Emmanuel Vargas
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Math 131
022D

Assignment 5

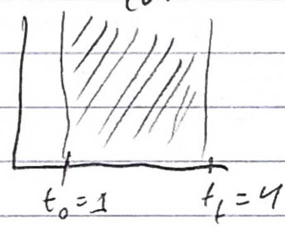
4. IVP well-posed? If so, find Lipschitz constant

$$y'(t) = -\frac{4}{t^3}y - t\ln t \quad 1 \le t \le 4 \quad y(1) = -1$$

- If IVP is well posed then it must be defined on a convex set and satisfy the lipschitz condition.

$$\left|\frac{\partial f}{\partial y}\right| \le L$$

- Is domain convex? ✓



$\boxed{\text{yes}}$

domain is convex

- Does it satisfy lipschitz cond $\left|\frac{\partial f}{\partial y}\right| \le L$? yes

-Find $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = -\frac{4}{t^3}$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|-\frac{4}{t^3}\right|$$

$$\left|\frac{\partial f}{\partial y}\right| = \left|-\frac{4}{1}\right| = |-4| = 4$$

- On interval $1 \le t \le 4$ thus, $\frac{\partial f}{\partial y}$ will be largest at $t = 1$, so this is the worst case scenario

- The Lipschitz constant L, $\approx \left|\frac{\partial f}{\partial y}\right|$

$$\left|\frac{\partial f}{\partial y}\right| = -\frac{4}{t^3}$$

, worst case at t worst case!

$L \ge 0$, valid constant ✓

$$\left|\frac{\partial f}{\partial y}\right| = \left|-\frac{4}{1}\right| = |-4| = 4 \approx L$$

$L = 4$, $\left|\frac{\partial f}{\partial y}\right| = 4$

$4 \le 4$ ✓

The lipschitz condition is satisfied w/ $\boxed{L = 4}$ so the IVP is well posed.

5. Find Taylor Method of order 3 and its LTE. What order is it?

- First start w/ a 3rd order taylor polynomial

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + err$$

- To turn into Taylor method, evaluate taylor polynomial of y at t_i evaluated at $t_i + \Delta t$
  - replace f to y $\quad x_0$ to t_i $\quad x$ to $t_i + \Delta t$

$$y(t_i+\Delta t) = y(t_i) + y'(t_i)(t_i+\Delta t - t_i) + \frac{y''(t_i)}{2}(t_i+\Delta t - t_i)^2 + \frac{y'''(t_i)}{6}(t_i-\Delta t - t_i)^3$$

- Simplify

$$\boxed{y(t_i+\Delta t) = y(t_i) + y'(t_i)\Delta t + \frac{y''(t_i)}{2}\Delta t^2 + \frac{y'''(t_i)}{6}\Delta t^3} + err$$

Find LTE

- The error term for taylor polynomial of order 3

$$\to \frac{y''''(\xi)}{4!}\Delta t^4 = \frac{y''''(\xi)}{24}\Delta t^4$$

- Solve for the error term when added to the taylor method

$$y(t_i+\Delta t) - y(t_i) - y'(t_i)\Delta t - \frac{y''(t_i)}{2}\Delta t^2 - \frac{y'''(t_i)}{6}\Delta t^3 = \frac{y''''(\xi)}{24}\Delta t^4$$

- must try to get left term to contain an approx of $y'(t_i)$
- To do this divide by Δt

$$\underbrace{\frac{y(t_i+\Delta t) - y(t_i)}{\Delta t} - y'(t_i)}_{\text{This is approx of } y(t_i)} - \frac{y''(t_i)}{2}\Delta t - \frac{y'''(t_i)}{6}\Delta t^2 = \underbrace{\frac{y''''(\xi)}{24}\Delta t^3}$$

- LTE is now the right hand side of the equation

$$\boxed{LTE = \frac{y'''(\xi)}{24}\Delta t^3}$$

- from the LTE we can see this method is $\boxed{O(\Delta t^3)}$