Some Results in Thermodynamic Formalism

Defense for the completion of a PhD in Mathematics.

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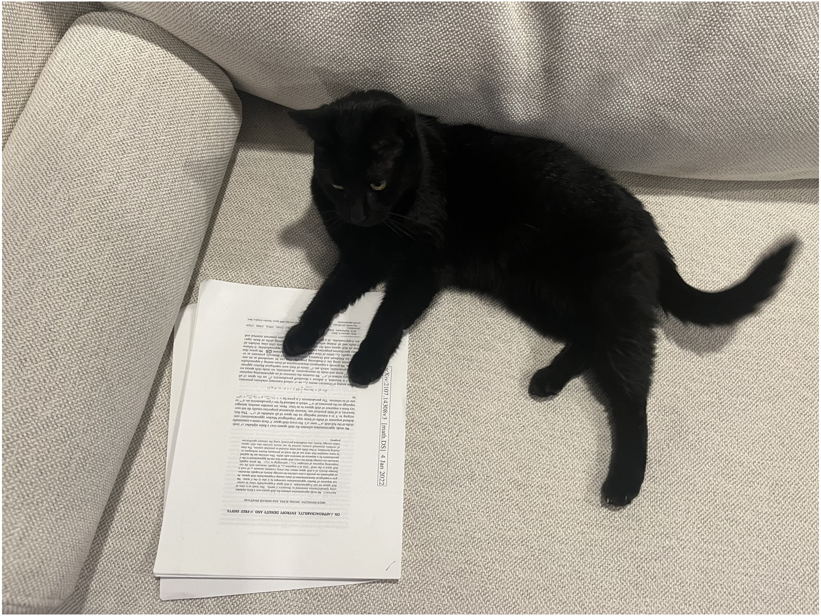
Presentation Outline

1. Introduction to Dynamical Systems
2. Preliminaries and Background
   - Basic Concepts and Examples
   - Entropy and Pressure
   - Equilibrium States
3. Results on Equilibrium States and Gibbs Property
4. Computability in Dynamical Systems
   - Preliminaries on Computability Theory
   - Results
5. Freezing Phase Transitions
   - Background and Motivation
   - Results

Intermittent breaks to stare into the void.

Void 1



Dynamical Systems

For countless domains, it is essential to understand the time evolution behavior of a system.

- Physics, chemistry, biology, sociology, economics, etc.

It is of interest then to understand these systems in general. For that we turn to the field of Dynamical Systems.

Dynamical Systems and (Semi)-Group Actions

- X is a compact metric space.
- A (semi)-group G acts continuously on X by $\mathcal{T}$.
- $(X, \mathcal{T})$ is a **Dynamical System**.

Examples:

- $\mathbb{R}$ Action: N particles confined to a box
- $\mathbb{N}$ Action: $2x \mod 1$ on the Torus
- $\mathbb{Z}^d$ action: Ferro-magnetic solids (Ising model and generalizations)

Symbolic Dynamics

We start with a finite set $\mathcal{A}$ called the **alphabet**. (Think $\{0, 1\}$)

- $\mathcal{A}^{\mathbb{Z}}$ (bi-infinite sequences of our symbols).
- The **left shift map** $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_n = x_{n+1}.$$

- We call a closed, σ-invariant $X \subset \mathcal{A}^{\mathbb{Z}}$ a **subshift**.
- We define a **word** w to be a finite sequence that appears in X, e.g. 001.
- For a given word w, we define the **cylinder set**

$$[w] = \{x \in X : x_{[0:n-1]} = w\}.$$

Golden Mean Subshift

Example:

- $X = \{x \in \{0,1\}^{\mathbb{Z}} : \forall i \in \mathbb{Z},\ x_{i,i+1} \neq 11\}$.
- $\ldots 100.010 \ldots = x \in X$.
- $\sigma(x) = \ldots 1000.10 \ldots$.
- $[000] = \{\ldots - -.000 - - \ldots\}$.

Statistical Framework

Taking the view of statistical physics:

- Each $x \in X$ represents a **microstate**.
- We **do not** have access to x directly.
  - Ex: All the positions and momentums of every particle in this room.
- We **do** have access to probability distributions on X.
  - Ex: Distributions of positions and momentums of the particles in this room.

