Composing rotations in 2-D and 3-D

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Introduction

A rotation can be characterised by an an angle and something to rotate around – an axis (in 3-D) or a point (in 2-D). If we apply two rotations in succession to an object, i.e., if we compose them, what can we say about the result? What single transformation could have taken the starting object to the final destination?

In some cases the answer is easy, for example if we are working in 2-D and both rotations are about the same point. A harder example is when the rotations are in 3-D about non-intersecting non-parallel (skew) axes. Such an example is shown in Figure 1 where we start with the green object on the flat grey plane. This is then rotated (by about a quarter turn) about the line through P and then rotated about the line through Q (again by about a quarter turn) to give the orange shape.

If we want to go from the green to the orange shape directly, is there a single rotation that would work? If not, is it possible with by combining a rotation and a translation? In either case, what can we say about the rotation (and any possible 'extra' translation)?

These notes look at different cases in turn and try to answer these questions. If we know the characteristics of the two rotations, i.e. their axes and angles, *can* we characterise their composition as a rotation? If so, what are its axis and angle? If not, what kind of transformation is the result?

In what follows, I try and get as far as possible in answering the questions about composition above using basic geometry. This will take up Chapter 1. After that, a more algebraic approach is given in Chapter 2 to cover what I could not solve with basic geometry. Code that implements the methods described is available at github.com/evariste/isometries

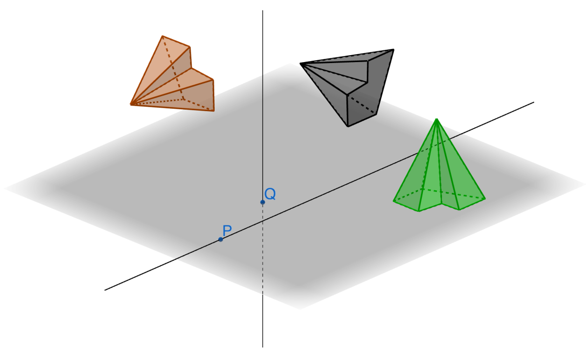

FIGURE 1. The green object on the plane is rotated about the line through P to give the grey object. This is then rotated about the line through Q to give the brown one. Can we directly go from start to end in a single rotation?

Why not use matrices or quaternions?

It is possible to model rotations, using matrices, which are an excellent way to represent linear transformations. Using homogeneous coordinates, we can use matrices to represent both rotations about the origin, rotations about points other than the origin and, more generally, any isometries[1]. There are formulas for converting between axis-angle representations and matrix representations and the composition of rotations can be reduced to matrix multiplication.

Quaternions can also be used to represent rotations in 3-D in a very powerful way that also reduces composition to multiplication. Both matrices and quaternions are used for efficient computations involving rotations and their usefulness is widely recognised. However, there is a sense of a 'black-box' about their use. In applying a rule to convert rotations to a structure such as a matrix or quaternion, then carrying out operations on these structures and finally retrieving the result, the geometry of the how rotations interact is hidden to us. These notes aim try to keep the descriptions of operations close to the simple geometric properties of the transformations.

Further links

Some further reading:

Composition of rotations about non-intersecting axes:

math.stackexchange.com/questions/2964608

Can composing a pair of 3-D rotations about skew axes lead to a rotation?

math.stackexchange.com/questions/4999941

Link to interactive Geogebra sheet to demonstrate composition of rotations in 2-D.

This page looks at composition of two rotations in 3-D with axes that *do* intersect at a common point.

[1]Transformations that preserve distances.

Contents

CHAPTER 1

Basic geometry

1.1. Rotations in 2-D about the same point.

The easiest case is when the rotations are in 2-D and both are about the same point. This is illustrated in Figure 1 (left) where the object (an L-shape) starts at the position marked a. It is then rotated about point O by an angle α, to reach position b. Then a second rotation about O, with angle β, is applied to the object to reach the position labelled c.

Clearly, the single transformation that can replace these two rotations is single rotation about O by an angle of $\alpha + \beta$.

In this case (2-D, same point), the order of rotations does not matter (they commute). If we first apply the rotation by angle β, followed by the rotation by α, the result is still the same as shown in Figure 1 (right).

