In Figure 4, the green circle denotes the origin and the red triangle denotes the final position of the particle at $t = 1000\Delta t$.

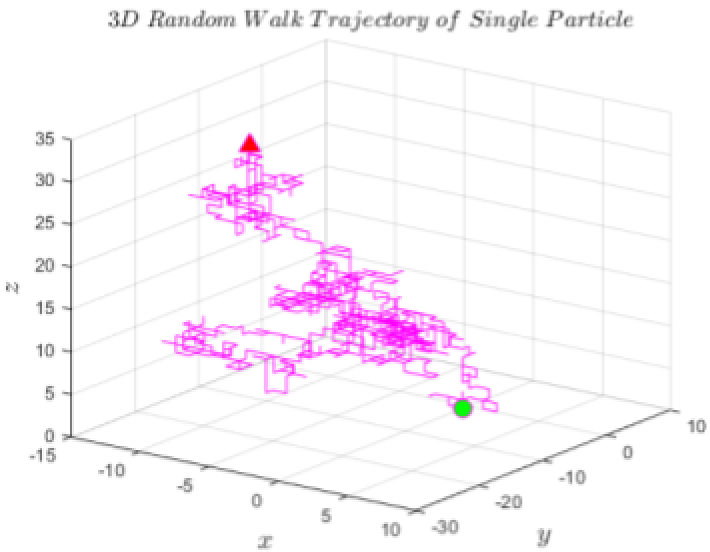

Figure 4

By repeating the process described above with many particles, a Gaussian distribution can be observed. The sphere in Figure 5 shows the results of repeating this process when 10^7 particles are used.

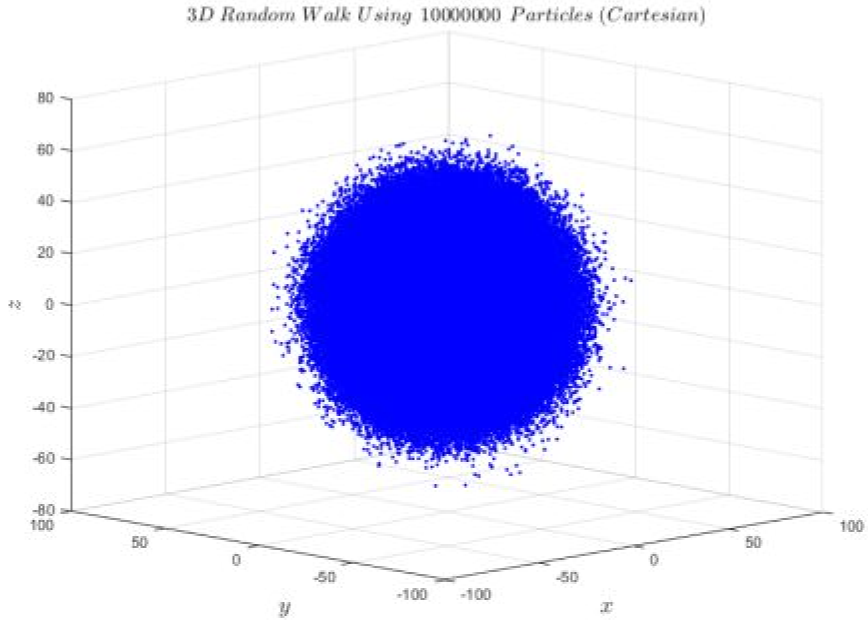

Figure 5

To convert this discrete random walk process into a continuous mathematical equation, let *u(x,y,z,t)* give the probability that a particle is at *(x,y,z)* at time *t*. The particle has a 1/6 probability of moving in any of the six unit directions in three dimensional space. Therefore, the probability of moving a certain direction in 3D space, in a discrete step of time (t + Δt) can be given as:

$$\begin{aligned} u(x,y,z,t+\Delta t) = \frac{1}{6}u(x+\Delta x,y,z,t) + \frac{1}{6}u(x-\Delta x,y,z,t) \\ +\frac{1}{6}u(x,y+\Delta y,z,t) + \frac{1}{6}u(x,y-\Delta y,z,t) \\ +\frac{1}{6}u(x,y,z+\Delta z,t) + \frac{1}{6}u(x,y,z-\Delta z,t) \end{aligned}$$

Taylor expansion around t + Δt → t:

$$u(x,y,z,t+\Delta t) \Rightarrow u(x,y,z,t) + u_t(x,y,z,t)(t+\Delta t-t) + ...$$

