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Representation of hypergeometric products of higher nesting depths in difference rings

Evans Doe Ocansey [a,b,c], Carsten Schneider [b]

[a] Johannes Kepler University Linz, Institute for Algebra, Linz, Austria
[b] Johannes Kepler University Linz, Research Institute for Symbolic Computation (RISC), Linz, Austria
[c] RISC Software GmbH, Softwarepark 32a, 4232 Hagenberg, Austria

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ABSTRACT

A non-trivial symbolic machinery is presented that can rephrase algorithmically a finite set of nested hypergeometric products in appropriately designed difference rings. As a consequence, one obtains an alternative representation in terms of one single product defined over a root of unity and nested hypergeometric products which are algebraically independent among each other. In particular, one can solve the zero-recognition problem: the input expression of nested hypergeometric products evaluates to zero if and only if the output expression is the zero expression. Combined with available symbolic summation algorithms in the setting of difference rings, one obtains a general machinery that can represent (and simplify) nested sums defined over nested products.

1. Introduction

An important problem in symbolic summation is the simplification of sums defined over products to expressions in terms of simpler sums and products; in the best case, one might find an expression without sums. A first milestone was Gosper's algorithm (Gosper, 1978) and Zeilberger's groundbreaking application for creative telescoping (Zeilberger, 1991) where the summand is given by one hypergeometric product. Further extensions have been accomplished for q-hypergeometric and multibasic products (Paule and Riese, 1997), their mixed versions (Bauer and Petkovšek, 1999)

E-mail addresses: evans.ocansey@risc.jku.at (E.D. Ocansey), cschneid@risc.jku.at (C. Schneider).

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and (q–)multi-summation (Riese, 2003; Wegschaider, 1997; Wilf and Zeilberger, 1992). In addition, structural properties and further insight in this setting have been elaborated, e.g., in Chen et al. (2011, 2013); Paule (1995). More generally, the holonomic system approach (Zeilberger, 1990) and their refinements (Blümlein et al., 2018; Chyzak, 2000; Koutschan, 2013) represent, e.g., multi-sums over (q–)hypergeometric products by systems of linear recurrence relations.

In particular, Karr's difference field approach (Karr, 1981, 1985) paved the way for a general framework to represent rather complicated product expressions in a formal way. Here the generators of his $\Pi\Sigma$-field construction enable one to model (up to a certain level) indefinite nested products of the form

$$P(n)=\prod_{k_1=\ell_1}^{n} f_1(k_1)\prod_{k_2=\ell_2}^{k_1} f_2(k_2)\cdots\prod_{k_m=\ell_m}^{k_{m-1}} f_m(k_m) \tag{1}$$

where the multiplicands $f_i(n)=\frac{p_i(n)}{q_i(n)}$ for all i with $1\leq i\leq m$ are built by polynomial expressions $p_i(n)$ and $q_i(n)$ in terms of indefinite nested products that are again of the form (1). In Karr's seminal works (Karr, 1981, 1985), often considered as the discrete version of Risch's integration algorithm (Risch, 1969), a sophisticated algorithm is provided that enables one to test if a given product representation (and sums defined over such products) are expressed properly in his $\Pi\Sigma$-field setting. As a bonus, his toolbox and refinements in Schneider (2007a, 2008, 2015) enable one to decide constructively if a given summand represented in a $\Pi\Sigma$-field has a solution in the same field or in an appropriate extension of it.

The appropriate representation of products in the setting of difference fields and rings turns out to be much harder. First important results have been derived to simplify *one single-nested* hypergeometric product $P(n)=\prod_{k=\ell}^{n} f(k)$ with a rational function $f(x)\in\mathbb{K}(x)^*$ and $\ell\in\mathbb{Z}_{\geq 0}$ chosen properly (i.e., $P(n)$ is well defined and nonzero for all $n\in\mathbb{Z}_{\geq 0}$). In Abramov and Petkovšek (2002) algorithms have been developed to compute a hypergeometric product $Q(n)=\prod_{k=\ell'}^{n} h(k)$ with $h(x)\in\mathbb{K}(x)^*$, $\ell'\in\mathbb{Z}_{\geq 0}$ and $r(x)\in\mathbb{K}(x)^*$ such that

$$P(n)=r(n)\prod_{k=\ell'}^{n} h(k) \tag{2}$$

holds for all n sufficiently large and the degrees of the numerator and denominator of the multiplicand $h(x)$ are minimal among all the possible solutions; these ideas have been generalized in Schneider (2005) for a multiplicand that is given in Karr's general difference field setting. As observed in Abramov et al. (2003) (and later extended in Abramov and Petkovšek (2010) to the general difference field setting) this representation is not unique and provides insight for further optimizations: among all the possible choices of $Q(n)$ (with minimal degrees) also the factor $r(x)\in\mathbb{K}(x)$ can be optimized by minimizing certain combinations of the degrees of the numerator and denominator. Furthermore, it has been shown in Schneider (2005) that such a product $P(n)$ can be represented in a $\Pi\Sigma$-field if and only if it cannot be rewritten in the form $P(n)=r(n)\,\zeta^n$ where $r(x)\in\mathbb{K}(x)$ and $\zeta\in\mathbb{K}$ is a primitive λ-th root of unity with $\lambda>1$. As a consequence, a hypergeometric product $P(n)$ in general cannot be represented in a difference field but only in difference rings modeling elements ζ^n with the relation $(\zeta^n)^\lambda=1$. In particular, zero-divisors are introduced coming from $(1+\zeta^n+\dots+(\zeta^n)^{\lambda-1})(1-\zeta^n)=0$.

Motivated by this observation (among others) a refined difference ring theory has been elaborated in Schneider (2016, 2017) that combines big parts of Karr's general framework together with generators of the form ζ^n. In this way, not only one hypergeometric product $P(n)$, but more generally any polynomial expression in terms of several hypergeometric products $P_1(n),\dots P_e(n)$ can be represented in the class of so-called R$\Pi\Sigma$-extensions. The first algorithms derived in Schneider (2005, 2014) require that the input products are defined over $\mathbb{K}(x)$ where $\mathbb{K}=\mathbb{Q}(\kappa_1,\dots,\kappa_u)$ with $u\geq 0$ is a rational function field defined over the rational numbers $\mathbb{Q}$. More generally, a complete algorithm has been elaborated in Ocansey and Schneider (2018) (utilizing ideas from Ge (1993b); Schneider (2014)) that can represent a finite set of hypergeometric products over $\mathbb{K}(x)$ where $\mathbb{K}=K(y_1,\dots,y_r)$

is a rational function field defined over an algebraic number field K. In addition, using algorithms from Bauer and Petkovšek (1999) it can deal also with q-hypergeometric, multibasic and mixed hypergeometric products. Finally, a general framework has been elaborated in Schneider (2020) that considers single nested products defined over a general class of difference fields; an extra bonus is that the latter approach constructs a difference ring in which the given products are rephrased optimally with the following property: the number of generators of the ring and the order λ of the used ζ^n are minimal.

A remarkable feature of the above algorithms (Ocansey and Schneider, 2018; Schneider, 2005, 2014, 2020) is that the given input expression of hypergeometric products (and their generalized versions) are rephrased in terms of a finite set of alternative products $Q_1(n), \ldots, Q_s(n)$ together with a distinguished root of unity product ζ^n such that the sequences produced by $Q_1(n), \ldots, Q_s(n)$ are algebraically independent among each other. For this result we rely on ideas of Schneider (2017) that are inspired by Hardouin and Singer (2008); Schneider (2010a); compare also Chen et al. (2011). We remark further that these results are also connected to Kauers and Zimmermann (2008) that can compute all algebraic relations of C-finite solutions (i.e., solutions of homogeneous recurrences with constant coefficients).

We emphasize that the algorithms from Abramov and Petkovšek (2010); Ocansey and Schneider (2018); Schneider (2005, 2014, 2020, 2023) can be utilized to simplify hypergeometric solutions (Abramov et al., 2021; Van Hoeij, 1999; Petkovšek, 1992) of linear difference equations and can be combined with symbolic summation algorithms (Schneider, 2007a, 2008, 2015) to simplify more general solutions, such as d'Alembertian solutions (Abramov and Petkovšek, 1994; Abramov and Zima, 1996) and Liouvillian solutions (Hendriks and Singer, 1999; Petkovšek and Zakrajšek, 2013).

In this article we aim at extending this toolbox significantly for the class of nested hypergeometric products that can be defined as follows; in this article all fields and rings have characteristic 0.

Definition 1.1. Let $\mathbb{K}(x)$ be a rational function field and let $f_1(x), \ldots, f_m(x) \in \mathbb{K}(x)^*$. Furthermore, let $\ell_1, \ldots, \ell_m \in \mathbb{Z}_{\geq 0}$ such that for all i with $1 \leq i \leq m$, $f_i(j)$ is non-zero and has no pole for all $j \in \mathbb{Z}_{\geq 0}$ with $j \geq \ell_i$. Then the indefinite product expression (1) is called a *hypergeometric product in n of nesting depth m*. The vector $(f_1(x), \ldots, f_m(x)) \in (\mathbb{K}(x)^*)^m$ is also called the *multiplicand representation* of $P(n)$. If $f_i(x) \in \mathbb{K}^*$ for $1 \leq i \leq m$, then we call (1) a *constant* or *geometric product in n of nesting depth m*. Further, we define the set of ground expressions with $\mathbb{K}(n) = \{f(n) \mid f(x) \in \mathbb{K}(x)\}$; their elements are considered as expressions that can be evaluated for sufficiently large $n \in \mathbb{Z}_{\geq 0}$. Moreover, we define $\mathrm{Prod}_\mathrm{n}(\mathbb{G})$ with $\mathbb{G} \subseteq \mathbb{K}(x)$ as the set of all such products where the multiplicand representations are taken from $\mathbb{G}$. Furthermore, the set of all product monomials $\mathrm{ProdM}_\mathrm{n}(\mathbb{G})$ is defined by all elements

$$a(n)P_1(n)^{\nu_1} \cdots P_e(n)^{\nu_e}$$

with $a(x) \in \mathbb{G}$, $e \in \mathbb{Z}_{\geq 0}$, $\nu_1, \ldots, \nu_e \in \mathbb{Z}$ and $P_1(n), \ldots, P_e(n) \in \mathrm{Prod}_\mathrm{n}(\mathbb{G})$. Finally, we introduce the set of product expressions $\mathrm{ProdE}_\mathrm{n}(\mathbb{G})$ as the set of all elements

$$A(n) = \sum_{\boldsymbol{v}=(\nu_1,\ldots,\nu_e)\in S} a_{\boldsymbol{v}}(n)\, P_1(n)^{\nu_1} \cdots P_e(n)^{\nu_e} \tag{3}$$

with $e \in \mathbb{Z}_{\geq 0}$, $S \subseteq \mathbb{Z}^e$ finite, $a_{\boldsymbol{v}}(x) \in \mathbb{G}$ for $\boldsymbol{v} \in S$ and $P_1(n), \ldots, P_e(n) \in \mathrm{Prod}_\mathrm{n}(\mathbb{G})$. Note that $\mathrm{Prod}_\mathrm{n}(\mathbb{G}) \subseteq \mathrm{ProdM}_\mathrm{n}(\mathbb{G}) \subseteq \mathrm{ProdE}_\mathrm{n}(\mathbb{G})$.

Based on the algorithms from Ocansey and Schneider (2018) we will obtain enhanced algorithms that can rephrase expressions from $\mathrm{ProdE}_\mathrm{n}(\mathbb{K}(x))$ in the setting of $R\Pi\Sigma$-extensions. As a consequence we will solve the following problem; for details see Theorem 6.2 and Corollary 6.2 and its summary in Algorithm 6.1 below.

Problem RPE: Representation of Product Expressions.

Let $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ be a rational function field with $u \geq 0$ over an algebraic number field K. *Given* $A(n) \in \mathrm{ProdE_n}(\mathbb{K}(x))$. *Find* $B(n) \in \mathrm{ProdE_n}(\tilde{\mathbb{K}}(x))$ with $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \ldots, \kappa_u)$ where $\tilde{K}$ is an algebraic field extension of K, and a non-negative integer $\nu \in \mathbb{Z}_{\geq 0}$ with the following properties:

(1) $A(n) = B(n)$ for all $n \in \mathbb{Z}_{\geq 0}$ with $n \geq \nu$;
(2) All the products $Q_1(n), \ldots, Q_s(n) \in \mathrm{Prod_n}(\tilde{\mathbb{K}}(x))$ arising in $B(n)$ (apart from the distinguished product ζ^n with ζ a root of unity) are algebraically independent among each other.
(3) The zero-recognition property holds, i.e., $A(n) = 0$ holds for all n from a certain point on if and only if $B(n)$ is the zero-expression.

The full machinery has been implemented within Ocansey's Mathematica package `NestedProducts` whose functionality will be illustrated in Section 6.3 below; for additional aspects we refer also to Ocansey (2019). We expect that this implementation will open up new applications, e.g., in combinatorics, such as non-trivial evaluations of determinants (Krattenthaler, 2001; Mills et al., 1983; Zeilberger, 1996). In particular, in interaction with the symbolic summation algorithms available in the package `Sigma` (Schneider, 2007b) one obtains a fully automatic toolbox to tackle nested sums defined over nested hypergeometric products.

The outline of the article is as follows. In Section 2 we will introduce rewrite rules that enable one to transform expressions from $\mathrm{ProdE_n}(\mathbb{K}(x))$ to a more suitable form (see Proposition 2.2 below) to solve Problem RPE. Given this tailored form, we show in Section 3 how such expressions can be rephrased straightforwardly in terms of multiple-chain AP-extensions. In order to solve Problem RPE, we have to refine this difference ring construction. Namely, in Section 4 we introduce RΠ-extensions: these are AP-extensions where during the construction the set of constants remains unchanged. In particular, we will elaborate that such rings can be straightforwardly embedded into the ring of sequences and will provide structural theorems that will prepare the ground to solve Problem RPE. With these results we will present in Section 5 the main steps how nested products can be represented in RΠ-extensions. In Section 6 we will combine all these ideas in Theorem 6.2 and Corollary 6.2 that yield a complete algorithm for Problem RPE that is summarized in Algorithm 6.1. In addition, we will illustrate with non-trivial examples how one can solve Problem RPE with the new Mathematica package `NestedProducts`. The conclusions are given in Section 7.

2. Preprocessing hypergeometric products of finite nesting depth

In order to support our machinery to solve Problem RPE, the arising products $P(n)$ in $A(n) \in \mathrm{ProdE_n}(\mathbb{K}(x))$ (e.g., given in (3)) will be transformed to a particularly nice form. We will illustrate each preprocessing step with an example and summarize the derived result in Proposition 2.2 below.

Let $\mathbb{K}(x)$ be a rational function field and define the *zero-function* (in short *Z-function*) by

$$Z(p) = \max\left(\{k \in \mathbb{Z}_{\geq 0} \mid p(k) = 0\}\right) + 1 \;\text{ for any } p \in \mathbb{K}[x] \tag{4}$$

with $\max(\emptyset) = -1$. We call $\mathbb{K}$ *computable* if all basic field operations are computable. Note that if $\mathbb{K}$ is a rational function field over an algebraic number field, then $\mathbb{K}$ and also its Z-function are computable.

We start with the hypergeometric product $P(n) \in \mathrm{Prod_n}(\mathbb{K}(x))$ in n of nesting depth $m \in \mathbb{Z}_{\geq 0}$ given by (1) where $f_i(x) \in \mathbb{K}(x)^*$ and $\ell_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq m$. Note that by definition $P(n) \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. In particular, no poles arise for any evaluation at $n \in \mathbb{Z}_{\geq 0}$. We remark that the Z-function can be used to specify the lower bounds $\ell_i \in \mathbb{Z}_{\geq 0}$ such that this property holds. Then $P(n)$ in $\mathrm{Prod_n}(\mathbb{K}(x))$ is preprocessed as follows.

2.1. Transformation of indefinite products to product factored form

The first transformation is based on the following simple observation.

Proposition 2.1. *Let $P(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ as given in (1) with $f_1, \dots, f_m \in \mathbb{K}(x)^*$. Then for*

$$Q(n) = \left(\prod_{k_1=\ell_1}^{n} f_1(k_1)\right)\left(\prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1} f_2(k_2)\right)\cdots\left(\prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1}\cdots\prod_{k_m=\ell_m}^{k_{m-1}} f_m(k_m)\right) \in \mathrm{ProdM_n}(\mathbb{K}(x)) \quad (5)$$

we have $P(n) = Q(n)$ for all $n \in \mathbb{Z}_{\geq 0}$.

Definition 2.1. The product $Q(n)$ in (5) is also called a *product factored form* of $P(n)$. Moreover,

$$P'(n) = \prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1}\cdots\prod_{k_m=\ell_m}^{k_{m-1}} c(k_m) \in \mathrm{Prod_n}(\mathbb{K}(x)) \quad (6)$$

is also called a *product in factored form*. We also call $c(x) \in \mathbb{K}(x)^*$ (instead of $(1, \dots, 1, c(x))$) the *multiplicand representation* of $P'(n)$.

Further, for $1 \leq i \leq m$ write

$$f_i = u_i\, f_{i,1}^{e_{i,1}} \cdots f_{i,r_i}^{e_{i,r_i}} \in \mathbb{K}(x) \quad (7)$$

in its complete factorization. This means that f_i can be decomposed by $u_i \in \mathbb{K}^*$ and irreducible monic polynomials $f_{i,j} \in \mathbb{K}[x] \setminus \mathbb{K}$ with $e_{i,j} \in \mathbb{Z}$ for some $1 \leq j \leq r_i$ with $r_i \in \mathbb{Z}_{\geq 0}$. Substituting (7) into the right-hand side of (5) and expanding the product quantifiers over each factor in (7) we get

$$P(n) = A_1(n)\, A_2(n) \cdots A_m(n)$$

for all $n \in \mathbb{Z}_{\geq 0}$ where

$$A_i(n) = \left(\prod_{k_1=l_1}^{n}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} u_i\right)\left(\prod_{k_1=\ell_1}^{n}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} f_{i,1}(k_i)\right)^{e_{i,1}}\cdots\left(\prod_{k_1=\ell_1}^{n}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} f_{i,r_i}(k_i)\right)^{e_{i,r_i}} \in \mathrm{ProdM_n}(\mathbb{K}(x)) \quad (8)$$

for all $1 \leq i \leq m$. In particular, the first product on the right hand side in (8) with innermost multiplicand $u_i \in \mathbb{K}^*$ is a geometric product of nesting depth i in $\mathrm{Prod_n}(\mathbb{K})$, while the rest are nesting depth i hypergeometric products in $\mathrm{ProdM_n}(\mathbb{K}(x))$ which are not geometric products.

Example 2.1. Let $\mathbb{K} = \mathbb{Q}(\sqrt{3})$ and $\mathbb{K}(x)$ be the rational function field over $\mathbb{K}$ with the Z-function (4). Suppose we are given the nesting depth 2 hypergeometric product

$$P(n) = \prod_{k=1}^{n} \frac{24k+1}{-\sqrt{3}} \prod_{j=3}^{k} \frac{-2\,(j^3-3\,j+2)}{5\,(j^2-j-2)} \in \mathrm{Prod_n}(\mathbb{K}(x)). \quad (9)$$

Then with

$$
\begin{aligned}
A_1(n) &= \left(\prod_{k=1}^{n} -1\right)\left(\prod_{k=1}^{n}\sqrt{3}\right)^{-1}\left(\prod_{k=1}^{n}24\right)\left(\prod_{k=1}^{n}\left(k+\tfrac{1}{24}\right)\right) \in \mathrm{ProdM_n}(\mathbb{K}(x)),\\
A_2(n) &= \left(\prod_{k=1}^{n}\prod_{j=3}^{k} -1\right)\left(\prod_{k=1}^{n}\prod_{j=3}^{k}5\right)^{-1}\left(\prod_{k=1}^{n}\prod_{j=3}^{k}2\right)\left(\prod_{k=1}^{n}\prod_{j=3}^{k}(j-2)\right)^{-1}\left(\prod_{k=1}^{n}\prod_{j=3}^{k}(j-1)\right)^{2}\\
&\quad \left(\prod_{k=1}^{n}\prod_{j=3}^{k}(j+1)\right)^{-1}\left(\prod_{k=1}^{n}\prod_{j=3}^{k}(j+2)\right) \in \mathrm{ProdM_n}(\mathbb{K}(x))
\end{aligned}
\tag{10}
$$

we get $P(n) = A_1(n)\,A_2(n)$ for all $n \in \mathbb{Z}_{\geq 0}$ where the multiplicand representations of the products in $A_1(n)$ and $A_2(n)$ are either from $\mathbb{K}$ or are irreducible polynomials from $\mathbb{K}[x]$.

2.2. Synchronization of lower bounds

Another transformation will guarantee that all arising products have the same lower bound, i.e., that the expression is δ-refined for some $\delta \in \mathbb{Z}_{\geq 0}$.

Definition 2.2. Let $\mathbb{K}(x)$ be a rational function field over a field $\mathbb{K}$ and $\delta \in \mathbb{Z}_{\geq 0}$. $H(n) \in \mathrm{ProdE_n}(\mathbb{K}(x))$ is said to be δ-*refined* if the lower bounds in all the arising products of $H(n)$ are δ.

Such a transformation of a given product expression to a δ-refined version can be accomplished by taking δ to be the maximum of all arising lower bounds within the given expression.

Example 2.2 *(Cont. Ex. 2.1).* In $P(n)$ (resp. $A_1(n)$ and $A_2(n)$) of Example 2.1 we choose $\delta = 3$.

Namely, for all $1 \leq i \leq m$, rewrite each product in (8) such that the lower bounds are synchronized to δ. More precisely we apply the formula

$$
\begin{aligned}
\prod_{k_1=\ell_1}^{n}\prod_{k_2=\ell_2}^{k_1}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} h(k_i) &= \left(\prod_{k_1=\ell_1}^{\delta-1}\prod_{k_2=\ell_2}^{k_1}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} h(k_i)\right)\left(\prod_{k_1=\delta}^{n}\prod_{k_2=\ell_2}^{\delta-1}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} h(k_i)\right)\\
&\quad \left(\prod_{k_1=\delta}^{n}\prod_{k_2=\delta}^{k_1}\prod_{k_3=\ell_3}^{\delta-1}\cdots\prod_{k_i=\ell_i}^{k_{i-1}} h(k_i)\right)\cdots\left(\prod_{k_1=\delta}^{n}\prod_{k_2=\delta}^{k_1}\cdots\prod_{k_{i-1}=\delta}^{k_{i-2}}\prod_{k_i=\ell_i}^{\delta-1} h(k_i)\right)\left(\prod_{k_1=\delta}^{n}\prod_{k_2=\delta}^{k_1}\cdots\prod_{k_i=\delta}^{k_{i-1}} h(k_i)\right)
\end{aligned}
\tag{11}
$$

to each of the products in (8). Note that the first product on the right-hand side in (11) evaluates to a constant in $\mathbb{K}^*$, the last product is from $\mathrm{Prod_n}(\mathbb{K}(x))$, and all the remaining products (after all finite multiplications are carried out) are from $\mathrm{Prod_n}(\mathbb{K})$. Since δ is chosen as the maximum among all lower bounds of the input expression, no poles or zero-evaluations will be introduced. As a consequence, the obtained result is again an element from $\mathrm{ProdM_n}(\mathbb{K}(x))$ and evaluates to the same values as the input expression $P(n) \in \mathrm{Prod_n}(\mathbb{K}(x))$ for all $n \geq \max(\delta - 1, 0)$. Since geometric products never introduce poles or zeroes, we can bring each geometric product to a 1-refined form by using a similar formula as given in (11). By rearranging the arising products in product factored form we obtain the decomposition

$$
P'(n) = c\,G(n)\,H(n) \tag{12}
$$

with $c \in \mathbb{K}^*$ where the products in $G(n) \in \mathrm{ProdM_n}(\mathbb{K})$ are geometric and 1-refined and in $H(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ are hypergeometric (and not geometric) and δ-refined.

Example 2.3 *(Cont. Example 2.2).* Applying the formula (11) to $P(n) = A_1(n)\,A_2(n)$ with (10) and synchronizing the lower bounds of the arising geometric product to 1 we get

$$c = -\frac{245}{288}, \tag{13}$$

$$\begin{aligned} G(n) = {}& \left(\prod_{k=1}^{n} -1\right) \left(\prod_{k=1}^{n} \sqrt{3}\right)^{-1} \left(\prod_{k=1}^{n} 4\right)^{-1} \left(\prod_{k=1}^{n} 24\right) \left(\prod_{k=1}^{n} 25\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} -1\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} 5\right)^{-1} \\ & \times \left(\prod_{k=1}^{n}\prod_{j=1}^{k} 2\right), \end{aligned} \tag{14}$$

$$\begin{aligned} H(n) = {}& \left(\prod_{k=3}^{n} \left(k + \tfrac{1}{24}\right)\right) \left(\prod_{k=3}^{n}\prod_{j=3}^{k} (j-2)\right)^{-1} \left(\prod_{k=3}^{n}\prod_{j=3}^{k} (j-1)\right)^{2} \left(\prod_{k=3}^{n}\prod_{j=3}^{k} (j+1)\right)^{-1} \\ & \times \left(\prod_{k=3}^{n}\prod_{j=3}^{k} (j+2)\right) \end{aligned} \tag{15}$$

such that $P(n) = c\,G(n)\,H(n)$ holds for all $n \geq 2$.

2.3. Shift-coprime representation

Finally, we turn our focus to the class of hypergeometric products given in factored form, and whose innermost multiplicands are irreducible monic polynomials. In order to reduce this class of products further, we will need the following definition.

Definition 2.3. Two nonzero polynomials $f(x)$ and $h(x)$ in the polynomial ring $\mathbb{K}[x]$ are said to be *shift-coprime* if for all $k \in \mathbb{Z}$ we have that $\gcd(f(x), h(x+k)) = 1$. Furthermore, $f(x)$ and $h(x)$ are called *shift-equivalent* if there is a $k \in \mathbb{Z}$ such that $\frac{f(x+k)}{h(x)} \in \mathbb{K}$.

Let $\mathbb{K}(x)$ be a rational function field and let $H_1(n), \ldots, H_e(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ where the arising hypergeometric products are in factored form. We say that $H_1(n), \ldots, H_e(n)$ are in *shift-coprime product representation form* if

(1) the multiplicand representation of each product in $H_i(n)$ for $1 \leq i \leq e$ is an irreducible monic polynomial in $\mathbb{K}[x] \setminus \mathbb{K}$;
(2) the multiplicand representations in $H_1(n), \ldots, H_e(n)$ are shift-coprime among each other.

It is immediate that the shift-equivalence in Definition 2.3 induces an equivalence relation on the set of all irreducible polynomials. Let $\mathcal{D} = \{f_1, \ldots, f_e\} \subseteq \mathbb{K}[x]$ where all elements are irreducible and shift equivalent among each other. Then we call $f_i \in \mathcal{D}$ with $i \in \{1, 2, \ldots, e\}$ the *leftmost polynomial* in $\mathcal{D}$ if for all $h \in \mathcal{D}$ there is a $k \in \mathbb{Z}_{\geq 0}$ with $\frac{f_i(x+k)}{h(x)} \in \mathbb{K}$. It is well known that $\frac{f_i(x+k)}{h(x)} \in \mathbb{K}$ iff $k \in \mathbb{Z}$ is a root of $p(z) = \mathrm{res}_x(h(x), f_i(x+z)) \in \mathbb{K}[z]$; compare Petkovšek et al. (1996, Sec. 5.3). In particular, if $\mathbb{K}$ is computable and one can factorize univariate polynomials over $\mathbb{K}$, one can determine all integer roots of $p(z)$ and thus can decide constructively if there is a $k \in \mathbb{Z}$ with $\frac{f_i(x+k)}{h(x)} \in \mathbb{K}$. We remark that k is precisely the dispersion introduced in Abramov (1971, 1989). All the above properties (and slight generalizations) play a crucial role in symbolic summation; compare Abramov et al. (2021); Abramov (1971, 1989); Abramov and Petkovšek (2010); Abramov and Zima (1996); Bauer and Petkovšek (1999); Blümlein et al. (2018); Bronstein (2000); Chen et al. (2011, 2013); Chyzak (2000); Gosper (1978); Karr (1981, 1985); Koutschan (2013); Paule (1995); Paule and Riese (1997); Petkovšek et al. (1996); Riese (2003); Schneider (2005); Wegschaider (1997); Wilf and Zeilberger (1992); Zeilberger (1991). Further links will be given in Remark 2.1 below. In particular, the following simple lemma is heavily used within symbolic summation; see also Schneider (2005, Lemma 4.12). The proof is constructive and the underlying algorithm will be essential to rewrite hypergeometric products to a shift-coprime product representation in Lemma 2.2 below.

Lemma 2.1. *Let $\mathbb{K}(x)$ be a rational function field and let $f(x), h(x) \in \mathbb{K}[x] \setminus \mathbb{K}$ be monic irreducible polynomials that are shift equivalent. Then there is a $g(x) \in \mathbb{K}(x)^*$ with $h(x) = \frac{g(x+1)}{g(x)} f(x)$ where all the monic irreducible factors in $g(x)$ are shift equivalent to $f(x)$ (resp. $h(x)$). If $\mathbb{K}$ is computable and one can factorize polynomials over $\mathbb{K}$, then such a g can be computed.*

Proof. Since $f(x)$ and $h(x)$ are shift equivalent and monic, there is a $k \in \mathbb{Z}$ with $f(x+k) = h(x)$. If $k \geq 0$, we take the factorial polynomial $g(x) := \prod_{i=0}^{k-1} f(x+i)$. Then $\frac{g(x+1)}{g(x)} = \frac{f(x+k)}{f(x)} = \frac{h(x)}{f(x)}$. On the other hand, if $k < 0$, set $g(x) := \prod_{i=1}^{-k} \frac{1}{f(x-i)}$. Then $\frac{g(x+1)}{g(x)} = \frac{1/f(x)}{1/f(x+k)} = \frac{f(x+k)}{f(x)} = \frac{h(x)}{f(x)}$. By construction all irreducible monic factors in $g(x)$ are shift equivalent to $f(x)$. Furthermore, k can be computed if $\mathbb{K}$ is computable and one can factorize polynomials over $\mathbb{K}$. $\square$

Example 2.4 *(Cont. Example 2.3).* Let $\mathbb{K}(x)$ be a rational function field as defined in Example 2.1. Let $\mathcal{D}$ be the set defined by the multiplicand representations of the products in factored form given in (15). That is, $\mathcal{D} = \{f_1(x), f_2(x), \dots, f_5(x)\} \subseteq \mathbb{K}[x] \setminus \mathbb{K}$ where

$$f_1(x) = x-2, \quad f_2(x) = x-1, \quad f_3(x) = x+1, \quad f_4(x) = x+2 \quad \text{and} \quad f_5(x) = x + \tfrac{1}{24}.$$

Since $f_1(x)$ is shift equivalent with $f_2(x), f_3(x), f_4(x)$, i.e., $f_1(x+1) = f_2(x)$, $f_1(x+3) = f_3(x)$ and $f_1(x+4) = f_4(x)$, they fall into the same equivalence class $\mathcal{E}_1 = \{f_1(x), f_2(x), f_3(x), f_4(x)\}$. The other equivalence class is $\mathcal{E}_2 = \{f_5(x)\}$. For each of these equivalence classes $\mathcal{E}_1$ and $\mathcal{E}_2$, take their leftmost elements: $f_1(x)$ and $f_5(x)$ respectively. Then by Lemma 2.1, we can express the elements of each equivalence class in terms of the leftmost polynomial $f_1(x)$ or $f_5(x)$. More precisely, we get the relations

$$f_{i+1}(x) = \frac{g_i(x+1)}{g_i(x)} f_1(x) \tag{16}$$

for $i = 1, 2, 3$ with $g_1(x) = (x-2)$, $g_2(x) = (x-2)(x-1)x$ and $g_3(x) = (x-2)(x-1)x(x+1)$. Finally, we reduce each component of the hypergeometric product expression $H(n)$ given by (15). We will begin with the nesting depth 2 hypergeometric products in factored form whose innermost multiplicand corresponds to the polynomial $f_4(x)$. Using (16) with $i = 1$ we get

$$\begin{aligned}\prod_{k=3}^{n}\prod_{j=3}^{k} f_4(j) &= \prod_{k=3}^{n}\prod_{j=3}^{k}(j+2) = \left(\prod_{k=3}^{n}\prod_{j=3}^{k}\frac{g_3(j+1)}{g_3(j)}\right)\prod_{k=3}^{n}\prod_{j=3}^{k} f_1(j) = \left(\prod_{k=3}^{n}\frac{g_3(k+1)}{g_3(3)}\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k} f_1(j)\right)\\ &= \left(\prod_{k=3}^{n}\frac{1}{24}\right)\left(\prod_{k=3}^{n}(k-1)\right)\left(\prod_{k=3}^{n}k\right)\left(\prod_{k=3}^{n}(k+1)\right)\left(\prod_{k=3}^{n}(k+2)\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right)\\ &= 576\left(\prod_{k=1}^{n}\frac{1}{24}\right)\left(\prod_{k=3}^{n}(k-1)\right)\left(\prod_{k=3}^{n}k\right)\left(\prod_{k=3}^{n}(k+1)\right)\left(\prod_{k=3}^{n}(k+2)\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right).\end{aligned} \tag{17}$$

Using (16) for $i = 2, 3$, a similar reduction can be achieved for the nesting depth 2 hypergeometric products in factored form arising in $H(n)$ whose innermost multiplicands correspond to the polynomials $f_3(x)$ and $f_2(x)$ respectively. In particular, we have the following:

$$\prod_{k=3}^{n}\prod_{j=3}^{k} f_2(j) = \left(\prod_{k=3}^{n}(k-1)\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right),$$

$$\prod_{k=3}^{n}\prod_{j=3}^{k} f_3(j) = 36\left(\prod_{k=1}^{n}\frac{1}{6}\right)\left(\prod_{k=3}^{n}(k-1)\right)\left(\prod_{k=3}^{n}k\right)\left(\prod_{k=3}^{n}(k+1)\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right). \tag{18}$$

The above constructions of the example can be summarized with the following lemma.

Lemma 2.2. *Let $f(x) \in \mathbb{K}[x]$ be irreducible and monic and take $\delta \in \mathbb{Z}_{\geq 0}$ with $\delta \geq Z(f)$. Further take $k \in \mathbb{Z}_{\geq 0}$ and define the monic polynomial $h(x) = f(x+k) \in \mathbb{K}[x]$ with $Z(f) \geq Z(h)$. Then there are*

(1) a nonzero polynomial $a(x) \in \mathbb{K}[x]$ with $\max(\delta-1, 0) \geq Z(a)$,
(2) $G(n) \in \mathrm{ProdM_n}(\mathbb{K})$ being a product of 1-refined products in product factored form with depth smaller than m, and
(3) $H(n) \in \mathrm{ProdM_n}(\mathbb{K}[x])$ being a product of δ-refined products in product factored form with depth smaller than m and with multiplicand representations of the form $f(x+r)$ with $r \in \mathbb{Z}_{\geq 0}$

such that for all $n \geq \max(\delta-1, 0)$ we have

$$\prod_{k_1=\delta}^{n} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} \prod_{k_m=\delta}^{k_{m-1}} h(k_m) = a(n)\, G(n)\, H(n) \cdot \prod_{k_1=\delta}^{n} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} \prod_{k_m=\delta}^{k_{m-1}} f(k_m). \tag{19}$$

If $\mathbb{K}$ is computable, the components a, G and H can be computed.