1.2. Rotations in 2-D about different points.

Remaining in 2-D, but allowing the rotations to be about different points makes things a bit more complex. An example illustration is given in Figure 2 which shows a shape starting at position a and rotated about point O to give the shape at b. The shape at b is then rotated about P to give the shape at C.

We can go from a to c directly with a single rotation about the point X. In this case, if we are given the points O and P and the *signed angles* of rotation about these points, can we find the centre, X, for the single rotation and the angle of rotation needed?

Convention: an angle is positive if it is anti-clockwise when viewed in the 2-D plane.

Sometimes, composing a pair of rotations in 2-D can lead to a translation. This is shown in Figure 3. The rotations change both the position and the orientation

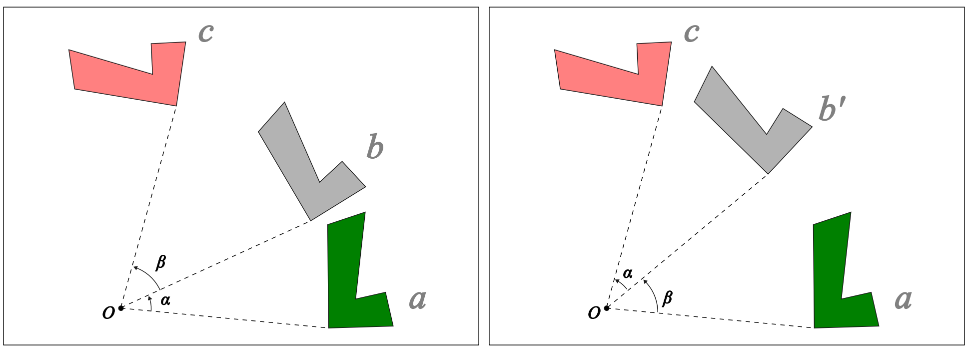

FIGURE 1. Left: Two rotations about the same point in 2-D. Right: Reversing the order of the rotations.

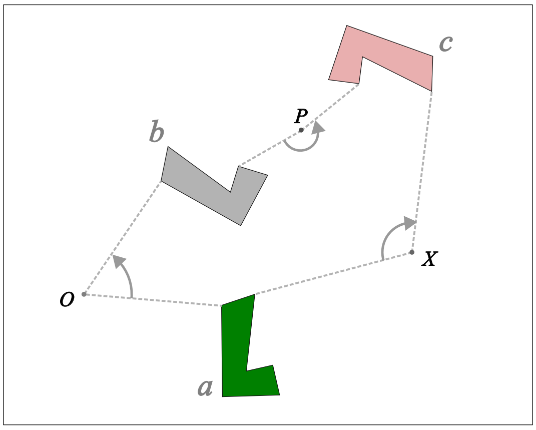

FIGURE 2. Composing two 2-D rotations with different centres. Shape a is rotated about O to give shape b which, in turn, is rotated about P to give shape c. A single rotation about X achieves the same result.

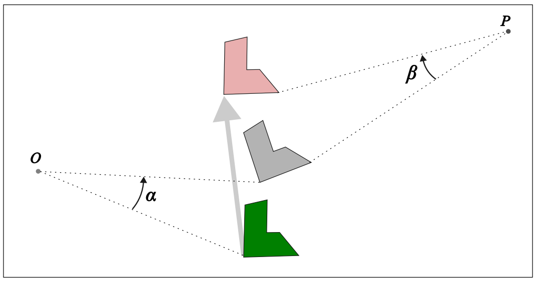

FIGURE 3. Composing two rotations about different centres can sometimes lead to a translation.

of an object and, in this case, the change in orientation due to the first rotation is 'undone' by the second rotation. The resulting object therefore only differs from the starting one by a change in position – i.e., a translation.

Before looking at how to characterise the composition of rotations in 2-D with different centres, we will look next at how a *single* rotation can be represented by reflections.

1.2.1. A single rotation can be expressed as two reflections.

If we have a pair of lines through a point P, we can treat them as mirror lines and reflect in each line successively. This is shown in Figure 4 (Left) where the object a is first reflected in the solid line through P to give shape b, which is then reflected in the dashed line (also through P) to give c.