Taylor expansion around x + Δx → x:

$$\frac{1}{6}u(x+\Delta x,y,z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_x(x,y,z,t)(x+\Delta x-x)+\frac{1}{2}u_{xx}(x,y,z,t)(x+\Delta x-x)^2+...]$$

Taylor expansion around x - Δx → x:

$$\frac{1}{6}u(x-\Delta x,y,z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_x(x,y,z,t)(x-\Delta x-x)+\frac{1}{2}u_{xx}(x,y,z,t)(x-\Delta x-x)^2+...]$$

Taylor expansion around y + Δy → y:

$$\frac{1}{6}u(x,y+\Delta y,z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_y(x,y,z,t)(y+\Delta y-y)+\frac{1}{2}u_{yy}(x,y,z,t)(y+\Delta y-y)^2+...]$$

Taylor expansion around y - Δy → y:

$$\frac{1}{6}u(x,y-\Delta y,z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_y(x,y,z,t)(y-\Delta y-y)+\frac{1}{2}u_{yy}(x,y,z,t)(y-\Delta y-y)^2+...]$$

Taylor expansion around z + Δz → z:

$$\frac{1}{6}u(x,y,z+\Delta z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_z(x,y,z,t)(z+\Delta z-z)+\frac{1}{2}u_{zz}(x,y,z,t)(z+\Delta z-z)^2+...]$$

Taylor expansion around z - Δz → z:

$$\frac{1}{6}u(x,y,z-\Delta z,t) \Rightarrow \frac{1}{6}[u(x,y,z,t)+u_z(x,y,z,t)(z-\Delta z-z)+\frac{1}{2}u_{zz}(x,y,z,t)(z-\Delta z-z)^2+...]$$

Assuming Δx = Δy = Δz, lets set all the increments in space to a common value *h*. Then after combining everything into the original and after simplifying, the partial differential equation describing the particle's movement is given by:

$$u_t(x,y,z,t) = \frac{h^2}{6\Delta t}[u_{xx}(x,y,z) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t)]$$

It is noted that this is the same equation that is used to model the diffusion process.

Since the desired PDF of a distribution of distance traveled from the origin, and because the final positions of the particles form a sphere shape, it is convenient to convert to the spherical coordinate system. This will allow the PDF to be just of the radial distance parameter (*r*) in spherical coordinates, instead of three separate probability distributions for (x, y, z) in the coordinate system.

Thus, the last step is creating a histogram of all the radial distances as seen in Figure 6. The resulting histogram is the PDF of radial distance, and clearly resembles a Gaussian distribution.