Proof. Denote by $P'(n)$ the product on the left-hand side of (19); note that $P'(n)$ is in product factored form with depth m and with the multiplicand representation $h(x)$. By Lemma 2.1 we can take $g := \prod_{i=0}^{k-1} f(x+i) \in \mathbb{K}[x]$ with $Z(g) = Z(f) \geq \delta$ such that $h(x) = \frac{g(x+1)}{g(x)} f(x)$ holds. Thus

$$P'(n) = \prod_{k_1=\delta}^{n} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} \prod_{k_m=\delta}^{k_{m-1}} \frac{g(k_m+1)}{g(k_m)} f(k_m) = \prod_{k_1=\delta}^{n} \prod_{k_2=\delta}^{k_1} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} \frac{g(k_{m-1}+1)}{g(\delta)} \prod_{k_m=\delta}^{k_{m-1}} f(k_m)$$
$$= \underbrace{\left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} g(\delta)\right)^{-1}}_{=G'(n)} \underbrace{\left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_{m-1}=\delta}^{k_{m-2}} g(k_{m-1}+1)\right)}_{=H'(n)} \left(\prod_{k_1=\delta}^{n} \cdots \prod_{k_m=\delta}^{k_{m-1}} f(k_m)\right).$$

First we assume that $m \geq 2$. Then $H'(n)$ is a hypergeometric product in factored form of nesting depth less than m where for the multiplicand representation $h'(x) := g(x+1)$ we have that $h'(n) \neq 0$ for all $n \geq \max(\delta-1, 0)$. In particular, $h'(x)$ consists of monic irreducible factors which are again shift-equivalent to $f(x)$. Repeating the steps in Subsection 2.1 to $H'(n)$ yields a product $H(n)$ of products in factored form with depth smaller than m and multiplicand representations of the form $f(x+r)$ with $r \in \mathbb{Z}_{\geq 0}$ such that $H'(n) = H(n)$ holds for all $n \geq \max(\delta-1, 0)$. Note further that also the new geometric product $G'(n)$ occurs with lower bound δ. Thus we apply the transformations introduced in Section 2.2 in order to turn it to a product $G(n)$ of 1-refined products of depth smaller than m such that $G'(n) = G(n)$ holds for all $n \geq \max(\delta-1, 0)$; extra contributions of constants are moved to $a(x) \in \mathbb{K}^*$. This yields (19).

Otherwise, if $m = 1$, we define $a(x) = \frac{g(x+1)}{g(\delta)} \in \mathbb{K}[x] \setminus \{0\}$. Since $\delta \geq Z(f) = Z(g)$, we have $\max(\delta - 1, 0) \geq Z(a)$. Thus for all $n \geq \max(\delta-1, 0)$ we get $H'(n)\, G'(n) = \frac{g(n+1)}{g(\delta)} = a(n)$. Taking in addition $G(n) = H(n) = 1$ we obtain again (19) which holds for all $n \geq \max(\delta-1, 0)$. $\square$

In particular, applying this lemma iteratively leads to the following proposition.

Proposition 2.2. *Let $\mathbb{K}(x)$ be a rational function field and suppose that we are given the hypergeometric products $P_1(n), \dots, P_e(n) \in \mathrm{Prod}_n(\mathbb{K}(x))$ of nesting depth at most $d \in \mathbb{Z}_{\geq 0}$. Then there is a $\delta \in \mathbb{Z}_{\geq 0}$ and there are*

(1) for all $1 \leq \ell \leq e$ rational functions $\tilde{r}_\ell(x) \in \mathbb{K}(x)^*$;
(2) geometric product expressions $\tilde{G}_1(n), \dots, \tilde{G}_e(n) \in \mathrm{ProdM}_n(\mathbb{K})$ with depth at most d which are all 1-refined and in factored form;
(3) hypergeometric product expressions $\tilde{H}_1(n), \dots, \tilde{H}_e(n) \in \mathrm{ProdM}_n(\mathbb{K}(x))$ (which are not built by geometric products) with depth at most d which are δ-refined and are in shift-coprime product representation form

such that for $1 \leq \ell \leq e$ and for all $n \geq \max(0, \delta - 1)$ we have

$$P_\ell(n) = \tilde{r}_\ell(n)\, \tilde{G}_\ell(n)\, \tilde{H}_\ell(n) \neq 0. \tag{20}$$

If $\mathbb{K}$ is computable and polynomials can be factored in $\mathbb{K}$, then δ, $\tilde{r}_\ell(n)$, $\tilde{G}_\ell(n)$, $\tilde{H}_\ell(n)$ can be computed.

Proof. We apply the steps as carried out in Algorithm 2.1. After the rewriting of the products in product-factored form in Step 1, one applies iteratively Steps (2) to (3) and rewrites the products to a shift-coprime representation by considering first the products with highest nesting depth and treating those with lower in the next iteration. Here Lemma 2.2 is applied iteratively until all multiplicands are written in terms of the elements in $\mathcal{E}$ which are the leftmost elements of each equivalence class. Due to Lemma 2.2 the new products coming from $H(n)$ in (19) have the multiplicands $f(x+k)$ with $k \in \mathbb{Z}_{\geq 0}$ with $f \in \mathcal{E}$. Thus the leftmost elements arising in $\mathcal{E}$ do not change; only new elements in step (2) might arise when considering products with lower nesting depth. Thus the already transformed products do not have to be reconsidered later and the reduction to lower depth in each iteration (2)–(3) works properly. As a consequence, one gets the desired result (20) for all $n \geq \max(0, \delta - 1)$. In particular, we obtain for each component an extra factor $\tilde{r}_i(x) = \frac{p(x)}{q(x)} \in \mathbb{K}(x)$ with $p, q \in \mathbb{K}[x]$ where for $n \in \mathbb{Z}_{\geq 0}$ with $n \geq \max(0, \delta - 1)$ we have $p(n) \neq 0 \neq q(n)$. This construction proves the existence the claimed representation. If $\mathbb{K}$ is computable and one can factorize polynomials over $\mathbb{K}$, all the steps given in Algorithm 2.1 can be carried out explicitly and the construction turns to an algorithm. □

Algorithm 2.1. (Compute a shift-coprime product representation)
Input: hypergeometric products $P_1(n), \dots P_e(n) \in \mathrm{Prod}_n(\mathbb{K}(x))$ as given in (1) with $m \leq d$.
Output: $\delta \in \mathbb{Z}_{\geq 0}$, $\tilde{r}_1(x), \dots, \tilde{r}_e(x) \in \mathbb{K}(x)^*$, $\tilde{G}_1(n), \dots, \tilde{G}_e(n) \in \mathrm{ProdM}_n(\mathbb{K})$ and $\tilde{H}_1(n), \dots, \tilde{H}_e(n) \in \mathrm{ProdM}_n(\mathbb{K}(x))$ as stated in Proposition 2.2.

(1) Following the construction in Sections 2.1 and (2.2) we bring each $P_i(n)$ to the form (12) with $P'_i(n) := c_i\, G_i(n)\, H_i(n)$ with $c_i \in \mathbb{K}^*$, $G_i(n) \in \mathrm{ProdM}_n(\mathbb{K})$ and $H_i(n) \in \mathrm{ProdM}_n(\mathbb{K}(x))$ such that $P_i(n) = P'_i(n)$ holds for all $n \geq \max(0, \delta - 1)$ for some $\delta \in \mathbb{Z}_{\geq 0}$. Here all the products in $H_1(n), \dots, H_e \in \mathrm{ProdM}_n(\mathbb{K}(x))$ and $G_1(n), \dots, G_e \in \mathrm{ProdM}_n(\mathbb{K}(x))$ are in factored form with depth at most d where the $H_i(n)$ are δ-refined and the $G_i(n)$ are 1-refined. In particular, all the multiplicand representations in $H_i(n)$ are monic irreducible polynomials in $\mathbb{K}[x]$.
Set $\tilde{r}_i := c_i$, $\tilde{G}_i := G_i$ and $\tilde{H}_i := H_i$.
(2) Let $\mathcal{D} \subseteq \mathbb{K}[x] \setminus \mathbb{K}$ be the set of all the irreducible monic polynomials in the products arising in $\tilde{H}_i$. Among all the elements in $\mathcal{D}$, compute the shift equivalence classes say, $\mathcal{E}_1, \dots, \mathcal{E}_v$ with respect to the automorphism $\sigma(x) = x + 1$, and let $\mathcal{R}$ be the set of the leftmost polynomial of each equivalence class. Thus, the elements of the set $\mathcal{R}$ are shift-coprime among each other and each element represents exactly one equivalence class. Among all products in $\tilde{H}_i$ let $m \in \mathbb{Z}_{\geq 0}$ with $m \leq d$ be the maximal depth of those products whose multiplicands are not elements from $\mathcal{R}$. If no product has to be treated (i.e., $m = 0$), go to Step (5) and stop.

(3) Take all products of nesting depth m and reduce them with the elements in $f(x) \in \mathcal{R}$ following the construction of Lemma 2.2. During this rewriting one replaces the left-hand sides of (19) with the right-hand sides of (19) and produces besides the desired hypergeometric product of depth m with the multiplicand $f(x)$ extra geometric and hypergeometric products in factored form with depth less than m that are collected accordingly. In addition, the arising polynomials $a(x) \in \mathbb{K}[x]$ with $a(n) \neq 0$ for all $n \geq \max(0, \delta - 1)$ are moved to $\tilde{r}_i(x)$; note that the contribution in $\tilde{r}_i(x)$ is either $a(x)$ or $1/a(x)$ depending on the situation if the product under consideration occurs in the numerator or denominator of $\tilde{H}_i$.
(4) Go to step (2).
(5) Return $\tilde{r}_i \in \mathbb{K}(x)^*$, $\tilde{H}_i(n) \in \mathrm{ProdM_n}(\mathbb{K})$ and $\tilde{G}_i(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ where the corresponding contributions of rational functions, 1-refined geometric product in factored form and the δ-refined hypergeometric products in shift-coprime product representation are collected accordingly.

Example 2.5 *(Cont. of Ex. 2.4).* In Example 2.4 the steps (1)–(3) of Algorithm 2.1 have been carried out and all nesting depth 2 hypergeometric products are transformed accordingly. We repeat the steps (2)–(3) of the algorithm for the products of nesting depth $m = 1$. Here the new polynomial $f_6(x) = x$ emerge. It falls into the equivalence class $\mathcal{E}_1$, and the leftmost polynomial of this equivalence class remains unchanged. We get $f_6(x) = \frac{g_4(x+1)}{g_4(x)} f_1(x)$ with $g_4(x) = (x-2)(x-1)$ by Lemma 2.1. Using the relations (16) with $i = 1, 2, 3$ we can reduce all nesting depth 1 hypergeometric products whose multiplicand representations are $f_4(x)$, $f_3(x)$, $f_6(x)$, and $f_2(x)$ respectively. More precisely we have the following:

$$\prod_{k=3}^{n} f_4(k) = \frac{(n-1)\,n\,(n+1)\,(n+2)}{24} \prod_{k=3}^{n} (k-2), \qquad \prod_{k=3}^{n} f_3(k) = \frac{(n-1)\,n\,(n+1)}{6} \prod_{k=3}^{n} (k-2)$$
$$\prod_{k=3}^{n} f_6(k) = \frac{(n-1)\,n}{2} \prod_{k=3}^{n} (k-2), \qquad \prod_{k=3}^{n} f_2(k) = (n-1) \prod_{k=3}^{n} (k-2). \tag{21}$$

Substituting (21) into (17) and (18) and afterwards into the expression (15) gives

$$\hat{H}(n) = \frac{2}{3}\,(n-1)^3\, n\,(n+1)\,(n+2) \left(\prod_{k=1}^{n} 24\right)^{-1} \left(\prod_{k=1}^{n} 6\right) \left(\prod_{k=3}^{n} (k-2)\right)^{3}$$
$$\times \left(\prod_{k=3}^{n} \left(k + \tfrac{1}{24}\right)\right) \left(\prod_{k=3}^{n}\prod_{j=3}^{k} (j-2)\right). \tag{22}$$

Note that $H(n) = \hat{H}(n)$ for all $n \geq 2$ where the two different multiplicands $(x-2)$ and $(x + \frac{1}{24})$ are monic, irreducible and shift-coprime. Since all products are treated, we quit the algorithm and return the corresponding result. Namely, putting (13) and (14) in Example 2.3 and (22) together, we obtain $\tilde{P}(n) = \tilde{r}(n)\, \tilde{G}(n)\, \tilde{H}(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ with

$$\tilde{r}(n) = -\frac{254}{432}\,(n-1)^3\, n\,(n+1)\,(n+2), \tag{23}$$

$$\tilde{G}(n) = \left(\prod_{k=1}^{n} -1\right) \left(\prod_{k=1}^{n} \sqrt{3}\right)^{-1} \left(\prod_{k=1}^{n} 2\right)^{-1} \left(\prod_{k=1}^{n} 3\right) \left(\prod_{k=1}^{n} 25\right) \left(\prod_{k=1}^{n}\prod_{j=1}^{k} -1\right)$$
$$\times \left(\prod_{k=1}^{n}\prod_{j=1}^{k} 5\right)^{-1} \left(\prod_{k=1}^{n}\prod_{j=1}^{k} 2\right), \tag{24}$$

$$\tilde{H}(n) = \left(\prod_{k=3}^{n}(k-2)\right)^3 \left(\prod_{k=3}^{n}\left(k+\tfrac{1}{24}\right)\right) \left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right). \tag{25}$$

where $P(n) = \tilde{P}(n)$ holds for all $n \in \mathbb{Z}_{\geq 0}$ with $n \geq 2$.

Remark 2.1. Alternatively to this construction, one may compute first the σ-factorization which has been introduced in Karr (1981) and is nowadays also called shift-orbit decomposition; compare (Bronstein, 2000). More precisely, one can compute a refined factorization of all the arising rational functions $f_i(x)$ in (7) where all shift equivalent factors in one class $\mathcal{E}_i$ are collected by taking one representative of $\mathcal{R}$ (not necessarily the left most polynomial) together with the necessary shifts to produce the other shift equivalent factors; this collection is also called a shift orbit. Applying Lemma 2.1 to these refined factorizations one finally gets $f_i(x) = h(x)\frac{g(x+1)}{g(x)}$ with $h, g \in \mathbb{K}(x)$ where the factors in the numerator and denominator of h are all shift-coprime; h is also called a saturated rational function. Observe that this h is precisely a solution of the representation (2) with $r(x) = c\, g(x+1)$ for some $c \in \mathbb{K}^*$ as elaborated in Abramov et al. (2003); Abramov and Petkovšek (2002, 2010); Schneider (2005). We remark further that this construction is closely related to Paule's greatest factorial factorization (Paule, 1995). Finally, one can split the products further into its factored form. We remark that the construction introduced above (first producing the factored form and afterwards computing the shift-coprime form using Lemma 2.2) turns out to be more straightforward in order to get Proposition 2.2.

From now on, we assume that the arising hypergeometric products $P_1(n), \ldots, P_e(n) \in \mathrm{Prod}_n(\mathbb{K}(x))$ have undergone the preprocessing steps given in Algorithm 2.1 yielding the representation given in (20). In general, there are still algebraic relations among the products that occur in the derived expressions (20), i.e., statements (2) and (3) of Problem RPE do not hold yet. In order to accomplish this task, extra insight from difference ring theory will be utilized. More precisely, we will show that the hypergeometric products coming from the $\tilde{H}_\ell$ are already algebraically independent, but the representation of the geometric products has to be improved to establish a solution of Problem RPE.

3. A formal representation in difference rings

Inspired by Karr (1981); Schneider (2016, 2017), this section focuses on an algebraic setting of difference rings (resp. fields) in which expressions of $\mathrm{ProdE}_n(\mathbb{K}(x))$ can be naturally rephrased.

3.1. Difference fields and difference rings

A *difference ring* (resp. *field*) $(\mathbb{A}, \sigma)$ is a ring (resp. field) $\mathbb{A}$ together with a ring (resp. field) automorphism $\sigma : \mathbb{A} \to \mathbb{A}$. Subsequently, all rings (resp. fields) are commutative with unity; in addition they contain the set of rational numbers $\mathbb{Q}$, as a subring (resp. subfield). The multiplicative group of units of a ring (resp. field) $\mathbb{A}$ is denoted by $\mathbb{A}^*$. A difference ring (resp. field) $(\mathbb{A}, \sigma)$ is called *computable* if $\mathbb{A}$ and σ are both computable. In the following we will introduce AP-extensions that will be the foundation to represent hypergeometric products of finite nesting depth in difference rings.

A-extensions will be used to cover objects like ζ^k where ζ is a root of unity. In general, let $(\mathbb{A}, \sigma)$ be a difference ring and let $\zeta \in \mathbb{A}^*$ be a λ-th root of unity with $\lambda > 1$ (i.e., $\lambda \in \mathbb{Z}_{\geq 2}$ with $\zeta^\lambda = 1$). Take the uniquely determined difference ring extension $(\mathbb{A}[y], \sigma)$ of $(\mathbb{A}, \sigma)$ where y is transcendental over $\mathbb{A}$ and $\sigma(y) = \zeta\, y$. Now consider the ideal $I := \langle y^\lambda - 1\rangle$ and the quotient ring $\mathbb{E} := \mathbb{A}[y]/I$. Since I is closed under σ and σ^{-1} i.e., I is a reflexive difference ideal, we can define the map $\sigma : \mathbb{E} \to \mathbb{E}$ with $\sigma(h + I) = \sigma(h) + I$ which forms a ring automorphism. Note that by this construction the ring $\mathbb{A}$ can naturally be embedded into the ring $\mathbb{E}$ by identifying $a \in \mathbb{A}$ with $a + I \in \mathbb{E}$, i.e., $a \mapsto a + I$. Now set $\vartheta := y + I$. Then $(\mathbb{A}[\vartheta], \sigma)$ is a difference ring extension of $(\mathbb{A}, \sigma)$ subject to the relations $\vartheta^\lambda = 1$ and $\sigma(\vartheta) = \zeta\, \vartheta$. This extension is called an *algebraic extension* (in short A-*extension*) of order λ. The generator ϑ is called an A-*monomial* with its order $\lambda = \min\{n > 0 \mid \vartheta^n = 1\}$. Note that the ring $\mathbb{A}[\vartheta]$ is not an integral domain (i.e., it has zero-divisors) since $(\vartheta - 1)(\vartheta^{\lambda-1} + \cdots + \vartheta + 1) = 0$ but

$(\vartheta-1)\neq 0\neq(\vartheta^{\lambda-1}+\cdots+\vartheta+1)$. In this setting, the A-monomial ϑ with the relations $\vartheta^{\lambda}=1$ and $\sigma(\vartheta)=\zeta\,\vartheta$ with $\zeta:=\mathrm{e}^{\frac{2\pi\,\mathrm{i}}{\lambda}}=(-1)^{\frac{2}{\lambda}}$, models ζ^k subject to the relations $(\zeta^k)^{\lambda}=1$ and $\zeta^{k+1}=\zeta\,\zeta^k$.

In addition, we define P-extensions in order to treat products of finite nesting depth whose multiplicands are not given by roots of unity. Let $(\mathbb{A},\sigma)$ be a difference ring, $\alpha\in\mathbb{A}^*$ be a unit, and consider the ring of Laurent polynomials $\mathbb{A}[t,t^{-1}]$ (i.e., t is transcendental over $\mathbb{A}$). Then there is a unique difference ring extension $(\mathbb{A}[t,t^{-1}],\sigma)$ of $(\mathbb{A},\sigma)$ with $\sigma(t)=\alpha\,t$ and $\sigma(t^{-1})=\alpha^{-1}\,t^{-1}$. The extension here is called a *product-extension* (in short P-*extension*) and the generator t is called a P-*monomial*.

We introduce the following notations for convenience. Let $(\mathbb{E},\sigma)$ be a difference ring extension of $(\mathbb{A},\sigma)$ with $t\in\mathbb{E}$. $\mathbb{A}\langle t\rangle$ denotes the ring of Laurent polynomials $\mathbb{A}[t,\frac{1}{t}]$ (i.e., t is transcendental over $\mathbb{A}$) if $(\mathbb{A}[t,\frac{1}{t}],\sigma)$ is a P-extension of $(\mathbb{A},\sigma)$. Lastly, $\mathbb{A}\langle t\rangle$ denotes the ring $\mathbb{A}[t]$ with $t\notin\mathbb{A}$ but subject to the relation $t^{\lambda}=1$ if $(\mathbb{A}[t],\sigma)$ is an A-extension of $(\mathbb{A},\sigma)$ of order λ.

We say that the difference ring extension $(\mathbb{A}\langle t\rangle,\sigma)$ of $(\mathbb{A},\sigma)$ is an AP-*extension* (and t is an AP-*monomial*) if it is an A- or a P-extension. Finally, we call $(\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle,\sigma)$ a *(nested)* A-/P-/AP-*extension* of $(\mathbb{A},\sigma)$ it is built by a tower of such extensions.

In the following we will restrict to the subclass of ordered simple AP-extension. Here, the following definitions are useful.

Definition 3.1. Let $(\mathbb{E},\sigma)$ be a (nested) AP-extension of $(\mathbb{A},\sigma)$ with $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$ where $\sigma(t_i)=\alpha_i\,t_i$ for $1\leq i\leq e$. We define the *depth* function of elements of $\mathbb{E}$ over $\mathbb{A}$, $\mathfrak{d}:\mathbb{E}\to\mathbb{Z}_{\geq 0}$ as follows:

(1) For any $h\in\mathbb{A}$, $\mathfrak{d}_{\mathbb{A}}(h)=0$.
(2) If $\mathfrak{d}_{\mathbb{A}}$ is defined for $(\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle,\sigma)$ with $i>1$, then we define $\mathfrak{d}_{\mathbb{A}}(t_i):=\mathfrak{d}_{\mathbb{A}}(\alpha_i)+1$ and for $f\in\mathbb{A}\langle t_1\rangle\dots\langle t_i\rangle$, we define $\mathfrak{d}_{\mathbb{A}}(f):=\max\big(\{\mathfrak{d}_{\mathbb{A}}(t_i)\,|\,t_i\text{ occurs in }f\}\cup\{0\}\big)$.

The *extension depth* of $(\mathbb{E},\sigma)$ over $\mathbb{A}$ is given by $\mathfrak{d}_{\mathbb{A}}(\mathbb{E}):=\big(\mathfrak{d}_{\mathbb{A}}(t_1),\dots,\mathfrak{d}_{\mathbb{A}}(t_e)\big)$. We call such an extension *ordered*, if $\mathfrak{d}_{\mathbb{A}}(t_1)\leq\mathfrak{d}_{\mathbb{A}}(t_2)\leq\cdots\leq\mathfrak{d}_{\mathbb{A}}(t_e)$. In particular, we say that $(\mathbb{E},\sigma)$ is of *monomial depth* m if $m=\max(0,\mathfrak{d}_{\mathbb{A}}(t_1),\dots,\mathfrak{d}_{\mathbb{A}}(t_e))$. If $\mathbb{A}$ is clear from the context, we write $\mathfrak{d}_{\mathbb{A}}$ as $\mathfrak{d}$.

Now, let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$ be a nested A-/P-/AP-extension of a difference ring $(\mathbb{A},\sigma)$ and let G be a multiplicative subgroup of $\mathbb{A}^*$. Following (Schneider, 2016, 2017) we call

$$G_{\mathbb{A}}^{\mathbb{E}}:=\{g\,t_1^{v_1}\cdots t_e^{v_e}\,|\,g\in G,\text{ and }v_i\in\mathbb{Z}\}$$

the *product group* over G with respect to A-/P-/AP-monomials for the nested A-/P-/AP-extension $(\mathbb{E},\sigma)$ of $(\mathbb{A},\sigma)$. In the following we will restrict ourselves to the following subclass of AP-extensions.

Definition 3.2. Let $(\mathbb{A},\sigma)$ be a difference ring and let $\mathbb{G}$ be a subgroup of $\mathbb{A}^*$. Let $(\mathbb{E},\sigma)$ be an A-/P-/AP-extension of $(\mathbb{A},\sigma)$ with $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$ with $\sigma(t_i)=\alpha_i\,t_i$ for $1\leq i\leq e$. Then this extension is called $\mathbb{G}$-*simple* if for all $1\leq i\leq e$,

$$\alpha_i=\frac{\sigma(t_i)}{t_i}\in\mathbb{G}_{\mathbb{A}}^{\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle}.$$

In addition such a $\mathbb{G}$-simple extension is called $\mathbb{G}$-basic,[1] if for any A-monomial t_i we have that $\sigma(\alpha_i)=\alpha_i$ (i.e., α_i is a constant; see also the definition (40) below) and for any P-monomial t_i we have that $\alpha_i\in\mathbb{G}_{\mathbb{A}}^{\mathbb{A}\langle t_1\rangle\dots\langle t_{i-1}\rangle}$ is free of A-monomials. If $\mathbb{G}=\mathbb{A}^*$, such extensions are also called *simple* (resp. *basic*) instead of $\mathbb{A}^*$-simple ($\mathbb{A}^*$-basic).

In particular, we will work with the following class of simple A-/P-/AP-extensions that are closely related to the products in factored form given in (5); for concrete constructions see Example 3.1 below.

[1] In other words, products whose multiplicands are roots of unity have nesting depth 1 (and the roots of unity are from the constant field), while the remaining products do not depend on these products over roots of unity.

Definition 3.3. Let $(\mathbb{A},\sigma)$ be a difference ring and $\mathbb{G}$ be a subgroup of $\mathbb{A}^*$. We call $(\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle,\sigma)$ a *single chain* A-/P-/AP-*extension of* $(\mathbb{A},\sigma)$ *over* $\mathbb{G}$ if for all $1\le k\le e$,

$$\sigma(t_k)=c_k\,t_1\cdots t_{k-1}\,t_k,\qquad \text{with } c_k\in\mathbb{G}.$$

We call c_1 also the *base* of the single chain A-/P-/AP-extension. If $\mathbb{G}=\mathbb{A}^*$, we also say that $(\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle,\sigma)$ is a *single chain* A-/P-/AP-*extension* of $(\mathbb{A},\sigma)$. Further, we call $(\mathbb{E},\sigma)$ a *multiple chain* A-/P-/AP-*extension* of $(\mathbb{A},\sigma)$ over $\mathbb{G}$ with base $(c_1,\dots,c_m)\in\mathbb{G}^m$ if it is a tower of m single chain A-/P-extensions over $\mathbb{G}$ with the bases $c_1,\dots,c_m$, respectively. If $\mathbb{G}=\mathbb{A}^*$, we simply call it a multiple chain A-/P-/AP-extension.

Remark 3.1. Let $(\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle,\sigma)$ be a single chain A-/P-/AP-extension of $(\mathbb{A},\sigma)$ as given in Definition 3.3 and let $\mathfrak{d}:\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle\to\mathbb{Z}_{\ge 0}$ be the depth function over $\mathbb{A}$. Then we have $\mathfrak{d}(t_k)=k$ for all $1\le k\le e$. In particular, the extension is ordered, its extension-depth is $(1,2,\dots,e)$ and the monomial depth is e. Furthermore observe that for $2\le i\le e$ we have

$$\sigma(t_i)=\sigma(t_{i-1})\,t_i\quad\Leftrightarrow\quad\frac{t_i}{\sigma^{-1}(t_i)}=t_{i-1}.$$

3.2. Ring of sequences

For a field $\mathbb{K}$ we denote by $\mathbb{K}^{\mathbb{Z}_{\ge 0}}$ the set of all sequences

$$\langle a(n)\rangle_{n\ge 0}=\langle a(0),a(1),a(2),\dots\rangle \tag{26}$$

whose terms are in $\mathbb{K}$. Equipping $\mathbb{K}^{\mathbb{Z}_{\ge 0}}$ with component-wise addition and multiplication, we get a commutative ring. In this ring, the field $\mathbb{K}$ can be naturally embedded into $\mathbb{K}^{\mathbb{Z}_{\ge 0}}$ as a subring, by identifying any $c\in\mathbb{K}$ with the constant sequence $\langle c,c,c,\dots\rangle\in\mathbb{K}^{\mathbb{Z}_{\ge 0}}$. Following the construction in (Petkovšek et al., 1996, Section 8.2), we turn the shift operator $S:\mathbb{K}^{\mathbb{Z}_{\ge 0}}\to\mathbb{K}^{\mathbb{Z}_{\ge 0}}$ with

$$S:\langle a(0),\,a(1),\,a(2),\,\dots\rangle\mapsto\langle a(1),\,a(2),\,a(3),\,\dots\rangle \tag{27}$$

into a ring automorphism by introducing an equivalence relation $\sim$ on sequences in $\mathbb{K}^{\mathbb{Z}_{\ge 0}}$. Two sequences $A=\langle a(n)\rangle_{n\ge 0}$ and $B=\langle b(n)\rangle_{n\ge 0}$ are said to be equivalent (in short $A\sim B$) if and only if there exists a non-negative integer δ such that

$$\forall n\ge\delta: a(n)=b(n).$$

The set of equivalence classes form a ring again with component-wise addition and multiplication which we will denote by $\mathcal{S}(\mathbb{K}):=\mathbb{K}^{\mathbb{Z}_{\ge 0}}/\sim$. Now it is obvious that $S:\mathcal{S}(\mathbb{K})\to\mathcal{S}(\mathbb{K})$ with (27) is bijective and thus a ring automorphism. We call $(\mathcal{S}(\mathbb{K}),S)$ also the *difference ring of sequences over* $\mathbb{K}$. For simplicity, we denote the elements of $\mathcal{S}(\mathbb{K})$ by the usual sequence notation as in (26) above.

We will follow the convention introduced in Paule and Schneider (2019) to illustrate how the indefinite products of finite nesting depth covered in this article are modeled by expressions in a difference ring.

Definition 3.4. Let $(\mathbb{A},\sigma)$ be a difference ring with a constant field $\mathbb{K}=\text{const}(\mathbb{A},\sigma)$. An *evaluation function* $\text{ev}:\mathbb{A}\times\mathbb{Z}_{\ge 0}\to\mathbb{K}$ *for* $(\mathbb{A},\sigma)$ is a function which satisfies the following three properties:

(i) for all $c\in\mathbb{K}$, there is a natural number $\delta\ge 0$ such that

$$\forall n\ge\delta:\ \text{ev}(c,n)=c;$$

(ii) for all $f,g\in\mathbb{A}$ there is a natural number $\delta\ge 0$ such that

$$\forall n\ge\delta:\ \text{ev}(f\,g,n)=\text{ev}(f,n)\ \text{ev}(g,n),$$
$$\forall n\ge\delta:\ \text{ev}(f+g,n)=\text{ev}(f,n)+\text{ev}(g,n);$$

(iii) for all $f \in \mathbb{A}$ and $i \in \mathbb{Z}$, there is a natural number $\delta \geq 0$ such that

$$\forall n \geq \delta : \operatorname{ev}(\sigma^i(f), n) = \operatorname{ev}(f, n+i).$$

We say a sequence $\langle F(n) \rangle_{n \geq 0} \in \mathcal{S}(\mathbb{K})$ is *modeled* by $f \in \mathbb{A}$ in the difference ring $(\mathbb{A}, \sigma)$, if there is an evaluation function ev such that

$$F(k) = \operatorname{ev}(f, k)$$

holds for all $k \in \mathbb{Z}_{\geq 0}$ from a certain point on.

In this article, our base field is a rational function field $\mathbb{K}(x)$ which is equipped with the evaluation function $\operatorname{ev} : \mathbb{K}(x) \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined as follows. For $f = \frac{g}{h} \in \mathbb{K}(x)$ with $h \neq 0$ where g and h are coprime (if $g = 0$ we take $h = 1$) we have

$$\operatorname{ev}(f, k) := \begin{cases} 0 & \text{if } h(k) = 0, \\ \frac{g(k)}{h(k)} & \text{if } h(k) \neq 0. \end{cases} \tag{28}$$

Here, $g(k)$ and $h(k)$ are the usual polynomial evaluation at some natural number k.

Then given a tower of AP-extension defined over $(\mathbb{K}(x), \sigma)$, one can define an appropriate evaluation function by iterative applications of the following lemma that is implied by Schneider (2017, Lemma 5.4).

Lemma 3.1. *Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$ and let $\operatorname{ev} : \mathbb{A} \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ be an evaluation function for $(\mathbb{A}, \sigma)$. Let $(\mathbb{A}\langle t \rangle, \sigma)$ be an AP-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ ($\alpha \in \mathbb{A}^*$) and suppose that there is a $\delta \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{ev}(\alpha, n) \neq 0$ for all $n \geq \delta$. Further, take $u \in \mathbb{K}^*$; if $t^\lambda = 1$ for some $\lambda > 1$, we further assume that $u^\lambda = 1$ holds. Consider the map $\operatorname{ev}' : \mathbb{A}\langle t \rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined by*

$$\operatorname{ev}'(\sum_i h_i\, t^i, n) = \sum_i \operatorname{ev}(h_i, n)\, \operatorname{ev}'(t, n)^i$$

with $\operatorname{ev}'(t, n) = u \prod_{k=\delta}^{n} \operatorname{ev}(\alpha, k-1)$. Then ev' is an evaluation function for $(\mathbb{A}\langle t \rangle, \sigma)$.

We can carry out the following construction to model a given number of hypergeometric products $P_1(n), \dots, P_e(n) \in \operatorname{Prod}_n(\mathbb{K}(x))$ in a P-extension. Here we start with the difference field $(\mathbb{K}(x), \sigma)$ given by $\sigma|_{\mathbb{K}} = \mathrm{id}$ and $\sigma(x) = x + 1$ which is equipped with the evaluation function ev given in (28) and the zero-function (4) and apply the following algorithm.

Algorithm 3.1. (Modeling hypergeometric products in a multiple chain P-extension)
Input: hypergeometric products $P_1(n), \dots P_e(n) \in \operatorname{Prod}_n(\mathbb{K}(x))$ as given in (1) with $m \leq d$.
Output: $\delta \in \mathbb{Z}_{\geq 0}$ and a multiple chain P-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\mathbb{A} = \mathbb{K}(x)\langle t_1 \rangle \dots \langle t_v \rangle$ together with an evaluation function $\operatorname{ev} : \mathbb{A} \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ with the following property: for any i with $1 \leq i \leq e$ there is a j with $1 \leq j \leq v$ such that $\operatorname{ev}(t_j, n) = P_i(n)$ for all $n \geq \delta$.