Shape c is the mirror image of a mirror image of a, it can be reached from a by a rotation about the point P and the angle of rotation is twice the angle between the lines of reflection. This is illustrated in Figure 4 (Right).

The rotation that we get when reflecting in a pair of lines through a point P depends on:

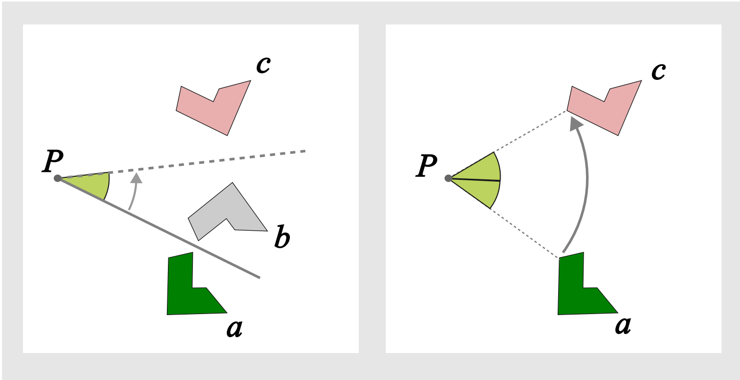

FIGURE 4. Two reflections that lead to a rotation. Left: Reflecting shape a in the solid line gives shape b, which is reflected in the dashed line to give c. Right: a can be mapped directly to c with a rotation about P by double the angle between the original lines.

- The position of the point P.
- The angle turned when going from the first line to the second.

The rotation *does not* depend on the orientations of the lines, only on the angle between them. This is shown in Figure 5 where a centre point P is fixed and two different pairs of lines through P are chosen, each with the same angle from the first (solid) line to the second (dashed) line. In each case, successively reflecting in the solid line then the dashed one leads to the same rotation, one that takes shape a to shape c. For an illustration of why two reflections in intersecting lines give a rotation by double the angle between them, see Section 3.4.

For completeness, we can look at the case where lines used for the two reflections *do not* intersect at a point, i.e., they are parallel. This is illustrated in Figure 6 where two reflections, one in the solid line, followed by a reflection in the (parallel) dashed line can be replaced with a translation from shape a to shape c by a distance that is double the distance between the lines[1].

CHECKPOINT 1. *Given two lines l and m in 2-D, we can define a transformation T by carrying out a reflection in l followed by a reflection in m. There are two cases:*

- *The lines l and m intersect at some point P: In this case, T is a rotation about P by two times the angle from l to m.*
- *The lines l and m are parallel: In this case, T is a translation in the direction from l to m by a distance that is two times the distance between the lines.*

1.2.2. Composed rotations via reflections. As shown in Section 1.2.1, a rotation about a point can be expressed as a pair of reflections. We also note that the lines used for the reflection can point in any direction we like, as long as the

[1]More detail on why the distance of translation is double the distance between the lines is given in Section 1.2.2 and Figure 9

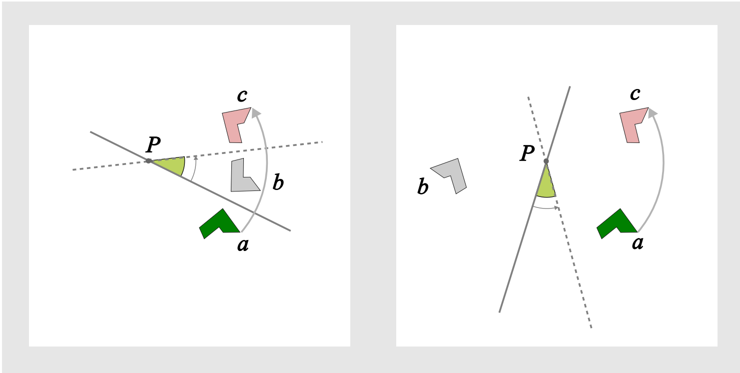

FIGURE 5. If we fix a point P and reflect successively in two lines through P, then we get the same rotation even if the orientation of the lines is varied. In each diagram above, the successive reflections (in the solid then dashed line) can be replaced with the same single rotation to take a to c. Note that, even though they have different orientations, the (signed) angle from the solid to the dashed line is the same for each pair.