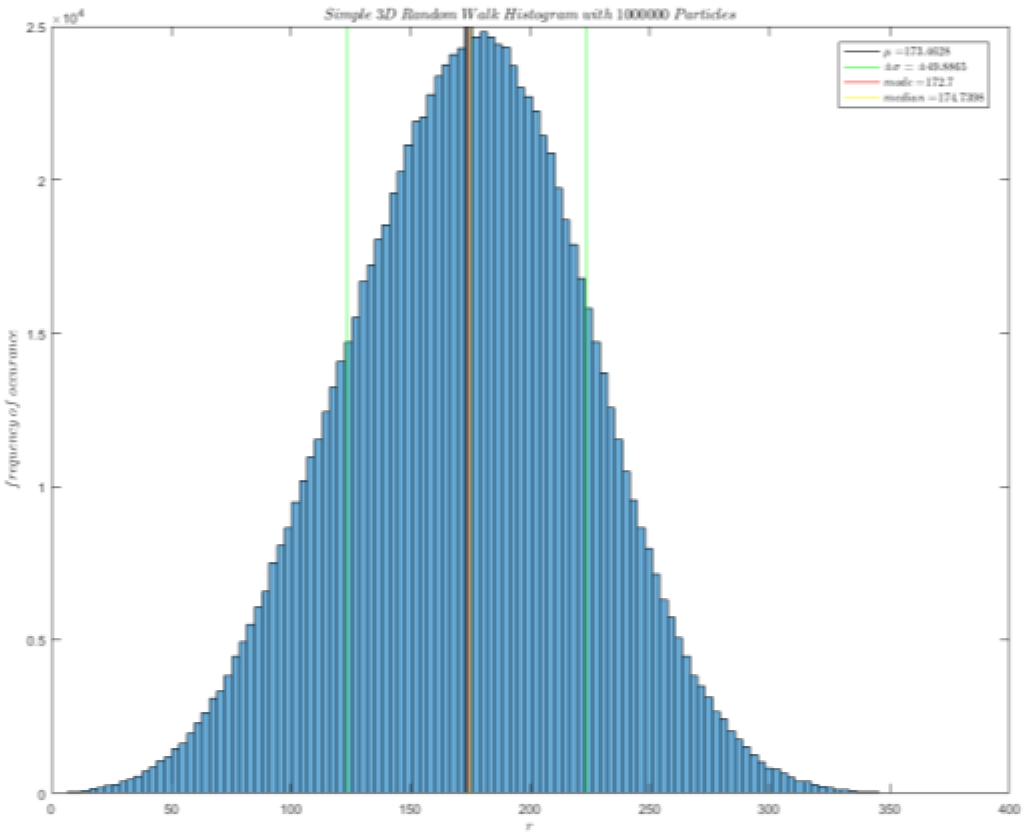

Figure 6

To further demonstrate that this distribution is Gaussian, it is noted that histogram in Figure 6 has a mean value of 173.4628, a mode value of 172.2 and a median value of 174.7398. These values are all almost equal, and since an ideal Gaussian distribution has the same values for mean, median, and mode, this distribution can be considered normally distributed. Of course, the distribution in Figure 6 does not have an equal mean, median, and mode, but in order to get the perfect Gaussian distribution an infinite amount of particles would be needed in the random walk. Furthermore, the last thing to note about Figure 6 is that it is nearly symmetrical. In its first lower standard deviation (-σ) the histogram contains 3515625 particles and the upper first standard deviation (+σ) there are 3545420 which are fairly similar, thus indicating symmetry.

III. Generating a Skewed Gaussian Distribution from a 3D Random Walk

To create a skewed Gaussian distribution from a 3D random walk, the random walk process must be changed so that the PDF for particle distance from the origin has a different mean and mode. A simple answer to obtaining a different mean and mode, is to add conditions to the random walk so that some the directional movement probabilities for some particles become biased in a certain direction. This will cause some particles to move much further (away from the origin) resulting in a much larger radial distance for those particles when compared to all of the other particles.

Specifically, this was done using an adjustment parameter (α) to alter the probabilities for directional movement, depending on past directional movements. Every time the particle moves in a certain direction, the probability range corresponding to that directional movement is slightly increased by some value α, and at the same time, all the other directional movement probability ranges are decreased by $\alpha/5$ in order to keep the total range of values between 0 and 1. The adjusting of directional probability ranges according to the particles random movements essentially creates the potential for a positive feedback loop, in that, once a particle moves in a certain it is more likely to move in that same direction again If the particle does happen to move in that same direction again, the probability range for moving in that particular direction starts to snowball, eventually resulting in a very large probability for the particle to move in that one direction and a very small probability that the particle moves in any other direction The adjusting probability ranges in response to particle movements can be visualized in Figures 7-10.

At t = 0, when the particle has not taken any steps yet, the directional probability ranges can be seen in Figure 7, which is the same as they are in the simple random walk.

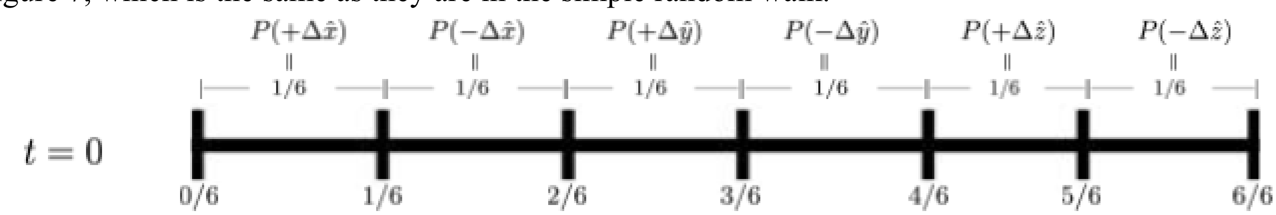

Figure 7

If the first step generates a random number of 0.05, the particle moves in the $+x$ direction because the random number is in the directional probability range between 0 and 1/6. Now, because the particle moved in the $+x$ direction, the adjustment parameter increases the probability range of $+x$ by α and pushes all the other probability ranges (except the last one) to the right by $\alpha/5$. Figure 8 shows the adjusted probability ranges at the beginning of the next step.