(1) Following Sections 2.1 and (2.2) we rewrite $P_i(n)$ for $1 \leq i \leq e$ such that it is composed multiplicatively by products in factored form and such that all geometric products are 1-refined and the remaining hypergeometric products are δ-refined for some $\delta \in \mathbb{Z}_{\geq 0}$; if all products are geometric, we set $\delta = 1$. Let $\mathcal{P}$ be the set of all products that occur in the rewritten $P_i(n)$.
(2) Among all nested products of $\mathcal{P}$ with the same multiplicand representation $c(x) \in \mathbb{K}(x)$, take one of the products, say $P'(n)$ of the form (6) where $\ell := \ell_1 = \dots = \ell_m \in \{1, \delta\}$, with the highest nesting depth m.
(3) Construct a single chain P-extension $(\mathbb{K}(x)\langle p_1 \rangle \dots \langle p_m \rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with base $c(x+1)$ and extend the evaluation function by Lemma 3.1 such that $\operatorname{ev}(p_m, n) = P'(n)$ for all $n \geq \ell$.
(4) Remove all the products from $\mathcal{P}$ which have the same multiplicand $c(x)$.

(5) Repeat step (2) until $\mathcal{P}$ is the empty set.
(6) Combine the constructed single chain P-extensions of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}(x)$ to obtain a multiple chain P-extension $(\mathbb{A},\sigma)$ of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}(x)$. In addition, combine the evaluation functions to one extended version.

Proposition 3.1. *Algorithm 3.1 can be executed if $\mathbb{K}$ is computable and returns the correct result.*

Proof. If $\mathbb{K}$ is computable, all the described steps can be carried out. In particular, the arising products with highest depth are modeled by the top extension of each single chain extension. Namely, in step (2) the product $P'(n)$ is modeled by p_m. However, any arising product with the same multiplicand representation $c(x)$ and depth $\mu<m$ is modeled by p_μ. Thus any product in $\mathcal{P}$ is modeled by the generators in $\mathbb{A}$ and the statement is proven. $\square$

Remark 3.2. (1) In Step (1) of Algorithm 3.1 one may rewrite the arising products using Algorithm 2.1. In this way all hypergeometric products (which are not geometric) can be given even in shift-coprime representation.
(2) Furthermore, one may refine Step (3) further and can construct single chain A-extensions (instead of P-extensions) whenever $c(x)$ is a root of unity in $\mathbb{K}$. This construction will be exemplified in Example 3.1 below.
(3) More generally, one can tackle any expression $A(n)$ of the form (3) in terms of the products $P_1(n),\dots,P_e(n)$ where $a_{\boldsymbol{v}}=\frac{\alpha_{\boldsymbol{v}}}{\beta_{\boldsymbol{v}}}\in\mathbb{K}(x)$ with $\alpha_{\boldsymbol{v}},\beta_{\boldsymbol{v}}\in\mathbb{K}[x]$ for $\boldsymbol{v}\in S$. Namely, define $\lambda=\max\{Z(\beta_{\boldsymbol{v}})\mid \boldsymbol{v}\in S\}$, i.e., the evaluation $a_{\boldsymbol{v}}(n)$ does not introduce poles for any $n\in\mathbb{Z}_{\geq0}$ with $n\geq\lambda$. Then replacing the rewritten products in $A(n)$ and afterwards replacing the involved products by the corresponding P-monomials produces an $a\in\mathbb{A}$ with $\operatorname{ev}(a,n)=A(n)$ for all $n\geq\max(\delta,\lambda)$.

We summarize the above constructions with the following example.

Example 3.1. Let $\mathbb{K}=\mathbb{Q}(\sqrt{3})$ and take the difference field $(\mathbb{K}(x),\sigma)$ where the automorphism is defined by $\sigma(x)=x+1$ and $\sigma|_{\mathbb{K}}=\mathrm{id}$. Furthermore, take the evaluation function $\operatorname{ev}:\mathbb{K}(x)\times\mathbb{Z}_{\geq0}\to\mathbb{K}$ given by (28). Then we can construct the following single chain extensions of $(\mathbb{K}(x),\sigma)$ in order to model the geometric products in $\tilde{G}(n)$ and the hypergeometric products in $\tilde{H}(n)$ given in (24) and (25).

(1) We define the single chain A-extension $(\mathbb{K}(x)\langle\vartheta_{1,1}\rangle\langle\vartheta_{1,2}\rangle,\sigma)$ of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}$ of order 2, based at -1 where the automorphism is given by

$$\sigma(\vartheta_{1,1})=-\vartheta_{1,1},\qquad \sigma(\vartheta_{1,2})=-\vartheta_{1,1}\,\vartheta_{1,2}. \tag{29}$$

By applying Lemma 3.1 twice we extend the evaluation function to $\operatorname{ev}:\mathbb{K}(x)\langle\vartheta_{1,1}\rangle\langle\vartheta_{1,2}\rangle\times\mathbb{Z}_{\geq0}\to\mathbb{K}$ with

$$\operatorname{ev}(\vartheta_{1,1},n)=\prod_{k=1}^{n}-1,\qquad \operatorname{ev}(\vartheta_{1,2},n)=\prod_{k=1}^{n}\prod_{j=1}^{k}-1. \tag{30}$$

(2) Similarly, define the single chain P-extension $(\mathbb{K}(x)\langle y_{1,1}\rangle,\sigma)$ of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}$ based at $\sqrt{3}$ together with the evaluation function $\operatorname{ev}:\mathbb{K}(x)\langle y_{1,1}\rangle\times\mathbb{Z}_{\geq0}\to\mathbb{K}$ (using Lemma 3.1) by

$$\sigma(y_{1,1})=\sqrt{3}\,y_{1,1},\qquad\text{and}\qquad \operatorname{ev}(y_{1,1},n)=\prod_{k=1}^{n}\sqrt{3}. \tag{31}$$

(3) Define the single chain P-extension $(\mathbb{K}(x)\langle y_{2,1}\rangle\langle y_{2,2}\rangle,\sigma)$ of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}$ based at 2 equipped with the evaluation function $\operatorname{ev}:\mathbb{K}(x)\langle y_{2,1}\rangle\langle y_{2,2}\rangle\times\mathbb{Z}_{\geq0}\to\mathbb{K}$ by

$$\begin{aligned} &\sigma(y_{2,1}) = 2\, y_{2,1}, && \sigma(y_{2,2}) = 2\, y_{2,1}\, y_{2,2}, \quad \text{and} \\ &\mathrm{ev}(y_{2,1}, n) = \prod_{k=1}^{n} 2, && \mathrm{ev}(y_{2,2}, n) = \prod_{k=1}^{n}\prod_{j=1}^{k} 2. \end{aligned} \tag{32}$$

(4) Define the single chain P-extension $(\mathbb{K}(x)\langle y_{3,1}\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ based at 3 together with the evaluation function $\mathrm{ev} : \mathbb{K}(x)\langle y_{3,1}\rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ by

$$\sigma(y_{3,1}) = 3\, y_{3,1}, \qquad \text{and} \qquad \mathrm{ev}(y_{3,1}, n) = \prod_{k=1}^{n} 3. \tag{33}$$

(5) Define the single chain P-extension $(\mathbb{K}(x)\langle y_{4,1}\rangle\langle y_{4,2}\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ with 5 as its base accompanied with the evaluation function $\mathrm{ev} : \mathbb{K}(x)\langle y_{4,1}\rangle\langle y_{4,2}\rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ by

$$\begin{aligned} &\sigma(y_{4,1}) = 5\, y_{4,1}, && \sigma(y_{4,2}) = 5\, y_{4,1}\, y_{4,2}, \quad \text{and} \\ &\mathrm{ev}(y_{4,1}, n) = \prod_{k=1}^{n} 5, && \mathrm{ev}(y_{4,1}, n) = \prod_{k=1}^{n}\prod_{j=1}^{k} 5. \end{aligned} \tag{34}$$

(6) Define the single chain P-extension $(\mathbb{K}(x)\langle y_{5,1}\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ with 25 as its base together with the evaluation function $\mathrm{ev} : \mathbb{K}(x)\langle y_{5,1}\rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ by

$$\sigma(y_{5,1}) = 25\, y_{5,1}, \qquad \text{and} \qquad \mathrm{ev}(y_{5,1}, n) = \prod_{k=1}^{n} 25. \tag{35}$$

(7) Define the single chain P-extension $(\mathbb{K}(x)\langle z_{1,1}\rangle\langle z_{1,2}\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ based at $(x-1)$ and the evaluation function $\mathrm{ev} : \mathbb{K}(x)\langle z_{1,1}\rangle\langle z_{1,2}\rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ by

$$\begin{aligned} &\sigma(z_{1,1}) = (x-1)\, z_{1,1}, && \sigma(z_{1,2}) = (x-1)\, z_{1,1}\, z_{1,2}, \quad \text{and} \\ &\mathrm{ev}(z_{1,1}, n) = \prod_{k=3}^{n} (k-2), && \mathrm{ev}(z_{1,2}, n) = \prod_{k=3}^{n}\prod_{j=3}^{k} (j-2). \end{aligned} \tag{36}$$

(8) Define the single chain P-extension $(\mathbb{K}(x)\langle z_{2,1}\rangle, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ with $\left(x + \frac{25}{24}\right)$ as its base together with the evaluation function $\mathrm{ev} : \mathbb{K}(x)\langle z_{2,1}\rangle \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ by

$$\sigma(z_{2,1}) = \left(x + \tfrac{25}{24}\right) z_{2,1}, \qquad \text{and} \qquad \mathrm{ev}(z_{1,1}, n) = \prod_{k=3}^{n} \left(k + \tfrac{1}{24}\right). \tag{37}$$

Putting everything together, we have constructed the multiple chain AP-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ with

$$\mathbb{A} = \mathbb{K}(x)\langle\vartheta_{1,1}\rangle\langle\vartheta_{1,2}\rangle\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{2,2}\rangle\langle y_{3,1}\rangle\langle y_{4,1}\rangle\langle y_{4,2}\rangle\langle y_{5,1}\rangle\langle z_{1,1}\rangle\langle z_{1,2}\rangle\langle z_{2,1}\rangle \tag{38}$$

based at $\left(-1,\, -1,\, \sqrt{3},\, 2,\, 2,\, 3,\, 5,\, 5,\, 25,\, x-2,\, x-1,\, x + \frac{25}{24}\right)$ where $(1, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1)$ is the extension depth. In this ring, the geometric product expression $\tilde{G}(n)$ and the hypergeometric product expression $\tilde{H}(n)$ defined in (24) and (25) are modeled by

$$g = \frac{\vartheta_{1,1}\, y_{3,1}\, y_{5,1}\, \vartheta_{1,2}\, y_{2,2}}{y_{1,1}\, y_{2,1}\, y_{4,2}} \qquad \text{and} \qquad h = z_{1,1}^{3}\, z_{2,1}\, z_{1,2} \tag{39}$$

respectively. That is, $\tilde{G}(n) = \mathrm{ev}(g, n)$ holds for all $n \geq 1$ and $\tilde{H}(n) = \mathrm{ev}(h, n)$ holds for all $n \geq 2$. As a consequence, the indefinite hypergeometric product expression $\tilde{P}(n) = \tilde{r}(n)\, \tilde{G}(n)\, \tilde{H}(n)$ with $\tilde{r}(n)$, $\tilde{G}(n)$ and $\tilde{H}(n)$ given in (23), (24) and (25) is modeled by the expression

$$\tilde{p} = \frac{254\,(x-1)^3\,x\,(x+1)\,(x+2)}{432}\,g\,h \in \mathbb{A}.$$

This means that $\tilde{P}(n) = \mathrm{ev}(\tilde{p}, n)$ holds for all $n \in \mathbb{Z}_{\geq 0}$ with $n \geq 2$.

4. A refined difference ring approach: RΠ-extensions

In general, the naive construction of an (ordered) multiple chain P-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}(x), \sigma)$ following Algorithm 3.1 or a slightly refined construction of an AP-extension like in Example 3.1 introduce algebraic relations among the monomials. In order to tackle Problem RPE above, we will refine AP-extensions further to the class of RΠ-extensions. In this regard, the set of constants

$$\mathrm{const}(\mathbb{A}, \sigma) = \{c \in \mathbb{A} \mid \sigma(c) = c\} \tag{40}$$

of a difference ring (field) $(\mathbb{A}, \sigma)$ plays a decisive role. In general it forms a subring (subfield) of $\mathbb{A}$ which contains the rational numbers $\mathbb{Q}$ as a subfield. In this article, $\mathrm{const}(\mathbb{A}, \sigma)$ will always be a field also called the *constant field* of $(\mathbb{A}, \sigma)$, which we will also denote by $\mathbb{K}$. We note further that one can decide if $c \in \mathbb{A}$ is a constant if $(\mathbb{A}, \sigma)$ is computable.

We are now ready to refine AP-extensions as follows.

Definition 4.1. Let $(\mathbb{A}\langle t\rangle, \sigma)$ be an A-/P-/AP-extension of $(\mathbb{A}, \sigma)$. Then it is called an R-/Π-/RΠ-*extension* if $\mathrm{const}(\mathbb{A}\langle t\rangle, \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$ holds. Depending on the type of extension, we call the generator t an R-/Π-/RΠ-*monomial*, respectively. A (nested) A-/P-/AP-extension $(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma)$ of $(\mathbb{A}, \sigma)$ with $\mathrm{const}(\mathbb{A}\langle t_1\rangle \dots \langle t_e\rangle, \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$ is also called a *(nested)* R-/Π-/RΠ–*extension*.

Given an A-/P-/AP-extension, there exist criteria that enable one to check if it is an R-/Π-/RΠ-extension. We refer the reader to see Schneider (2016, Theorem 2.12) for further details and proofs.

Theorem 4.1. *Let $(\mathbb{A}, \sigma)$ be a difference ring. Then the following statements hold.*

1. *Let $(\mathbb{A}[t, \frac{1}{t}], \sigma)$ be a* P*-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\,t$ where $\alpha \in \mathbb{A}^*$. Then this is a* Π*-extension (i.e., $\mathrm{const}(\mathbb{A}[t, \frac{1}{t}], \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$) iff there are no $g \in \mathbb{A} \setminus \{0\}$ and $v \in \mathbb{Z} \setminus \{0\}$ with $\sigma(g) = \alpha^v\,g$.*
2. *Let $(\mathbb{A}[\vartheta], \sigma)$ be an* A*-extension of $(\mathbb{A}, \sigma)$ of order $\lambda > 1$ with $\sigma(\vartheta) = \zeta\,\vartheta$ where $\zeta \in \mathbb{A}^*$. Then this is an* R*-extension (i.e., $\mathrm{const}(\mathbb{A}[\vartheta], \sigma) = \mathrm{const}(\mathbb{A}, \sigma)$) iff there are no $g \in \mathbb{A} \setminus \{0\}$ and $v \in \{1, \dots, \lambda - 1\}$ with $\sigma(g) = \zeta^v\,g$. If it is an* R*-extension, ζ is a primitive λ-th root of unity.*

We remark that the above definitions and also Theorem 4.1 are inspired by Karr's ΠΣ-field extensions (Karr, 1981; Schneider, 2001). Since we will use this notion later (see Theorem 5.1 and 5.2 below) we will introduce them already here.

Definition 4.2. Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of a difference field $(\mathbb{F}, \sigma)$ with t transcendental over $\mathbb{F}$ and $\sigma(t) = \alpha\,t + \beta$ with $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. This extension is called a Σ-*field extension* if $\alpha = 1$ and $\mathrm{const}(\mathbb{F}(t), \sigma) = \mathrm{const}(\mathbb{F}, \sigma)$, and it is called a Π-*field extension* if $\beta = 0$ and $\mathrm{const}(\mathbb{F}(t), \sigma) = \mathrm{const}(\mathbb{F}, \sigma)$. A difference field $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ is called a ΠΣ-*field over* $\mathbb{K}$ if $(\mathbb{K}(t_1) \dots (t_i), \sigma)$ is a ΠΣ-extension of $(\mathbb{K}(t_1) \dots (t_{i-1}), \sigma)$ for $1 \leq i \leq e$ with $\mathrm{const}(\mathbb{K}, \sigma) = \mathbb{K}$.

Throughout this article, our base case difference field is $(\mathbb{K}(x), \sigma)$ with the automorphism $\sigma(x) = x + 1$ and $\sigma|_{\mathbb{K}} = \mathrm{id}$ which in fact is a Σ-extension of $(\mathbb{K}, \sigma)$, i.e., $\mathrm{const}(\mathbb{K}(x), \sigma) = \mathbb{K}$. In particular $(\mathbb{K}(x), \sigma)$ is a ΠΣ-field over $\mathbb{K}$. We conclude this subsection by observing that the check if an A-extension is an R-extension (see part 2 of Theorem 4.1) is not necessary if the ground field is a ΠΣ-field; compare Ocansey and Schneider (2018, Lemma 2.1).

Lemma 4.1. *Let $(\mathbb{F}, \sigma)$ be a ΠΣ-field over $\mathbb{K}$. Then any* A*-extension $(\mathbb{F}[\vartheta], \sigma)$ of $(\mathbb{F}, \sigma)$ with order $\lambda > 1$ is an* R*-extension.*

4.1. Embedding into the ring of sequences

In this subsection, we will discuss the connection between RΠ-extensions and the difference ring of sequences. More precisely, we will elaborate how RΠ-extensions can be embedded into the difference ring of sequences (Schneider, 2017); compare also van der Put and Singer (1997). This feature will enable us to handle condition (2) of Problem RPE in the sections below.

Definition 4.3. Let $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ be two difference rings. The map $\tau : \mathbb{A} \to \mathbb{A}'$ is called a *difference ring homomorphism* if τ is a ring homomorphism, and for all $f \in \mathbb{A}$, $\tau(\sigma(f)) = \sigma'(\tau(f))$. If τ is injective, then it is called a *difference ring monomorphism* or a *difference ring embedding*.[2] If τ is a bijection, then it is a *difference ring isomorphism* and we say $(\mathbb{A}, \sigma)$ and $(\mathbb{A}', \sigma')$ are isomorphic; we write $(\mathbb{A}, \sigma) \simeq (\mathbb{A}', \sigma')$. Let $(\mathbb{E}, \sigma)$ and $(\tilde{\mathbb{E}}, \tilde{\sigma})$ be difference ring extensions of $(\mathbb{A}, \sigma)$. Then a difference ring-homomorphism/isomorphism/monomorphism $\tau : \mathbb{E} \to \tilde{\mathbb{E}}$ is called an $\mathbb{A}$-*homomorphism*/$\mathbb{A}$-*isomorphism*/$\mathbb{A}$-*monomorphism*, if $\tau|_{\mathbb{A}} = \mathrm{id}$.

Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$. A difference ring homomorphism (resp. monomorphism) $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K})$ is called a $\mathbb{K}$-*homomorphism* (resp. -*monomorphism*) if for all $c \in \mathbb{K}$ we have that $\tau(c) = \boldsymbol{c} := \langle c, c, c, \dots \rangle$.

The following results provide the key property that will enable us to embed RΠ-extensions into the ring of sequences. First, we recall that the evaluation function of a difference ring establishes naturally a $\mathbb{K}$-homomorphism. More precisely, by Schneider (2001, Lemma 2.5.1) we get

Lemma 4.2. *Let $(\mathbb{A}, \sigma)$ be a difference ring with constant field $\mathbb{K}$. Then the map $\tau : \mathbb{A} \to \mathcal{S}(\mathbb{K})$ is a $\mathbb{K}$-homomorphism if and only if there is an evaluation function* $\mathrm{ev} : \mathbb{A} \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ *for $(\mathbb{A}, \sigma)$ (see Definition 3.4) with* $\tau(f) = \langle \mathrm{ev}(f, 0), \mathrm{ev}(f, 1), \dots \rangle$.

Starting with our ΠΣ-field $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}$ and the evaluation function (28) we can construct for an AP-extension an appropriate evaluation function by iterative application of Lemma 3.1. In particular this yields a $\mathbb{K}$-homomorphism from the given AP-extension into the ring of sequences by Lemma 4.2. Finally, we utilize the following result from Schneider (2017); compare Ocansey and Schneider (2018, Lemma 2.2).

Theorem 4.2. *Let $(\mathbb{A}, \sigma)$ be a difference field with constant field $\mathbb{K}$ and let $(\mathbb{E}, \sigma)$ be a basic* RΠ*-extension of $(\mathbb{A}, \sigma)$. Then any $\mathbb{K}$-homomorphism $\tau : \mathbb{E} \to \mathcal{S}(\mathbb{K})$ is injective.*

In other words, if we succeed in modeling our nested products within a basic RΠ-extension (in particular, a multiple chain RΠ-extension) over $(\mathbb{K}(x), \sigma)$ with an appropriate evaluation function, then we automatically obtain a $\mathbb{K}$-embedding.

4.2. A structural theorem for multiple chain Π-extensions

In part 1 of Theorem 4.1 a criterion is given that enables one to check with, e.g., the algorithms from Karr (1981); Schneider (2016) whether a P-extension is a Π-extension. In Ocansey and Schneider (2018, Lemma 5.1) (based on Schneider (2010a, 2017)) this criterion has been generalized for “single nested” P-extensions as follows.

Lemma 4.3. *Let $(\mathbb{F}, \sigma)$ be a difference field and let $f_1, \dots, f_s \in \mathbb{F}^*$. Then the following statements are equivalent.*

[2] In this case, $(\tau(\mathbb{A}), \sigma)$ is a sub-difference ring of $(\mathbb{A}', \sigma')$ where $(\mathbb{A}, \sigma)$ and $(\tau(\mathbb{A}), \sigma)$ are the same up to renaming with respect to τ.

(1) There do not exist $(v_1,\dots,v_s)\in\mathbb{Z}^s\setminus\{\mathbf{0}_s\}$ *and* $g\in\mathbb{F}^*$ *such that* $\frac{\sigma(g)}{g}=f_1^{v_1}\cdots f_s^{v_s}$ *holds.*
(2) The P*-extension* $(\mathbb{F}[z_1,z_1^{-1}]\dots[z_s,z_s^{-1}],\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* $\sigma(z_i)=f_i\,z_i$ *for* $1\le i\le s$ *is a* Π*-extension.*
(3) The difference field extension $(\mathbb{F}(z_1)\dots(z_s),\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* z_i *transcendental over* $\mathbb{F}(z_1)\dots(z_{i-1})$ *and* $\sigma(z_i)=f_i\,z_i$ *for* $1\le i\le s$ *is a* Π*-field extension.*

In Theorem 4.3 we will extend this result further to multiple-chain P-extensions. Here we utilize that a solution for a certain class of homogeneous first-order difference equations has a particularly simple form; this result is a specialization of Schneider (2016, Corollary 4.6).

Corollary 4.1. *Let* $(\mathbb{E},\sigma)$ *be a* Π*-extension of a difference field* $(\mathbb{A},\sigma)$ *with* $\mathbb{E}=\mathbb{A}\langle t_1\rangle\dots\langle t_e\rangle$*. Then for any* $g\in\mathbb{E}\setminus\{0\}$ *with* $\sigma(g)=u\,t_1^{z_1}\cdots t_e^{z_e}\,g$ *for some* $u\in\mathbb{A}^*$ *and* $z_i\in\mathbb{Z}$ *we have* $g=h\,t_1^{v_1}\cdots t_e^{v_e}$ *with* $h\in\mathbb{A}^*$ *and* $v_i\in\mathbb{Z}$*.*

Now we are ready to prove a general criterion that enables one to check if a multiple chain P-extension forms a Π-extension. This result will be heavily used within the next two sections.

Theorem 4.3. *Let* $(\mathbb{H},\sigma)$ *be a difference field and let* $(\mathbb{H}_\ell,\sigma)$ *with* $\mathbb{H}_\ell=\mathbb{H}\langle t_{\ell,1}\rangle\dots\langle t_{\ell,s_\ell}\rangle$ *for* $1\le\ell\le m$ *be single chain* P*-extensions of* $(\mathbb{H},\sigma)$ *over* $\mathbb{H}$ *with base* $c_\ell\in\mathbb{H}^*$ *where* $s_1\ge s_2\ge\cdots\ge s_m$*. In particular, the automorphisms are given by*

$$\sigma(t_{\ell,k})=\alpha_{\ell,k}\,t_{\ell,k}\quad\text{where}\quad\alpha_{\ell,k}=c_\ell\,t_{\ell,1}\cdots t_{\ell,k-1}\in(\mathbb{H}^*)_{\mathbb{H}}^{\mathbb{H}\langle t_{\ell,1}\rangle\dots\langle t_{\ell,k-1}\rangle}.\tag{41}$$

Let $(\mathbb{A},\sigma)$ *be the ordered multiple chain* P*-ext. of* $(\mathbb{H},\sigma)$ *with* $\mathbb{A}=\mathbb{H}\langle t_{1,1}\rangle\dots\langle t_{w_1,1}\rangle\dots\langle t_{1,d}\rangle\dots\langle t_{w_d,d}\rangle$ *of monomial depth* $d:=s_1$ *with* $m=w_1\ge w_2\ge\cdots\ge w_d$ *composed by the single chain* Π*-extensions* $(\mathbb{H}_\ell,\sigma)$ *of* $(\mathbb{H},\sigma)$ *with the automorphism* (41)*. Then* $(\mathbb{A},\sigma)$ *is a* Π*-extension of* $(\mathbb{H},\sigma)$ *if and only if*[3] *there does not exist a* $g\in\mathbb{H}^*$ *and* $(v_1,\dots,v_m)\in\mathbb{Z}^m\setminus\{\mathbf{0}_m\}$ *such that*

$$\frac{\sigma(g)}{g}=c_1^{v_1}\cdots c_m^{v_m}.\tag{42}$$

Proof. "$\Longrightarrow$" Suppose that $(\mathbb{A},\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$. Then, it is a tower of Π-extensions $(\mathbb{A}_i,\sigma)$ of $(\mathbb{H},\sigma)$ where $\mathbb{A}_i=\mathbb{A}_{i-1}\langle t_{1,i}\rangle\dots\langle t_{w_i,i}\rangle$ for $1\le i\le d$ with $\mathbb{A}_0=\mathbb{H}$. Since $(\mathbb{A}_1,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$, it follows by Lemma 4.3 that there does not exist a $g\in\mathbb{H}$ and $(v_1,\dots,v_{w_1})\in\mathbb{Z}^{w_1}\setminus\{\mathbf{0}_{w_1}\}$ with $w_1=m$ such that (42) holds.
"$\Longleftarrow$" Conversely, suppose that there does not exist a $g\in\mathbb{H}$ and $(v_1,\dots,v_{w_1})\in\mathbb{Z}^{w_1}\setminus\{\mathbf{0}_{w_1}\}$ with $w_1=m$ such that (42) holds. Let $(\mathbb{A}_1,\sigma)$ with $\mathbb{A}_1=\mathbb{H}\langle t_{1,1}\rangle\dots\langle t_{w_1,1}\rangle$ be a P-extension of $(\mathbb{H},\sigma)$ with $\sigma(t_{j,1})=\alpha_{j,1}\,t_{j,1}$ for all $1\le j\le w_1$. By Lemma 4.3, $(\mathbb{A}_1,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$. Let $(\mathbb{A}_i,\sigma)$ with $\mathbb{A}_i=\mathbb{A}_{i-1}\langle t_{1,i}\rangle\dots\langle t_{w_i,i}\rangle$ be the multiple chain P-extension of $(\mathbb{H},\sigma)$ with $\mathfrak{d}(t_{1,i})=\cdots=\mathfrak{d}(t_{w_i,i})$ for all $1\le i\le d$ with the automorphism (41). Assume that $(\mathbb{A}_k,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$ for all $1\le k\le\delta$ with $d>\delta\ge 1$ and that $(\mathbb{A}_{\delta+1},\sigma)$ is not a Π-extension of $(\mathbb{A}_\delta,\sigma)$. Then by Lemma 4.3, we can take a $g\in\mathbb{A}_\delta\setminus\{0\}$ and $(\upsilon_1,\upsilon_2,\dots,\upsilon_{w_{\delta+1}})\in\mathbb{Z}^{w_{\delta+1}}\setminus\{\mathbf{0}_{w_{\delta+1}}\}$ such that

$$\sigma(g)=\alpha_{1,\delta+1}^{\upsilon_1}\,\alpha_{2,\delta+1}^{\upsilon_2}\cdots\alpha_{w_{\delta+1},\delta+1}^{\upsilon_{w_{\delta+1}}}\,g\tag{43}$$

holds. By Corollary 4.1, it follows that $g=h\,t_{1,1}^{v_{1,1}}\,t_{2,1}^{v_{2,1}}\cdots t_{w_1,1}^{v_{w_1,1}}\cdots t_{1,\delta}^{v_{1,\delta}}\,t_{2,\delta}^{v_{2,\delta}}\cdots t_{w_\delta,\delta}^{v_{w_\delta,\delta}}$ with $h\in\mathbb{H}^*$ and $v_{i,j}\in\mathbb{Z}$. For the left hand side of (43) we have that

$$\sigma(g)=\gamma\,t_{1,\delta}^{v_{1,\delta}}\,t_{2,\delta}^{v_{2,\delta}}\cdots t_{w_\delta,\delta}^{v_{w_\delta,\delta}}$$

where $\gamma\in\mathbb{A}_{\delta-1}$ and for the right hand side of (43) we have that

[3] Note that $(c_1,\dots,c_m)=(\alpha_{1,1},\dots,\alpha_{w_1,1})$.

$$\alpha_{1,\delta+1}^{\upsilon_1}\alpha_{2,\delta+1}^{\upsilon_2}\cdots\alpha_{w_{\delta+1},\delta+1}^{\upsilon_{w_{\delta+1}}}\,g=\omega\,t_{1,\delta}^{v_{1,\delta}+\upsilon_1}\,t_{2,\delta}^{v_{2,\delta}+\upsilon_2}\cdots t_{w_{\delta+1},\delta}^{v_{w_{\delta+1},\delta}+\upsilon_{w_{\delta+1}}}\,t_{w_{\delta+1}+1,\delta}^{v_{w_{\delta+1}+1,\delta}}\cdots t_{w_\delta,\delta}^{v_{w_\delta,\delta}}$$

where $\omega\in\mathbb{A}_{\delta-1}$. Consequently, $v_{k,\delta}=v_{k,\delta}+\upsilon_k$ and thus $\upsilon_k=0$ for all $1\leq k\leq w_{\delta+1}$ which is a contradiction to the assumption that $(\upsilon_1,\ldots,\upsilon_{w_{\delta+1}})\neq\mathbf{0}_{w_{\delta+1}}$. Thus $(\mathbb{A}_d,\sigma)$ is a Π-extension of $(\mathbb{H},\sigma)$. $\square$

5. The main building blocks to represent nested products in RΠ-extensions

Suppose that we are given a finite set of hypergeometric products of finite nesting depth which have been brought into the form as given in Proposition 2.2. In the following we will show in Sections 5.1 and 5.2 how these hypergeometric and geometric products can be modeled in RΠ-extensions. For the treatment of geometric products one has to deal in addition with products defined over roots of unity of finite nesting depth. This extra complication will be treated in Section 5.3. Finally, in Section 6 below we will combine all these techniques to represent the full class of hypergeometric products of finite nesting depth in RΠ-extensions.

5.1. *Nested hypergeometric products with shift-coprime multiplicands*

In Proposition 2.2 we showed that a finite set of hypergeometric products of finite nesting depth can be brought in a shift-coprime product representation form. In the setting of ΠΣ-field extensions the underlying Definition 2.3 can be generalized as follows.

Definition 5.1. Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-field extension of $(\mathbb{F},\sigma)$. We call two polynomials $f,g\in\mathbb{F}[t]$ shift-coprime (or σ-coprime) if for all $k\in\mathbb{Z}$ we have that $\gcd(f,\sigma^k(g))=1$.

Inspired by Schneider (2005, 2014) we refined Lemma 4.3 in Ocansey and Schneider (2018, Theorem 5.3) to the following result that was the key tool to represent hypergeometric products of nesting depth 1 in a Π-extension.

Theorem 5.1. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$. Let $f_1,\ldots,f_s\in\mathbb{F}[t]\setminus\mathbb{F}$ be irreducible monic polynomials. Then the following statements are equivalent.*

(1) For all i,j with $1\leq i<j\leq s$, f_i and f_j are shift-coprime.
(2) There does not exist $(v_1,\ldots,v_s)\in\mathbb{Z}^s\setminus\{\mathbf{0}_s\}$ and $g\in\mathbb{F}(t)^*$ with $\frac{\sigma(g)}{g}=f_1^{v_1}\cdots f_s^{v_s}$.
(3) The P*-extension $(\mathbb{F}(t)[z_1,z_1^{-1}]\ldots[z_s,z_s^{-1}],\sigma)$ of $(\mathbb{F}(t),\sigma)$ with $\sigma(z_i)=f_i\,z_i$ for $1\leq i\leq s$ is a Π-extension.*

With this result we are now in the position to refine Theorem 4.3 in order to construct a Π-extension in which we can model hypergeometric products of finite nesting depth that are in shift-coprime representation form.

Theorem 5.2. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$. Let $\boldsymbol{f}=(f_1,\ldots,f_m)\in(\mathbb{F}[t]\setminus\mathbb{F})^m$ be irreducible monic polynomials. For all $1\leq\ell\leq m$, let $(\mathbb{F}_\ell,\sigma)$ with $\mathbb{F}_\ell:=\mathbb{F}(t)\langle z_{\ell,1}\rangle\ldots\langle z_{\ell,s_\ell}\rangle$ be a single chain* P*-extension of $(\mathbb{F}(t),\sigma)$ with base $f_\ell\in\mathbb{F}[t]\setminus\mathbb{F}$ with the automorphism*

$$\sigma(z_{\ell,k})=\alpha_{\ell,k}\,z_{\ell,k}\quad\text{where}\quad\alpha_{\ell,k}=f_\ell\,z_{\ell,1}\cdots z_{\ell,k-1}\in(\mathbb{F}^*)_{\mathbb{F}}^{\mathbb{F}\langle z_{\ell,1}\rangle\ldots\langle z_{\ell,k-1}\rangle}$$

and with $s_1\geq s_2\geq\cdots\geq s_m\geq1$. Let $(\mathbb{H}_b,\sigma)$ with

$$\mathbb{H}_b=\mathbb{F}(t)\langle\boldsymbol{z_1}\rangle\ldots\langle\boldsymbol{z_b}\rangle=\mathbb{F}(t)\langle z_{1,1}\rangle\ldots\langle z_{w_1,1}\rangle\ldots\langle z_{1,b}\rangle\ldots\langle z_{w_b,b}\rangle$$

be an ordered multiple chain P*-extension of $(\mathbb{F}(t),\sigma)$ of monomial depth $b=s_1$ with bases $f_1,\ldots,f_m$ where $m=w_1\geq w_2\geq\cdots\geq w_b\geq1$ which is composed by the single chain* P*-extensions $(\mathbb{F}_\ell,\sigma)$ of $(\mathbb{F}(t),\sigma)$. Then $(\mathbb{H}_b,\sigma)$ is a Π-extension of $(\mathbb{F}(t),\sigma)$ if and only if for all i,j with $1\leq i<j\leq m$ the f_i and f_j are shift-coprime.*

Proof. "$\Longrightarrow$" If $(\mathbb{H}_b, \sigma)$ is a Π-extension of $(\mathbb{F}(t), \sigma)$, then by Theorem 4.3 there does not exist a $g \in \mathbb{F}(t)^*$ such that $\frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_m^{v_m}$ holds, and by Theorem 5.1 for all i, j with $1 \le i < j \le m$, f_i and f_j are shift-coprime.
"$\Longleftarrow$" Conversely, if for all i, j with $1 \le i < j \le m$, f_i and f_j are shift-coprime, then by Theorem 5.1 there does not exist a $g \in \mathbb{F}(t)^*$ such that $\frac{\sigma(g)}{g} = f_1^{v_1} \cdots f_m^{v_m}$ holds, and by Theorem 4.3 $(\mathbb{H}_b, \sigma)$ is a Π-extension of $(\mathbb{F}(t), \sigma)$. $\square$

Summarizing, we obtain the following crucial result.