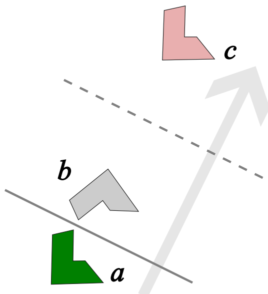

FIGURE 6. Left: Reflecting shape a in the solid line gives shape b and reflecting b in the dashed line gives c. Right: a can be mapped directly to c with a translation perpendicular to the lines by double the distance between them.

angle from the first reflection line to the second is fixed. If the angle between the lines is θ then the angle of rotation will be 2θ.

Figure 7 (Left), illustrates the composition of two rotations. The first, R_O, is about point O by an angle of 2α and the second, R_P, is about the point P by an angle of 2β. We write the angles in the form 2α, 2β for convenience as these will be twice the angles between line pairs for the rotations.

The composition, $R_Q = R_P \circ R_O$, where R_O is applied first, is about the point Q by an angle of γ.

We can select any pair of reflection lines to represent R_O, as long as they go through O and the angle from the first to the second is α. A similar argument applies to R_P.

Specifically, we can choose reflection lines for the rotations so that one is shared by them – this needs to be the line through O and P, call this line m (See Figure 7, right). We can then ensure that m is the second reflection line for R_O and the first reflection line for R_P. This determines the first line, l, for R_O. It must go through

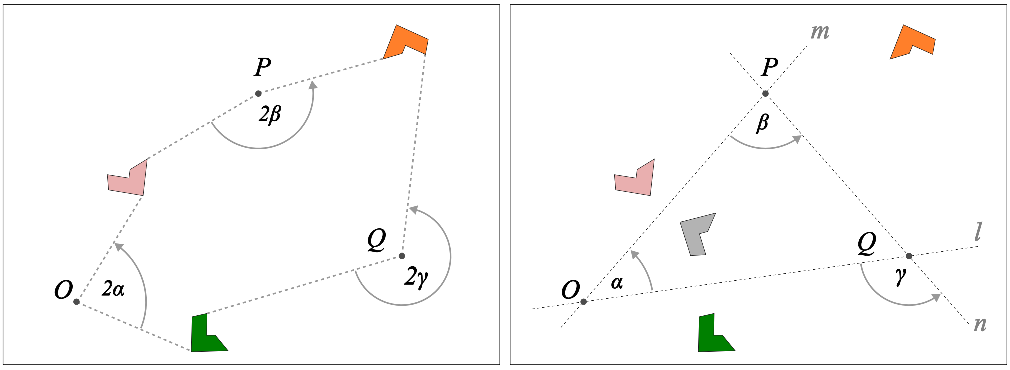

FIGURE 7. Left: A rotation R_O about O followed by a rotation R_P about P is equivalent to the rotation R_Q. Right: We can choose lines of reflection for the rotations such that one of them, m, is shared for both rotations. Applying the pairs of reflections for the rotations R_O then R_P takes us through the shapes in the order: *green* $\to$ *grey* $\to$ *pink* $\to$ *grey* $\to$ *orange*. This provides lines of reflection for the composition R_Q: Reflecting in l followed by m gives the shape sequence *green* $\to$ *grey* $\to$ *orange*.

O, such that the angle from l to m is α. Similarly, the second line, n, for R_P also becomes fixed, it must go through P so that the angle from m to n is β.

Let S_m represent a reflection in the line m with the same notation for the other lines.

We can now write $R_O = S_m \circ S_l$ and $R_P = S_n \circ S_m$.

The composition is therefore $R_Q = R_P \circ R_O = S_n \circ S_m \circ S_m \circ S_l$.

Let's drop the circle notation and simply write $R_Q = S_n\, S_m\, S_m\, S_l$.