Invariant Measures

Let $M_\sigma(X)$ be the collection of all σ-invariant measures:

$$\mu(\sigma^{-1}(A)) = \mu(A).$$

In the subshift setting, probability measures can be completely described by viewing the probability of finite configurations:

- $\mu([000])$ represents the probability of seeing a 000.

Measure Theoretic Entropy

- For each $\mu \in M_\sigma(X)$ we can compute the Kolmogorov-Sinai entropy of μ by

$$h(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{w\in\mathcal{A}^n} -\mu([w]) \log \mu([w])$$

where $0 \log 0 = 0$.

In a sense, $h(\mu)$ measures the total disorder of a system, or how unpredictable it is.

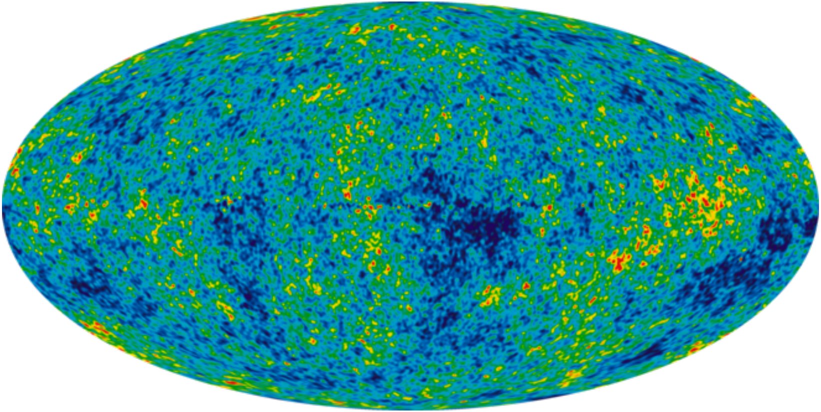
$h(\mu)$ is low when we can easily predict the next letter, and high when the next letter is hard to predict.

Entropy for Subshifts

Example on $X = \{0,1\}^{\mathbb{Z}}$:

- Let δ assign full probability to 0^∞. Then
$$h(\delta) = 0$$
- Let ν be i.i.d. assigning $(1/3, 2/3)$ to $[0]$ and $[1]$ respectively. Then
$$h(\nu) \approx 0.27$$
- Let μ be i.i.d. assigning $(1/2, 1/2)$ to $[0]$ and $[1]$ respectively. Then
$$h(\mu) \approx 0.3$$

Void 2



Equilibrium and Free Energy

In physics, the Gibbs Free Energy of a system is defined as:

$$G = E - Th$$

Where E is the total internal energy of the system, T is the temperature, and h is the entropy.

A system is said to be at equilibrium when G is minimized.

Potentials

- Given a subshift X, a continuous, real valued $\phi \in C(X)$ is called a **potential**.
- Potentials assign *energy* to points in x.
- By additionally considering the energy of points, we can construct a generalization of topological entropy: **topological pressure**.

We will use this generalization to understand equilibrium states.

Pressure and Equilibrium States

We define **topological pressure** of a potential ϕ to be:

$$P_{top}(\phi) = \sup\{h(\mu) + \int \phi d\mu : \mu \in M_\sigma(X)\}.$$

We say μ is an **equilibrium state** for ϕ if

$$h(\mu) + \int \phi d\mu = P_{top}(\phi).$$

In the subshift setting, at least one equilibrium state exists.

Finite Equilibrium State

Let our system $X = \{x, y\}$. Let $\phi(x) = 0$, $\phi(y) = 1$.

Let μ_p assign probability p to x.

$$h(\mu_p) = -p \log p - (1-p)\log(1-p).$$
$$\int \phi \, d\mu_p = (1-p).$$

$$h(\mu_p) + \int \phi \, d\mu_p \text{ is maximized when } p = \frac{1}{1+e}$$

The Gibbs Property

$X = \{x, y\}$. Let $\phi(x) = 0$, $\phi(y) = 1$.

Notice the equilibrium state μ satisfies:

$$\mu(x) = \frac{e^{\phi(x)}}{e^{\phi(x)} + e^{\phi(y)}} \text{ and } \mu(y) = \frac{e^{\phi(y)}}{e^{\phi(x)} + e^{\phi(y)}}.$$

This kind of property is called the **Gibbs Property** and it can be extended to the infinite case.