Corollary 5.1. *Let $(\mathbb{K}(x), \sigma)$ be a rational difference field with the automorphism $\sigma(x) = x + 1$ and the evaluation function $\mathrm{ev} : \mathbb{K}(x) \times \mathbb{Z}_{\ge 0} \to \mathbb{K}$ given by (28). Let $\tilde{H}_1(n), \dots, \tilde{H}_e(n)$ be hypergeometric products in $\mathrm{Prod}_n(\mathbb{K}(x))$ of nesting depth at most b which are all in shift-coprime representation form (see Definition 2.3) and which are all δ-refined for some $\delta \in \mathbb{Z}_{\ge 0}$. Then one can construct an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with*

$$\tilde{\mathbb{H}}_b = \mathbb{K}(x)\langle \tilde{\boldsymbol{z}}_{\boldsymbol{1}} \rangle \dots \langle \tilde{\boldsymbol{z}}_{\boldsymbol{b}} \rangle = \mathbb{K}(x)\langle \tilde{z}_{1,1} \rangle \dots \langle \tilde{z}_{p_1,1} \rangle \dots \langle \tilde{z}_{1,b} \rangle \dots \langle \tilde{z}_{p_b,b} \rangle \tag{44}$$

which is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_\ell, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \dots \langle \tilde{z}_{\ell,s_\ell} \rangle$ with

(1) the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ defined by

$$\sigma(\tilde{z}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{z}_{\ell,k} \quad \text{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{f}_\ell\, \tilde{z}_{\ell,1} \cdots \tilde{z}_{\ell,k-1} \in (\mathbb{K}(x)^*)_{\mathbb{K}(x)}^{\mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \dots \langle \tilde{z}_{\ell,k-1} \rangle}$$

for $1 \le \ell \le p_1$ and $1 \le k \le s_\ell$ where $\tilde{f}_\ell \in \mathbb{K}[x] \setminus \mathbb{K}$ is an irreducible monic polynomial, and
(2) the evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{Z}_{\ge 0} \to \mathbb{K}$ given by $\tilde{\mathrm{ev}}|_{\mathbb{K}(x)} = \mathrm{ev}$ with (28) and

$$\tilde{\mathrm{ev}}(\tilde{z}_{\ell,k}, n) = \prod_{j=\delta}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,k}, j-1) \tag{45}$$

for $1 \le \ell \le p_1$ and $1 \le k \le s_\ell$

with the following property: for all $1 \le i \le e$ there are k, j such that

$$\mathrm{ev}(\tilde{z}_{k,j}, n) = \tilde{H}_i(n), \quad \forall n \ge \max(0, \delta - 1). \tag{46}$$

Furthermore, for all $g \in \tilde{\mathbb{H}}_b$, the map $\tilde{\tau} : \tilde{\mathbb{H}}_b \to \mathcal{S}(\mathbb{K})$ defined by

$$\tilde{\tau}(g) = \left\langle \tilde{\mathrm{ev}}(g, n) \right\rangle_{n \ge 0} \tag{47}$$

is a $\mathbb{K}$-embedding. If $\mathbb{K}$ is computable, the above construction can be given explicitly.

Proof. By the procedure outlined in Algorithm 3.1 (skipping step (1) since the input is already in the right form) we obtain the ordered multiple chain P-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with (44) and (45) such that (46) holds for all $1 \le i \le e$ for some j, k. Since the bases $\tilde{f}_1, \dots, \tilde{f}_{p_1}$ of the single chain P-extensions $(\tilde{\mathbb{F}}_1, \sigma), \dots, (\tilde{\mathbb{F}}_{p_1}, \sigma)$ that composes $(\tilde{\mathbb{H}}_b, \sigma)$ are shift-coprime, it follows by Theorem 5.2 that $(\tilde{\mathbb{H}}_b, \sigma)$ is a Π-extension of $(\mathbb{K}(x), \sigma)$. Since $(\tilde{\mathbb{H}}_b, \sigma)$ is a basic Π-extension of the rational difference field $(\mathbb{K}(x), \sigma)$, it follows by Theorem 4.2 that the $\mathbb{K}$-homomorphism $\tilde{\tau} : \tilde{\mathbb{H}}_b \to \mathcal{S}(\mathbb{K})$ defined by (47) is injective. Since $\mathbb{K}$ is computable, all the above ingredients can be constructed explicitly. $\square$

Example 5.1 *(Cont. Example 3.1).* Consider the ordered multiple chain P-extension $(\tilde{\mathbb{H}}, \sigma)$ of the rational difference field $(\mathbb{K}(x), \sigma)$ of monomial depth 2 with $\tilde{\mathbb{H}} = \mathbb{K}(x)\langle z_{1,1} \rangle \langle z_{2,1} \rangle \langle z_{1,2} \rangle$ where $(\tilde{\mathbb{H}}, \sigma)$

is composed by the single chain Π-extensions of $(\mathbb{K}(x),\sigma)$ constructed in parts (7) and (8) of Example 3.1. Since the bases of $(\tilde{\mathbb{H}},\sigma)$ given by $(x-2)$ and $\left(x+\frac{1}{24}\right)$ are shift-coprime with respect to the automorphism $\sigma(x)=x+1$, it follows that the ordered multiple chain P-extension $(\tilde{\mathbb{H}},\sigma)$ of the rational difference field $(\mathbb{K}(x),\sigma)$ of monomial depth 2 is a Π-extension. Furthermore, it follows by Theorem 4.2 that the map $\tilde{\tau}:\tilde{\mathbb{H}}\to\mathcal{S}(\mathbb{K})$ defined by $\tilde{\tau}(f)=\langle\tilde{\text{ev}}(f,n)\rangle_{n\geq 0}$ for all $f\in\tilde{\mathbb{H}}$ is a $\mathbb{K}$-embedding where $\tilde{\text{ev}}=\text{ev}$ defined in (36), and (37). In particular, for the expression h given by (39) we have that $\tilde{H}(n)=\tilde{\text{ev}}(h,n)$ holds for all $n\geq 2$.

5.2. Geometric products

In Karr's algorithm (Karr, 1981) and all the improvements (Abramov and Petkovšek, 2010; Kauers and Schneider, 2006; Schneider, 2007a, 2008, 2015, 2016, 2017) one relies on certain algorithmic properties of the constant field $\mathbb{K}$. Among those, one needs to solve the following general orbit problem.

Problem GO for $\alpha_1,\dots,\alpha_w\in K^*$
Given a field K and $\alpha_1,\dots,\alpha_w\in K^*$. Compute a basis of the submodule $$\mathbb{V}:=\left\{(u_1,\dots,u_w)\in\mathbb{Z}^w\,\middle|\,\prod_{i=1}^{w}{\alpha_i}^{u_i}=1\right\}\text{ of }\mathbb{Z}^w\text{ over }\mathbb{Z}.$$

In our approach Problem GO is crucial to solve Problem RPE, but one has to solve it not only in a given field K (compare the definition of σ-computable in Kauers and Schneider (2006); Schneider (2005)) but one must be able to solve it in any finite algebraic field extension of K. This gives rise to the following definition.

Definition 5.2. A field K is *strongly σ-computable* if the standard operations in K can be performed, multivariate polynomials can be factored over K and Problem GO can be solved for K and any finite algebraic field extension of K.

Note that Ge's algorithm (Ge, 1993a) (see also Kauers (2005, Algorithm 7.16, page 84)) solves Problem GO over an arbitrary number field K. Since any finite algebraic extension of an algebraic number field is again an algebraic number field, we obtain the following result; for a weaker result see Schneider (2005, Theorem 3.5).

Lemma 5.1. *An algebraic number field K is strongly σ-computable.*

By Ocansey and Schneider (2018, Theorem 5.4) and the consideration of Ocansey and Schneider (2018, pg. 204) (see also Ocansey, 2019, Lemma 5.2.2) we provided an algorithm that enabled us to handle geometric products of nested depth 1. More precisely, given a P-extension that models such products, Lemma 5.2 states that one can construct an RΠ-extension in which the products can be rephrased.

Lemma 5.2. *Let $\mathbb{K}=K(\kappa_1,\dots,\kappa_u)$ be a rational function field over a field K and $(\mathbb{K},\sigma)$ be a difference field with $\sigma(c)=c$ for all $c\in\mathbb{K}$. Let $(\mathbb{K}\langle x_1\rangle\dots\langle x_e\rangle,\sigma)$ be a* P*-extension of $(\mathbb{K},\sigma)$ with $\sigma(x_i)=\gamma_i\,x_i$ for $1\leq i\leq e$ where $\gamma_i\in\mathbb{K}^*$. Let $\text{ev}:\mathbb{K}\langle x_1\rangle\dots\langle x_e\rangle\times\mathbb{Z}_{\geq 0}\to\mathbb{K}$ be the evaluation function defined by $\text{ev}(x_i,n)=\gamma_i^n$ for $1\leq i\leq e$. Then:*

(1) One can construct an $R\Pi$*-extension* $(\tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle\dots\langle\tilde{y}_s\rangle,\sigma)$[4] *of* $(\tilde{\mathbb{K}},\sigma)$ *with*

$$\sigma(\vartheta)=\zeta\,\vartheta \qquad \text{and} \qquad \sigma(\tilde{y}_k)=\alpha_k\,\tilde{y}_k$$

for $1\le k\le s$ *where* $\tilde{\mathbb{K}}=\tilde{K}(\kappa_1,\dots,\kappa_u)$ *and* $\tilde{K}$ *is a finite algebraic field extension of* K *with* $\zeta\in\tilde{K}$ *being a primitive* λ*-th root of unity and* $\alpha_k\in\tilde{\mathbb{K}}^*$*;*

(2) one can construct the evaluation function $\tilde{\mathrm{ev}}:\tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle\dots\langle\tilde{y}_s\rangle\times\mathbb{Z}_{\ge0}\to\tilde{\mathbb{K}}$ *defined as*

$$\tilde{\mathrm{ev}}(\vartheta,n)=\zeta^n \qquad \text{and} \qquad \tilde{\mathrm{ev}}(\tilde{y}_k,n)=\alpha_k^n;$$

(3) one can construct a difference ring homomorphism $\varphi:\mathbb{K}\langle x_1\rangle\dots\langle x_e\rangle\to\tilde{\mathbb{K}}\langle\vartheta\rangle\langle\tilde{y}_1\rangle\dots\langle\tilde{y}_s\rangle$ *with*

$$\varphi(x_i)=\vartheta^{\mu_i}\,\tilde{\boldsymbol{y}}^{\boldsymbol{v_i}}=\vartheta^{\mu_i}\,\tilde{y}_1^{v_{i,1}}\cdots\tilde{y}_s^{v_{i,s}} \tag{48}$$

for $1\le i\le e$ *where* $0\le\mu_i<\lambda$ *and* $v_{i,k}\in\mathbb{Z}$ *for* $1\le k\le s$

such that for all $f\in\mathbb{K}\langle x_1\rangle\dots\langle x_e\rangle$ *and for all* $n\in\mathbb{Z}_{\ge0}$*,*

$$\mathrm{ev}(f,n)=\tilde{\mathrm{ev}}(\varphi(f),n)$$

holds. If K *is strongly* σ*-computable, then the above constructions are computable.*

Using this result we will derive an extended version in Lemma 5.4 that deals with the class of ordered multiple chain AP-extensions that models geometric products of arbitrary but finite nesting depth.

In the following let $m\in\mathbb{Z}_{\ge1}$, and for $1\le\ell\le m$ let $(\mathbb{K}_\ell,\sigma)$ with $\mathbb{K}_\ell=\mathbb{K}\langle\boldsymbol{y}_\ell\rangle=\mathbb{K}\langle y_{\ell,1}\rangle\dots\langle y_{\ell,s_\ell}\rangle$ be a single chain P-extension of $(\mathbb{K},\sigma)$ with base $h_\ell\in\mathbb{K}^*$ where

$$\sigma(y_{\ell,i})=\alpha_{\ell,i}\,y_{\ell,i} \quad \text{with} \quad \alpha_{\ell,i}=h_\ell\,y_{\ell,1}\cdots y_{\ell,i-1}\in(\mathbb{K}^*)_{\mathbb{K}}^{\mathbb{K}\langle y_{\ell,1}\rangle\dots\langle y_{\ell,i-1}\rangle}. \tag{49}$$

In particular, we assume that $s_1\ge s_2\ge\dots\ge s_m$. Let $\mathrm{ev}:\mathbb{K}_\ell\times\mathbb{Z}_{\ge0}\to\mathbb{K}$ be the evaluation function defined by

$$\mathrm{ev}(y_{\ell,i},n)=\prod_{j=1}^{n}\mathrm{ev}(\alpha_{\ell,i},j-1)=\prod_{j=1}^{n}\alpha_{\ell,i}; \tag{50}$$

in particular, for all $c\in\mathbb{K}$ and $n\ge0$ we set $\mathrm{ev}(c,n)=c$. Let $(\mathbb{A},\sigma)$ be the multiple chain P-extension of $(\mathbb{K},\sigma)$ built by the single chain Π-extensions $(\mathbb{K}_\ell,\sigma)$ of $(\mathbb{K},\sigma)$ over $\mathbb{K}$. That is,

$$\mathbb{A}=\mathbb{K}\langle\boldsymbol{y_1}\rangle\langle\boldsymbol{y_2}\rangle\dots\langle\boldsymbol{y_m}\rangle=\mathbb{K}\langle y_{1,1}\rangle\dots\langle y_{1,s_1}\rangle\langle y_{2,1}\rangle\dots\langle y_{2,s_2}\rangle\dots\langle y_{m,1}\rangle\dots\langle y_{m,s_m}\rangle.$$

We emphasize that all the $y_{\ell,i}$ model 1-refined geometric products in product factored form of an arbitrary but finite nesting depth. Depending on the context, $\boldsymbol{y}_\ell$ denotes $(y_{\ell,1},\dots,y_{\ell,s_\ell})$ or $y_{\ell,1},\dots,y_{\ell,s_\ell}$ or $y_{\ell,1}\cdots y_{\ell,s_\ell}$. Note that the P-monomials $y_{\ell,i}$ can be ordered in increasing order of their depths. Namely, take the depth function $\mathfrak{d}:\mathbb{A}\to\mathbb{Z}_{\ge0}$ over $\mathbb{K}$ of $(\mathbb{A},\sigma)$ and let $d=\max(s_1,s_2,\dots,s_m)$ be the maximal depth. Then taking $\mathbb{A}_0=\mathbb{K}$ we can consider the tower of P-extensions $(\mathbb{A}_i,\sigma)$ of $(\mathbb{A}_{i-1},\sigma)$ with

$$\mathbb{A}_i=\mathbb{A}_{i-1}\langle\boldsymbol{y_i}\rangle=\mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle\dots\langle y_{w_i,i}\rangle$$

for $1\le i\le d$ where $m=w_1\ge w_2\ge\dots\ge w_d$ and with the automorphism (49) for $1\le\ell\le w_i$. In this way, the P-monomials at the i-th extension have the depth $\mathfrak{d}(y_{1,i})=\mathfrak{d}(y_{2,i})=\dots\mathfrak{d}(y_{w_i,i})=i$. Further,

[4] For concrete instances the R-monomial ϑ might be obsolete. In particular, if $\mu_i=0$ for all $1\le i\le e$ in (48) it can be removed.

the ring $\mathbb{A}_d$ is isomorphic to $\mathbb{A}$ up to reordering of the P-monomials. In particular, $(\mathbb{A}_d, \sigma)$ is an *ordered multiple chain* P*-extension* of $(\mathbb{K}, \sigma)$ of monomial depth at most d induced by the single chain Π-extensions $(\mathbb{K}_\ell, \sigma)$ of $(\mathbb{K}, \sigma)$ for $1 \le \ell \le m$ with (49) and (50). Observe that since $\mathbb{A}_d \simeq \mathbb{A}$, the evaluation function $\mathrm{ev} : \mathbb{A}_i \times \mathbb{Z}_{\ge 0} \to \mathbb{K}$ for all i with $1 \le i \le d$ is also defined by (50).

In order to derive the main result of this subsection in Lemma 5.4, we need following simple construction.

Lemma 5.3. *Let $(\mathbb{A}\langle t\rangle, \sigma)$ be a Π-extension of $(\mathbb{A}, \sigma)$ with $\sigma(t) = \alpha\, t$ and let $(\mathbb{H}, \sigma)$ be a difference ring. Let $\tilde{\rho} : \mathbb{A} \to \mathbb{H}$ be a difference ring homomorphism and let $\rho : \mathbb{A}\langle t\rangle \to \mathbb{H}$ be a ring homomorphism defined by $\rho\,|_{\mathbb{A}} = \tilde{\rho}$ and $\rho(t) = g$ for some $g \in \mathbb{H}$. If $\sigma(g) = \rho(\alpha)\, g$, then ρ is a difference ring homomorphism.*

Proof. Suppose that $\sigma(g) = \rho(\alpha)\, g$ holds. Then $\sigma(\rho(t)) = \sigma(g) = \rho(\alpha)\, g = \rho(\alpha\, t) = \rho(\sigma(t))$. Consequently, $\sigma(\rho(f)) = \rho(\sigma(f))$ for all $f \in \mathbb{A}\langle t\rangle$. $\square$

Lemma 5.4. *For $1 \le \ell \le m$, let $(\mathbb{K}_\ell, \sigma)$ with $\mathbb{K}_\ell = \mathbb{K}\langle y_{\ell,1}\rangle \dots \langle y_{\ell,s_\ell}\rangle$ be single chain* P*-extensions of $(\mathbb{K}, \sigma)$ over a rational function field $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ with base $h_\ell \in \mathbb{K}^*$, the automorphisms (49) and the evaluation functions (50). Let $d := \max(s_1, \dots, s_m)$ and $\mathbb{A}_0 = \mathbb{K}$. Consider the tower of difference ring extensions $(\mathbb{A}_i, \sigma)$ of $(\mathbb{A}_{i-1}, \sigma)$ where $\mathbb{A}_i = \mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle \dots \langle y_{w_i,i}\rangle$ for $1 \le i \le d$ with $m = w_1 \ge \dots \ge w_d$, the automorphism (49) and the evaluation function (50). In particular, one gets $(\mathbb{A}_d, \sigma)$ as the ordered multiple chain* P*-extension of $(\mathbb{K}, \sigma)$ of monomial depth at most d composed by the single chain* P*-extensions $(\mathbb{K}_\ell, \sigma)$ of $(\mathbb{K}, \sigma)$ for $1 \le \ell \le m$ with (49) and (50). Then one can construct*

(a) an ordered multiple chain AP*-extension $(\mathbb{G}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth at most d with $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$ where $\tilde{K}$ is a finite algebraic field extension of K, with*

$$\mathbb{G}_d = \tilde{\mathbb{K}}\langle \vartheta_{1,1}\rangle \dots \langle \vartheta_{\upsilon_1,1}\rangle\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \vartheta_{1,d}\rangle \dots \langle \vartheta_{\upsilon_d,d}\rangle\langle \tilde{y}_{1,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle \tag{51}$$

where[5] $\upsilon_i \ge 0, e_i \ge 0$. Here the automorphism is defined for the A*-monomials by*

$$\sigma(\vartheta_{\ell,k}) = \gamma_{\ell,k}\, \vartheta_{\ell,k} \quad \text{where} \quad \gamma_{\ell,k} = \zeta^{\mu_\ell}\, \vartheta_{\ell,1} \cdots \vartheta_{\ell,k-1} \in \mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}[\vartheta_{\ell,1}]\dots[\vartheta_{\ell,k-1}]} \tag{52}$$

for $1 \le k \le d$ and $1 \le \ell \le \upsilon_k$ where $\mathbb{U} = \langle \zeta \rangle$ is a multiplicative cyclic subgroup of $\tilde{K}^*$ generated by a primitive λ-th root of unity $\zeta \in \tilde{K}^*$, and the automorphism is defined for the P*-monomials by*

$$\sigma(\tilde{y}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{y}_{\ell,k} \quad \text{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,k-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1}\rangle \dots \langle \tilde{y}_{\ell,k-1}\rangle} \tag{53}$$

for $1 \le k \le d$ and $1 \le \ell \le e_k$;

(b) an evaluation function $\tilde{\mathrm{ev}} : \mathbb{G}_d \times \mathbb{Z}_{\ge 0} \to \tilde{\mathbb{K}}$ defined by[6]

$$\tilde{\mathrm{ev}}(\vartheta_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\mathrm{ev}}(\gamma_{\ell,k}, j-1) \quad \text{and} \quad \tilde{\mathrm{ev}}(\tilde{y}_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,k}, j-1); \tag{54}$$

(c) a difference ring homomorphism $\rho_d : \mathbb{A}_d \to \mathbb{G}_d$ defined by $\rho_d|_{\mathbb{K}} = \mathrm{id}_{\mathbb{K}}$ and[7]

$$\rho_d(y_{\ell,k}) = \boldsymbol{\vartheta}_{\boldsymbol{k}}^{\boldsymbol{\mu}_{\ell,\boldsymbol{k}}}\, \tilde{\boldsymbol{y}}_{\boldsymbol{k}}^{\boldsymbol{\nu}_{\ell,\boldsymbol{k}}} = \vartheta_{1,k}^{\mu_{\ell,1,k}} \cdots \vartheta_{\upsilon_k,k}^{\mu_{\ell,\upsilon_k,k}}\, \tilde{y}_{1,k}^{\nu_{\ell,1,k}} \cdots \tilde{y}_{e_k,k}^{\nu_{\ell,e_k,k}} \tag{55}$$

for $1 \le \ell \le m$ and $1 \le k \le s_\ell$ with $\mu_{\ell,i,k} \in \mathbb{Z}_{\ge 0}$ for $1 \le i \le \upsilon_k$ and $\nu_{\ell,i,k} \in \mathbb{Z}$ for $1 \le i \le e_k$

such that the following properties hold:

[5] Note that if $\upsilon_i = 0$ or $e_i = 0$, then there is no A-monomial or P-monomial of depth i, respectively.

[6] For all $c \in \mathbb{K}$, we set $\tilde{\mathrm{ev}}(c, n) = c$ for all $n \ge 0$.

[7] Note that any P-monomial $y_{\ell,k}$ with depth k is mapped to a power product of AP-monomials having all depth k.

(1) There does not exist a $(v_1,\dots,v_{e_1})\in\mathbb{Z}^{e_1}\setminus\{\mathbf{0}_{e_1}\}$ *with* $\tilde{h}_1^{v_1}\dots\tilde{h}_{e_1}^{v_{e_1}}=1$.

(2) The P*-extension* $(\tilde{\mathbb{A}}_d,\sigma)$ *of* $(\tilde{\mathbb{K}},\sigma)$ *with*

$$\tilde{\mathbb{A}}_d=\tilde{\mathbb{K}}\langle\tilde{\boldsymbol{y}}_{\mathbf{1}}\rangle\langle\tilde{\boldsymbol{y}}_{\mathbf{2}}\rangle\dots\langle\tilde{\boldsymbol{y}}_{\boldsymbol{d}}\rangle=\tilde{\mathbb{K}}\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\langle\tilde{y}_{2,1}\rangle\dots\langle\tilde{y}_{e_2,2}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle \tag{56}$$

and the automorphism given in (53) is a Π*-extension. In particular, it is an ordered multiple chain* Π*-extension of monomial depth* d*.*

(3) For all $f\in\mathbb{A}_d$ *and for all* $n\in\mathbb{Z}_{\geq 0}$ *we have*

$$\mathrm{ev}(f,n)=\tilde{\mathrm{ev}}(\rho_d(f),n). \tag{57}$$

If K *is strongly* σ*-computable, then the above constructions are computable.*

Proof. Let $(\mathbb{A}_d,\sigma)$ with $\mathbb{A}_d=\mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle\dots\langle y_{w_d,d}\rangle$ be the ordered multiple chain P-extension of $(\mathbb{K},\sigma)$ of monomial depth $d\in\mathbb{Z}_{\geq 0}$ as described above with the automorphism (49) and the evaluation function (50). We prove the lemma by induction on the monomial depth d.

If $d=1$, statements (2) and (3) of the lemma hold by Lemma 5.2. Hence by Lemma 4.3 there are no $g\in\tilde{\mathbb{K}}^*$ and $(v_1,\dots,v_{e_1})\in\mathbb{Z}^{e_1}\setminus\{\mathbf{0}_{e_1}\}$ with $\tilde{h}_1^{v_1}\dots\tilde{h}_{e_1}^{v_{e_1}}=\frac{\sigma(g)}{g}=1$ and thus also statement (1) of the lemma holds.

Now let $d\geq 2$ and suppose that the lemma holds for $d-1$. That is, we can construct $(\mathbb{G}_{d-1},\sigma)$ with

$$\mathbb{G}_{d-1}=\tilde{\mathbb{K}}\langle\vartheta_{1,1}\rangle\dots\langle\vartheta_{\upsilon_1,1}\rangle\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\vartheta_{1,d-1}\rangle\dots\langle\vartheta_{\upsilon_{d-1},d-1}\rangle\langle\tilde{y}_{1,d-1}\rangle\dots\langle\tilde{y}_{e_{d-1},d-1}\rangle$$

which is an ordered multiple chain AP-extension of $(\tilde{\mathbb{K}},\sigma)$ of monomial depth at most $d-1$ with the automorphism given by (52) for $1\leq k\leq d-1$ and $1\leq\ell\leq\upsilon_k$ and given by (53) for $1\leq k\leq d-1$ and $1\leq\ell\leq e_k$. In addition, we get the evaluation function $\tilde{\mathrm{ev}}:\mathbb{G}_{d-1}\times\mathbb{Z}_{\geq 0}\to\tilde{\mathbb{K}}$ defined as (54) and the difference ring homomorphism $\rho_{d-1}:\mathbb{A}_{d-1}\to\mathbb{G}_{d-1}$ defined by $\rho_{d-1}|_{\mathbb{K}}=\mathrm{id}_{\mathbb{K}}$ and (55) such that statements (1), (2), and (3) of the lemma hold. We prove the lemma for the ordered multiple chain P-extension $(\mathbb{A}_d,\sigma)$ of $(\mathbb{K},\sigma)$ with $\mathbb{A}_d=\mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle\dots\langle y_{w_d,d}\rangle$ where $\mathfrak{d}(y_{1,d})=\dots=\mathfrak{d}(y_{w_d,d})=d$. Since the shift quotient of these P-monomials is contained in $\mathbb{A}_{d-1}$, i.e.,

$$\frac{\sigma(y_{\ell,d})}{y_{\ell,d}}=\alpha_{\ell,d}\in\mathbb{A}_{d-1}^*,$$

we can iteratively apply the difference ring homomorphism $\rho_{d-1}:\mathbb{A}_{d-1}\to\mathbb{G}_{d-1}$ to rephrase each $\alpha_{\ell,d}$ in $\mathbb{G}_{d-1}$. In particular, by Remark 3.1 we have $\sigma^{-1}(\alpha_{\ell,d})=y_{\ell,d-1}$ and thus by (55) we get

$$h_{\ell,d}:=\rho_{d-1}(\sigma^{-1}(\alpha_{\ell,d}))=\rho_{d-1}(y_{\ell,d-1})=\boldsymbol{\vartheta}_{\boldsymbol{d-1}}^{\boldsymbol{\mu_{\ell,d-1}}}\,\tilde{\boldsymbol{y}}_{\boldsymbol{d-1}}^{\boldsymbol{v_{\ell,d-1}}} \tag{58}$$

where $\boldsymbol{\vartheta}_{\boldsymbol{d-1}}^{\boldsymbol{u_{\ell,d-1}}}=\vartheta_{1,d-1}^{u_{\ell,1,d-1}}\cdots\vartheta_{\upsilon_{d-1},d-1}^{u_{\ell,\upsilon_{d-1},d-1}}$ and $\tilde{\boldsymbol{y}}_{\boldsymbol{d-1}}^{\boldsymbol{v_{\ell,d-1}}}=\tilde{y}_{1,d-1}^{v_{\ell,1,d-1}}\cdots\tilde{y}_{e_{d-1},d-1}^{v_{\ell,e_{d-1},d-1}}$ for $1\leq\ell\leq w_d$ with $\mu_{\ell,k,d-1}\in\mathbb{Z}_{\geq 0}$ for $1\leq k\leq\upsilon_{d-1}$ and $v_{\ell,k,d-1}\in\mathbb{Z}$ for $1\leq k\leq e_{d-1}$.

If $h_{\ell,d}=1$, it follows with (57) (d replaced by $d-1$) that for all $n\in\mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned}\mathrm{ev}(y_{\ell,d},n)&=\prod_{j=1}^{n}\mathrm{ev}(\alpha_{\ell,d},j-1)=\prod_{j=1}^{n}\mathrm{ev}(\sigma^{-1}(\alpha_{\ell,d}),j)=\prod_{j=1}^{n}\tilde{\mathrm{ev}}(\rho_{d-1}(\sigma^{-1}(\alpha_{\ell,d})),j)\\&=\prod_{j=1}^{n}\tilde{\mathrm{ev}}(h_{\ell,d},j)=1.\end{aligned}$$

In particular, if $h_{\ell,d}=1$ holds for all $1\leq\ell\leq w_d$, we can set $\mathbb{G}_d:=\mathbb{G}_{d-1}$ and extend ρ_{d-1} to $\rho_d:\mathbb{A}_d\to\mathbb{G}_{d-1}$ with $\rho_d(y_{\ell,d})=1$ for $1\leq\ell\leq w_d$. Thus the lemma is proven.

Otherwise, take all AP-monomials in (58) for $1\leq\ell\leq w_d$ with non-zero integer exponents. Then they belong to at least one of the single chain AP-extensions of $(\tilde{\mathbb{K}},\sigma)$ in $(\mathbb{G}_{d-1},\sigma)$. Suppose there are $e_d \geq 0$ of these single chains Π-extensions and $\upsilon_d \geq 0$ of them that are single chain A-extensions $(\mathbb{H}_b, \sigma)$; note that we have $e_d + \upsilon_d \geq 1$. By appropriate reordering of $(\mathbb{G}_{d-1}, \sigma)$ we may suppose that these e_d single chain Π-extensions $(\mathbb{F}_r, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with $1 \leq r \leq e_d$ are given by $\mathbb{F}_r = \tilde{\mathbb{K}}\langle \tilde{y}_{r,1}\rangle\langle \tilde{y}_{r,2}\rangle \dots \langle \tilde{y}_{r,d-1}\rangle$ and the υ_d A-extensions $(\mathbb{H}_b, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with $1 \leq b \leq \upsilon_d$ can be given by $\mathbb{H}_b = \tilde{\mathbb{K}}\langle \vartheta_{b,1}\rangle\langle \vartheta_{b,2}\rangle \dots \langle \vartheta_{b,d-1}\rangle$. Now adjoin the P-monomials $\tilde{y}_{r,d}$ to $\mathbb{F}_r$ with (53) where $k = d$ and $\ell = r$ yielding the single chain P-extensions $(\mathbb{F}'_r, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth d where

$$\mathbb{F}'_r = \mathbb{F}_r\langle \tilde{y}_{r,d}\rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{r,1}\rangle\langle \tilde{y}_{r,2}\rangle \dots \langle \tilde{y}_{r,d-1}\rangle\langle \tilde{y}_{r,d}\rangle$$

and adjoin the A-monomial $\vartheta_{b,d}$ with (52) where $k = d$ and $\ell = b$ yielding the single chain A-extensions $(\mathbb{H}'_b, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth d where

$$\mathbb{H}'_b = \mathbb{H}_b\langle \vartheta_{b,d}\rangle = \tilde{\mathbb{K}}\langle \vartheta_{b,1}\rangle\langle \vartheta_{b,2}\rangle \dots \langle \vartheta_{b,d-1}\rangle\langle \vartheta_{b,d}\rangle.$$

Furthermore extend the evaluation functions $\tilde{\mathrm{ev}} : \mathbb{F}'_r \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ and $\tilde{\mathrm{ev}} : \mathbb{H}'_b \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ with (54) where $k = d, \ell = r$ or $k = d, \ell = b$, respectively. Now consider the multiple chain P-extension $(\tilde{\mathbb{A}}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\tilde{\mathbb{A}}_d = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1}\rangle \dots \langle \tilde{y}_{e_1,1}\rangle \dots \langle \tilde{y}_{1,d-1}\rangle \dots \langle \tilde{y}_{e_{d-1},d-1}\rangle\langle \tilde{y}_{1,d}\rangle\langle \tilde{y}_{2,d}\rangle \dots \langle \tilde{y}_{e_d,d}\rangle$$

which one gets by taking all P-monomials in $\mathbb{G}_{d-1}$ and the new P-monomials in $\mathbb{F}'_r = \mathbb{F}_r\langle \tilde{y}_{r,d}\rangle$ with $1 \leq r \leq e_d$. Here the automorphism is given by (52) and (53) and the equipped evaluation function is given by (54). Since there does not exist a $g \in \tilde{\mathbb{K}}$ and $(v_1, \dots, v_d) \in \mathbb{Z}^d \setminus \{\mathbf{0}_d\}$ with $\frac{\sigma(g)}{g} = \tilde{h}_1^{v_1} \cdots \tilde{h}_d^{v_d}$, it follows by Theorem 4.3 that $(\tilde{\mathbb{A}}_d, \sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. In particular, it is an ordered multiple chain Π-extension of monomial depth d by construction. Thus statements (1) and (2) of the lemma hold. Finally, take the AP-extension $(\mathbb{G}_d, \sigma)$ of $(\mathbb{G}_{d-1}, \sigma)$ with $\mathbb{G}_d = \mathbb{G}_{d-1}\langle \vartheta_{1,d}\rangle\langle \vartheta_{2,d}\rangle \cdots \langle \vartheta_{\upsilon_d,d}\rangle\langle \tilde{y}_{1,d}\rangle\langle \tilde{y}_{2,d}\rangle \cdots \langle \tilde{y}_{e_d,d}\rangle$.