We can see this sequence of four reflections in Figure 7 (Right) by tracking the shapes starting with the green one. Applying the reflections gives the following order:

$$\textit{green} \xrightarrow{S_l} \textit{grey} \xrightarrow{S_m} \textit{pink} \xrightarrow{S_m} \textit{grey} \xrightarrow{S_n} \textit{orange}$$

Applying a reflection then applying the same reflection gets us back to where we started, because a reflection is its own inverse, it 'cancels' itself out. So we have $S_m\, S_m = I$, the identity transformation and we get

$$R_Q = S_n\, S_m\, S_m\, S_l = S_n\, I\, S_l = S_n\, S_l. \tag{1}$$

In summary, the composition can be written as two reflections in lines through Q: $R_Q = S_n\, S_l$. The first reflection is in line l and the second is in line n, so the direction of the rotation can be taken as going from l to n. The order of the shapes for R_Q is

$$\textit{green} \xrightarrow{S_l} \textit{grey} \xrightarrow{S_n} \textit{orange}$$

In the right of Figure 7, we can also see the relationship between the angles: $\gamma = \alpha + \beta$. Because the angles of rotation are double those between the reflection lines, we can see that the angle of $R_Q = R_P R_O$ is simply the sum of the signed angles for the two rotations R_O and R_P: $2\gamma = 2\alpha + 2\beta$. Recall that we take as a convention that an anti-clockwise turn is positive and a clockwise turn is negative.

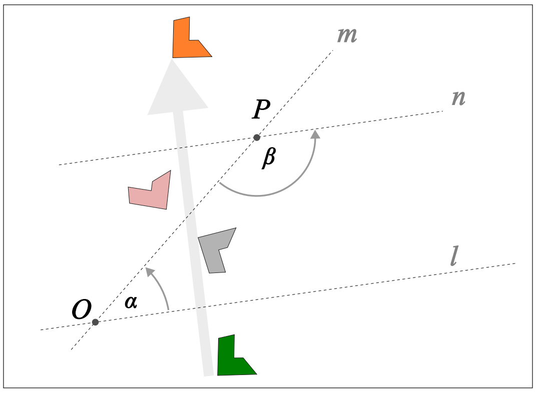

FIGURE 8. The reflections for a rotation about O followed by those for a rotation about P result in a translation from the first shape (green) to the last (orange). This is because the lines of reflection for the composition are parallel (See Figure 6).

The centre, Q, of the composed rotation can be found by calculating the intersection of lines l and n.

One special case we need to consider is when the non-shared lines for the reflections l and n are parallel - in this case there is no intersection point to act as a centre of rotation. This is illustrated in Figure 8.

Also, in this case, given that the angles between the line pairs are either complementary or alternating with respect to the two parallel lines, their sum will be a multiple of π (including zero). Therefore, the rotation angles, being twice the angles between the line pairs, will sum to a multiple of 2π (including zero).

However, the composition of the rotations can still be obtained by applying the same pair of reflections, i.e., $R_Q = S_n\ S_l$, but as l and n are parallel, the net effect is a translation in a direction perpendicular to l (and n).

If we track the signed distance of a point going through such a pair of reflections, $S_n\ S_l$, we can see that the distance translated is twice the distance d between the two lines, given by the sum $d - x + d - x + x + x = 2d$ as shown in Figure 9.

1.2.3. Summary: A method for 2-D rotations. We are given a pair of rotations in 2-D, R_A and R_B about the points A and B, where R_A has a signed angle of θ and R_B has an angle of ϕ.

When trying to characterise the composition, we have three cases:

(1) Both rotations are about the same point, A and B are identical.
(2) Centres A and B are different and ...
   (a) $\theta + \phi = 2\pi k$ for some integer k.
   (b) $\theta + \phi$ is some other value.

Case 1: Same centre. The composed rotation is about the same point by an angle of $\theta + \phi$.

Cases 2: Different centres. First carry out the following:

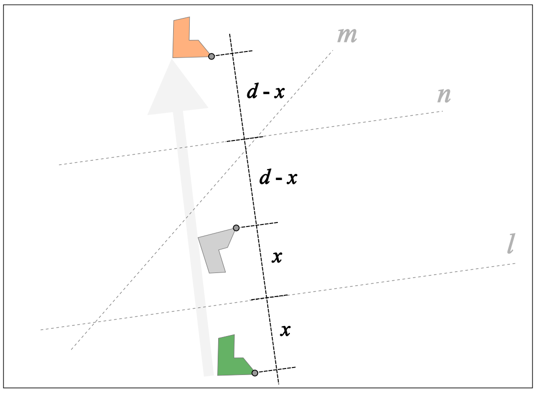

FIGURE 9. Reflecting in a pair of parallel lines that are a distance d apart. A starting point on the green object is at a distance x from the first reflection line. Tracking the results after the reflections leads to a total translated distance of $2d$.