Extended DLR Theorems

For nice subshifts $X \subset \mathcal{A}^G$, and sufficiently regular potentials $\phi \in C(X)$, the following are equivalent:

- μ is an equilibrium state for ϕ
- μ is a σ-invariant Gibbs measure for ϕ.

Outside of $\mathbb{Z}$, uniqueness is out the window (see Ising model in $\mathbb{Z}^2$).

What about when X isn't a nice subshift?

$\mathbb{Z}^d$ MMEs

For words $v, w \in L_F(X)$, we denote $v \to w$ to mean v can be replaced by w.

Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift and μ be a measure of maximal entropy.

- **Pavlov, Garcia-Ramos 2019**: If $v, w \in L_F(X)$ such that $v \to w$, then
$$\mu([v]) \leq \mu([w]).$$

G Equilibrium States

Let G be a countable, amenable group, $X \subset \mathcal{A}^G$ be a subshift, and $\phi \in C(X)$ have summable variation, and μ_ϕ an equilibrium state for ϕ.

- **H., 2024**: If $v, w \in L_F(X)$ such that $v \to w$, then

$$\mu_\phi([v]) \leq \mu_\phi([w]) \cdot \sup_{x \in [v]} exp \left(\sum_{g \in G} \phi(\sigma_g(x)) - \phi(\sigma_g(\xi_{v,w}(x))) \right).$$

G Equilibrium States

- **H., 2024**: If $v, w \in L_F(X)$ such that $v \to w$, then then $\mu_\phi \circ \xi_{v,w}$ is absolutely continuous with respect to μ_ϕ when restricted to $[w]$ and for μ_ϕ-almost every $x \in [w]$,

$$\frac{d(\mu_\phi \circ \xi_{v,w})}{d\mu_\phi}(x) \leq exp\left(\sum_{g \in G} \phi(\sigma_g(\xi_{v,w}(x))) - \phi(\sigma_g(x))\right).$$

Void 3

Understanding P_{top}

It is also of interest to understand the properties of the map:

$$P_{top}(\phi) = \sup\{h(\mu) + \int \phi d\mu : \mu \in M_\sigma(X)\}.$$

We know P_{top} is:

- Lipschitz,
- Convex,
- Gateaux differentiable at ϕ if and only if there exists a unique equilibrium state for ϕ,
- P_{top} is a computable function in certain situations,
- etc.

Computability Preliminaries

Computability Theory is the study of what can be accomplished by algorithms/computers.

We say $\alpha \in \mathbb{R}$ is **computable** if there exists an algorithm $T : \mathbb{N} \to \mathbb{Q}$ so that:

$$|T(n) - \alpha| < 2^{-n}.$$

We say $\alpha \in \mathbb{R}$ is **computable from above** if there exists an algorithm $T : \mathbb{N} \to \mathbb{Q}$ so that:

$$T(n) \searrow \alpha.$$

Computable Function Definition

Definition

Let X be a computable metric space and $f : X \to \mathbb{R}$. f is **computable** if there exists an algorithm T so that for all $x \in X$, for any oracle λ for x, for any $n \in \mathbb{N}$:

$$|T(\lambda, n) - f(x)| < 2^{-n}.$$

$P_X : C(\mathcal{A}^G) \to \mathbb{R}$ is **computable** if there exists an algorithm that upon inputs of approximations for $\phi \in C(\mathcal{A}^G)$, outputs approximations of $P_X(\phi)$.

Existing Results

- Spandl, 2007, 2008
  - If X is a $\mathbb{Z}$-SFT or sofic shift, P_X is computable.
- Burr, Das, Wolf, Yang, 2022
  - If X is coded subshift, P_X is computable.
- Hochman and Meyerovitch, 2011
  - $\alpha \geq 0$ is computable from above if and only if it is the entropy of a $\mathbb{Z}^2$ SFT.
- Marcus and Pavlov, 2013
  - If $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is an SI SFT, P_X is computable.