Let $1 \leq \ell \leq w_d$ and consider $h_{\ell,d}$ in (58). If $h_{\ell,d} = 1$, we define $g_\ell = 1$. In this case, $1 = h_{\ell,d} = \rho_{d-1}(\sigma^{-1}(\alpha_{\ell,d})) = \sigma^{-1}(\rho_{d-1}(\alpha_{\ell,d}))$, and thus $\frac{\sigma(g_\ell)}{g_\ell} = 1 = \rho_{d-1}(\alpha_{\ell,d})$ holds. Otherwise, if $h_{\ell,d} \neq 1$, we set

$$g_\ell := \vartheta_{1,d}^{\mu_{\ell,1,d}} \cdots \vartheta_{\upsilon_d,d}^{\mu_{\ell,\upsilon_d,d}}\, \tilde{y}_{1,d}^{v_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{v_{\ell,e_d,d}} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\mathbb{G}_d}.$$

Then based on the Remark 3.1 it follows that $\frac{\tilde{y}_{j,d}}{\sigma^{-1}(\tilde{y}_{j,d})} = \tilde{y}_{j,d-1}$ and $\frac{\vartheta_{j,d}}{\sigma^{-1}(\vartheta_{j,d})} = \vartheta_{j,d-1}$ and thus

$$\frac{g_\ell}{\sigma^{-1}(g_\ell)} = \vartheta_{1,d-1}^{\mu_{\ell,1,d-1}} \cdots \vartheta_{\upsilon_{d-1},d-1}^{\mu_{\ell,\upsilon_{d-1},d-1}}\, \tilde{y}_{1,d-1}^{v_{\ell,1,d-1}} \cdots \tilde{y}_{e_{d-1},d-1}^{v_{\ell,e_{d-1},d-1}} = \boldsymbol{\vartheta}_{\mathbf{d-1}}^{\boldsymbol{\mu}_{\boldsymbol{\ell},\mathbf{d-1}}}\, \tilde{\boldsymbol{y}}_{\mathbf{d-1}}^{\boldsymbol{v}_{\boldsymbol{\ell},\mathbf{d-1}}} = \rho_{d-1}(y_{\ell,d-1}).$$

Hence also in this case we get

$$\frac{\sigma(g_\ell)}{g_\ell} = \sigma\left(\tfrac{g_\ell}{\sigma^{-1}(g_\ell)}\right) = \sigma(\rho_{d-1}(y_{\ell,d-1})) = \rho_{d-1}(\sigma(y_{\ell,d-1})) = \rho_{d-1}(\alpha_{\ell,d}).$$

By iterative application of Lemma 5.3, the difference ring homomorphism $\rho_{d-1} : \mathbb{A}_{d-1} \to \mathbb{G}_{d-1}$ can be extended to

$$\rho_d : \mathbb{A}_{d-1}\langle y_{1,d}\rangle\langle y_{2,d}\rangle \cdots \langle y_{w_d,d}\rangle \to \mathbb{G}_{d-1}\langle \vartheta_{1,d}\rangle\langle \vartheta_{2,d}\rangle \cdots \langle \vartheta_{\upsilon_d,d}\rangle\langle \tilde{y}_{1,d}\rangle\langle \tilde{y}_{2,d}\rangle \cdots \langle \tilde{y}_{e_d,d}\rangle$$

with $\rho_d|_{\mathbb{A}_{d-1}} = \rho_{d-1}$ and $\rho_d(y_{\ell,d}) = g_\ell$ for $1 \leq \ell \leq w_d$. Finally, we show that for all $f \in \mathbb{A}_d$ and $n \in \mathbb{Z}_{\geq 0}$ we have $\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\rho_d(f), n)$. First note that for all $n \geq 0$ we have

$$\mathrm{ev}(y_{\ell,d}, n+1) = \mathrm{ev}(\sigma(y_{\ell,d}), n) = \mathrm{ev}(\alpha_{\ell,d}, n)\; \mathrm{ev}(y_{\ell,d}, n). \tag{59}$$

On the other hand, since ρ_d is a difference ring homomorphism, we have that

$$\sigma(\rho_d(y_{\ell,d})) = \rho_d(\sigma(y_{\ell,d})) = \rho_d(\alpha_{\ell,d})\, \rho_d(y_{\ell,d}) = \rho_{d-1}(\alpha_{\ell,d})\, \rho_d(y_{\ell,d}) \tag{60}$$

for all $n \geq 0$. Thus we get

$$\tilde{\text{ev}}(\rho_d(y_{\ell,d}), n+1) = \tilde{\text{ev}}(\sigma(\rho_d(y_{\ell,d})), n) \overset{(60)}{=} \tilde{\text{ev}}(\rho_{d-1}(\alpha_{\ell,d}), n)\, \tilde{\text{ev}}(\rho_d(y_{\ell,d}), n). \tag{61}$$

By the induction hypothesis, $\text{ev}(\alpha_{\ell,d}, n) = \tilde{\text{ev}}(\rho_{d-1}(\alpha_{\ell,d}), n)$ holds for all $n \in \mathbb{Z}_{\geq 0}$. Therefore with (59) and (61) it follows that $\text{ev}(y_{\ell,d}, n)$ and $\tilde{\text{ev}}(\rho(y_{\ell,d}), n)$ satisfy the same first-order recurrence relation. With $\text{ev}(y_{\ell,d}, 0) = 1$ and

$$\begin{aligned}\tilde{\text{ev}}(\rho_d(y_{\ell,d}), 0) &= \tilde{\text{ev}}(g_\ell, 0)\\ &= \tilde{\text{ev}}(\vartheta_{1,d}, 0)^{\mu_{\ell,1,d}} \cdots \tilde{\text{ev}}(\vartheta_{\upsilon_d,d}, 0)^{\mu_{\ell,\upsilon_d,d}}\, \tilde{\text{ev}}(\tilde{y}_{1,d}, 0)^{\nu_{\ell,1,d}} \cdots \tilde{\text{ev}}(\tilde{y}_{e_d,d}, 0)^{\nu_{\ell,e_d,d}} = 1\end{aligned}$$

it follows then that $\text{ev}(y_{\ell,d}, n) = \tilde{\text{ev}}(\rho_d(y_{\ell,d}), n)$ holds for all $n \geq 0$. Together with the induction hypothesis $\text{ev}(f, n) = \tilde{\text{ev}}(\rho_{d-1}(f), n)$ for all $f \in \mathbb{A}_{d-1}$ and $n \in \mathbb{Z}_{\geq 0}$ we get (57) for all $f \in \mathbb{A}_d$ and for all $n \geq 0$. Consequently also statement (3) of the lemma holds.

Finally, if K is strongly σ-computable, the base case $d = 1$ can be executed explicitly by activating Lemma 5.2. In particular the induction step can be performed iteratively and thus the difference ring $(\mathbb{G}_d, \sigma)$ with (51) together with (52), (53) and (54) can be computed. In addition, the difference ring $(\mathbb{G}_d, \sigma)$, the difference ring homomorphism $\rho_d : \mathbb{A}_d \to \mathbb{G}_d$ defined by (55) and the evaluation function $\tilde{\text{ev}}$ can be computed. This completes the proof. □

Remark 5.1. Note that the generators of $\mathbb{G}_d$ with (51) constructed in Lemma 5.4 can be rearranged to get the AΠ-extension $(\tilde{\mathbb{K}}[\vartheta_{1,1}]\dots[\vartheta_{\upsilon_1,1}]\dots[\vartheta_{1,d}]\dots[\vartheta_{\upsilon_d,d}]\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$. Furthermore, a consequence of statement (3) of Lemma 5.4 is that the diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\psi} & \mathcal{S}(\mathbb{K}) \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\rho'} \\ \mathbb{G}_d & \xrightarrow{\tilde{\tau}} & \mathcal{S}(\tilde{\mathbb{K}}) \end{array}$$

commutes where $\mathbb{A} = \mathbb{A}_d$, $\rho = \rho_d$, $\rho' = \text{id}$ and the difference ring homomorphism $\tilde{\tau}$ and ψ are defined by $\tilde{\tau}(f) = \langle\tilde{\text{ev}}(f, n)\rangle_{n\geq 0}$ and $\psi(g) = \langle\text{ev}(g, n)\rangle_{n\geq 0}$ respectively.

Example 5.2 *(Cont. Example 3.1).* Take the ordered multiple chain AP-extension $(\mathbb{A}', \sigma)$ of $(\mathbb{K}, \sigma)$ with monomial depth 2 with $\mathbb{A}' = \mathbb{K}\langle\vartheta_{1,1}\rangle\langle y_{1,1}\rangle\langle y_{2,1}\rangle\langle y_{3,1}\rangle\langle y_{4,1}\rangle\langle y_{5,1}\rangle\langle\vartheta_{1,2}\rangle\langle y_{2,2}\rangle\langle y_{4,2}\rangle$ where $(\mathbb{A}', \sigma)$ is composed by the single chain AP-extensions of $(\mathbb{K}, \sigma)$ constructed in parts (1), (2), (3), (4), (5), and (6) of Example 3.1. By Lemma 5.4 and Remark 5.1 we can construct the AP-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{K}, \sigma)$ where

$$\mathbb{G} = \mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle \tag{62}$$

with the automorphism σ and evaluation function $\tilde{\text{ev}} : \mathbb{G} \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ given by (29), (30) and

$$\begin{gathered}\sigma(\tilde{y}_{1,1}) = \sqrt{3}\,\tilde{y}_{1,1}, \qquad \sigma(\tilde{y}_{2,1}) = 2\,\tilde{y}_{2,1}, \qquad \sigma(\tilde{y}_{3,1}) = 5\,\tilde{y}_{3,1}, \qquad \sigma(\tilde{y}_{2,2}) = 2\,\tilde{y}_{2,1}\,\tilde{y}_{2,2},\\ \sigma(\tilde{y}_{3,2}) = 5\,\tilde{y}_{3,1}\,\tilde{y}_{3,2},\\ \tilde{\text{ev}}(\tilde{y}_{1,1}, n) = \prod_{k=1}^{n}\sqrt{3}, \qquad \tilde{\text{ev}}(\tilde{y}_{2,1}, n) = \prod_{k=1}^{n}2, \qquad \tilde{\text{ev}}(\tilde{y}_{3,1}, n) = \prod_{k=1}^{n}5, \qquad \tilde{\text{ev}}(\tilde{y}_{2,2}, n) = \prod_{k=1}^{n}\prod_{j=1}^{k}2,\\ \tilde{\text{ev}}(\tilde{y}_{3,2}, n) = \prod_{k=1}^{n}\prod_{j=1}^{k}5\end{gathered} \tag{63}$$

with the following properties. By part (2) of Lemma 5.4, the sub-difference ring $(\tilde{\mathbb{D}}, \sigma)$ of the difference ring $(\mathbb{G}, \sigma)$ with $\tilde{\mathbb{D}} = \mathbb{K}\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle$ is an ordered multiple chain Π-extension of $(\mathbb{K}, \sigma)$ with the automorphism σ and the evaluation function $\tilde{\text{ev}} : \tilde{\mathbb{D}} \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined in (63). In addition by part (3) of Lemma 5.4 we get the difference ring homomorphism $\rho : \mathbb{A}' \to \mathbb{G}$ defined by $\rho|_{\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}]} = \text{id}_{\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}]}$ and

$$\begin{aligned} &\rho(y_{1,1}) = \tilde{y}_{1,1}, && \rho(y_{2,1}) = \tilde{y}_{2,1}, && \rho(y_{3,1}) = \tilde{y}_{1,1}^2, && \rho(y_{4,1}) = \tilde{y}_{3,1}, \\ &\rho(y_{5,1}) = \tilde{y}_{3,1}^2, && \rho(y_{2,2}) = \tilde{y}_{2,2}, && \rho(y_{4,2}) = \tilde{y}_{3,2} && \end{aligned} \tag{64}$$

such that $\tilde{\text{ev}}(\rho(f), n) = \text{ev}(f, n)$ holds for all $n \in \mathbb{Z}_{\geq 0}$ and $f \in \mathbb{A}'$,

5.3. Nested products with roots of unity

Throughout this subsection, K is a field containing $\mathbb{Q}$, $\mathbb{K}_m$ is a splitting field for the polynomial $x^m - 1$ over K (i.e., all roots of the polynomial $x^m - 1$ are in $\mathbb{K}_m$) for some $m \in \mathbb{Z}_{\geq 2}$ and $\mathbb{U}_m$ is the set of all m-th roots of unity over K in $\mathbb{K}_m$ (which forms a multiplicative subgroup of K^*). Then $\text{Prod}_{\text{n}}(\mathbb{U}_m)$ is the set of all geometric products over roots of unity in $\mathbb{U}_m$. For $\mathbb{G} \subseteq \mathbb{K}_m(x)$ we define $\text{ProdE}_{\text{n}}(\mathbb{G}, \mathbb{U}_m)$ as the set of all elements

$$\sum_{\boldsymbol{v}=(v_1,\ldots,v_e)\in S} a_{\boldsymbol{v}}(n)\, P_1(n)^{v_1} \cdots P_e(n)^{v_e}$$

with $e \in \mathbb{Z}_{\geq 0}$, $S \subseteq \mathbb{Z}^e_{\geq 0}$ finite, $a_{\boldsymbol{v}}(x) \in \mathbb{G}$ for $\boldsymbol{v} \in S$ and $P_1(n), \ldots, P_e(n) \in \text{Prod}_{\text{n}}(\mathbb{U}_m)$.

The main result of this subsection in Theorem 5.3 states that products over roots of unity with finite nesting depth can be represented by the single product ζ_λ^n where $\zeta_\lambda \in \mathbb{U}_\lambda$ for some $\lambda \geq 2$ is a primitive λ-th root of unity.

Theorem 5.3. *Suppose we are given the geometric products $A_1(n), \ldots, A_e(n) \in \text{Prod}_{\text{n}}(\mathbb{U}_m)$ in n of nesting depth $r_i \in \mathbb{Z}_{\geq 0}$ with*

$$A_i(n) = \prod_{k_1=\ell_{i,1}}^{n} \zeta_{i,1} \prod_{k_2=\ell_{i,2}}^{k_1} \zeta_{i,2} \cdots \prod_{k_{r_i}=\ell_{i,r_i}}^{k_{r_i-1}} \zeta_{i,r_i} \tag{65}$$

for $1 \leq i \leq e$ where $\zeta_{i,j} \in \mathbb{U}_m$, $\ell_{i,j} \in \mathbb{Z}_{\geq 0}$ for $1 \leq j \leq r_i$. Then there exist a $\lambda \in \mathbb{Z}_{\geq 2}$ with $m \mid \lambda$ and a primitive λ-th root of unity $\zeta_\lambda \in \mathbb{K}_\lambda^*$ satisfying the following property. For all $1 \leq i \leq e$ there exist $f_{i,j} \in \mathbb{K}_\lambda$ for $0 \leq j < \lambda$ such that for

$$B_i(n) = \sum_{j=0}^{\lambda-1} f_{i,j}\, (\zeta_\lambda^n)^j \in \text{ProdE}_{\text{n}}(\mathbb{K}_\lambda, \mathbb{U}_\lambda) \tag{66}$$

we have

$$A_i(n) = B_i(n) \tag{67}$$

for all $n \geq \max(\ell_{i,1}, \ldots, \ell_{i,r_i}) - 1$. In particular, if $\mathbb{K}$ is computable and one can solve Problem O (see below), the above construction can be given explicitly.

5.3.1. The period and algorithmic aspects

For the treatment of Theorem 5.3 we will introduce the period of a difference ring element introduced in Karr (1981). In particular, we will use the algorithms from Schneider (2016) that enable one to calculate the period within nested R-extensions, resp. A-extensions.

Definition 5.3. Let $(\mathbb{A}, \sigma)$ be a difference ring. The *period* of $\alpha \in \mathbb{A}^*$ is defined by

$$\text{per}(\alpha) = \begin{cases} 0 & \text{if } \nexists n > 0 \text{ s.t. } \sigma^n(\alpha) = \alpha \\ \min\{n > 0 \mid \sigma^n(\alpha) = \alpha\} & \text{otherwise.} \end{cases}$$

As it turns out, this task is connected to compute the order of a ring.

Definition 5.4. Let $\mathbb{A}$ be a ring and let $\alpha \in \mathbb{A} \setminus \{0\}$. Then the order of α is defined by

$$\operatorname{ord}(\alpha) = \begin{cases} 0, & \text{if } \nexists n > 0 \text{ with } \alpha^n = 1. \\ \min\{n > 0 \,|\, \alpha^n = 1\}, & \text{otherwise.} \end{cases}$$

Namely, if we can solve the following problem (which is Problem GO with $w = 1$):

Problem O for $\alpha \in K^*$

Given a field K and $\alpha \in K^*$. *Find* $\operatorname{ord}(\alpha)$.

then we can also compute the period by the following lemma. The underlying algorithms of this lemma are highly recursive and we refer the reader to Schneider (2016, Proposition 5.5) (compare Ocansey, 2019, Proposition 6.2.20) for statement (1) and to Schneider (2016, Corollary 5.6) (compare Ocansey, 2019, Corollary 6.2.21) for statement (2). Some further details are given in the Example 5.3 below.

Lemma 5.5. *Let* $(\mathbb{E}, \sigma)$ *with* $\mathbb{E} = \mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]$ *be a simple* A*-extension of* $(\mathbb{K}_m, \sigma)$*. Then the following statements hold.*

(1) $\operatorname{per}(\vartheta_i) > 0$ *for all* $1 \leq i \leq e$.
(2) If $\mathbb{K}_m$ *is computable and* Problem O *is solvable, then* $\operatorname{per}(\vartheta_i)$ *is computable for all* $1 \leq i \leq e$.

Example 5.3 *(Cont. Example 5.2).* Consider the sub-difference ring $(\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}], \sigma)$ of $(\mathbb{G}, \sigma)$ with (62) with the automorphism and the evaluation function defined in (29) and (30). By construction it is a simple A-extension of the difference field $(\mathbb{K}, \sigma)$. Since $\operatorname{per}(c) = 1$ for all $c \in \mathbb{K}^*$, it follows that $\operatorname{per}(-1) = 1$. We applying the algorithms from Schneider (2016) (see the comments in the proof of Lemma 5.5) to compute the periods of $\vartheta_{1,1}$ and $\vartheta_{1,2}$. The output $\operatorname{per}(\vartheta_{1,1}) = 2$ is immediate. For $\vartheta_{1,2}$ with $\sigma(\vartheta_{1,2}) = \alpha\,\vartheta_{1,2}$ where $\alpha = -\vartheta_{1,1}$ the algorithm proceeds as follows. First one computes the minimal $n \geq 1$ such that $\alpha\,\sigma(\alpha)\dots\sigma^n(\alpha) = 1$ holds (the so-called factorial order) by using another algorithm described in Schneider (2016) (that relies on the orders and periods of the monomials below). Here we obtain $n = 4$. Finally, we use the fact that the period of $\vartheta_{1,2}$ equals the factorial order n of the multiplicand α, i.e., $\operatorname{per}(\vartheta_{1,2}) = 4$.

5.3.2. Idempotent representation of single RP*-extensions*

In order to prove Theorem 5.3 we rely on the property that elements in basic RP-extensions can be expressed by idempotent elements. All the ideas below are inspired by van der Put and Singer (1997). We start with the following simple fact given, e.g., in Schneider (2017, Sec. 4).[8]

Lemma 5.6. *Let* $\mathbb{F}$ *be a field and let* ζ *be a primitive* λ*-th root of unity. Let* $\mathbb{F}[\vartheta]$ *be a polynomial ring subject to the relation* $\vartheta^\lambda = 1$*. Then the following statements hold.*

[8] Note that the idempotent elements e_i are given implicitly in van der Put and Singer (1997, Corollary 1.16) by observing that $(\mathbb{F}[\vartheta], \sigma)$ is a simple difference field, i.e., any ideal which is closed under the shift automorphism is either the ring itself or the zero ideal. We further note that the e_i are related to Lagrangian interpolation.

(1) The elements $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{\lambda-1} \in \mathbb{F}[\vartheta]$ *with*

$$\boldsymbol{e}_k = \boldsymbol{e}_k(\vartheta) := \prod_{\substack{i=0 \\ i \neq \lambda-1-k}}^{\lambda-1} \frac{\vartheta - \zeta^i}{\zeta^{\lambda-1-k} - \zeta^i} \tag{68}$$

are idempotent and for all $0 \leq k < \lambda$ *we have*

$$\boldsymbol{e}_k(\zeta^j) = \begin{cases} 1 & \text{if } j = \lambda-1-k \\ 0 & \text{if } j \neq \lambda-1-k \end{cases} \qquad \text{and} \qquad \boldsymbol{e}_k(\zeta\,\vartheta) = \boldsymbol{e}_{k+1 \mod \lambda}. \tag{69}$$

(2) The idempotent elements defined in (68) *are pairwise orthogonal and* $\boldsymbol{e}_0 + \dots + \boldsymbol{e}_{\lambda-1} = 1$.

In van der Put and Singer (1997, Corollary 1.16) and Hardouin and Singer (2008, Lemma 6.8) a general theory is developed that shows that simple difference rings can be decomposed with such idempotent elements. In Schneider (2017, Theorem 4.3) it has been shown that this applies also to certain classes of R$\Pi\Sigma$-extensions. In Proposition 5.1 we present a specialized version for simple RP-extensions that is relevant for this article.

Proposition 5.1. *Let* $(\mathbb{E}, \sigma)$ *with* $\mathbb{E} = \mathbb{F}[\vartheta]\langle t_1\rangle \dots \langle t_e\rangle$ *be an* RP*-extension of a difference field* $(\mathbb{F}, \sigma)$ *where* ϑ *is an* R*-monomial of order* λ *with* $\zeta = \frac{\sigma(\vartheta)}{\vartheta} \in \text{const}(\mathbb{F}, \sigma)^*$ *and the* t_i *are* P*-monomials. Let* $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{\lambda-1}$ *be the idempotent, pairwise orthogonal elements in* (68) *that sum up to one. Then the following statement holds:*

(1) The ring $\mathbb{E}$ *can be written as the direct sum* $\mathbb{E} = \boldsymbol{e}_0\mathbb{E} \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{E}$ *where* $\boldsymbol{e}_k\mathbb{E}$ *forms for all* $0 \leq k < \lambda$ *a ring with* $\boldsymbol{e}_k$ *being the multiplicative identity element.*
(2) We have that $\boldsymbol{e}_k\mathbb{E} = \boldsymbol{e}_k\tilde{\mathbb{E}}$ *for* $0 \leq k < \lambda$ *where* $\tilde{\mathbb{E}} = \mathbb{F}\langle t_1\rangle \dots \langle t_e\rangle$.

We are now ready to obtain the following key result; for the corresponding result for nested A-extensions of monomial depth 1 we refer to Schneider (2017, Lemma 2.22).

Theorem 5.4. *Let* $m \in \mathbb{Z}_{\geq 2}$ *and take a primitive* m*-th root of unity* $\zeta_m \in \mathbb{K}_m^*$. *Let* $(\mathbb{K}_m[\vartheta_1] \dots [\vartheta_e], \sigma)$ *be a simple* A*-extension of* $(\mathbb{K}_m, \sigma)$ *with* $\sigma(\vartheta_i) = \alpha_i\,\vartheta_i$ *for* $1 \leq i \leq e$ *where* $\alpha_i = \zeta_m^{u_i}\,\vartheta_1^{z_{i,1}} \cdots \vartheta_{i-1}^{z_{i,i-1}}$ *with* $u_i, z_{i,j} \in \mathbb{Z}_{\geq 0}$. *Furthermore, let* $\text{ev}_m : \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e] \times \mathbb{Z}_{\geq 0} \to \mathbb{K}_m$ *be the evaluation function defined by*

$$\text{ev}_m(\vartheta_i, n) = \prod_{j=1}^{n} \text{ev}_m(\alpha_i, j-1), \tag{70}$$

and let $\tau_m : \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e] \to \mathcal{S}(\mathbb{K})$ *be the* $\mathbb{K}_m$*-homomorphism given by* $\tau_m(f) = \langle \text{ev}_m(f, n) \rangle_{n \geq 0}$ *Then the following statements hold.*

(1) Define $\lambda := \text{lcm}(m, \text{per}(\vartheta_1), \dots, \text{per}(\vartheta_e)) > 1$. *Then there is an* R*-extension* $(\mathbb{K}_\lambda[\vartheta], \sigma)$ *of* $(\mathbb{K}_\lambda, \sigma)$ *of order*[9] λ *with* $\zeta = \frac{\sigma(\vartheta)}{\vartheta} \in \mathbb{K}_\lambda^*$ *such that*

$$\varphi : \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e] \to \mathbb{K}_\lambda[\vartheta] = \boldsymbol{e}_0\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda \tag{71}$$

defined with

$$\varphi(f) = f_0\boldsymbol{e}_0 + \dots + f_{\lambda-1}\boldsymbol{e}_{\lambda-1} \tag{72}$$

where $f_i = \text{ev}_m(f, \lambda-1-i) \in \mathbb{K}_m \subseteq \mathbb{K}_\lambda$ *for* $0 \leq i < \lambda$ *is a difference ring homomorphism; here the* $\boldsymbol{e}_k$ *are the idempotent orthogonal elements defined in* (68). *In particular,* $\varphi|_{\mathbb{K}_m} = \text{id}_{\mathbb{K}_m}$.

[9] $\mathbb{K}_\lambda$ is a finite algebraic extension of $\mathbb{K}_m$ and $\zeta \in \mathbb{K}_m$ is a primitive λ-th root of unity.

(2) *Take the evaluation function* $\mathrm{ev}_\lambda : \mathbb{K}_\lambda[\vartheta] \times \mathbb{Z}_{\geq 0} \to \mathbb{K}_\lambda$ *defined by* $\mathrm{ev}_\lambda|_{\mathbb{K}_\lambda} = \mathrm{id}$ *and* $\mathrm{ev}_\lambda(\vartheta, n) = \zeta^n$ *and consider the* $\mathbb{K}_\lambda$*-homomorphism* $\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda)$ *defined by* $\tau_\lambda(f) = \langle \mathrm{ev}_\lambda(f, n)\rangle_{n\geq 0}$. *Then for the pairwise orthogonal idempotent elements* $\boldsymbol{e}_k$ *defined in* (68) *with* $0 \leq k < \lambda$*, we have that*

$$\mathrm{ev}_\lambda(\boldsymbol{e}_k, n) = \begin{cases} 1 & \text{if } \lambda \mid n+k+1, \\ 0 & \text{if } \lambda \nmid n+k+1. \end{cases} \tag{73}$$

(3) *The* $\mathbb{K}_\lambda$*-homomorphism* $\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda)$ *with the evaluation function defined in part* (2) *is injective.*

(4) *The diagram*

$$\begin{tikzcd} \mathbb{K}_m[\vartheta_1]\dots[\vartheta_e] \arrow[r, "\tau_m"] \arrow[d, "\varphi"'] & \mathcal{S}(\mathbb{K}_m) \arrow[d, "\varphi'"] \\ \mathbb{K}_\lambda[\vartheta] \simeq \boldsymbol{e}_0\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda \arrow[r, "\tau_\lambda"] & \mathcal{S}(\mathbb{K}_\lambda) \end{tikzcd} \tag{74}$$

commutes where $\varphi' : \mathcal{S}(\mathbb{K}_m) \to \mathcal{S}(\mathbb{K}_\lambda)$ *is the injective difference ring homomorphism defined by* $\varphi'(a) = a$.

If $\mathbb{K}_m$ *is computable and Problem O is solvable in* $\mathbb{K}_m$*, then the above constructions are computable.*

Proof. (1) Since $\zeta_m^{u_i} \in \mathbb{K}_m^*$, $\mathrm{per}(\zeta_m^{u_i}) = 1 > 0$ for all $1 \leq u_i \leq e$. In addition, it follows by statement (1) of Lemma 5.5 that $\mathrm{per}(\vartheta_i) > 0$ for all $1 \leq i \leq e$. Define $\lambda := \mathrm{lcm}(m, \mathrm{per}(\vartheta_1), \dots, \mathrm{per}(\vartheta_e)) > 1$. Note that $m \mid \lambda$, i.e., $\mathbb{K}_\lambda$ is an algebraic field extension of $\mathbb{K}_m$. Finally, take $\zeta := \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{\lambda}} = (-1)^{\frac{2}{\lambda}} \in \mathbb{K}_\lambda^*$ and construct the A-extension $(\mathbb{K}_\lambda[\vartheta], \sigma)$ of $(\mathbb{K}_\lambda, \sigma)$ with $\sigma(\vartheta) = \zeta\,\vartheta$. By Lemma 4.1 it follows that $(\mathbb{K}_\lambda[\vartheta], \sigma)$ is an R-extension of $(\mathbb{K}_\lambda, \sigma)$. By Proposition 5.1 we have that $\mathbb{K}_\lambda[\vartheta] = \boldsymbol{e}_0\mathbb{K}_\lambda \oplus \dots \oplus \boldsymbol{e}_{\lambda-1}\mathbb{K}_\lambda$ where the $\boldsymbol{e}_k$ for $0 \leq k < \lambda$ are the orthogonal idempotent elements defined in (68). Now consider the map (71) defined by (72). We will now show that φ is a ring homomorphism. Observe that for any $c \in \mathbb{K}_m$, $\mathrm{ev}_m(c, i) = c$ for all $i \in \mathbb{Z}_{\geq 0}$ and with statement (2) of Lemma 5.6 we have that

$$\varphi(c) = c\,\boldsymbol{e}_0 + \dots + c\,\boldsymbol{e}_{\lambda-1} = c\,(\boldsymbol{e}_0 + \dots + \boldsymbol{e}_{\lambda-1}) = c.$$

Further, let $f, g \in \mathbb{K}_m[\vartheta_1]\dots[\vartheta_e]$ with $f := a\,\vartheta_1^{v_1}\cdots\vartheta_e^{v_e}$ and $g := b\,\vartheta_1^{z_1}\cdots\vartheta_e^{z_e}$ where $a, b \in \mathbb{K}_m$ and $v_i, z_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq e$. Define $f_k := \mathrm{ev}_m(f, \lambda-1-k)$ and $g_k := \mathrm{ev}_m(g, \lambda-1-k)$ for $0 \leq k < \lambda$. Then,

$$\begin{aligned} \varphi(f+g) &= \mathrm{ev}_m(f+g, \lambda-1)\,\boldsymbol{e}_0 + \dots + \mathrm{ev}_m(f+g, 0)\,\boldsymbol{e}_{\lambda-1} \\ &= \big(\mathrm{ev}_m(f, \lambda-1) + \mathrm{ev}_m(g, \lambda-1)\big)\,\boldsymbol{e}_0 + \dots + \big(\mathrm{ev}_m(f, 0) + \mathrm{ev}_m(g, 0)\big)\,\boldsymbol{e}_{\lambda-1} \\ &= \big(\mathrm{ev}_m(f, \lambda-1)\,\boldsymbol{e}_0 + \dots + \mathrm{ev}_m(f, 0)\,\boldsymbol{e}_{\lambda-1}\big) \\ &\quad + \big(\mathrm{ev}_m(g, \lambda-1)\,\boldsymbol{e}_0 + \dots + \mathrm{ev}_m(g, 0)\,\boldsymbol{e}_{\lambda-1}\big) \\ &= \big(f_0\,\boldsymbol{e}_0 + \dots + f_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big) + \big(g_0\,\boldsymbol{e}_0 + \dots + g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big) \\ &= \varphi(f) + \varphi(g). \end{aligned}$$

Similarly,

$$\begin{aligned} \varphi(f\,g) &= \mathrm{ev}_m(f\,g, \lambda-1)\,\boldsymbol{e}_0 + \dots + \mathrm{ev}_m(f\,g, 0)\,\boldsymbol{e}_{\lambda-1} \\ &= \big(\mathrm{ev}_m(f, \lambda-1)\ \mathrm{ev}_m(g, \lambda-1)\big)\,\boldsymbol{e}_0 + \dots + \big(\mathrm{ev}_m(f, 0)\ \mathrm{ev}_m(g, 0)\big)\,\boldsymbol{e}_{\lambda-1} \\ &= f_0\,g_0\,\boldsymbol{e}_0 + f_1\,g_1\,\boldsymbol{e}_1 + \dots + f_{\lambda-1}\,g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1} \\ &= \big(f_0\,\boldsymbol{e}_0 + \dots + f_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big)\big(g_0\,\boldsymbol{e}_0 + \dots + g_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\big) \\ &= \varphi(f)\,\varphi(g). \end{aligned}$$

The last but one equality follows since the $\boldsymbol{e}_i$ are idempotent and orthogonal. Thus, φ is a ring homomorphism. Next we show by induction on the number of A-monomials, $e \in \mathbb{Z}_{\geq 0}$, that φ is a difference ring homomorphism. For the base case, i.e., $e = 0$, there are no A-monomials. Since $\sigma(\varphi(c)) = \sigma(c) = c = \varphi(c) = \varphi(\sigma(c))$ for all $c \in \mathbb{K}_m$, φ is a difference ring-homomorphism. Now assume that the statement holds for all A-monomials ϑ_i with $0 \leq i < e$, and consider an A-monomial ϑ_e with $\sigma(\vartheta_e) = \tilde{\alpha}\,\vartheta_e$ where $\tilde{\alpha} \in (\mathbb{K}_m^*)_{\mathbb{K}_m}^{\mathbb{K}_m[\vartheta_1]\cdots[\vartheta_{e-1}]}$. Then we will show that

$$\sigma(\varphi(\vartheta_e)) = \varphi(\sigma(\vartheta_e)) \tag{75}$$

holds. For the left hand side of (75), we have that $\varphi(\vartheta_e) = \gamma_0\,\boldsymbol{e}_0 + \cdots + \gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}$ where $\gamma_i = \mathrm{ev}_m(\vartheta_e, \lambda - 1 - i) \in \mathbb{K}_m$ for $0 \leq i < \lambda$ are λ-th roots of unity. Thus,

$$\sigma(\varphi(\vartheta_e)) = \sigma(\gamma_0)\,\sigma(\boldsymbol{e}_0) + \cdots + \sigma(\gamma_{\lambda-1})\,\sigma(\boldsymbol{e}_{\lambda-1}).$$

By (69) we have that $\sigma(\boldsymbol{e}_{\lambda-1}) = \boldsymbol{e}_0$ and $\sigma(\boldsymbol{e}_i) = \boldsymbol{e}_{i+1}$ for $0 \leq i < \lambda - 1$. In addition, for $1 \leq i < \lambda$ we get $\sigma(\gamma_i) = \mathrm{ev}_m(\vartheta_e, \lambda - i) = \gamma_{i-1}$. For $i = \lambda$ observe that $\mathrm{per}(\vartheta_e) \mid \lambda$ by definition and thus $\sigma^\lambda(\vartheta_e) = \vartheta_e$. Consequently $\sigma(\gamma_0) = \mathrm{ev}_m(\vartheta_e, \lambda) = \mathrm{ev}_m(\sigma^\lambda(\vartheta_e), 0) = \mathrm{ev}_m(\vartheta_e, 0) = \gamma_{\lambda-1}$. Therefore,

$$\sigma(\varphi(\vartheta_e)) = \tilde{\gamma}_0\,\boldsymbol{e}_0 + \cdots + \tilde{\gamma}_{\lambda-1}\,\boldsymbol{e}_{\lambda-1} \tag{76}$$

where $\tilde{\gamma}_0 = \gamma_{\lambda-1}$ and $\tilde{\gamma}_i = \gamma_{i-1}$ for $1 \leq i \leq \lambda - 1$.
For the right hand side of (75), we have

$$\begin{aligned}\varphi(\sigma(\vartheta_e)) = \varphi(\tilde{\alpha}\,\vartheta_e) = \varphi(\tilde{\alpha})\,\varphi(\vartheta_e)&\\ &= (\beta_0\,\boldsymbol{e}_0 + \cdots + \beta_{\lambda-1}\,\boldsymbol{e}_{\lambda-1})(\gamma_0\,\boldsymbol{e}_0 + \cdots + \gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1})\\ &= \beta_0\,\gamma_0\,\boldsymbol{e}_0 + \cdots + \beta_{\lambda-1}\,\gamma_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}\end{aligned} \tag{77}$$

where $\beta_i = \mathrm{ev}_m(\tilde{\alpha}, \lambda - 1 - i)$ and $\gamma_i = \mathrm{ev}_m(\vartheta_e, \lambda - 1 - i)$ for $0 \leq i < \lambda$ are λ-th roots of unity. Again (77) holds since the $\boldsymbol{e}_i$ are idempotent and orthogonal. Finally observe that for $0 \leq i < \lambda$,

$$\begin{aligned}\beta_i\,\gamma_i &= \mathrm{ev}_m(\vartheta_e,\, \lambda - 1 - i)\,\mathrm{ev}_m(\tilde{\alpha},\, \lambda - 1 - i) = \mathrm{ev}_m(\tilde{\alpha}\,\vartheta_e,\, \lambda - 1 - i)\\ &= \mathrm{ev}_m(\sigma(\vartheta_e),\, \lambda - 1 - i) = \mathrm{ev}_m(\vartheta_e,\, \lambda - i) = \tilde{\gamma}_i.\end{aligned}$$

With (76) we conclude that (75) holds. Thus, φ is a difference ring homomorphism.

