Equilibrium States are Freezing States

A new result in thermodynamic formalism for general dynamical systems.

C. Evans Hedges

Spring Topology and Dynamics Conference: 06 March 2025

Dynamical Systems

- Let X be a compact metric space.
- Let T be a continuous action by a (semi)-group G.
- (X, T) forms a **Dynamical System**.

Examples

$\mathbb{R}$ Actions:

- N particles confined to a box
- Celestial mechanics

$\mathbb{N}$ Action:

- $2x \mod 1$ on the Torus
- Axiom-A Diffeomorphism

$\mathbb{Z}^d$ action:

- Ferro-magnetic solids (Ising model and generalizations)
- Quasi-crystals

Statistical Framework

Taking the view of statistical physics:

- Each $x \in X$ represents a **microstate**.
- We **do not** have access to x directly.
  - Ex: All the positions and momentums of every particle in this room.
- We **do** have access to probability distributions on X.
  - Ex: Distributions of positions and momentums of the particles in this room.

T-Invariant Probability Measures

Let $M_T(X)$ be the set of Borel probability measures such that:

- For all $g \in G$, for all measurable $A \subset X$,

$$\mu(T_g^{-1}A) = \mu(A).$$

If G is **amenable** then $M_T(X)$ is non-empty.

Properties of $M_T(X)$

$M_T(X)$ is:

- compact in the weak-$*$ topology,
- convex, and
- its extreme points are exactly the ergodic measures.

(It is a Choquet simplex)

Entropy

For any $\mu \in M_T(X)$,

- the **entropy** of μ is denoted $h(\mu)$.

$h : M_T(X) \to [0, \infty]$ is affine:

$$h(\alpha\mu + (1-\alpha)\nu) = \alpha h(\mu) + (1-\alpha)h(\nu).$$

For the remainder of this talk we assume h is bounded.

Energy (Potentials)

We call a continuous $\phi : X \to \mathbb{R}$ a **potential**.

For a microstate $x \in X$, ϕ assigns *energy* to x.

For a macrostate $\mu \in M_T(X)$, we can talk about the average energy by:

$$\int \phi d\mu.$$

Pressure

Given a $\phi \in C(X)$, define the **measure theoretic pressure** of $\mu \in M_T(X)$ by:

$$h(\mu) + \int \phi d\mu.$$

The **topological pressure** of (X, T) with ϕ is:

$$P_{top}(\phi) = \sup\{h(\mu) + \int \phi d\mu : \mu \in M_T(X)\}.$$

Equilibrium States

We say $\mu \in M_T(X)$ is an **equilibrium state** if

$$h(\mu) + \int \phi d\mu = P_{top}(\phi)$$

(Inverse) Temperature

Let $\beta > 0$ represent the inverse temperature of our system.

New system: (X, T) with $\beta\phi$.

Since we care about

$$h(\mu) + \int \beta\phi d\mu,$$

as $\beta \to \infty$ the "energy" term dominates.

Cooling Our System

Let μ_β be an equilibrium state for $\beta\phi$.

We observe the system cool by looking at μ_β as $\beta \to \infty$.

Freezing

If there exists some $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, μ is an equilibrium state for $\beta\phi$, then we say ϕ **freezes** on μ (at β_0).

Our system stops changing after we cool below $T = 1/\beta_0$.

Trivially freezing is a stronger condition than being an equilibrium state: If ϕ freezes on μ, then μ is an equilibrium state for $\beta_0\phi$.

Previous Results for Attaining Equilibrium States

Fact

(Ruelle 1978) If $h : M_T(X) \to \mathbb{R}$ is upper semi-continuous and μ is ergodic, then there exists $\phi \in C(X)$ such that μ is the unique equilibrium state for ϕ.

Fact

(Jenkinson 2006) If $h : M_T(X) \to \mathbb{R}$ is upper semi-continuous and $\mathcal{F} \subset M_{erg}(X)$ is a closed collection of ergodic measures, then there exists $\phi \in C(X)$ such that $\overline{co}(\mathcal{F})$ is the collection of equilibrium states for ϕ.

Previous Results for Attaining Freezing States

In the symbolic setting $X = \mathcal{A}^{\mathbb{Z}}$

- Hofbauer, 1977:
  - ϕ freezes on δ_1
- Lopez, 1993:
  - ϕ freezes on $\overline{co}(\{\delta_1, \delta_0\})$
- Kucherenko, Quas, and Wolf, 2021:
  - ϕ freezes on the set of MMEs for any subshift X.

Attaining Freezing States

Fact

(H. 2024) Let $\mathcal{F} \subset M_T(X)$ be any subset such that:

- *h is constant on $\mathcal{F}$ and*
- *$\mathcal{F}$ is the collection of equilibrium states for some potential ϕ.*

Then there exists $\psi \in C(X)$ such that ψ freezes on $\mathcal{F}$ at $\beta_0 = 1$.

Corollary

(H. 2024) Let (X, T) be a dynamical system and let $\mu \in M_{erg}(X)$. Then μ is an equilibrium state for some potential if and only if μ is the freezing state for some potential.

Attaining Freezing States when h is u.s.c.

In the case where h is upper semi-continuous:

Fact

(H. 2024) Let $\mathcal{E} \subset M_{erg}(X)$ be any closed subset of ergodic measures where h is constant on $\mathcal{E}$. Then there exists $\psi \in C(X)$ such that ψ freezes on $\overline{co}(\mathcal{E})$ at $\beta_0 = 1$.

Freezing Potentials are Dense

Fact

(H. 2024) Let (X, T) be a dynamical system where h is u.s.c., then the set of potentials that freeze is dense in $C(X)$.

Thank you!

Questions?