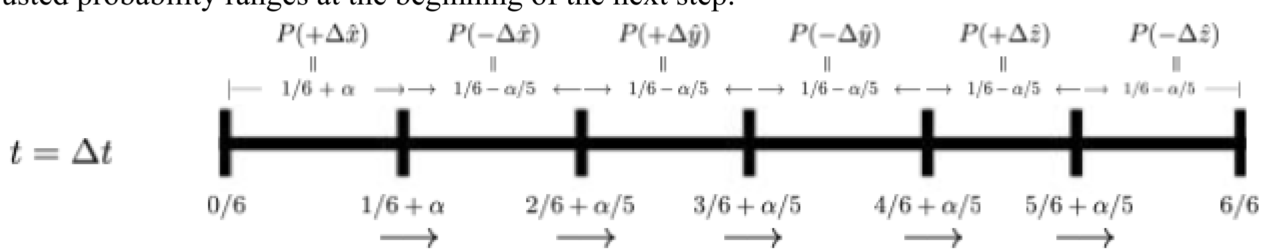

Figure 8

To further demonstrate how the adjusting probabilities works, Figures 9 and 10 also show how the probability ranges are changed in response to different random variables for the t = 0 step.

Figure 9 shows how the probability ranges change if the first random variable causes the particle to move in the -*x* direction. It is noted that the bounds for the -*x* direction probability (1/6 and 2/6) are not adjusted equally in order to keep the total probability interval between 0 and 1.

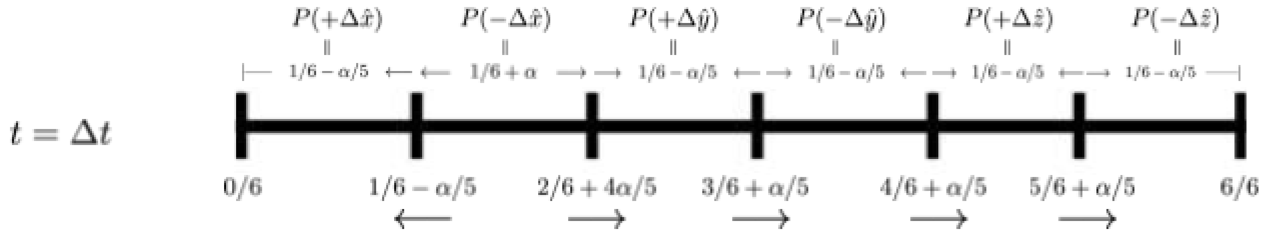

Figure 9

Figure 10 shows how the probability ranges change if the first random variable causes the particle to move in the +y direction.

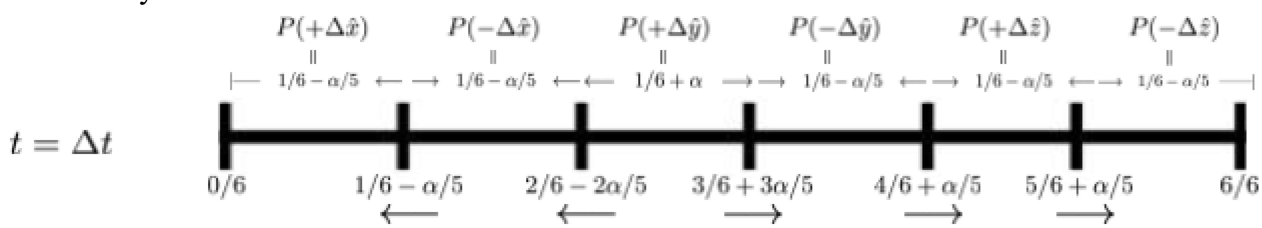

Figure 10

IV. Results

The results from the skewed random walk can be seen in Figures 11 and 12. The skewed random walk was done using 10^6 particles and with the adjustment parameter α = 0.015.