(2) By Lemma 3.1 we can define the evaluation function $\mathrm{ev}_\lambda : \mathbb{K}_\lambda[\vartheta] \times \mathbb{Z}_{\geq 0} \to \mathbb{K}_\lambda$ with $\mathrm{ev}_\lambda|_{\mathbb{K}_\lambda} = \mathrm{id}$ and $\mathrm{ev}_\lambda(\vartheta, n) = \zeta^n$ and by Lemma 4.2 we get the $\mathbb{K}_\lambda$-homomorphism $\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda)$ defined by $\tau_\lambda(f) = \langle \mathrm{ev}_\lambda(f, n) \rangle_{n \geq 0}$. Statement (73) follows by (69).

(3) Since $(\mathbb{K}_\lambda[\vartheta], \sigma)$ is an R-extension of a difference field $(\mathbb{K}_\lambda, \sigma)$ it follows by Theorem 4.2 that τ_λ is injective.

(4) Let $\alpha \in \mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e]$ and let ev_m, ev_λ be evaluation functions for $\mathbb{K}_m[\vartheta_1]\ldots[\vartheta_e]$ and $\mathbb{K}_\lambda[\vartheta]$ defined by (70) and (73), respectively. We will show

$$\varphi'(\tau_m(\alpha)) = \tau_\lambda(\varphi(\alpha)). \tag{78}$$

For the left hand side of (78), we have

$$\varphi'(\tau_m(\alpha)) = \tau_m(\alpha) = \langle \mathrm{ev}_m(\alpha, n) \rangle_{n \geq 0} \in \mathcal{S}(\mathbb{K}_m) \subseteq \mathcal{S}(\mathbb{K}_\lambda).$$

For the right hand side of (78) we note by (72) that $\varphi(\alpha) = \beta_0\,\boldsymbol{e}_0 + \cdots + \beta_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}$ holds where $\beta_i = \mathrm{ev}_m(\alpha, \lambda - 1 - i) \in \mathbb{K}_m \subseteq \mathbb{K}_\lambda$ for $0 \leq i < \lambda$. Thus,

$$\begin{aligned}\tau_\lambda(\varphi(\alpha)) &= \langle \mathrm{ev}_\lambda(\beta_0\,\boldsymbol{e}_0 + \cdots + \beta_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}, n) \rangle_{n \geq 0}\\ &= \langle \mathrm{ev}_\lambda(\beta_0\,\boldsymbol{e}_0, n) \rangle_{n \geq 0} + \cdots + \langle \mathrm{ev}_\lambda(\beta_{\lambda-1}\,\boldsymbol{e}_{\lambda-1}, n) \rangle_{n \geq 0}\\ &= \beta_0 \langle \mathrm{ev}_\lambda(\boldsymbol{e}_0, n) \rangle_{n \geq 0} + \cdots + \beta_{\lambda-1} \langle \mathrm{ev}_\lambda(\boldsymbol{e}_{\lambda-1}, n) \rangle_{n \geq 0}\\ &= \langle \mathrm{ev}_m(\alpha, n) \rangle_{n \geq 0}.\end{aligned}$$

The last equality follows by (73). This implies that the diagram (74) commutes.

Finally, if $\mathbb{K}_m$ is computable and Problem O is solvable in $\mathbb{K}_m$, then by statement (2) of Lemma 5.5 $\operatorname{per}(\vartheta_i)$ is computable for all $1 \le i \le e$. Consequently, the R-extension $(\mathbb{K}_\lambda[\vartheta], \sigma)$ of $(\mathbb{K}_\lambda, \sigma)$, the evaluation function $\operatorname{ev}_\lambda : \mathbb{K}_\lambda[\vartheta] \times \mathbb{Z}_{\ge 0} \to \mathbb{K}_\lambda$ given in statement (2) and the injective $\mathbb{K}_\lambda$-homomorphism $\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda)$ given in statement (4) can be constructed explicitly. $\square$

Remark 5.2. By statement (4) of Theorem 5.4 and (73) we observe that for a fixed $k \in \mathbb{Z}_{\ge 0}$ and $\alpha \in \mathbb{K}_m[\vartheta_1] \dots [\vartheta_e]$ we get

$$\begin{aligned}\operatorname{ev}_m(\alpha, k) &= \operatorname{ev}_\lambda(\varphi(\alpha), k) = \beta_0 \operatorname{ev}_\lambda(\boldsymbol{e}_0, k) + \cdots + \beta_{\lambda-1} \operatorname{ev}_\lambda(\boldsymbol{e}_{\lambda-1}, k) \\ &= \beta_j \operatorname{ev}_\lambda(\boldsymbol{e}_j, k) = \beta_j = \operatorname{ev}_m(\alpha, \lambda - 1 - j)\end{aligned}$$

for some $j \in \{0, 1, \dots, \lambda - 1\}$ with $\lambda \mid k + j + 1$. In other words, the sequence repeats periodically.

Example 5.4 *(Cont. Example 5.3).* Consider the $\mathbb{U}_2$-simple A-extension $(\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}], \sigma)$ of $(\mathbb{K}, \sigma)$ with the automorphism and the evaluation function given in (29) and (30), which was constructed in Examples 5.2 and 5.3 with $\mathbb{K} = \mathbb{Q}(\sqrt{3})(= \mathbb{K}_2)$. From Example 5.3 we already know the periods of the A-monomials $\vartheta_{1,1}$ and $\vartheta_{1,2}$ in $\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}]$. Set $\lambda = \operatorname{lcm}(m, \operatorname{per}(\vartheta_{1,1}), \operatorname{per}(\vartheta_{1,2})) = 4$ with $m = 2$, take a primitive 4th root of unity, say $\zeta := \mathbb{e}^{\frac{\pi \mathbb{i}}{2}} = (-1)^{\frac{1}{2}} = \mathbb{i}$ and define $\tilde{\mathbb{K}} = \mathbb{Q}(\mathbb{i}, \sqrt{3})(= \mathbb{K}_4)$. Then by statement (1) of Theorem 5.4 there is an R-extension $(\tilde{\mathbb{K}}[\vartheta], \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of order 4 with the automorphism $\sigma : \tilde{\mathbb{K}}[\vartheta] \to \tilde{\mathbb{K}}[\vartheta]$ and evaluation function $\tilde{\operatorname{ev}} : \tilde{\mathbb{K}}[\vartheta] \times \mathbb{Z}_{\ge 0} \to \tilde{\mathbb{K}}$ given by

$$\sigma(\vartheta) = \mathbb{i}\,\vartheta \qquad \text{and} \qquad \tilde{\operatorname{ev}}(\vartheta, n) = \prod_{k=1}^{n} \mathbb{i} = \mathbb{i}^n. \tag{79}$$

We have $\tilde{\mathbb{K}}[\vartheta] = \boldsymbol{e}_0\tilde{\mathbb{K}} \oplus \boldsymbol{e}_1\tilde{\mathbb{K}} \oplus \boldsymbol{e}_2\tilde{\mathbb{K}} \oplus \boldsymbol{e}_3\tilde{\mathbb{K}}$ where the idempotent elements $\boldsymbol{e}_k$ for $0 \le k \le 3$ are defined by

$$\begin{aligned}\boldsymbol{e}_0 &= \tfrac{\mathbb{i}}{4}\left(\vartheta^3 + \mathbb{i}\,\vartheta^2 - \vartheta - \mathbb{i}\right), & \boldsymbol{e}_1 &= \tfrac{1}{4}\left(1 - \vartheta + \vartheta^2 - \vartheta^3\right), \\ \boldsymbol{e}_2 &= \tfrac{\mathbb{i}}{4}\left(-\vartheta^3 + \mathbb{i}\,\vartheta^2 + \vartheta - \mathbb{i}\right), & \boldsymbol{e}_3 &= \tfrac{1}{4}\left(1 + \vartheta + \vartheta^2 + \vartheta^3\right)\end{aligned} \tag{80}$$

with $\boldsymbol{e}_0 + \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = 1$. Furthermore, the ring homomorphism $\varphi : \mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}] \to \tilde{\mathbb{K}}[\vartheta]$ defined by $\varphi|_{\mathbb{K}} = \mathrm{id}_{\mathbb{K}}$ and $\varphi(\vartheta_{1,i}) = \beta_{i,0}\boldsymbol{e}_0 + \beta_{i,1}\boldsymbol{e}_1 + \beta_{i,2}\boldsymbol{e}_2 + \beta_{i,3}\boldsymbol{e}_3$ where $\beta_{i,j} = \operatorname{ev}(\vartheta_{i,j}, 3 - j)$ for $i \in \{1, 2\}$ and $0 \le j \le 3$ is a difference ring homomorphism. More precisely, for the A-monomials we have that

$$\begin{aligned}\varphi(\vartheta_{1,1}) &= -\boldsymbol{e}_0 + \boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3 = \vartheta^2, \\ \varphi(\vartheta_{1,2}) &= -\boldsymbol{e}_0 - \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = \tfrac{(1-\mathbb{i})}{2}\,\vartheta\,(\vartheta^2 + \mathbb{i}).\end{aligned} \tag{81}$$

Given ev and $\tilde{\operatorname{ev}}$ we obtain the difference ring homomorphisms $\tau_2 : \mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}] \to \mathcal{S}(\mathbb{K})$ defined by $\tau_2(f) = (\operatorname{ev}(f, n))_{n \ge 0}$ and $\tau_4 : \tilde{\mathbb{K}}[\vartheta] \to \mathcal{S}(\tilde{\mathbb{K}})$ defined by $\tau_4(f) = (\tilde{\operatorname{ev}}(f, n))_{n \ge 0}$. In particular, by statement (3) of Theorem 5.4 τ_4 is injective. Finally, by defining the embedding $\varphi' : \mathcal{S}(\mathbb{K}) \to \mathcal{S}(\tilde{\mathbb{K}})$ with $\varphi'(a) = a$ for all $a \in \mathcal{S}(\mathbb{K})$ we conclude by statement (4) of the Theorem 5.4 that the following diagram commutes

$$\begin{array}{ccc} \mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}] & \xrightarrow{\quad\tau_2\quad} & \mathcal{S}(\tilde{\mathbb{K}}) \\ \varphi\big\downarrow & & \big\downarrow\varphi' \\ \tilde{\mathbb{K}}[\vartheta] = \boldsymbol{e}_0\tilde{\mathbb{K}} \oplus \boldsymbol{e}_1\tilde{\mathbb{K}} \oplus \boldsymbol{e}_2\tilde{\mathbb{K}} \oplus \boldsymbol{e}_3\tilde{\mathbb{K}} & \xrightarrow{\tau_4} & \mathcal{S}(\tilde{\mathbb{K}}). \end{array}$$

We are finally ready to obtain the proof of Theorem 5.3.

Proof of Theorem 5.3. Suppose we are given the geometric products $A_1(n), \dots, A_e(n) \in \mathrm{Prod}_n(\mathbb{U}_m)$ in n with (65) where $\zeta_{i,r_i} \neq 1$ for $1 \le i \le e$. As elaborated in Section 2.2 we can rewrite each $A_i(n)$ as

$$A_i(n) = u_i\, \tilde{A}_i(n) \quad \text{where} \quad \tilde{A}_i(n) = \prod_{k_1=1}^{n} \zeta_{i,1} \prod_{k_2=1}^{k_1} \zeta_{i,2} \cdots \prod_{k_{r_i}=1}^{k_{r_i-1}} \zeta_{i,r_i} \tag{82}$$

and $u_i \in \mathbb{U}_m$ which holds for all $n \ge \max(\ell_{i,1}, \dots, \ell_{i,r_i}) - 1 =: \delta_i$. Similar to Algorithm 3.1 (see Remark 3.2.(2)) we can rephrase the products in a simple A-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}_m, \sigma)$ with $\mathbb{A} = \mathbb{K}_m[\vartheta_{1,1}] \dots [\vartheta_{1,r_1}], \dots, [\vartheta_{e,1}] \dots [\vartheta_{e,r_e}]$ where $\alpha_{i,j} := \frac{\sigma(\vartheta_{i,j})}{\vartheta_{i,j}} = \zeta_m^{u_{i,j}}\, \vartheta_{i,1} \cdots \vartheta_{i,j-1}$ for $1 \le i \le e$ and $1 \le j \le r_i$ with $u_{i,i} \in \mathbb{Z}_{\ge 0}$ equipped with the evaluation function $\mathrm{ev}_m : \mathbb{A} \times \mathbb{Z}_{\ge 0} \to \mathbb{K}_m$ defined by $\mathrm{ev}_m(\vartheta_{i,j}, n) = \prod_{k=1}^{n} \mathrm{ev}_m(\alpha_{i,j}, k-1)$ with the following property. For all i with $1 \le i \le e$, there are ν_i, μ_i such that the geometric product $\tilde{A}_i(n)$ is modeled by ϑ_{ν_i,μ_i}, i.e.,

$$\mathrm{ev}_m(\vartheta_{\nu_i,\mu_i}, n) = \tilde{A}_i(n) \tag{83}$$

holds for all $n \ge 0$. In particular, we get the $\mathbb{K}$-homomorphism $\tau_m : \mathbb{A} \to \mathcal{S}(\mathbb{K}_\lambda)$. By Theorem 5.4, there is a single R-extension $(\mathbb{K}_\lambda[\vartheta], \sigma)$ of $(\mathbb{K}_\lambda, \sigma)$ subject to the relations $\vartheta^\lambda = 1$ and $\sigma(\vartheta) = \zeta_\lambda\, \vartheta$ where $\lambda := \mathrm{lcm}(m, \mathrm{per}(\gamma_1), \dots, \mathrm{per}(\gamma_s)) > 0$, $\zeta_\lambda := \mathrm{e}^{\frac{2\pi\,\mathrm{i}}{\lambda}} = (-1)^{\frac{2}{\lambda}} \in \mathbb{K}_\lambda$ and $\mathbb{K}_\lambda$ is some algebraic extension of $\mathbb{K}_m$ with $m \,|\, \lambda$. Furthermore, there is a difference ring homomorphism $\varphi : \mathbb{A} \to \mathbb{K}_\lambda[\vartheta]$ and a $\mathbb{K}_\lambda$-embedding $\tau_\lambda : \mathbb{K}_\lambda[\vartheta] \to \mathcal{S}(\mathbb{K}_\lambda)$ with $\tau_\lambda(\vartheta) = \langle \zeta_\lambda^n \rangle_{n \ge 0}$ such that $\tau_m(f) = \tau_\lambda(\varphi(f))$ holds for all $f \in \mathbb{A}$. In particular, we get

$$\mathrm{ev}_m(\vartheta_{\nu_i,\mu_i}, n) = \mathrm{ev}_\lambda(\varphi(\vartheta_{\nu_i,\mu_i}), n) \tag{84}$$

for $1 \le i \le e$. Now define $g_{i,k} \in \mathbb{K}_\lambda$ by $\varphi(\vartheta_{\nu_i,\mu_i}) = \sum_{k=0}^{\lambda-1} g_{i,k}\, \vartheta^k \in \mathbb{K}_\lambda[\vartheta]$. Then for $G_i(n) := \sum_{k=0}^{\lambda-1} g_{i,k}\, (\zeta_\lambda^n)^k$ with $1 \le i \le e$ we get

$$\mathrm{ev}_m(\vartheta_{\nu_i,\mu_i}, n) \overset{(84)}{=} \mathrm{ev}_\lambda(\varphi(\vartheta_{\nu_i,\mu_i}),\, n) = G_i(n) \quad \forall n \ge 0.$$

With (82) and (83) we conclude that

$$A_i(n) = u_i \tilde{A}_i(n) = u_i\, G_i(n)$$

holds for all $n \ge \delta_i$. In particular, for $B_i(n)$ given in (66) with $f_{i,k} := u_i\, g_{i,k} \in \mathbb{K}_\lambda$ we get (67).

If $\mathbb{K}_m$ is computable and Problem O can be solved, Theorem 5.4 is constructive and all the above ingredients can be given explicitly. □

Example 5.5 *(Cont. Example 5.4).* Consider the product expression

$$A(n) = \sqrt{3} \prod_{i=1}^{n} (-1) + 2 \prod_{k=1}^{n} \prod_{i=1}^{k} (-1) + 3 \left(\prod_{i=1}^{n} (-1) \right) \prod_{k=1}^{n} \prod_{i=1}^{k} (-1) \in \mathrm{ProdE}_n\left(\mathbb{Q}(\sqrt{3}),\, \mathbb{U}_2\right).$$

For this instance we follow the construction in Example 5.4 and get $f = \sqrt{3}\, \vartheta_{1,1} + 2\, \vartheta_{1,2} + 3\, \vartheta_{1,1}\, \vartheta_{1,2} \in \mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}]$ with $\mathrm{ev}(f, n) = A(n)$ for all $n \in \mathbb{Z}_{\ge 0}$. As a consequence we obtain $\tilde{f} := \varphi(f) = \left(\frac{1}{2} - \frac{\mathrm{i}}{2}\right) \vartheta \left((2 + 3\,\mathrm{i})\vartheta^2 + (1 + \mathrm{i})\, \sqrt{3}\, \vartheta + (3 + 2\,\mathrm{i})\right) \in \tilde{\mathbb{K}}[\vartheta]$ yielding for $n \in \mathbb{Z}_{\ge 0}$ the identity

$$A(n) = \mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\tilde{f}, n) = \left(\tfrac{1}{2} - \tfrac{\mathrm{i}}{2}\right) \mathrm{i}^n \left((2 + 3\mathrm{i})(\mathrm{i}^n)^2 + (1 + \mathrm{i})\sqrt{3}\mathrm{i}^n + (3 + 2\mathrm{i})\right).$$

In particular, as claimed in Theorem 5.3, each of the products in $A(n)$ can be expressed in terms of i^n. Namely, for all $n \in \mathbb{Z}_{\ge 0}$ we obtain

$$\prod_{k=1}^{n}(-1)=\mathrm{ev}(\vartheta_{1,1},n)=\tilde{\mathrm{ev}}(\varphi(\vartheta),n)=\tilde{\mathrm{ev}}(\vartheta^2,n)=(\mathbb{i}^n)^2,$$

$$\prod_{k=1}^{n}\prod_{i=1}^{k}(-1)=\mathrm{ev}(\vartheta_{1,2},n)=\tilde{\mathrm{ev}}(\varphi(\vartheta),n)=\tilde{\mathrm{ev}}(\tfrac{(1-\mathbb{i})}{2}\,\vartheta\,(\vartheta^2+\mathbb{i}),n)=\frac{(1-\mathbb{i})}{2}\,\mathbb{i}^n\,((\mathbb{i}^n)^2+\mathbb{i}).$$

6. A complete solution of Problem RPE

We are now ready to combine the building blocks from the previous section to solve Problem RPE in Sections 6.1 and 6.2 below. Afterwards we apply in Section 6.3 the machinery implemented within the package `NestedProducts` to concrete examples.

6.1. The difference ring setting for nested geometric products

First we combine Lemma 5.4 discussed in Subsection 5.2 and Theorem 5.4 discussed in Subsection 5.3. As a consequence, we will obtain the necessary difference ring tools for the full treatment of geometric products of arbitrary but finite nesting depth.

Theorem 6.1. *For $1 \le \ell \le m$, let $(\mathbb{K}_\ell,\sigma)$ with $\mathbb{K}_\ell=\mathbb{K}\langle y_{\ell,1}\rangle\dots\langle y_{\ell,s_\ell}\rangle$ be the single chain* P*-extensions of $(\mathbb{K},\sigma)$ over $\mathbb{K}=K(\kappa_1,\dots,\kappa_u)$ with base $h_\ell\in\mathbb{K}^*$ for $1\le\ell\le m$, the automorphisms (49) and the evaluation functions (50). Let $d:=\max(s_1,\dots,s_m)$ and $\mathbb{A}_0=\mathbb{K}$. Consider the tower of difference ring extensions $(\mathbb{A}_i,\sigma)$ of $(\mathbb{A}_{i-1},\sigma)$ where $\mathbb{A}_i=\mathbb{A}_{i-1}\langle y_{1,i}\rangle\langle y_{2,i}\rangle\dots\langle y_{w_i,i}\rangle$ for $1\le i\le d$ with $m=w_1\ge w_2\ge\dots\ge w_d$ and the automorphism (49) and the evaluation function (50). This yields the ordered multiple chain* P*-extension $(\mathbb{A}_d,\sigma)$ of $(\mathbb{K},\sigma)$ of monomial depth at most d composed by the single chain Π-extensions $(\mathbb{K}_\ell,\sigma)$ of $(\mathbb{K},\sigma)$ for $1\le\ell\le m$ with (49) and (50). Then one can construct*

(1) an RΠ*-extension $(\mathbb{D},\sigma)$ of $(\mathbb{K}',\sigma)$ with*[10]

$$\mathbb{D}=\mathbb{K}'[\vartheta]\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\dots\langle\tilde{y}_{1,d}\rangle\dots\langle\tilde{y}_{e_d,d}\rangle,$$

where $\mathbb{K}'=K'(\kappa_1,\dots,\kappa_u)$ and K' is a finite algebraic field extension of K and with the automorphism

$$\sigma(\vartheta)=\zeta'\,\vartheta\qquad\text{and}\qquad\sigma(\tilde{y}_{\ell,k})=\tilde{\alpha}_{\ell,k}\,\tilde{y}_{\ell,k}$$

where $\zeta'\in K'$ is a λ'-th root of unity and

$$\tilde{\alpha}_{\ell,k}=\tilde{h}_\ell\,\tilde{y}_{\ell,1}\cdots\tilde{y}_{\ell,k-1}\in(\mathbb{K}'^*)_{\mathbb{K}'}^{\mathbb{K}'\langle\tilde{y}_{\ell,1}\rangle\dots\langle\tilde{y}_{\ell,k-1}\rangle}$$

for $1\le k\le d$ and $1\le\ell\le e_k$;

(2) an evaluation function $\tilde{\mathrm{ev}}:\mathbb{D}\times\mathbb{Z}_{\ge0}\to\mathbb{K}'$ defined as[11]

$$\tilde{\mathrm{ev}}(\vartheta,n)=\prod_{j=1}^{n}\zeta'\qquad\text{and}\qquad\tilde{\mathrm{ev}}(\tilde{y}_{\ell,k},n)=\prod_{j=1}^{n}\tilde{\mathrm{ev}}(\tilde{\alpha}_{\ell,k},j-1);\tag{85}$$

(3) a difference ring homomorphism $\varphi:\mathbb{A}_d\to\mathbb{D}$ defined by $\varphi|_{\mathbb{K}}=\mathrm{id}_{\mathbb{K}}$ and

$$\varphi(y_{\ell,k})=\gamma_{\ell,k}\,\tilde{y}_{1,k}^{\nu_{\ell,1,k}}\cdots\tilde{y}_{e_k,k}^{\nu_{\ell,e_k,k}}\tag{86}$$

for $1\le\ell\le m$ and $1\le k\le s_\ell$ with $\gamma_{\ell,k}\in\mathbb{K}'[\vartheta]$ and $\nu_{\ell,i,k}\in\mathbb{Z}$ for $1\le i\le e_k$

[10] For concrete instances the R-monomial ϑ might not be needed. In particular, if $\nu_{\ell,k}=0$ in (86), it can be removed.
[11] Note that for all $c\in\mathbb{K}'$, $\tilde{\mathrm{ev}}(c,n)=c$ for all $n\ge0$.

such that for all $f \in \mathbb{A}_d$ and $n \in \mathbb{Z}_{\geq 0}$ we have

$$\mathrm{ev}(f, n) = \tilde{\mathrm{ev}}(\varphi(f), n). \tag{87}$$

If $\mathbb{K}$ is strongly σ-computable, then the constructions above are computable.

Proof. Let $(\mathbb{A}_d, \sigma)$ be an ordered multiple chain P-extension of $(\mathbb{K}, \sigma)$ of monomial depth at most d with the automorphism $\sigma : \mathbb{A}_d \to \mathbb{A}_d$ defined by (49) and the evaluation function $\mathrm{ev} : \mathbb{A}_d \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined by (50). Then by Lemma 5.4 we can construct an ordered multiple chain AP-extension $(\mathbb{G}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of monomial depth at most d with $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$, $\tilde{K}$ being a finite algebraic field extension of K where $\mathbb{G}_d$ is given by (51) with the automorphism (52) and (53), the evaluation function $\mathrm{ev}' : \mathbb{G}_d \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ with (54) (where $\tilde{\mathrm{ev}}$ is replaced by ev'), and the difference ring homomorphism $\rho_d : \mathbb{A}_d \to \mathbb{G}_d$ defined by $\rho_d|_{\mathbb{K}} = \mathrm{id}_{\mathbb{K}}$ and (55) with the following properties: the sub-difference ring $(\tilde{\mathbb{A}}_d, \sigma)$ of $(\mathbb{G}_d, \sigma)$ where $\tilde{\mathbb{A}}_d$ is given by (56) is a Π-extension of $(\tilde{\mathbb{K}}, \sigma)$. Furthermore, for all $f \in \mathbb{A}_d$ and for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$\mathrm{ev}(f, n) = \mathrm{ev}'(\rho_d(f), n). \tag{88}$$

If $\upsilon_1 = 0$ (and thus $\upsilon_2 = \dots = \upsilon_d = 0$), i.e., no A-monomials are involved, we are essentially done. We simply adjoin a redundant R-monomial (compare the footnote in Theorem 6.1). Otherwise $\upsilon_1 \geq 1$ and by Remark 5.1 the generators in $\mathbb{G}_d$ can be rearranged to get the AΠ-extension $(\tilde{\mathbb{G}}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ where

$$\tilde{\mathbb{G}} = \tilde{\mathbb{K}}[\vartheta_{1,1}] \dots [\vartheta_{\upsilon_1,1}] \dots [\vartheta_{1,d}] \dots [\vartheta_{\upsilon_d,d}] \langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle \tag{89}$$

with the automorphism given by (52) and (53) and the evaluation function given by (54) (where $\tilde{\mathrm{ev}}$ is replaced by ev') satisfying properties (1) and (2) of Lemma 5.4. Now consider the sub-difference ring $(\mathbb{L}, \sigma)$ of $(\tilde{\mathbb{G}}, \sigma)$ with $\mathbb{L} = \tilde{\mathbb{K}}[\vartheta_{1,1}] \dots [\vartheta_{\upsilon_1,1}] \dots [\vartheta_{1,d}] \dots [\vartheta_{\upsilon_d,d}]$, which is a difference ring extension of $(\tilde{\mathbb{K}}, \sigma)$, with the automorphism defined by

$$\sigma(\vartheta_{\ell,k}) = \gamma_{\ell,k}\, \vartheta_{\ell,k} \quad \text{where} \quad \gamma_{\ell,k} = \zeta^{\mu_\ell}\, \vartheta_{\ell,1} \cdots \vartheta_{\ell,k-1} \in \mathbb{U}_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}[\vartheta_{\ell,1}] \dots [\vartheta_{\ell,k-1}]} \tag{90}$$

for $1 \leq k \leq d$ and for $1 \leq \ell \leq \upsilon_k$ where $\mathbb{U} = \langle \zeta \rangle$ is the multiplicative cyclic subgroup of $\tilde{K}$ generated by a primitive λ-th root of unity, $\zeta \in \tilde{K}^*$. Observe that the difference ring extension $(\mathbb{L}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with (90) is a simple A-extension to which statement (1) of Theorem 5.4 can be applied. Thus there is an R-extension $(\mathbb{K}'[\vartheta], \sigma)$ of $(\mathbb{K}', \sigma)$ with

$$\sigma(\vartheta) = \zeta'\, \vartheta \tag{91}$$

of order λ' where $\mathbb{K}' = K'(\kappa_1, \dots, \kappa_u)$, ζ' is a primitive λ'-th root of unity in K' and K' is a finite algebraic field extension of $\tilde{K}$. Note that the difference ring $(\tilde{\mathbb{D}}, \sigma)$ where $\tilde{\mathbb{D}}$ is given by

$$\tilde{\mathbb{D}} = \mathbb{K}' \langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle$$

with the automorphism defined by (53) is a Π-extension of $(\mathbb{K}', \sigma)$. Thus by Lemma 4.1 it follows that the A-extension $(\tilde{\mathbb{D}}[\vartheta], \sigma)$ of $(\tilde{\mathbb{D}}, \sigma)$ with (91) of order λ' is an R-extension. Note that the generators in the ring $\tilde{\mathbb{D}}[\vartheta]$ can be rearranged to get $(\mathbb{D}, \sigma)$ where $\mathbb{D} = \mathbb{K}'[\vartheta] \langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle$ and σ is defined by (91) and (53). Since this rearrangement does not change the set of constants, $(\mathbb{D}, \sigma)$ is an RΠ-extension of $(\mathbb{K}', \sigma)$. By statement (1) of Proposition 5.1 $\mathbb{D} = \boldsymbol{e}_0 \mathbb{D} \oplus \dots \oplus \boldsymbol{e}_{\lambda'-1} \mathbb{D}$ and by statement (2) of the same proposition, $\boldsymbol{e}_k \mathbb{D} = \boldsymbol{e}_k \tilde{\mathbb{D}}$ for $0 \leq k < \lambda'$. Thus $\mathbb{D} = \boldsymbol{e}_0 \tilde{\mathbb{D}} \oplus \boldsymbol{e}_1 \tilde{\mathbb{D}} \oplus \dots \oplus \boldsymbol{e}_{\lambda'-1} \tilde{\mathbb{D}}$ holds. Now we show that $\phi : \tilde{\mathbb{G}} \to \mathbb{D}$ defined by $\phi|_{\tilde{\mathbb{K}}} = \mathrm{id}_{\tilde{\mathbb{K}}}$ with

$$\phi(\tilde{y}_{\ell,k}) = \tilde{y}_{\ell,k}, \tag{92}$$

$$\phi(\vartheta_{\ell,k}) = \beta_{\ell,k,0} \boldsymbol{e}_0 + \dots + \beta_{\ell,k,\lambda'-1} \boldsymbol{e}_{\lambda'-1} \tag{93}$$

where $\beta_{\ell,k,i} = \mathrm{ev}(\vartheta_{\ell,k}, \lambda' - 1 - i)$ for $0 \le i < \lambda'$ is a difference ring homomorphism. By statement (1) of Theorem 5.4, $\phi|_{\mathbb{L}}$ which is defined by (93) is a difference ring homomorphism. Since ϕ maps $\tilde{y}_{\ell,k}$ to itself, also ϕ is a difference ring homomorphism. Furthermore, for all $f \in \tilde{\mathbb{G}}$ and for all $n \in \mathbb{Z}_{\ge 0}$, we have

$$\mathrm{ev}'(f, n) = \tilde{\mathrm{ev}}(\phi(f), n). \tag{94}$$

Putting everything together, the map $\varphi : \mathbb{A}_d \to \mathbb{D}$ with $\varphi = \phi \circ \rho_d$ is a difference ring homomorphism. It is uniquely determined by $\varphi|_{\mathbb{K}} = \mathrm{id}_{\mathbb{K}}$ and

$$\varphi(y_{\ell,d}) = \phi(\rho_d(y_{\ell,d})) = \gamma_{\ell,d}\, \tilde{y}_{1,d}^{\,v_{\ell,1,d}} \cdots \tilde{y}_{e_d,d}^{\,v_{\ell,e_d,d}}$$

with $\gamma_{\ell,d} = \beta_{\ell,d,0}\boldsymbol{e}_0 + \cdots + \beta_{\ell,d,\lambda'-1}\boldsymbol{e}_{\lambda'-1} \in \mathbb{K}'[\vartheta]$. Furthermore, by (88) and (94) it follows that for all $f \in \mathbb{A}_d$ and for all $n \in \mathbb{Z}_{\ge 0}$ we get

$$\mathrm{ev}(f, n) = \mathrm{ev}'(\rho_d(f), n) = \tilde{\mathrm{ev}}(\phi(\rho_d(f)), n) = \tilde{\mathrm{ev}}(\varphi(f), n).$$

Finally if K is strongly σ-computable, then by Lemma 5.4 the difference ring $(\tilde{\mathbb{G}}, \sigma)$ with (89) together with the automorphism (52) and (53), the evaluation function (54) (where $\tilde{\mathrm{ev}}$ is replaced by ev') and the difference ring homomorphism $\rho_d : \mathbb{A}_d \to \tilde{\mathbb{G}}$ with (55) can be computed. Further, by Theorem 5.4 the difference ring $(\tilde{\mathbb{D}}[\vartheta], \sigma)$ with the automorphism $\sigma(\vartheta) = \zeta'\,\vartheta$ and (53), the evaluation function (85) and the difference ring homomorphism $\phi : \tilde{\mathbb{G}} \to \mathbb{D}$ given by (92) and (93) can be computed. In particular, φ and all the components stated in the theorem can be given explicitly. $\square$

Example 6.1 *(Cont. Example 5.2, 5.4).* Take the AΠ-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{K}, \sigma)$ with (62) constructed in Example 5.2 with the automorphism defined in (29) and (63), and consider the sub-difference ring $(\mathbb{K}[\vartheta_{1,1}][\vartheta_{1,2}], \sigma)$ of $(\mathbb{G}, \sigma)$ with the automorphism σ given in (29), which is a simple A-extension of $(\mathbb{K}, \sigma)$ where $\mathbb{K} = \mathbb{Q}(\sqrt{3})$. Now we refine the construction from Example 5.2 by utilizing Example 5.4. Namely, we take the R-extension $(\tilde{\mathbb{K}}[\vartheta], \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ of order 4 with the automorphism and the evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{K}}[\vartheta] \times \mathbb{Z}_{\ge 0} \to \tilde{\mathbb{K}}$ given by (79) where $\tilde{\mathbb{K}} = \mathbb{Q}(\mathbb{i}, \sqrt{3})$. Furthermore, take the P-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}[\vartheta], \sigma)$ where $\mathbb{D} = \tilde{\mathbb{K}}[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle$ with the automorphism and evaluation function given by (63) for the P-monomials $\tilde{y}_{\ell,k}$. By Theorem 6.1 $(\mathbb{D}, \sigma)$ is an RΠ-extension of $(\tilde{\mathbb{K}}, \sigma)$ where the ring $\mathbb{D}$ can be written as the direct sum $\mathbb{D} = \boldsymbol{e}_0\tilde{\mathbb{D}} \oplus \boldsymbol{e}_1\tilde{\mathbb{D}} \oplus \boldsymbol{e}_2\tilde{\mathbb{D}} \oplus \boldsymbol{e}_3\tilde{\mathbb{D}}$ with $\tilde{\mathbb{D}} = \tilde{\mathbb{K}}\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle$; here the idempotent elements $\boldsymbol{e}_k$ for $0 \le k \le 3$ are defined by (80). Furthermore, the ring homomorphism $\phi : \mathbb{G} \to \mathbb{D}$ defined by $\phi|_{\tilde{\mathbb{D}}} = \mathrm{id}_{\tilde{\mathbb{D}}}$ and (81) is a difference ring homomorphism.

