Modal Logic Notes

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Notes

From last time

$\tau = (\mathbf{O}, \rho), \rho : \mathbf{O} \mapsto \mathbf{N}, \mathbf{O} \neq \emptyset$
$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \triangle(\phi_1, \ldots, \phi_n)$ where $n = \rho(\triangle)$
$ML(\Phi, \tau)$

Complexity and Modal Complexity

Complexities are always finite numbers. To prove formulas have certain properties, use proof by induction.

Modal complexity—count the maximum number of nested operators; something like $C_\triangle(p) = 0, C(p) = 0, C(\neg\phi) = C(\phi)+1$ or $C_\triangle(\phi\vee\psi) = max\{C_\triangle(\phi), C_\triangle(\psi)\}$. Also $C_\triangle(\triangle'(\phi_1, \ldots, \phi_n)) = max\{C_\triangle(\phi_1), \ldots, C_\triangle(\phi_n)) + 1$

Models

Given some Φ and a $\tau = (\mathbf{O}, \rho)$, we say a frame for (Φ, τ) is $\mathbf{F} = (W, R_\triangle)_{\triangle\in\mathbf{O}}$.

So if the arity of the diamond $\rho(\triangle) = n$, then $R_\triangle \subseteq W^{n+1}$

You can think of W as your set of possible worlds, dots, states, moments of time, etc. The relations $R_\triangle$ will also have different interpretations. Typically they are called accessibility relations, as they say which nodes in the graph can "access" which others.

To give an interpretation to the propositional variables, you need to extend the interpretation to tell you which p's are true at each world.

So we say that a model for (Φ, τ) is a frame $\mathbf{F}$ together with a valuation V.

$$\mathbf{F} = (W, R_\triangle)_{\triangle\in\mathbf{O}}, V : \Phi \mapsto \mathbf{P}(W)$$

So we say a model $M = (W, R_\triangle, V)_{\triangle\in\mathbf{O}}$. Note that a frame is represented in the definition.

Given some model, you could get something like $V(p) = \{w, v\}$ and $V(q) = \{w, u\}$, which means that p is true at worlds w, v and q is true at worlds w, u.

Think of p as "peaceful" and q as "happy".

Satisfaction (truth)

$$M, w \models \phi \;\Leftrightarrow\; w \models_M \phi \;\Leftrightarrow\; M \models_w \phi$$

We say $M, w \models p$ iff $w \in V(p)$. So $M, w \models \neg\phi$ iff $\neg(M, w \models \phi)$
Then $M, w \models \phi \vee \psi$ iff either $M, w \models \phi$ or $M, w \models \psi$ (or both).

$$M, w \models \triangle(\phi_1, \ldots, \phi_n) \quad \Leftrightarrow \quad \exists w_1, \ldots, w_n : (R_\triangle(w, w_1, \ldots, w_m) \text{ and } M, w_1 \models \phi_1, \ldots, M, w_n \models \phi_n)$$

Likewise, we can do the same thing for ∇. Recall the definition:

$$\nabla(\phi_1, \ldots, \phi_n) = \neg\triangle(\neg\phi_1, \ldots, \neg\phi_n)$$

So we get

$$M, w \models \nabla(\phi_1, \ldots, \phi_n) \quad \Leftrightarrow \quad \forall w_1, \ldots, w_n : R_\triangle(w, w_1, \ldots, w_n) \Rightarrow M, w_1 \models \phi_1, \ldots, M, w_n \models \phi_n$$

Basic Temporal Logic

Uses only unary modalities, but has more than one. We introduce two more modalities F and P, which can be thought of as "sometime in the future/past," respectively. You can also think of them as $\Diamond_f$ and $\Diamond_p$.

We introduce the dual of $\Diamond_f$, which we will denote by G. The other dual will be denoted by H.

A frame will assign a binary relation to both F and P. So we get something like

$$\mathbf{F} = \langle W, R_F, R_P \rangle; R_F, R_P \subseteq W^2$$

A bi-directional model/frame (respectively) is one s.t. $R_p = R_F^\cup$ where the $\cup$ denotes "converse."

Example: Suppose I have a binary relation $R \subseteq W^2$. Then we define the converse as:

$$R^\cup = \{(w, w') \in W^2 : (w', w) \in R\}$$

We also have <u>composition</u>, denoted by the semi-colon ;
Suppose I have two relations R and R'. We say that $f \circ g = g \,;\, f$.
So we have

$$R \,;\, R' = \{(w, w'') \in W^2 : \exists w' \in W \text{ s.t. } (w, w') \in R \text{ and } (w', w'') \in R'\}$$

We also define the diagnoal

$$I = \{(w, w) : w \in W\}$$

and the special diagonal

$$I_A = \{(w, w) : w \in A\}$$

We also define the transitive closure of R and denote it as R^+:

$$R^+ = R \cup R \,;R \cup R \,;R \,;R \cup \ldots = \bigcup_{n \geq 1} R^n$$

We also define R^* as the reflexive, transitive closure of R:

$$R^* = R^+ \cup I = I \cup R \cup R^2 \ldots$$

R^+ is the smallest relation s.t.

1. $R \subseteq R^+$
2. R^+ is transitive

R is the immediate successor, so $(n, m) \in R$ iff $m = n + 1$
Exercise: Define R^+ using R^* and composition, union.