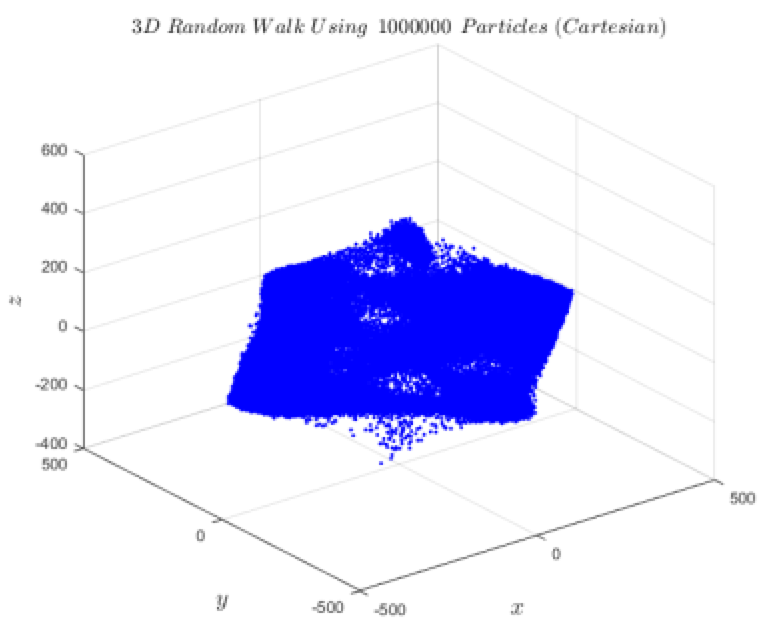

Figure 11

Figure 11 shows the final positions of the 10^6 particles. It is observed that the particles no longer form a sphere, but more of an octahedron shape.

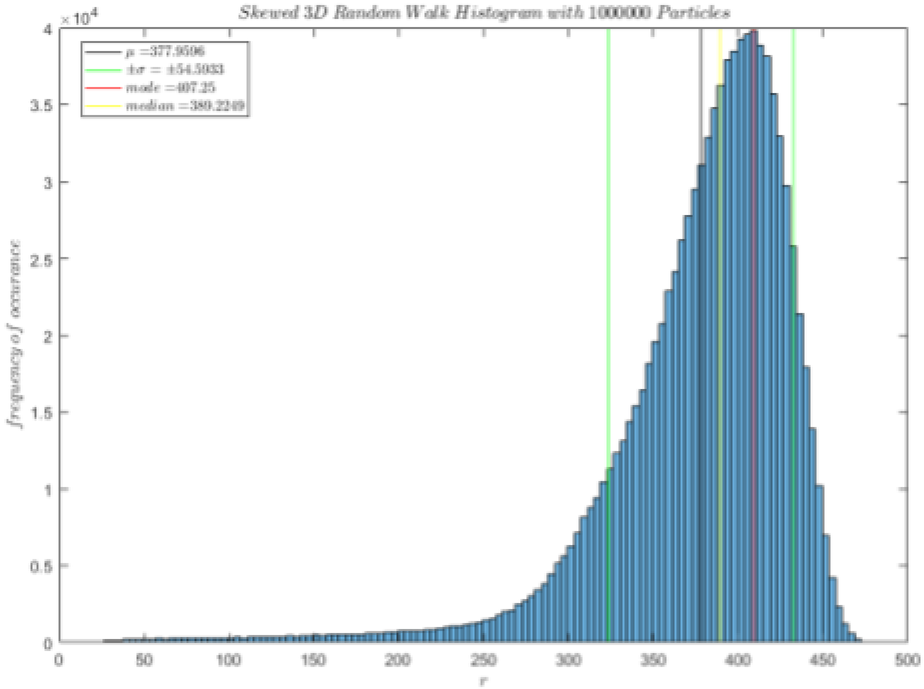

Figure 12

Figure 12 shows the histogram for the distance from the origin the 10^6 particles end up at in the skewed random walk. Additionally, it is observed that the amount of skew in the PDF depends on what value α is set to equal. Larger α values result in more skewed PDFs and a $\alpha = 0$ results in the normal distribution.

V. Discussion

The distribution in Figure 12 is observed to have a mean of 377.9596, a mode of 407.25, and a median of 389.2249. Thus, due to the significant difference in mean, median, and mode, the distribution in Figure 12 is a skewed Gaussian distribution. Specifically it is a negatively skewed distribution due to the mean being smaller than the mode. The asymmetry of this skewed distribution when the standard deviations are taken into account. In the lower standard deviation (- σ,) there is found to be 309045 particles and in the upper standard deviation (+ σ) there is found to be 534282 particles. Thus, due to the disproportionate particle density in the upper standard deviation, and due to the difference in mean, median, and mode, it can be concluded that the random walk with the adjustment parameter does result in a skewed Gaussian distribution.

Sources:

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