Assignment 2B

Blake Evans

1.4.1) Find the Minimum value for

(i) $P(x) = \frac{1}{2}(x_1^2 + x_2^2) - x_1 b_1 - x_2 b_2$

$P(x) = \frac{1}{2} x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - x^T \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Then, since $A^{-1} = A$ and $Ab = A^{-1}b$ which also equals $A^{-1}b = b = [b_1\ b_2]^T$ we can say

$$P(x_{min}) = -\frac{1}{2}(b^T A^{-1} b) \longrightarrow \boxed{P(x_{min}) = -\frac{1}{2}(b_1^2 + b_2^2)}$$

(ii) $P(x) = \frac{1}{2}(x_1^2 + 2x_1 x_2 + 2x_2^2) - x_1 + x_2$.

$P(x) = \frac{1}{2} x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x - x^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

And, like before, we can say

$$P(x_{min}) = -\frac{1}{2}(b^T A^{-1} b) = -\frac{1}{2} b^T x \longrightarrow \boxed{P(x_{min}) = -\frac{5}{2}}$$

1.4.2) What equations determine the minimizing x for

(i) $P = \frac{1}{2} x^T A x - x^T b$

From eq. 1E from the book, we know $Ax = b$. So,

$$P = \frac{1}{2} x^T A x - x^T b \rightarrow P = \frac{1}{2} x^T (b) - x^T b$$

$$P = -\frac{1}{2} x^T b$$

(ii) $P = \frac{1}{2} x^T A^T A x - x^T A^T b$

Also from eq. 1E, we can set $A = A^T A$ and $b = A^T b$. So that $Ax = b \rightarrow (A^T A) x = A^T b$. Then we can say

$$P = \frac{1}{2} x^T (A^T b) - x^T A^T b$$

$$P = -\frac{1}{2} x^T A^T b$$

(iii) $E = \|Ax - b\|^2$

$$E = \|Ax-b\|^2 = (Ax-b)^T(Ax-b)$$

$$E = x^TA^TAx - 2x^TA^Tb + b^Tb$$

Since b^Tb is a constant, we can ignore it. And, to minimize this, we can again use $A^TAx = A^Tb$.

$$E = x^T(A^Tb) - 2x^TA^Tb \longrightarrow \boxed{E = -x^TA^Tb}$$

1.4.4) Find the Least Squares Solution to

$$Ax = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = b$$

Graph 0,4,2 @ $t = 1,2,3$ and find line

$$A^TAx = A^Tb \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$A^TAx = A^Tb \longrightarrow \begin{bmatrix}3 & 6\\ 6 & 14\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix} = \begin{bmatrix}6\\ 14\end{bmatrix}$$

By inspection, $[C, D]^T = x$ must be $[0,1]^T$ to minimize $\|Ax-b\|^2$. Thus, the best straight line is $y = t$.

Let $x = C$ and $y = D$, $\left.\begin{array}{l} x + y = 0 \\ x + 2y = 4 \\ x + 3y = 2 \end{array}\right\} \begin{array}{l} y = -x \\ y = -\frac{1}{2}x + 2 \\ y = -\frac{1}{3}x + \frac{2}{3} \end{array}$