Equilibrium States are (at least somewhat) Gibbs

Some new results on equilibrium states for amenable group subshifts.

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Carolina Dynamics Symposium: 13 April 2024

Presentation Outline

- Introduction to Thermodynamic Formalism in $\mathbb{Z}$ Subshifts
  - Entropy
  - Pressure
  - Equilibrium States
- A jump to the amenable G subshift setting
- Overview of results.

$\mathbb{Z}$-Subshifts

- Finite set $\mathcal{A}$ called the **alphabet**.
- The **left shift map** $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_n = x_{n+1}.$$

- We call a closed, σ-invariant $X \subset \mathcal{A}^{\mathbb{Z}}$ a **subshift**.
- Let $L_n(X)$ denote the n-**language** of X. I.e. the words of length n that appear in X.
- For a given word $w \in L_n(X)$, we define the **cylinder set**

$$[w] = \{x \in X : x_{[0:n-1]} = w\}.$$

Golden Mean Subshift

Example:

- $X = \{x \in \{0,1\}^{\mathbb{Z}} : \forall i \in \mathbb{Z},\ x_{i,i+1} \neq 11\}$.
- $\ldots 100.010 \ldots\ = x \in X$.
- $\sigma(x) = \ldots 1000.10 \ldots$.
- $L_3(X) = \{000, 001, 010, 100, 101\}$.

Invariant Measures

For a subshift X, let $M_\sigma(X)$ be the collection of all σ-invariant Borel probability measures on X.

- $\mu(A) = \mu(\sigma^{-1}A)$ for all measurable $A \subset X$.

In the subshift setting, cylinder sets generate the Borel σ-algebra so we can restrict ourselves computing $\mu([w])$ for $w \in L(X)$.

Measure Theoretic Entropy

- For each $\mu \in M_\sigma(X)$ we can compute the Kolmogorov-Sinai entropy of μ by

$$h(\mu) = \lim_{n\to\infty} \frac{1}{n} \sum_{w\in L_n(X)} -\mu([w]) \log \mu([w]).$$

Measures of Maximal Entropy

We define topological entropy by:

$$h_{top}(X) = \sup_{\mu \in M_\sigma(X)} h(\mu).$$

For $\mu \in M_\sigma(X)$, if $h(\mu) = h_{top}(X)$, then μ is a Measure of Maximal Entropy.

By w^* upper semicontinuity of the entropy map $h : M_\sigma(X) \to \mathbb{R}$, we are guaranteed existence of such a measure.

Entropy for Subshifts

Example on $X = \{0,1\}^{\mathbb{Z}}$:

- Let ν be Bernoulli assigning $(1/3, 2/3)$ to [0] and [1] respectively. Then

$$h(\nu) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} < \log 2 = h_{top}(X).$$

- Let μ be Bernoulli assigning $(1/2, 1/2)$ to [0] and [1] respectively. Then

$$h(\mu) = -\log\frac{1}{2} = \log 2 = h_{top}(X)$$

Potentials

- Given a subshift X, a continuous, real valued $\phi \in C(X)$ is called a **potential**.
- Potentials assign *energy* to points in x.
- By additionally considering the energy of points, we can construct a generalization of topological entropy: **topological pressure**.

Pressure and Equilibrium States

We define **topological pressure** of a potential ϕ to be:

$$P_{top}(\phi) = \sup\{h(\mu) + \int \phi d\mu : \mu \in M_\sigma(X)\}.$$

We say μ is an **equilibrium state** for ϕ if

$$h(\mu) + \int \phi d\mu = P_{top}(\phi).$$

Existence of at least one equilibrium state is guaranteed in the subshift setting.

Equilibrium State Example

Example:

- Let $X = \{0, 1\}^{\mathbb{Z}}$,
- Define $\phi([0]) = 0$, $\phi([1]) = 1$.
- Find μ that maximizes $h(\mu) + \int \phi d\mu$.
- Balance the entropy and average energy.

Equilibrium State Example

- Let's look at μ_p Bernoulli $(p, 1-p)$
- $h(\mu_p) + \int \phi d\mu_p$.

$$h(\mu_p) = -p \log p - (1-p)\log(1-p),$$

$$\int \phi d\mu_p = 0 \cdot p + 1 \cdot (1-p) = 1 - p.$$

- ... some calculus later...
- Maximized at $p = \frac{1}{1+e}$.

