538 Riddler Classic

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Problem

King Auric adored his most prized possession: a set of perfect spheres of solid gold. There was one of each size, with diameters of 1 centimeter, 2 centimeters, 3 centimeters, and so on. Their brilliant beauty brought joy to his heart. After many years, he felt the time had finally come to pass the golden spheres down to the next generation — his three children.

He decided it was best to give each child precisely one-third of the total gold by weight, but he had a difficult time determining just how to do that. After some trial and error, he managed to divide his spheres into three groups of equal weight. He was further amused when he realized that his collection contained the *minimum* number of spheres needed for this division. How many golden spheres did King Auric have?

Extra credit: How many spheres would the king have needed to be able to divide his collection among other numbers of children: two, four, five, six or even more?

Solution

King Auric had 23 spheres. Each child received 13,295 cm^3 of gold, weighing 257 kg and worth \$14.31 million.

The table in Fig. 1 shows the solution to the extra credit: the minimum number of spheres for each number of children k.

k	minimum number of spheres
2	12
3	23
4	24
5	24

Figure 1: Extra credit solution

I also calculated the solution for gold in different dimensions (1-D wires, 2-D discs, etc.), although in its current state, my code did not perform well for 4-D. (Fig. 2)

k	1-D	2-D	3-D	4-D
2	3	7	12	16
3	5	13	23	?
4	7	15	24	?
5	9	19	24	?

Figure 2: Dimensionally-generalized solution

Justification

This problem is a specific instance of the 3-partition problem (or, for the extra credit, the k-partition problem). This problem asks, given a multiset of numbers (a set which can contain duplicate elements), how can those numbers be put in three "mutually exclusive and collectively exhaustive" subsets. I looked to Korf, 2009[1] for information and algorithms about this problem. (I will be using terminology from this paper in the remainder of this report.)

In the Riddler, the set to be partitioned, for each number of spheres n is $\{\rho\frac{\pi}{6}, \rho\frac{4\pi}{3}, ..., \rho\frac{4\pi}{3}(\frac{n}{2})^3\}$, where ρ is the density of gold. Of course, we can factor out ρ as well as a factor of $\frac{\pi}{6}$. This makes the set $1, 8, ..., n^3$, all of which are integers, and more comprehensible.

On to the partitioning algorithm. This could, of course, be done brute force, by simply checking every possible way to partition the values, but this is too slow and inefficient. Korf describes a new, very efficient method, but for a problem like this, a simpler algorithm should do the trick. The "Complete Greedy Algorithm" (CGA) is what I ended up using. It recursively searches a tree, where at each level, one member of the set to be partitioned is put in a subset, and the different branches at each node correspond to putting it in a different subset. Korf provides a few pruning rules to reduce the number of branches to be sorted. Since we're only looking for a perfect solution, in which all subsets must be equal (as opposed to the best approximation, if no perfect solution exists), I made a few slight modifications:

- Once a solution is found, return it and stop.
- If t is the sum of the original set, s the current largest subset sum, d the difference (between largest and smallest subset sums) of the best solution so far, and k the number of partitions (3 in the original problem), then stop if $s - \frac{t-s}{k-1} \geq d$.
- If, at any step, multiple of the partial subsets have the same sum, then only continue to search one of them.
- Of the new branches to check at any node, check from smallest partial subset sum to largest.

Following this algorithm, a solution to the problem is the following: (the numbers correspond to the radius of the sphere cubed, or, equivalently, $\frac{6}{\pi}$ times the weight.)

Child 1: 1, 64, 343, 512, 1728, 4096, 8000, 10648

Child 2: 8, 125, 729, 1331, 2744, 3375, 4913, 12167

Child 3: 27, 216, 1000, 2197, 5832, 6859, 9261

[1] https://www.ijcai.org/Proceedings/09/Papers/096.pdf