Finally, consider the AP-extension $(\mathbb{A}', \sigma)$ of $(\mathbb{K}, \sigma)$ as given in Example 5.2 and consider the difference ring homomorphism $\rho : \mathbb{A}' \to \mathbb{G}$ given in (64). Then with the difference ring homomorphism $\varphi : \mathbb{A}' \to \mathbb{D}$ defined by $\phi(\rho(f))$ for $f \in \mathbb{A}'$ we get (87) for all $n \in \mathbb{Z}_{\ge 0}$ and $f \in \mathbb{A}'$. Given this explicit construction we can choose for instance $g \in \mathbb{A}'$ defined in (39) that models $\tilde{G}(n)$ given in (24). This means that $\mathrm{ev}(g, n) = \tilde{G}(n)$ for all $n \ge 0$. Thus

$$\tilde{g} := \varphi(g) = \frac{\varphi(\vartheta_{1,1})\,\varphi(y_{3,1})\,\varphi(y_{5,1})\,\varphi(\vartheta_{1,2})\,\varphi(y_{2,2})}{\varphi(y_{1,1})\,\varphi(y_{2,1})\,\varphi(y_{4,2})} = \frac{(1-\mathbb{i})\,\vartheta\,(\mathbb{i}\,\vartheta^2+1)\,\tilde{y}_{1,1}\,\tilde{y}_{3,1}^2\,\tilde{y}_{2,2}}{2\,\tilde{y}_{2,1}\,\tilde{y}_{3,2}} \in \mathbb{D} \tag{95}$$

yields for $n \ge 0$ the identity

$$\tilde{G}(n) = \mathrm{ev}(g, n) = \tilde{\mathrm{ev}}(\tilde{g}, n) = \frac{1}{2}\,\frac{(1-\mathbb{i})\,(\mathbb{i})^n\,(\mathbb{i}\,(\mathbb{i}^n)^2+1)\,\left(\sqrt{3}\right)^n\,\left(5^n\right)^2\,2^{\binom{n+1}{2}}}{2^n\,5^{\binom{n+1}{2}}}.$$

6.2. The solution for nested hypergeometric products

So far we have treated hypergeometric products over monic irreducible polynomials of finite nesting depth, say b, that are δ-refined for some $\delta \in \mathbb{Z}_{\geq 0}$; see Definition 2.3. Given such hypergeometric products, it follows by Corollary 5.1 that we can construct an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $\mathbb{K} = K(\kappa_1, \dots, \kappa_u)$ and

$$\tilde{\mathbb{H}}_b = \mathbb{K}(x)\langle \tilde{\boldsymbol{z}}_{\boldsymbol{1}} \rangle \dots \langle \tilde{\boldsymbol{z}}_{\boldsymbol{b}} \rangle = \mathbb{K}(x)\langle \tilde{z}_{1,1} \rangle \dots \langle \tilde{z}_{p_1,1} \rangle \dots \langle \tilde{z}_{1,b} \rangle \dots \langle \tilde{z}_{p_b,b} \rangle. \tag{96}$$

In particular, $(\tilde{\mathbb{H}}_b, \sigma)$ is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_\ell, \sigma)$ of $(\mathbb{K}(x), \sigma)$ for $1 \leq \ell \leq p_1$ with

$$\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \langle \tilde{z}_{\ell,2} \rangle \dots \langle \tilde{z}_{\ell,s_\ell} \rangle, \qquad 1 \leq k \leq s_\ell$$

given by the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ defined by

$$\sigma(\tilde{z}_{\ell,k}) = \tilde{\alpha}_{\ell,k}\, \tilde{z}_{\ell,k} \quad \text{where} \quad \tilde{\alpha}_{\ell,k} = \tilde{f}_\ell\, \tilde{z}_{\ell,1} \cdots \tilde{z}_{\ell,k-1} \in (\mathbb{K}(x)^*)_{\mathbb{K}(x)}^{\mathbb{K}(x)\langle \tilde{z}_{\ell,1} \rangle \dots \langle \tilde{z}_{\ell,k-1} \rangle} \tag{97}$$

and the evaluation function $\tilde{\text{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined by

$$\tilde{\text{ev}}(\tilde{z}_{\ell,k}, n) = \prod_{j=\delta}^{n} \tilde{\text{ev}}(\tilde{\alpha}_{\ell,k}, j-1). \tag{98}$$

On the other hand, geometric products over the contents were treated in Subsection 5.2. In Theorem 6.1 we constructed a simple $R\Pi$-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with $\tilde{\mathbb{K}} = \tilde{K}(\kappa_1, \dots, \kappa_u)$ where $\tilde{K}$ is a finite algebraic field extension of K in which the geometric products can be modeled. To accomplish this task, we set up a ring of the form

$$\mathbb{D} = \tilde{\mathbb{K}}[\vartheta]\langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle \tag{99}$$

with

(a) the automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ defined by

$$\sigma(\vartheta) = \zeta\, \vartheta, \tag{100}$$

$$\sigma(\tilde{y}_{\ell,k}) = \tilde{\gamma}_{\ell,k}\, \tilde{y}_{\ell,k} \tag{101}$$

where $\zeta \in \tilde{K}^*$ is a λ-th root of unity and $\tilde{\gamma}_{\ell,k} = \tilde{h}_\ell\, \tilde{y}_{\ell,1} \cdots \tilde{y}_{\ell,k-1} \in (\tilde{\mathbb{K}}^*)_{\tilde{\mathbb{K}}}^{\tilde{\mathbb{K}}\langle \tilde{y}_{\ell,1} \rangle \dots \langle \tilde{y}_{\ell,k-1} \rangle}$ for $1 \leq k \leq d$ and $1 \leq \ell \leq e_k$ and

(b) the evaluation function $\tilde{\text{ev}} : \mathbb{D} \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ defined by

$$\tilde{\text{ev}}(\vartheta, n) = \prod_{j=1}^{n} \zeta, \tag{102}$$

$$\tilde{\text{ev}}(\tilde{y}_{\ell,k}, n) = \prod_{j=1}^{n} \tilde{\text{ev}}(\tilde{\gamma}_{\ell,k}, j-1). \tag{103}$$

In particular, by reordering we obtain the difference ring extension $(\tilde{\mathbb{A}}_d, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with

$$\tilde{\mathbb{A}}_d = \tilde{\mathbb{K}}\langle \tilde{\boldsymbol{y}}_{\boldsymbol{1}} \rangle \dots \langle \tilde{\boldsymbol{y}}_{\boldsymbol{d}} \rangle = \tilde{\mathbb{K}}\langle \tilde{y}_{1,1} \rangle \dots \langle \tilde{y}_{e_1,1} \rangle \dots \langle \tilde{y}_{1,d} \rangle \dots \langle \tilde{y}_{e_d,d} \rangle, \tag{104}$$

the automorphism (101) and the evaluation function (103) which is a sub-difference ring of $(\mathbb{D},\sigma)$. Furthermore, it is an ordered multiple chain Π-extension of $(\tilde{\mathbb{K}},\sigma)$ and is composed by the single chain Π-extensions $(\tilde{\mathbb{K}}_\ell,\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ where $\tilde{\mathbb{K}}_\ell=\tilde{\mathbb{K}}\langle\tilde{y}_{\ell,1}\rangle\langle\tilde{y}_{\ell,2}\rangle\dots\langle\tilde{y}_{\ell,\tilde{e}_\ell}\rangle$ for $1\le\ell\le e_1$.

Putting the two difference rings $(\tilde{\mathbb{H}}_b,\sigma)$ with (96) and $(\mathbb{D},\sigma)$ with (99) together, we will obtain a difference ring in which we can model any finite set of hypergeometric product expressions of finite nesting depth coming from $\mathrm{ProdE_n}(\mathbb{K}(x))$. Before we can complete this final argument, we have to take care that the two combined extensions yield again an RΠ-extension. Here we utilize the following result from Ocansey and Schneider (2018, Lemma 5.6) that holds for single nested RΠ-extensions.

Lemma 6.1. *Let $(\mathbb{K}(x),\sigma)$ be the rational difference field with $\sigma(x)=x+1$ and $(\mathbb{K}(x)\langle z_1\rangle\dots\langle z_s\rangle,\sigma)$ be a Π-extension of $(\mathbb{K}(x),\sigma)$ with $\frac{\sigma(z_k)}{z_k}\in\mathbb{K}[x]\setminus\mathbb{K}$. Further, let $\mathbb{K}'$ be an algebraic field extension of $\mathbb{K}$ and let $(\mathbb{K}'\langle y_1\rangle\dots\langle y_w\rangle,\sigma)$ be a Π-extension of $(\mathbb{K}',\sigma)$ with $\frac{\sigma(y_i)}{y_i}\in\mathbb{K}'\setminus\{0\}$. Then the difference ring $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{K}'(x)\langle y_1\rangle\dots\langle y_w\rangle\langle z_1\rangle\dots\langle z_s\rangle$ is a Π-extension of $(\mathbb{K}'(x),\sigma)$. Furthermore, the A-extension $(\mathbb{E}[\vartheta],\sigma)$ of $(\mathbb{E},\sigma)$ with $\sigma(\vartheta)=\zeta\,\vartheta$ of order λ is an R-extension.*

Namely, we can enhance the above lemma to nested RΠ-extensions.

Corollary 6.1. *Let $(\mathbb{K}(x),\sigma)$ be the rational difference field over $\mathbb{K}$ with $\sigma(x)=x+1$ and let the difference ring $(\tilde{\mathbb{H}}_b,\sigma)$ with (96) be an ordered multiple chain Π-extension of $(\mathbb{K}(x),\sigma)$ with the automorphism (97). Further, let $\tilde{\mathbb{K}}$ be an algebraic field extension of $\mathbb{K}$ and let the difference ring $(\tilde{\mathbb{A}}_d,\sigma)$ with (104) be the ordered multiple chain Π-extension of $(\tilde{\mathbb{K}},\sigma)$ with the automorphism (101). Then the difference ring $(\tilde{\mathbb{E}},\sigma)$ with*

$$\tilde{\mathbb{E}}=\tilde{\mathbb{K}}(x)\langle\tilde{\boldsymbol{y}}_{\mathbf{1}}\rangle\langle\tilde{\boldsymbol{z}}_{\mathbf{1}}\rangle\dots\langle\tilde{\boldsymbol{y}}_{\boldsymbol{\lambda}}\rangle\langle\tilde{\boldsymbol{z}}_{\boldsymbol{\lambda}}\rangle \tag{105}$$

with $\lambda=\max(b,d)$ where[12] *$\langle\tilde{\boldsymbol{y}}_{\boldsymbol{i}}\rangle=\langle\tilde{y}_{1,i}\rangle\dots\langle\tilde{y}_{e_i,i}\rangle$ and $\langle\tilde{\boldsymbol{z}}_{\boldsymbol{i}}\rangle=\langle\tilde{z}_{1,i}\rangle\dots\langle\tilde{z}_{p_i,i}\rangle$ for $1\le i\le\lambda$ is an ordered multiple chain Π-extension of $(\tilde{\mathbb{K}}(x),\sigma)$. Furthermore, the A-extension $(\mathbb{E},\sigma)$ of $(\tilde{\mathbb{E}},\sigma)$ where $\mathbb{E}=\tilde{\mathbb{E}}[\vartheta]$ with (100) of order λ is an R-extension.*

Proof. Take the Π-extensions $(\tilde{\mathbb{H}}_1,\sigma)$ of $(\mathbb{K}(x),\sigma)$ with $\tilde{\mathbb{H}}_1=\mathbb{K}(x)\langle z_{1,1}\rangle\dots\langle z_{p_1,1}\rangle$ and $(\tilde{\mathbb{A}}_1,\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ with $\tilde{\mathbb{A}}_1=\tilde{\mathbb{K}}\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle$ which are both of monomial depth 1. By Lemma 6.1 the difference ring $(\mathbb{E}_1,\sigma)$ with $\tilde{\mathbb{E}}_1=\tilde{\mathbb{K}}(x)\langle\tilde{y}_{1,1}\rangle\dots\langle\tilde{y}_{e_1,1}\rangle\langle\tilde{z}_{1,1}\rangle\dots\langle\tilde{z}_{p_1,1}\rangle$ is a Π-extension of $(\tilde{\mathbb{K}}(x),\sigma)$ of monomial depth 1. Consider the ordered multiple chain P-extension $(\tilde{\mathbb{E}},\sigma)$ of $(\tilde{\mathbb{K}}(x),\sigma)$ with (105) which is composed by the single chain Π-extensions in the ordered multiple chains $(\tilde{\mathbb{H}}_b,\sigma)$ and $(\tilde{\mathbb{A}}_d,\sigma)$. By Theorem 4.3 together with Lemma 4.3 it follows that $(\tilde{\mathbb{E}},\sigma)$ is a Π-extension of $(\tilde{\mathbb{K}}(x),\sigma)$. The quotient field of $\tilde{\mathbb{E}}$ gives the rational function field $\mathbb{H}=\mathbb{K}(x)(\tilde{\boldsymbol{y}}_{\mathbf{1}})(\tilde{\boldsymbol{z}}_{\mathbf{1}})\dots(\tilde{\boldsymbol{y}}_{\boldsymbol{\lambda}})(\tilde{\boldsymbol{z}}_{\boldsymbol{\lambda}})$ and one can extend the automorphism σ from $\tilde{\mathbb{E}}$ to $\mathbb{H}$ accordingly. Then by Lemma 4.3 ((2) $\Leftrightarrow$ (3)) it follows that $(\mathbb{H},\sigma)$ is a Π-field extension of $(\mathbb{K}(x),\sigma)$. In particular, $(\mathbb{H},\sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$. Thus by Lemma 4.1 the A-extension $(\mathbb{H}[\vartheta],\sigma)$ of $(\mathbb{H},\sigma)$ of order λ with the automorphism (100) is an R-extension. Therefore $\mathrm{const}(\mathbb{H}[\vartheta])=\mathbb{K}$ and with $\mathbb{K}\subseteq\tilde{\mathbb{E}}[\vartheta]\subseteq\mathbb{H}$ it follows that $\mathrm{const}(\tilde{\mathbb{E}}[\vartheta])=\mathbb{K}$. But this implies that the A-extension $(\tilde{\mathbb{E}}[\vartheta],\sigma)$ of $(\tilde{\mathbb{E}},\sigma)$ of order λ with the automorphism (100) is an R-extension. □

Finally, we arrive at the following main result.

Theorem 6.2. *Let $\mathbb{K}=K(\kappa_1,\dots,\kappa_u)$ be a rational function field, and let $(\mathbb{K}(x),\sigma)$ with $\sigma(x)=x+1$ be a rational difference field with the evaluation function $\mathrm{ev}:\mathbb{K}(x)\times\mathbb{Z}_{\ge0}\to\mathbb{K}$ defined by (28), and the Z-function given by (4). Suppose we are given a finite set of hypergeometric product expressions*

[12] Here we set $e_m=0$ and $p_n=0$ for all m,n with $d<m\le\lambda$ and $b<n\le\lambda$.

$$\{A_1(n), \ldots, A_m(n)\} \subseteq \mathrm{ProdE_n}(\mathbb{K}(x)) \tag{106}$$

of nesting depth at most d for some $d \in \mathbb{Z}_{\geq 0}$. Then there is a $\nu \in \mathbb{Z}_{\geq 0}$ and an $\mathrm{R}\Pi$-extension $(\mathbb{E}, \sigma)$ of $(\tilde{\mathbb{K}}(x), \sigma)$ of monomial depth at most d where $\tilde{\mathbb{K}}$ is a finite algebraic field extension of $\mathbb{K}$ equipped with an evaluation function $\tilde{\mathrm{ev}} : \mathbb{E} \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ with the following properties:

(1) The map $\tau : \mathbb{E} \to \mathcal{S}(\tilde{\mathbb{K}})$ with $\tau(f) = \langle \tilde{\mathrm{ev}}(f, n)\rangle_{n\geq 0}$ is a $\tilde{\mathbb{K}}$-embedding.
(2) There are elements $a_1, \ldots, a_m \in \mathbb{E}^*$ such that for j with $1 \leq j \leq m$ and for all $n \geq \nu$ we have

$$A_j(n) = \tilde{\mathrm{ev}}(a_j, n);$$

here no pole evaluations occur within $\tilde{\mathrm{ev}}(a_j, n)$ when one evaluates the rational functions from $\tilde{\mathbb{K}}(x)$, i.e., one does not enter in the first case of (28).

If K is a strongly σ-computable, such a ν, $(\mathbb{E}, \sigma)$ with $\tilde{\mathrm{ev}}$, and the $a_1, \ldots, a_m \in \mathbb{E}$ can be computed.

Proof. (a) We are given the hypergeometric product expressions in (106) where

$$A_j(n) = \sum_{\boldsymbol{\nu}=(\nu_1,\ldots,\nu_e)\in S_j} a_{j,\boldsymbol{\nu}}(n)\, P_1(n)^{\nu_1} \cdots P_e(n)^{\nu_e}$$

with $S_j \subseteq \mathbb{Z}^e$ finite, $a_{j,\boldsymbol{\nu}}(x) \in \mathbb{K}(x)$ and $P_1(n), \ldots, P_e(n) \in \mathrm{Prod_n}(\mathbb{K}(x))$. Now we follow the construction in Proposition 2.2, i.e., execute Algorithm 2.2. There we can take a $\delta \in \mathbb{Z}_{\geq 0}$ and construct for all $1 \leq j \leq e$, $c_j \in \mathbb{K}^*$, rational functions $\tilde{r}_j \in \mathbb{K}(x)^*$, 1-refined geometric product expressions $\tilde{G}_j(n) \in \mathrm{ProdM_n}(\mathbb{K})$ and δ-refined hypergeometric product expressions in shift-coprime product representation form $\tilde{H}_j(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ such that

$$P_j(n) = \tilde{r}_j(n)\, \tilde{G}_j(n)\, \tilde{H}_j(n) \neq 0 \tag{107}$$

holds for all $n \geq \max(0, \delta - 1)$.

(b) For the hypergeometric product expressions $\tilde{H}_1(n), \ldots, \tilde{H}_e(n)$ in (107) we have

$$\tilde{H}_i(n) = \tilde{H}_{i,1}(n)^{n_{i,1}} \cdots \tilde{H}_{i,l_i}(n)^{n_{i,l_i}}$$

for some $l_i \in \mathbb{Z}_{\geq 0}$ with $n_{i,j} \in \mathbb{Z}$ for $1 \leq j \leq l_i$ where all the arising hypergeometric products $\tilde{H}_{i,j}(n)$ are δ-refined and in shift-coprime product representation form. By Corollary 5.1 we can construct an ordered multiple chain Π-extension $(\tilde{\mathbb{H}}_b, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with (96) which is composed by the single chain Π-extensions $(\tilde{\mathbb{F}}_\ell, \sigma)$ of $(\mathbb{K}(x), \sigma)$ with $1 \leq \ell \leq p_1$ for some $p_1 \in \mathbb{Z}_{\geq 0}$ with $\tilde{\mathbb{F}}_\ell = \mathbb{K}(x)\langle \tilde{z}_{\ell,1}\rangle\langle \tilde{z}_{\ell,2}\rangle \ldots \langle \tilde{z}_{\ell,s_\ell}\rangle$, the automorphism $\sigma : \tilde{\mathbb{F}}_\ell \to \tilde{\mathbb{F}}_\ell$ given in (97) and the evaluation function $\tilde{\mathrm{ev}} : \tilde{\mathbb{F}}_\ell \times \mathbb{Z}_{\geq 0} \to \mathbb{K}$ defined by $\tilde{\mathrm{ev}}|_{\mathbb{K}(x)\times\mathbb{Z}_{\geq 0}} = \mathrm{ev}$ and (98). In particular, there are $\nu_{i,j}, \mu_{i,j}$ such that $\tilde{H}_{i,j}(n) = \tilde{\mathrm{ev}}(\tilde{z}_{\nu_{i,j},\mu_{i,j}}, n)$ holds for all $n \geq \max(0, \delta - 1)$. Thus we can take $\tilde{h}_i = \tilde{z}^{n_{i,1}}_{\nu_{i,1},\mu_{i,1}} \ldots \tilde{z}^{n_{i,l_i}}_{\nu_{i,l_i},\mu_{i,l_i}} \in \tilde{\mathbb{H}}_b$ with

$$\tilde{\mathrm{ev}}(\tilde{h}_j, n) = \tilde{H}_j(n) \quad \forall n \geq \delta. \tag{108}$$

(c) Next we treat the geometric product expressions $\tilde{G}_1(n), \ldots, \tilde{G}_e(n)$ in (107). Following Algorithm 3.1 we can construct a multiple chain P-extension $(\mathbb{A}, \sigma)$ of $(\mathbb{K}, \sigma)$ where the bases are from $\mathbb{K}^*$ such that there are $g_1, \ldots, g_e \in \mathbb{A}$ with $\mathrm{ev}(g_i, n) = \tilde{G}_i(n)$ for all $n \in \mathbb{Z}_{\geq 0}$. Then by Theorem 6.1 we can construct an $\mathrm{R}\Pi$-extension $(\mathbb{D}, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with (99) together with the automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ given in (100) and (101), with the evaluation function $\tilde{\mathrm{ev}} : \mathbb{D} \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ defined by $\tilde{\mathrm{ev}}(c, n) = c$ for all $c \in \tilde{K}$, $n \in \mathbb{Z}_{\geq 0}$, (102) and (103), and with a difference ring homomorphism $\varphi : \mathbb{A}_d \to \mathbb{D}$ s.t. for all $f \in \mathbb{A}$ and $n \in \mathbb{Z}_{\geq 0}$ we have (87). Thus for $\tilde{g}_j := \varphi(g_j)$ with $1 \leq j \leq e$ we get

$$\tilde{\mathrm{ev}}(\tilde{g}_j, n) = \tilde{ev}(\varphi(g_j), n) \stackrel{(87)}{=} \mathrm{ev}(g_j, n) = \tilde{G}_j(n) \quad \forall n \geq 0. \tag{109}$$

(d) By Corollary 6.1 we can merge these two difference rings to obtain an RΠ-extension $(\mathbb{E},\sigma)$ of $(\tilde{\mathbb{K}}(x),\sigma)$ with $\mathbb{E}=\tilde{\mathbb{E}}[\vartheta]$ and (105) equipped with the automorphism $\sigma:\mathbb{E}\to\mathbb{E}$ and the evaluation function $\tilde{\mathrm{ev}}:\mathbb{E}\times\mathbb{Z}_{\geq 0}\to\tilde{\mathbb{K}}$ defined accordingly. As $(\mathbb{E},\sigma)$ is an RΠ-extension of the rational difference field $(\tilde{\mathbb{K}}(x),\sigma)$, it follows by Theorem 4.2 that $\tilde{\tau}:\mathbb{E}\to\mathcal{S}(\tilde{\mathbb{K}})$ defined by (47) is a $\tilde{\mathbb{K}}$-embedding. For $1\leq j\leq e$, define

$$q_j:=\tilde{r}_j\,\tilde{g}_j\,\tilde{h}_j\in\mathbb{E}.$$

With $\tilde{\mathrm{ev}}(\tilde{r}_j,n)=\tilde{r}_j(n)$ and the evaluations of $\tilde{h}_j$ and $\tilde{g}_j$ given in (108) and (109) together with (107) it follows that for all $1\leq j\leq e$ and for all $n\geq\max(0,\delta-1)$ we have

$$P_j(n)=\tilde{r}_j(n)\,\tilde{G}_j(n)\,\tilde{H}_j(n)=\tilde{\mathrm{ev}}(\tilde{r}_j,n)\,\tilde{\mathrm{ev}}(\tilde{g}_j,n)\,\tilde{\mathrm{ev}}(\tilde{h}_j,n)=\tilde{\mathrm{ev}}(\tilde{r}_j\,\tilde{g}_j\,\tilde{h}_j,n)=\tilde{\mathrm{ev}}(q_j,n).$$

(e) Finally, we can define $a_j=\sum_{\boldsymbol{v}=(\nu_1,\dots,\nu_e)\in S_j}a_{j,\boldsymbol{v}}\,q_1^{\nu_1}\cdots q_e^{\nu_e}\in\mathbb{E}$ for $1\leq j\leq m$. In addition, with $a_{j,\boldsymbol{v}}=\frac{\alpha_{j,\boldsymbol{v}}}{\beta_{j,\boldsymbol{v}}}\in\mathbb{K}(x)$ where $\alpha_{j,\boldsymbol{v}},\beta_{j,\boldsymbol{v}}\in\mathbb{K}[x]$ we define $\lambda:=\max_{1\leq j\leq m,\boldsymbol{v}\in S_j}(Z(\beta_{j,\boldsymbol{v}}))$, i.e., the evaluation of $a_{j,\boldsymbol{v}}(n)$ with $n\in\mathbb{Z}_{\geq 0}$ and $n\geq\mu$ does not yield pole evaluations. Thus taking $\nu=\max(\delta-1,\lambda,0)$ we get $\mathrm{ev}(a_j,n)=A_j(n)$ for all $n\in\mathbb{Z}_{\geq 0}$ with $n\geq\nu$ as desired.

If $\mathbb{K}$ is strongly σ-computable, all the ingredients delivered by Corollary 5.1 and Theorem 6.1 can be computed. In particular, $(\mathbb{E},\sigma)$, τ with $\tilde{\mathrm{ev}}$, $\nu\in\mathbb{Z}_{\geq 0}$ and $a_1,\dots,a_m\in\mathbb{E}$ can be computed explicitly. □

As a consequence we are now in the position to solve Problem RPE as follows.

Corollary 6.2. *Let $A(n)\in\mathrm{ProdE_n}(\mathbb{K}(x))$ with (3). For $A_1(n)=A(n)$ with $m=1$ let $\nu\in\mathbb{Z}_{\geq 0}$, $(\mathbb{E},\sigma)$ with the evaluation function $\tilde{\mathrm{ev}}$ and the $a:=a_1\in\mathbb{E}$ be the ingredients as provided in Theorem 6.2. In particular, let $\mathbb{E}=\tilde{\mathbb{K}}(x)[\vartheta]\langle p_1\rangle\dots\langle p_s\rangle$ where ϑ is the R-monomial with $\sigma(\vartheta)=\zeta\,\vartheta$ and let $p_1,\dots,p_s$ be the Π-monomials. Furthermore, let $a=\sum_{\boldsymbol{v}=(\nu_0,\dots,\nu_s)\in\tilde{S}}b_{\boldsymbol{v}}(n)\,\vartheta^{\mu_0}\,p_1^{\mu_1}\cdots p_s^{\mu_s}$ with $\tilde{S}\subseteq\{0,\dots,\lambda-1\}\times\mathbb{Z}^s$ finite and $b_{\boldsymbol{v}}(x)\in\tilde{\mathbb{K}}(x)$ for $\boldsymbol{v}\in\tilde{S}$. Then the following holds:*

(1) $\mathrm{ev}(\vartheta,n)=\zeta^n$ for all $n\in\mathbb{Z}_{\geq 0}$; furthermore, for $1\leq i\leq s$ we have $\mathrm{ev}(p_i,n)=Q_i(n)$ for all $n\geq 0$ for some explicitly given $Q_i(n)\in\mathrm{Prod_n}(\tilde{\mathbb{K}}(x))$.

(2) $A(n)=B(n)$ for all $n\geq\nu$ with $B(n)=\sum_{\boldsymbol{v}=(\nu_0,\dots,\nu_s)\in\tilde{S}}b_{\boldsymbol{v}}(n)\,(\zeta^n)^{\mu_0}\,Q_1(n)^{\mu_1}\cdots Q_s(n)^{\mu_s}\in\mathrm{ProdE_n}(\tilde{\mathbb{K}}(x))$.

(3) The subring $\mathbb{S}=\tau(\tilde{\mathbb{K}}(x))[\langle\zeta^n\rangle_{n\geq 0}][\langle Q_1(n)\rangle_{n\geq 0},\langle Q_1(n)^{-1}\rangle_{n\geq 0}]\dots[\langle Q_s(n)\rangle_{n\geq 0},\langle Q_s(n)^{-1}\rangle_{n\geq 0}]$ of $\mathcal{S}(\tilde{\mathbb{K}})$ forms a Laurent polynomial ring extension of $\tau(\tilde{\mathbb{K}}(x))[\langle\zeta^n\rangle_{n\geq 0}]$. Thus the sequences produced by $Q_1(n),\dots,Q_s(n)$ are algebraically independent among each other over $\tau(\tilde{\mathbb{K}}(x))[\langle\zeta^n\rangle_{n\geq 0}]$.

(4) We have that $A(n)=0$ for all $n\geq d$ for some $d\in\mathbb{Z}_{\geq 0}$ if and only if $a=0$ if and only if $B(n)$ is the zero-expression. If this holds, $A(n)=0$ for all $n\geq\nu$.

Proof. (1) Note that $(\mathbb{E},\sigma)$ is a multiple chain Π-extension of $(\tilde{\mathbb{K}}(x)[\vartheta],\sigma)$ over $\tilde{\mathbb{K}}(x)$ equipped by an evaluation function given by the iterative application of Lemma 3.1. Thus statement (1) follows.

(2) By statement (2) of Theorem 6.2 we have $\tilde{\mathrm{ev}}(a,n)=A(n)$ for all $n\geq\nu$. By definition of the $\tilde{\mathrm{ev}}$ function we get $\tilde{ev}(a,n)=B(n)$ for all $n\geq\nu$; note that no pole evaluations arise in $b_{\boldsymbol{v}}(n)\in\tilde{\mathbb{K}}(x)$ of $B(n)$. Thus statement (2) holds.

(3) Since $\tau:\mathbb{E}\to\mathcal{S}(\tilde{\mathbb{K}})$ with $\tau(f)=\langle\tilde{\mathrm{ev}}(f,n)\rangle_{n\geq 0}$ is an injective difference ring homomorphism by statement (1) of Theorem 6.2, the ring $\mathbb{E}$ is isomorphic to $\tau(\mathbb{E})=\mathbb{S}$ and statement (3) follows.

(4) Since τ is injective, it follows that

$$A(n)=0 \text{ for all } n\geq d \text{ for some } d\in\mathbb{Z}_{\geq 0} \overset{\text{item (2)}}{\iff} 0=B(n)=\tilde{\mathrm{ev}}(a,n) \text{ for all } n\geq \max(d,\nu)$$
$$\iff \tau(a)=\mathbf{0}$$
$$\overset{\tau \text{ injective}}{\iff} a=0$$
$$\iff B(n) \text{ is the zero-expression.}$$

If this is the case, $A(n)=B(n)=0$ for all $n\geq\nu$. This proves the last statement. □

The above constructions in Corollary 6.2 (using Theorem 6.2 as a special case) can be summarized in Algorithm 6.1 and illustrated in Example 6.2 below.

Algorithm 6.1. (Representation of product expressions, Problem RPE)
Input: a hypergeometric product expression $A(n)\in\mathrm{ProdE_n}(\mathbb{K}(x))$ in the form (3) where $\mathbb{K}=K(\kappa_1,\dots,\kappa_u)$ is a rational function field over a strongly σ-computable field K.
Output: $\nu\in\mathbb{Z}_{\geq 0}$ and $B(n)\in\mathrm{ProdE_n}(\tilde{\mathbb{K}}(x))$ as given in Corollary 6.2 that solves Problem RPE; here $\tilde{\mathbb{K}}=\tilde{K}(\kappa_1,\dots,\kappa_u)$ where $\tilde{K}$ is an algebraic field extension of K.