Equilibrium State Example

- For $X = \{0,1\}^{\mathbb{Z}}$,
- $\phi([0]) = 0$, $\phi([1]) = 1$,
- μ Bernoulli $(\frac{1}{1+e}, \frac{e}{1+e})$,

μ satisfies the following equation for any $w \in L_n(X)$, $x \in [w]$:

$$\mu([w]) = \frac{exp(\phi_n(x))}{\sum_{v \in L_n(X)} exp(\phi_n(y))} \tag{1}$$

where $\phi_n(x) = \sum_{i=0}^{n-1} \phi(\sigma^i(x))$.

In general, (1) will not hold for an equilibrium state μ, and μ may not even be unique.

Uniqueness of Equilibrium States in $\mathbb{Z}$

Fact: Given a mixing SFT $X \subset \{0,1\}^{\mathbb{Z}}$ and a Hölder continuous potential $\phi \in C(X)$, there exists a unique equilibrium state $\mu_\phi \in M_\sigma(X)$ and μ_ϕ is Gibbs for ϕ.

μ_ϕ being Gibbs for ϕ means (1) holds up to a multiplicative constant.

But, for general ϕ, general subshift, or general G, this does not necessarily hold.

Extended DLR Theorems

For nice subshifts $X \subset \mathcal{A}^G$, and sufficiently regular potentials $\phi \in C(X)$, the following are equivalent:

- μ is an equilibrium state for ϕ
- μ is a σ-invariant Gibbs measure for ϕ.

Outside of $\mathbb{Z}$, uniqueness is out the window (see Ising model in $\mathbb{Z}^2$).

What about when X isn't a nice subshift?

$\mathbb{Z}^d$ Measures of Maximal Entropy

For words $v, w \in L_F(X)$, we denote $v \to w$ to mean v can be replaced by w.

- **Garcia-Ramos and Pavlov, 2019:** Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be any subshift, $w, v \in L_F(X)$, and μ be any m.m.e.. If $v \to w$, then

$$\mu([v]) \leq \mu([w]).$$

An equivalent way to express this would be for μ-a.e. $x \in [w]$,

$$\frac{d(\mu \circ \xi_{v,w})}{d\mu}(x) \leq 1.$$

G Equilibrium States

Let G is a countable, amenable group, $X \subset \mathcal{A}^G$ be a subshift, and $\phi \in C(X)$ have summable variation, and μ_ϕ an equilibrium state for ϕ.

- **H., 2024**: If $v, w \in L_F(X)$ such that $v \to w$, then

$$\mu_\phi([v]) \leq \mu_\phi([w]) \cdot \sup_{x \in [v]} exp \left(\sum_{g \in G} \phi(\sigma_g(x)) - \phi(\sigma_g(\xi_{v,w}(x))) \right).$$

G Measures of Maximal Entropy

As an immediate corollary when $\phi = 0$, we extend the results of Garcia-Ramos and Pavlov to the amenable group G subshift setting:

- **H., 2024**: Let $X \subset \mathcal{A}^G$ be any subshift, $w, v \in L_F(X)$, and μ be any m.m.e.. If $v \to w$, then

$$\mu([v]) \leq \mu([w]).$$

$\mathbb{Z}^d$ Bounds when $v \leftrightarrow w$

Let $X \subset \mathcal{A}^{\mathbb{Z}^d}$ be a subshift, $\phi \in C(X)$ be a potential with d-summable variation, and $\mu \in M_\sigma(X)$ an equilibrium state for ϕ.

- **Meyerovitch, 2013:** For any $v, w \in L_F(X)$ such that $v \leftrightarrow w$,

$$\frac{d(\mu \circ \xi_{v,w})}{d\mu}(x) = exp\left(\sum_{n\in\mathbb{Z}^d} \phi(\sigma_n(\xi_{v,w}(x))) - \phi(\sigma_n(x))\right).$$

G Bounds when $v \to w$

Let G is a countable, amenable group, $X \subset \mathcal{A}^G$ be a subshift, and $\phi \in C(X)$ have summable variation, and μ_ϕ an equilibrium state for ϕ.

- **H., 2024**: If $v, w \in L_F(X)$ such that $v \to w$, then for μ_ϕ-a.e. $x \in [w]$:

$$\frac{d(\mu \circ \xi_{v,w})}{d\mu}(x) \leq exp\left(\sum_{g \in G} \phi(\sigma_g(\xi_{v,w}(x))) - \phi(\sigma_g(x))\right).$$

G Bounds when $v \leftrightarrow w$

As a corollary we extend the results of Meyerovitch to the amenable group subshift setting:

- **H., 2024**: Let $X \subset \mathcal{A}^G$ be any subshift, let $\phi \in C(X)$ have summable variation, and let μ_ϕ be an equilibrium state for ϕ. Then μ_ϕ is topologically Gibbs.

Thank you!

Questions?