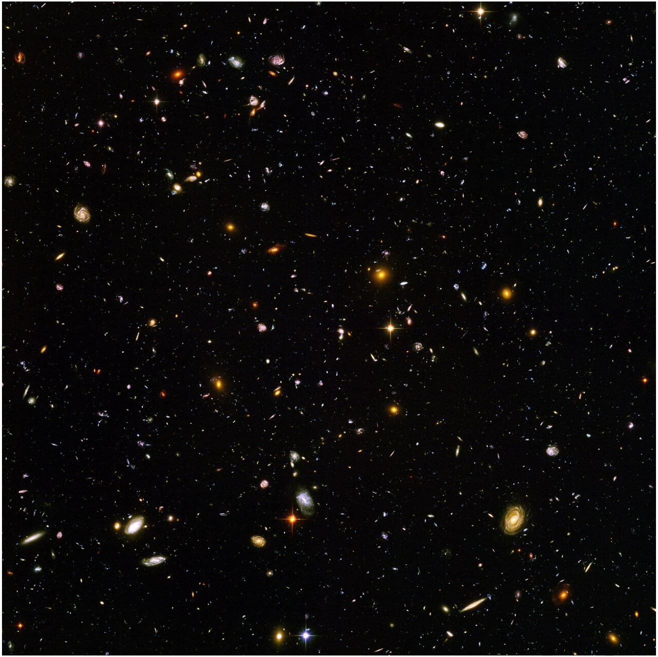
P_{top} is Computable (from Above)

- **Pavlov, H., 2024**: Let G be a finitely generated, amenable group with decidable word problem and let $X \subset \mathcal{A}^G$ be any subshift. Then $P_X : C(\mathcal{A}^G) \to \mathbb{R}$ is computable from above given an enumeration for a forbidden list for X.

P_{top} is Computable (SI SFT)

- **Pavlov, H., 2024**: Let G be a finitely generated, amenable group with decidable word problem. Let $X \subset \mathcal{A}^G$ be a strongly irreducible SFT. Then the pressure function $P_X : C(\mathcal{A}^G) \to \mathbb{R}$ is computable.

Void 4



Introducing (inverse) Temperature

Let $\beta > 0$ represent the inverse temperature of our system.

$$G = E - Th \implies \beta G = \beta E - h$$

So with a fixed energy function, we can learn about the properties of our system as we change the temperature through this modified pressure function:

$$P_\phi(\beta) = P_{top}(\beta\phi) = \sup\{h(\mu) + \int \beta\phi d\mu : \mu \in M_T(X)\}.$$

Cooling our System to Zero Temperature

For each $\beta > 0$, let μ_β be an equilibrium state for $\beta\phi$.

Definition

If μ_∞ is a weak-$*$ cluster point for $(\mu_\beta)_{\beta>0}$ as $\beta \nearrow \infty$, then we call μ_∞ a **ground state** or **zero temperature limit**.

- $\int \phi d\mu_\infty$ is the **ground state energy**.
- $h(\mu_\infty)$ the **ground state entropy** or **residual entropy**.

Question: Can we compute these values from ϕ?

Computability of Ground State Energy Map

- **Pavlov, H., 2024**: Let G be a countable, amenable group and let $X \subset \mathcal{A}^G$ be a subshift such that P_X is computable (computable from above). Then the function that sends ϕ to its ground state energy $\sup_{\nu \in M_\sigma(X)} \int \phi d\mu$ is computable (computable from above).

Computability of Ground State Entropy Map

- **Pavlov, H., 2024**: Let G be a countable, amenable group and let $X \subset \mathcal{A}^G$ be a subshift such that P_X is computable. Then, the function that sends ϕ to the ground state entropy of ϕ is computable from above.

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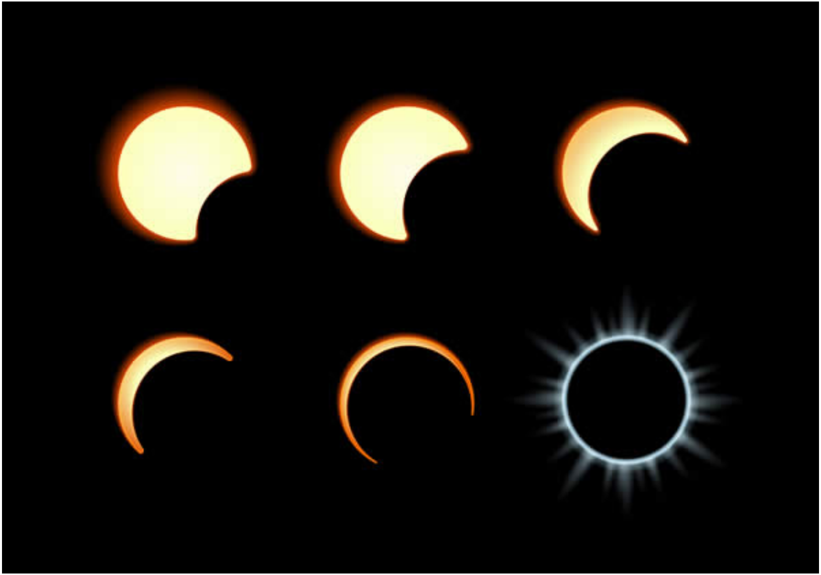