(1) Execute Algorithm 2.1 with the input $P_1(n)\dots,P_e(n)\in\mathrm{Prod_n}(\mathbb{K}(x))$ of the arising hypergeometric products in $A(n)$. As output we get (107) in factored form where in $\tilde{G}_i(n)$ the geometric products and in $\tilde{H}_i(n)$ the hypergeometric shift-coprime products are collected.
(2) Apply Corollary 5.1 to the hypergeometric products in $\tilde{H}_i(n)$ and compute $\delta\in\mathbb{Z}_{\geq 0}$ and a multiple chain Π-extension $(\mathbb{E}_H,\sigma)$ of $(\mathbb{K}(x),\sigma)$ over $\mathbb{K}(x)$ with $\mathbb{E}_H=\mathbb{K}(x)\langle z_1\rangle\dots\langle z_r\rangle$ equipped with an evaluation function $\mathrm{ev}:\mathbb{E}_H\times\mathbb{Z}_{\geq 0}\to\mathbb{K}$ together with $\tilde{h}_i\in\mathbb{E}_H$ for $1\leq i\leq e$ such that $\mathrm{ev}(\tilde{h}_i,n)=\tilde{H}_i(n)$ for all $n\geq\delta$; internally also Algorithm 3.1 is used.
(3) Execute Algorithm 3.1 to compute a multiple chain P-extension $(\mathbb{E}_G,\sigma)$ of $(\mathbb{K},\sigma)$ over $\mathbb{K}$ equipped with an evaluation function $\mathrm{ev}:\mathbb{E}_G\times\mathbb{Z}_{\geq 0}\to\mathbb{K}$ together with $g_i\in\mathbb{E}_G$ for $1\leq i\leq e$ such that $\mathrm{ev}(g_i,n)=\tilde{G}_i(n)$ for all $n\geq 1$.
(4) Apply Theorem 6.1 to $(\mathbb{E}_G,\sigma)$ to get a multiple chain Π-extension $(\tilde{\mathbb{E}}_G,\sigma)$ of $(\tilde{\mathbb{K}},\sigma)$ over $\tilde{\mathbb{K}}$ with $\tilde{\mathbb{E}}_G=\tilde{\mathbb{K}}\langle y_1\rangle\dots\langle y_\rho\rangle$ and an R-extension $(\tilde{\mathbb{E}}_G[\vartheta],\sigma)$ of $(\tilde{\mathbb{E}},\sigma)$ with $\sigma(\vartheta)/\vartheta=\zeta\in\tilde{\mathbb{K}}^*$ a primitive root of unity. In addition, one gets an evaluation function $\mathrm{ev}:\tilde{\mathbb{E}}_G[\vartheta]\times\mathbb{Z}_{\geq 0}\to\tilde{\mathbb{K}}$ with $\tilde{g}_i\in\tilde{\mathbb{E}}_G[\vartheta]$ for $1\leq i\leq e$ such that $\mathrm{ev}(\tilde{g}_i,n)=\tilde{G}_i(n)$ for all $n\geq 1$.
(5) Merge $(\mathbb{E}_H,\sigma)$ and $(\tilde{\mathbb{E}}_G[\vartheta],\sigma)$ to get the RΠ-ext. $(\mathbb{E},\sigma)$ of $(\tilde{\mathbb{K}}(x),\sigma)$ with $\mathbb{E}=\tilde{\mathbb{K}}(x)[\vartheta]\langle p_1\rangle\dots\langle p_s\rangle$ where $s=r+\rho$ and $(p_1,\dots,p_s)=(z_1,\dots,z_r,y_1,\dots,y_\rho)$ equipped with $\mathrm{ev}:\mathbb{E}\times\mathbb{Z}_{\geq 0}\to\tilde{\mathbb{K}}$ in which we get $\mathrm{ev}(q_i,n)=P_i(n)$ with $q_i=\tilde{r}_j\,\tilde{g}_i\,\tilde{h}_i$ for all $n\geq\delta$ and all $1\leq i\leq e$. In addition, one gets products $Q_i(n)\in\mathrm{Prod_n}(\tilde{\mathbb{K}}(x))$ in factored form with $\mathrm{ev}(p_i,n)=Q_i(n)$ for all $n\geq\delta$.
(6) Define $a=\sum_{\boldsymbol{\nu}=(\nu_1,\dots,\nu_e)\in S} a_{\boldsymbol{\nu}}\,q_1^{\nu_1}\cdots q_e^{\nu_e}$ for given $a_{\boldsymbol{\nu}}=\frac{\alpha_{\boldsymbol{\nu}}}{\beta_{\boldsymbol{\nu}}}$ with $\alpha_{\boldsymbol{\nu}},\beta_{\boldsymbol{\nu}}\in\mathbb{K}[x]$. Compute $\lambda=\max_{\boldsymbol{\nu}\in S}Z(\beta_{\boldsymbol{\nu}})$.
Write $a=\sum_{\boldsymbol{\nu}=(\nu_0,\dots,\nu_s)\in\tilde{S}} b_{\boldsymbol{\nu}}(n)\,\vartheta^{\mu_0}p_1^{\mu_1}\cdots p_s^{\mu_s}\in\mathbb{E}$ with $\tilde{S}\subseteq\{0,\dots,\lambda-1\}\times\mathbb{Z}^s$ finite and $b_{\boldsymbol{\nu}}(x)=\frac{\tilde{\alpha}_{\boldsymbol{\nu}}}{\tilde{\beta}_{\boldsymbol{\nu}}}$ with $\tilde{\alpha}_{\boldsymbol{\nu}},\tilde{\beta}_{\boldsymbol{\nu}}\in\tilde{\mathbb{K}}[x]$ for $\boldsymbol{\nu}\in\tilde{S}$; note that $\lambda\geq\max_{\boldsymbol{\nu}\in\tilde{S}}Z(\tilde{\beta}_{\boldsymbol{\nu}})$.
(7) Set $\nu=\max(0,\delta-1,\lambda)$; note that $\mathrm{ev}(a,n)=A(n)$ for all $n\geq\nu$.
(8) Define $B(n)=\sum_{\boldsymbol{\nu}=(\nu_0,\dots,\nu_s)\in\tilde{S}} b_{\boldsymbol{\nu}}(n)\,\vartheta^{\mu_0}Q_1^{\mu_1}\cdots Q_s(n)^{\mu_s}\in\mathrm{ProdE_n}(\tilde{\mathbb{K}(x)})$; note that $A(n)=B(n)$ for all $n\geq\nu$.
(9) return ν and $B(n)$.

Example 6.2 *(Cont. Examples 2.1, 2.4, 3.1, 5.1, 5.2).* We activate Algorithm 6.1 with $A(n)=P(n)$ for the given nesting depth 2 hypergeometric product (9) in Example 2.1.
Step 1: Let $(\mathbb{K}(x),\sigma)$ be the rational difference field with $\mathbb{K}=\mathbb{Q}(\sqrt{3})$ equipped with the field automorphism $\sigma:\mathbb{K}(x)\to\mathbb{K}(x)$ and the evaluation function $\mathrm{ev}:\mathbb{K}(x)\times\mathbb{Z}_{\geq 0}\to\mathbb{K}$ defined by $\sigma(x)=x+1$

and (28) respectively. In Example 2.4 we computed the δ-refined ($\delta = 3$) shift-coprime product expression $\tilde{P}(n) = \tilde{r}(n)\,\tilde{G}(n)\,\tilde{H}(n) \in \mathrm{ProdM_n}(\mathbb{K}(x))$ where $\tilde{r}(n)$, $\tilde{G}(n)$ and $\tilde{H}(n)$ are given in (23), (24), and (25) respectively such that $A(n) = P(n) = \tilde{P}(n)$ holds for all $n \in \mathbb{Z}_{\geq 0}$ with $n \geq \delta - 1 = 2$.
Step 2: In Example 5.1 we constructed the ordered multiple chain Π-extension $(\mathbb{E}_H, \sigma)$ of $(\mathbb{K}(x), \sigma)$ over $\mathbb{K}(x)$ with $\mathbb{E}_H = \tilde{\mathbb{H}} = \mathbb{K}(x)\langle z_{1,1}\rangle\langle z_{2,1}\rangle\langle z_{1,2}\rangle$ which is composed by the single chain Π-extensions of $(\mathbb{K}(x), \sigma)$ defined in items (7) and (8). The automorphism and the evaluation function were defined as given in (36) and (37). In particular, the hypergeometric product expression $\tilde{H}(n)$ is modeled by the expression $\tilde{h} = {z_{1,1}}^3\, z_{2,1}\, z_{1,2}$ where $\tilde{h} = h$ is taken from (39).
Step 3 and 4: Furthermore, we constructed the RΠ-extension $(\tilde{\mathbb{E}}_G, \sigma)$ of $(\tilde{\mathbb{K}}, \sigma)$ with $\tilde{\mathbb{E}}_G = \mathbb{D}$ and $\tilde{\mathbb{K}} = \mathbb{Q}(\sqrt{3}, \mathbb{i})$ equipped with the evaluation function $\tilde{\mathrm{ev}}$ from Ex. 6.1. There $\tilde{G}(n)$ is modeled by (95).
Step 5: Merging the two difference rings $(\mathbb{E}_H, \sigma)$ and $(\tilde{\mathbb{E}}_G, \sigma)$, we get the R$\Pi$-extension $(\mathbb{E}, \sigma)$ of $(\tilde{\mathbb{K}}(x), \sigma)$ with $\mathbb{E} = \tilde{\mathbb{K}}(x)[\vartheta]\langle\tilde{y}_{1,1}\rangle\langle\tilde{y}_{2,1}\rangle\langle\tilde{y}_{3,1}\rangle\langle z_{1,1}\rangle\langle z_{2,1}\rangle\langle\tilde{y}_{2,2}\rangle\langle\tilde{y}_{3,2}\rangle\langle z_{1,2}\rangle$ where the automorphism $\sigma : \mathbb{E} \to \mathbb{E}$ and the evaluation function $\tilde{\mathrm{ev}} : \mathbb{E} \times \mathbb{Z}_{\geq 0} \to \tilde{\mathbb{K}}$ are defined by (79), (63), (36), and (37). Following the proof of Theorem 6.2 we set $q := \tilde{r}\,\tilde{g}\,\tilde{h}$ and get for all $n \geq 2$ the simplification

$$P(n) = \tilde{\mathrm{ev}}(q, n) = -\frac{254}{432}\,(n-1)^3\, n\,(n+1)\,(n+2)\,\frac{1}{2}\,\frac{(1-\mathbb{i})\,(\mathbb{i})^n\,(\mathbb{i}\,(\mathbb{i}^n)^2+1)\,\left(\sqrt{3}\right)^n\,(5^n)^2\,2^{\binom{n+1}{2}}}{2^n\,5^{\binom{n+1}{2}}}$$
$$\left(\prod_{k=3}^{n}(k-2)\right)^3\left(\prod_{k=3}^{n}\left(k+\tfrac{1}{24}\right)\right)\left(\prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)\right). \tag{110}$$

Step 6,7,8: We return $B(n)$ as the right-hand side of (110) with $\nu = 2$.

Based on this representation in an RΠ-extension, we can extract the following extra property. Since $\tau : \mathbb{E} \to \mathcal{S}(\tilde{\mathbb{K}})$ is a $\tilde{\mathbb{K}}$-embedding, it follows that the sub-difference ring (R, S) of $(\mathcal{S}(\tilde{\mathbb{K}}), S)$ with

$$R = \tau(\tilde{\mathbb{K}}(x))[\langle\mathbb{i}^n\rangle_{n\geq 0}][\langle Q_1(n)\rangle_{n\geq 0}, \langle Q_1(n)^{-1}\rangle_{n\geq 0}]\ldots[\langle Q_8(n)\rangle_{n\geq 0}, \langle Q_8(n)^{-1}\rangle_{n\geq 0}]$$

and

$$Q_1(n) = \left(\sqrt{3}\right)^n, \qquad Q_2(n) = 2^n, \qquad Q_3(n) = 5^n, \qquad Q_4(n) = \prod_{k=3}^{n}(k-2),$$
$$Q_5(n) = \prod_{k=3}^{n}\left(k+\tfrac{1}{24}\right), \quad Q_6(n) = 2^{\binom{n+1}{2}}, \quad Q_7(n) = 5^{\binom{n+1}{2}}, \quad Q_8(n) = \prod_{k=3}^{n}\prod_{j=3}^{k}(j-2)$$

is isomorphic to $(\mathbb{E}, \sigma)$. In particular, we can conclude that R is a Laurent polynomial ring extension of the ring $G = \tau(\tilde{\mathbb{K}}(x))[\langle\mathbb{i}^n\rangle_{n\geq 0}]$. In a nutshell, the sequences generated by the products $Q_1(n), \ldots, Q_8(n)$ are algebraically independent among each other over the ring G.

Remark 6.1. Some remarks are appropriate concerning the efficiency of the proposed toolbox. In general, our algorithms rely on stable and efficient algorithms that carry out the field operations in the given field $\mathbb{K}$. In most applications coming, e.g., from combinatorial problems (Krattenthaler, 2001; Mills et al., 1983; Zeilberger, 1996) the ground field is $\mathbb{K} = \mathbb{Q}$ in which all algorithms are efficiently available within the existing computer algebra systems. In general, our algorithms work in a rational function field $\mathbb{K} = K(\kappa_1, \ldots, \kappa_u)$ over a strongly σ-computable field K; see Definition 5.2. In particular, K can be an algebraic number field, i.e., it is built by an algebraic field extension over $\mathbb{Q}$; see Lemma 5.1. Already here most computer algebra systems slow down to perform the standard field operations. In the light of this general observation the following extra challenges arise in Algorithm 6.1.
• In Step 1 one has to carry out Algorithm 2.1 where complete factorizations (7) of the multiplicands of the products (5) have to be calculated. In addition, one has to compute the dispersions of these

factors in case that they are shift-equivalent. For complicated fields $\mathbb{K}$ (i.e., the number of variables u is large and/or the defining irreducible polynomial for the algebraic field extension is big) the calculation of such a factorization can be a serious bottleneck.

• In Step 4 one has to carry out the algorithmic machinery described in the proof of Theorem 6.1 that depends on the algorithms hidden in Lemma 5.2. Internally one may compute several times a basis of the highly nontrivial Problem GO (see page 23). Utilizing Ge's algorithm (Ge, 1993a) (see also Kauers, 2005, Algorithm 7.16, page 84) one has to execute the LLL-algorithm (Lenstra et al., 1982) in a nontrivial way. As a consequence, if too many geometric products arise over different algebraic numbers, the calculation might slow down significantly.

Besides the challenges related to the ground field $\mathbb{K}$, another intrinsic problem arises.

• When we compute a shift-coprime representation of the products in Step 1 (see Algorithm 2.1) the expression swell might be enormous and can slow down the rewrite-rules executed in Algorithm 2.1. This can be seen already with the simple example $n! + (n+20)!$. Representing this object in a difference ring built by an RΠ-extension in terms of $n!$, one gets

$$\begin{aligned}
&n!\big(n^{20}+210n^{19}+20615n^{18}+1256850n^{17}+53327946n^{16}+1672280820n^{15}\\
&+40171771630n^{14}+756111184500n^{13}+11310276995381n^{12}+135585182899530n^{11}\\
&+1307535010540395n^{10}+10142299865511450n^{9}+63030812099294896n^{8}\\
&+311333643161390640n^{7}+1206647803780373360n^{6}+3599979517947607200n^{5}\\
&+8037811822645051776n^{4}+12870931245150988800n^{3}+13803759753640704000n^{2}\\
&+8752948036761600000n+2432902008176640001\big);
\end{aligned}$$

note that any other choice $(n+i)!$ with $i \in \mathbb{Z}$ would yield a similar large expression. This gets even worse, if one deals with nested products. E.g., representing $(n!)^6 \prod_{i=1}^{n} i! + \prod_{i=1}^{n} (6+i)!$ within an RΠ-extension in terms of $n!$ and $\prod_{i=1}^{n} i!$ gives

$$\begin{aligned}
&\frac{1}{24883200}(n!)^6\Big(\prod_{i=1}^{n} i!\Big)\Big(49766400+277447680n+1447365888n^2+4697205696n^3\\
&+10637255232n^4+17872240112n^5+23125043824n^6+23608674132n^7+19327952588n^8\\
&+12825741073n^9+6944399940n^{10}+3077867318n^{11}+1116881584n^{12}+330867999n^{13}\\
&+79518296n^{14}+15343384n^{15}+2339340n^{16}+275135n^{17}+24052n^{18}+1470n^{19}\\
&+56n^{20}+n^{21}\Big).
\end{aligned}$$

While this problem cannot be avoided in general, appropriate choices of the basis elements (in our setting we always chose the leftmost representant of the shift-equivalent factors) might lead to better representations. Similarly, one may avoid full factorizations and allow products with multiplicands that are not irreducible by using ideas as described in Schneider (2020). In this regard also the improvements in Abramov et al. (2003) introduced only for one single nested hypergeometric product could be explored further.

6.3. *The Mathematica package —* **`NestedProducts`**

In the following we will demonstrate how our tools can be activated with the help of the Mathematica package **`NestedProduct`**. We start with the nested hypergeometric product expression

$$A(n)=\frac{1}{2}\prod_{k=1}^{n-1}\frac{1}{36}\left(\prod_{i=1}^{k-1}\frac{(i+1)\,(i+2)}{4\,(2\,i+3)^2}\right)\in \mathrm{ProdE_n}(\mathbb{Q}(x)) \tag{111}$$

from Kauers (2018, Example 3) which was guessed using the Mathematica package RATE written by Christian Krattenthaler; see Krattenthaler (1997).

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After loading the package

In[1]:= **«NestedProducts.m**

NestedProducts — A package by Evans Doe Ocansey — © RISC

into Mathematica, we define the product with the command

In[2]:= $\mathbf{A=\frac{1}{2}FormalProduct\Big[\frac{1}{36}\Big(FormalProduct\Big[\frac{(i+1)\,(i+2)}{4\,(2i+3)^2},\{i,1,k-1\}\Big]\Big),\{k,1,n-1\}\Big];}$

Here `FormalProduct`[f,{k,a,n+b}] (as shortcut one can use `FProduct`) defines a nested product $\prod_{k=a}^{n+b} f$ where a,b are integers and the multiplicand f, free of n, must be an expression in terms of nested products whose outermost upper bounds are given by k or an integer shift of k. Then applying the command `ProductReduce` to `A` we solve Problem RPE in 0.19 seconds:

In[3]:= **B = ProductReduce[A]**

Out[3]= $$\frac{9\,(2^n)^5\left(\prod_{k=1}^{n}(k+\frac{3}{2})\right)^4\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(i+1)\right)^2}{(2\,n+3)^2(3^n)^2\left(2^{\binom{n+1}{2}}\right)^4\left(\prod_{k=1}^{n}(k+1)\right)^3\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(i+\frac{3}{2})\right)^2}$$

Internally, the package synchronizes in (111) the upper bounds to n and k, respectively. This yields

$$\frac{9\,(n+1)\,(n+2)}{2\,(2n+3)}\left(\prod_{j=1}^{n}\frac{4\,(2\,j+3)^2}{(j+1)\,(j+2)}\right)\left(\prod_{k=1}^{n}\frac{(2\,k+3)^2}{9\,(k+1)\,(k+2)}\left(\prod_{j=1}^{k}\frac{(j+1)\,(j+2)}{4\,(2\,j+3)^2}\right)\right).$$

Then this is reduced further to Out[3] in terms of the algebraically independent products

$$\begin{gathered}2^n,\qquad 3^n,\qquad \prod_{k=1}^{n}(k+\tfrac{3}{2}),\qquad \prod_{k=1}^{n}(k+1),\qquad 2^{\binom{n+1}{2}}=\prod_{k=1}^{n}\prod_{i=1}^{k}2,\\ \prod_{k=1}^{n}\prod_{i=1}^{k}(i+\tfrac{3}{2}),\qquad \prod_{k=1}^{n}\prod_{i=1}^{k}(i+1).\end{gathered}\tag{112}$$

Note that one could have represented the product in (111) directly within a Π-extension by simply taking the inner product as the first Π-monomial and the outermost product as the second Π-monomial. However, for more complicated expression such a representation can be rather challenging.

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The full capability of our machinery can be illustrated by combining, e.g., the expression (111) with the following related product (where one of the inner products is slightly modified):

In[4]:= $\mathbf{A_2 = A + FProduct\Big[\frac{4(3+2k)^4}{(k+1)^2(2k+1)^4(k+2)^2}FProduct[\frac{-(i+1)(i+2)}{4(2i-1)^2}, \{i,1,k\}], \{k,1,n\}\Big];}$

Then it is not immediate how this expression can be represented in an RΠ-extension. But applying our toolbox this task can be automatically accomplished in 0.22 seconds:

In[5]:= $\mathbf{B_2 = ProductReduce[A_2]}$

Out[5]= $$\frac{\left(81\left(n^2+3n+2\right)+(1+\mathbb{i})\,(2n+3)^4\,(\mathbb{i})^n+(1-\mathbb{i})\,(2n+3)^4\,((\mathbb{i})^n)^3\right)2^n\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(i+1)\right)^2}{81\,(n+2)\left(2^{\binom{n+1}{2}}\right)^4\left(\prod_{k=1}^{n}(k+1)\right)^3\left(\prod_{k=1}^{n}\prod_{i=1}^{k}(i-\frac{1}{2})\right)^2}$$

By solving Problem RPE the expression can be rephrased in an RΠ-extension with the R-monomial $\mathbb{i}^n$ and the Π-monomials given in (112). In short, together with $\mathbb{i}^n$ the expression can be reduced again in terms of the algebraic independent products (112).

Similar expressions as given in (111) arise during challenging evaluations of determinants; see, e.g., Krattenthaler (2001); Mills et al. (1983); Zeilberger (1996). We expect that the new tools elaborated in this article will prove beneficial in related but more complicated product expressions.

We conclude this section by combining our tools from above with the summation package **`Sigma`**.

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After loading in

In[6]:= **`<<Sigma.m`**

Sigma - A summation package by Carsten Schneider © RISC-Linz

we insert the sum

In[7]:= $\mathbf{mySum = SigmaSum[\Big(-1+(1+k)(2+k)^2\prod_{j=1}^{k}(1+j)^2\Big)\prod_{j=1}^{k}j\prod_{i=1}^{j}(1+i)^2}$
$\mathbf{\quad -\frac{4}{3}\Big(1+2(1+k)^2(3+k)\prod_{j=1}^{k}-j(2+j)\Big)\prod_{j=1}^{k}2j\prod_{i=1}^{j}-i(2+i), \{k,1,n\}];}$

Afterwards we can activate the available summation algorithms of `Sigma` with the function call `SigmaReduce` and succeed in eliminating the summation sign in 2.2 seconds:

In[8]:= $\mathbf{SigmaReduce[mySum]}$

Out[8]= $4-\frac{1}{3}(1+n)^5(2+n)^2\left(-3+(1+\mathbb{i})\left(-\mathbb{i}+(\mathbb{i}^n)^2\right)(3+n)\,\mathbb{i}^n\right)(n!)^5\left(\prod_{i=1}^{k}\prod_{j=1}^{i}j\right)^2$

In other words, we have derived the simplification

$$\sum_{k=1}^{n}\Bigg(\Big(-1+(1+k)\,(2+k)^2\prod_{j=1}^{k}(1+j)^2\Big)\prod_{j=1}^{k}j\prod_{i=1}^{j}(1+i)^2$$

$$-\frac{4}{3}\Big(1+2\,(1+k)^2\,(3+k)\prod_{j=1}^{k}-j\,(2+j)\Big)\prod_{j=1}^{k}2\,j\prod_{i=1}^{j}-i\,(2+i)\Big)$$
$$=4-\frac{1}{3}\,(1+n)^5\,(2+n)^2\Big(-3+(1+i)\big(-\mathbb{i}+(\mathbb{i}^n)^2\big)\,(3+n)\,\mathbb{i}^n\Big)\,(n!)^5\left(\prod_{i=1}^{k}\prod_{j=1}^{i}j\right)^2.$$

We emphasize that the summand given in In[8] has been transformed internally with the package `NestedProducts` to the form

$$\frac{1}{3}\,(1+k)^2\left(\prod_{i=1}^{k}i\right)^3\left(\prod_{i=1}^{k}\prod_{j=1}^{i}j\right)^2\Bigg(-3+(-1-\mathbb{i})\,(2+k)\,\mathbb{i}^k+(-1+\mathbb{i})\,(2+k)\,(\mathbb{i}^k)^3$$
$$+(1+k)^3\,(2+k)^2\Big(3+(-1+\mathbb{i})\,(3+k)\,\mathbb{i}^k+(-1-\mathbb{i})\,(3+k)\,(\mathbb{i}^k)^3\Big)\left(\prod_{i=1}^{k}i\right)^2\Bigg).$$

Then `Sigma` reads off the derived products and rephrases them directly to a tower of RΠ-extensions (without using the available tools in `Sigma` to check whether the constant field remains unchanged). Afterwards the underlying summation algorithms of `Sigma` are applied to derive the final result.

7. Conclusion

We enhanced non-trivially the ideas from Ocansey and Schneider (2018); Schneider (2005, 2014) (related also to Abramov et al. (2003); Abramov and Petkovšek (2010); Chen et al. (2011)) in order to solve Problem RPE in Theorem 6.2 and Corollary 6.2 above. There we cannot only reduce or simplify expressions in terms of hypergeometric products of nesting depth 1 but in terms of hypergeometric products of nesting depth ≥ 1. More precisely, the expression can be reduced to an expression in terms of one root of unity product of the form ζ^n and hypergeometric products $Q_1(n),\dots,Q_s(n)\in\mathrm{Prod}_n(\tilde{\mathbb{K}}(x))$ of arbitrary but finite nesting depth which are algebraically independent among each other. This latter property has been extracted from results elaborated in Schneider (2017) (which are inspired by van der Put and Singer (1997)). Combined with the existing difference ring algorithms for symbolic summation (Karr, 1981; Schneider, 2001, 2016) this yields a complete summation machinery to reduce and simplify nested sums over hypergeometric products of arbitrary but finite nesting depth.

A natural future task is to enhance this combined toolbox of the packages **`NestedProducts`** and **`Sigma`** further and to tackle, e.g., definite summation problems. In particular, the interaction with the available creative telescoping algorithms (Schneider, 2007a, 2008, 2010b, 2015) and recurrence solving algorithms (Abramov et al., 2021) should be explored further. In addition, following the ideas from Ocansey and Schneider (2018) one might extend the above machinery to the class of nested q-hypergeometric products covering also the multibasic and mixed case (Bauer and Petkovšek, 1999).

Another open task is to optimize the proposed methods in connection with Remark 6.1. For instance, one may combine the above ideas with contributions from Schneider (2020) (based on Smith normal form calculations) to find optimal representations of such nested products. This means that in the output expression the order λ of the primitive root of unity ζ in ζ^n and the number s of algebraically independent products $Q_1(n),\dots,Q_s(n)$ should be minimized. In particular, one may try to find representations where the expression swell is reduced using, e.g., ideas from Abramov et al. (2003).

Finally, it would be interesting to see if the class of hypergeometric products of finite nesting depth can be generalized further to products of the form (1) where in the multiplicands the products do not appear only in form of Laurent polynomial expressions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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References

Abramov, Sergei A., Bronstein, Manuel, Petkovšek, Marko, Schneider, Carsten, 2021. On rational and hypergeometric solutions of linear ordinary difference equations in $\Pi\Sigma^*$-field extensions. J. Symb. Comput. 107, 23–66 (English). arXiv:2005.04944 [cs.SC].

Abramov, Sergei A., 1971. On the summation of rational functions. USSR Comput. Math. Math. Phys. 11 (4), 324–330.

Abramov, Sergei A., 1989. Rational solutions of linear differential and difference equations with polynomial coefficients. USSR Comput. Math. Math. Phys. 29 (6), 7–12.

Abramov, Sergei A., Le, Haquang, Petkovšek, Marko, 2003. Rational canonical forms and efficient representations of hypergeometric terms. In: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, vol. ISSAC '03. Association for Computing Machinery, New York, NY, USA, pp. 7–14.

Abramov, Sergei A., Petkovšek, Marko, 1994. D'Alembertian Solutions of Linear Differential and Difference Equations. Citeseer.

Abramov, Sergei A., Petkovšek, Marko, 2002. Rational normal forms and minimal decompositions of hypergeometric terms. J. Symb. Comput. 33 (5), 521–543.

Abramov, Sergei A., Petkovšek, Marko, 2010. Polynomial ring automorphisms, rational (w, σ)-canonical forms, and the assignment problem. J. Symb. Comput. 45 (6), 684–708.

Abramov, Sergei A., Zima, Eugene V., 1996. D'Alembertian solutions of inhomogeneous linear equations (differential, difference, and some other). In: International Conference on Symbolic and Algebraic Computation: Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation, vol. 24, pp. 232–240.

Bauer, Andrej, Petkovšek, Marko, 1999. Multibasic and mixed hypergeometric Gosper-type algorithms. J. Symb. Comput. 28 (4–5), 711–736.

Blümlein, Johannes, Round, Mark, Schneider, Carsten, 2018. Refined holonomic summation algorithms in particle physics. In: Zima, E., Schneider, C. (Eds.), Advances in Computer Algebra. WWCA 2016. In: Springer Proceedings in Mathematics & Statistics, vol. 226. Springer, pp. 51–91. arXiv:1706.03677 [cs.SC] (English).

Bronstein, Manuel, 2000. On solutions of linear ordinary difference equations in their coefficient field. J. Symb. Comput. 29 (6), 841–877.

Chen, Shaoshi, Feng, Ruyong, Fu, Guofeng, Li, Ziming, 2011. On the structure of compatible rational functions. In: Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation. ACM, pp. 91–98.

Chen, Shaoshi, Jaroschek, Maximilian, Kauers, Manuel, Singer, Michael F., 2013. Desingularization explains order-degree curves for Ore operators. In: Kauers, Manuel (Ed.), Proceedings of ISSAC'13, pp. 157–164.

Chyzak, Frédéric, 2000. An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math. 217 (1), 115–134.

Ge, Guoqiang, 1993a. Algorithms Related to the Multiplicative Representation of Algebraic Numbers. Ph.D. thesis. University of California at Berkeley.

Ge, Guoqiang, 1993b. Testing equalities of multiplicative representations in polynomial time. In: 34th Annual Symposium on Foundations of Computer Science, 1993. Proceedings. IEEE, pp. 422–426.

Gosper, Ralph William, 1978. Decision procedures for indefinite hypergeometric summation. Proc. Natl. Acad. Sci. USA 75, 40–42.

Hardouin, Charlotte, Singer, Michael F., 2008. Differential Galois theory of linear difference equations. Math. Ann. 342 (2), 333–377.

Hendriks, Peter A., Singer, Michael F., 1999. Solving difference equations in finite terms. J. Symb. Comput. 27 (3), 239–259.

Van Hoeij, Mark, 1999. Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra 139 (1–3), 109–131.

Karr, Michael, 1981. Summation in finite terms. J. ACM 28 (2), 305–350.

Karr, Michael, 1985. Theory of summation in finite terms. J. Symb. Comput. 1 (3), 303–315.

Kauers, Manuel, 2005. Algorithms for Nonlinear Higher Order Difference Equations. Ph.D. thesis. RISC-Linz.

Kauers, Manuel, 2018. The guess-and-prove paradigm in action. Int. Math. News 237, 1–15.

Kauers, Manuel, Schneider, Carsten, 2006. Indefinite summation with unspecified summands. Discrete Math. 306 (17), 2073–2083.

Kauers, Manuel, Zimmermann, Burkhard, 2008. Computing the algebraic relations of c-finite sequences and multisequences. J. Symb. Comput. 43 (11), 787–803.

Koutschan, Christoph, 2013. Creative telescoping for holonomic functions. In: Computer Algebra in Quantum Field Theory. In: Texts Monogr. Symbol. Comput. Springer, Vienna, pp. 171–194.

Krattenthaler, Christian, 1997. RATE – a Mathematica guessing machine. Available at https://mat.univie.ac.at/~kratt/rate/rate.html.

Krattenthaler, Christian, 2001. Advanced determinant calculus. In: The Andrews Festschrift. Springer, pp. 349–426.

Lenstra, Arjen K., Lenstra, Hendrik W., Lovász, László, 1982. Factoring polynomials with rational coefficients. Math. Ann. 261, 515–534.

Mills, William H., Robbins, David P., Rumsey Jr., Howard, 1983. Alternating sign matrices and descending plane partitions. J. Comb. Theory, Ser. A 34 (3), 340–359.

Ocansey, Evans Doe, 2019. Difference Ring Algorithms for Nested Products. Ph.D. thesis. RISC, Johannes Kepler University Linz.

Ocansey, Evans Doe, Schneider, Carsten, 2018. Representing ($\boldsymbol{q}$-)hypergeometric products and mixed versions in difference rings. In: Waterloo Workshop on Computer Algebra. Springer, pp. 175–213.

Paule, Peter, 1995. Greatest factorial factorization and symbolic summation. J. Symb. Comput. 20 (3), 235–268.

Paule, Peter, Riese, Axel, 1997. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping. In: Ismail, M., Rahman, M. (Eds.), Special Functions, q-Series and Related Topics, vol. 14. AMS, pp. 179–210.

Paule, Peter, Schneider, Carsten, 2019. Towards a symbolic summation theory for unspecified sequences. In: Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory. Springer, pp. 351–390.

Petkovšek, Marko, 1992. Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symb. Comput. 14 (2–3), 243–264.

Petkovšek, Marko, Wilf, Herbert S., Zeilberger, Doron, 1996. $A = B$. A. K. Peters, Wellesley, MA.

Petkovšek, Marko, Zakrajšek, Helena, 2013. Solving linear recurrence equations with polynomial coefficients. In: Computer Algebra in Quantum Field Theory. Springer, pp. 259–284.

Riese, Axel, 2003. qMultisum - a package for proving q-hypergeometric multiple summation identities. J. Symb. Comput. 35, 349–377.

Risch, Robert H., 1969. The problem of integration in finite terms. Trans. Am. Math. Soc. 139, 167–189. MR 237477.

Schneider, Carsten, 2001. Symbolic Summation in Difference Fields. Ph.D. thesis. RISC, J. Kepler University Linz (Published as Technical report no. 01-17 in RISC Report Series.).

Schneider, Carsten, 2005. Product representations in $\Pi\Sigma$-fields. Ann. Comb. 9 (1), 75–99.

Schneider, Carsten, 2007a. Simplifying sums in $\Pi\Sigma^*$-extensions. J. Algebra Appl. 6 (03), 415–441.

Schneider, Carsten, 2007b. Symbolic summation assists combinatorics. Sémin. Lothar. Comb. 56, 1–36, B56b.

Schneider, Carsten, 2008. A refined difference field theory for symbolic summation. J. Symb. Comput. 43 (9), 611–644.

Schneider, Carsten, 2010a. Parameterized telescoping proves algebraic independence of sums. Ann. Comb. 14 (4), 533–552. Appeared earlier in Proc. of FPSAC 2007.

Schneider, Carsten, 2010b. Structural theorems for symbolic summation. Appl. Algebra Eng. Commun. Comput. 21 (1), 1–32 (English).

Schneider, Carsten, 2014. A streamlined difference ring theory: indefinite nested sums, the alternating sign and the parameterized telescoping problem. In: Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), 2014 15th International Symposium. IEEE Computer Society, pp. 26–33. arXiv:1412.2782 (English).

Schneider, Carsten, 2015. Fast algorithms for refined parameterized telescoping in difference fields. In: Computer Algebra and Polynomials. Springer, pp. 157–191.

Schneider, Carsten, 2016. A difference ring theory for symbolic summation. J. Symb. Comput. 72, 82–127.

Schneider, Carsten, 2017. Summation theory II: characterizations of R$\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80, 616–664.

Schneider, Carsten, 2020. Minimal representations and algebraic relations for single nested products. Program. Comput. Softw. 46, 133–161.

Schneider, Carsten, 2023. Refined telescoping algorithms in $R\Pi\Sigma$-extensions to reduce the degrees of the denominators. RISC Report Series 23-01. Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz. arXiv:2302.03563 [cs.SC].

van der Put, Marius, Singer, Michael F., 1997. Galois Theory of Difference Equations. Lecture Notes in Mathematics, vol. 1666. Springer-Verlag, Berlin.

Wegschaider, Kurt, 1997. Computer generated proofs of binomial multi-sum identities. Master's thesis. RISC, J. Kepler University.

Wilf, Herbert, Zeilberger, Doron, 1992. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. Invent. Math. 108, 575–633.

Zeilberger, Doron, 1990. A holonomic systems approach to special functions identities. J. Comput. Appl. Math. 32, 321–368.

Zeilberger, Doron, 1991. The method of creative telescoping. J. Symb. Comput. 11, 195–204.

Zeilberger, Doron, 1996. Proof of the alternating sign matrix conjecture. Electron. J. Comb. 3 (2), R13.