Freezing in General Dynamical Systems

Finally we pivot into the very general world of dynamical systems.

- X is a compact metric space
- G is a (semi)-group that acts continuous on X by $\mathcal{T}$

Assumptions:

- $M_T(X)$ is non-empty.
- $h : M_T(X) \to \mathbb{R}$ is bounded.

(This holds when G is amenable and $h_{top}(X, \mathcal{T}) < \infty$).

Freezing in General Dynamical Systems

For a fixed potential $\phi \in C(X)$, we say ϕ **freezes** on $\mu \in M_T(X)$ if there exists some $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, μ is an equilibrium state for $\beta\phi$.

Ex: If we weigh 1's heavily enough by ϕ, it could be the case that for $\beta \geq \beta_0 = 2$, any equilibrium state for $\beta\phi$ will have to be entirely supported on the point of all 1's.

Question: When does this happen? When can this happen? Is it rare?

Previous Results

In the symbolic setting of $\mathcal{A}^{\mathbb{Z}}$:

- Hofbauer 1977
  - ϕ freezes on δ_1
- Lopez 1993
  - ϕ freezes on $\overline{co}(\{\delta_1, \delta_0\})$
- Kucherenko, Quas, and Wolf 2021
  - ϕ freezes on the set of MMEs for any subshift $X \subset \mathcal{A}^{\mathbb{Z}}$
  - Extended in 2025 to $\mathcal{A}^{\mathbb{Z}^d}$

Equivalence of Freezing and Equilibrium State

- **H., 2024**: Let (X, T) be any dynamical system and let $\mu \in M_{erg}(X)$. Then μ is an equilibrium state for some potential if and only if μ is the freezing state for some potential.
- **H., 2024** If h is u.s.c. and $\mu \in M_{erg}(X)$, then there exists a potential ψ that freezes on μ.

Freezing Results

- **H., 2024**: Let (X, T) be any dynamical system and let $\mathcal{F}$ be a non-empty subset of $M_T(X)$ such that:
  - For all $\mu, \nu \in \mathcal{F}$, $h(\mu) = h(\nu)$ and
  - there exists some $\phi \in C(X)$ such that $\mathcal{F}$ is the collection of equilibrium states for ϕ.

  Then there exists $\psi \in C(X)$ such that for all $\beta \geq 1$, $\mathcal{F}$ is the collection of equilibrium states for $\beta\psi$.

Upper Semi-Continuous h Case

- **H., 2024**: Let (X, T) be any dynamical system such that the entropy map is upper semi-continuous. Let $\mathcal{E} \subset M_{erg}(X)$ be a non-empty collection of ergodic measures so that $\mathcal{E}$ is closed in $M_T(X)$ and suppose that h is constant on $\mathcal{E}$. Then there exists $\phi \in C(X)$ such that for all $\beta \geq 1$, $\overline{co}(\mathcal{E})$ is the collection of equilibrium states for $\beta\phi$.

Density of Freezing Potentials

- **H., 2024**: Let (X, T) be a dynamical system such that the entropy map is upper semi-continuous. Then the set of potentials that freeze is dense in $C(X)$ with the uniform topology.

- **H., 2025**: Let (X, T) be a dynamical system where T is a $\mathbb{Z}$ action and (X, T) has the specification property. Then the collection of potentials that do not freeze contains a dense G_δ.

Summary

1. Equilibrium states are (somewhat) Gibbs,
2. Pressure (and related functions) *are sometimes* computable,
3. If a measure is an equilibrium state, it is also a frozen state.

Thank you!


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