- Find the line through A and B, call this m.
- Rotate m by $-\theta/2$ about A to give line l
- Rotate m by $\phi/2$ about B to give line n

Now, consider the sum of the angles for the following two sub-cases

Case 2(a): $\theta + \phi = 2\pi k$, $k \in \mathbb{Z}$.

We get a translation (Figure 8) with lines l and n parallel.

We can find the details of the translation by following the steps:

- Find any vector perpendicular to l in the direction of n. Call this $\mathbf{t}$
- Find the perpendicular distance d between lines l and n.
- The translation obtained by composing the rotations is in the direction of $\mathbf{t}$ by a distance $2d$.

Case 2(b): $\theta + \phi \neq 2\pi k$.

The composition gives another rotation (Figure 7).

We can find the details of the rotation by following the steps:

- Lines l and n are not parallel. Find their intersection point C.
- The rotation we seek is about the point C through an angle of $\theta + \phi$.

CHECKPOINT 2. *Composing 2-D rotations.*

Given a pair of rotations in 2-D, consider the centre points of each one and their angles of rotation. In particular determine whether the centres are the same or different, and find the sum of the angles.

This will then determine which case to apply from Section 1.2.3 and will provide the details for the composition which will either be a translation or a rotation.

1.3. Rotations in 3-D

Now we look at composing a pair of rotations in 3-D. Some of the methods we used for 2-D rotations in previous sections can be adapted to 3-D. In particular, it is still possible to represent a rotation as a pair of reflections. For the 2-D case, the reflections are in lines that intersect at the centre of rotation (see Figure 4), while for the 3-D case, the reflections are in *planes* that intersect in a line. This line forms the axis of rotation as illustrated in Figure 10.

1.3.1. The reflections for a 3-D rotation.

As there was in the 2-D case, there is some choice in how we represent the rotation, namely a choice in the reflection planes used. We can select any pair of planes, with orientations of our choice, as long as:

- they intersect at the rotation axis
- the angle between them is half the rotation angle.
- we are consistent about the order in which we apply the reflections so that it matches the direction of the required angle

See Figure 5 for the 2-D illustration of how the orientation of the two lines can be freely chosen.

Instead of trying to visualise the planes of reflection directly, as in Figure 10, we can show them as great circles where they intersect a sphere that is placed so that its centre lies on the planes' line of intersection (the rotation axis). This is shown in Figure 11 where the first plane is represented by the blue circle and the second by the red circle. The axis of rotation goes through the intersections of the circles (or the intersection of the planes). This visualisation will be used later when we look at composing rotations in 3-D about intersecting axes.

In later sections, we look at different cases for composing 3-D rotations based on their axes. When both axes can be contained in a single plane (are co-planar), we can distinguish the sub-case where the axes intersect at a point from one in which they are parallel. Finally, there is the case where the axes are not co-planar, meaning they are neither parallel nor intersecting, i.e., they are skew.

1.3.2. Characterising a 3-D rotation with an axis and signed angle.

We can characterise the axis of a rotation simply by a unit vector because the direction of the axis is the only important piece of information. We use the hat notation for a unit vector, e.g., $\hat{\boldsymbol{u}}$.

With regard to the sign of the angle, in the 2-D case, we defined a positive angle of rotation as anti-clockwise when viewing the 'standard' xy-plane with the x axis going to the right and the y-axis going up. In 3-D, we need to do a bit more to define the direction of the rotation. We can adopt a standard convention of, say, the right-hand rule[2], in which the rotation is clockwise when viewing *along* the direction of the axis and that the clockwise direction is taken as positive.

The use of a direction vector and the right-hand rule means that the same rotation can be represented in two different ways, i.e., a rotation about an axis with direction $\hat{\boldsymbol{u}}$ and angle θ can also be represented by a rotation about the (anti-parallel) axis with direction $-\hat{\boldsymbol{u}}$ and angle $-\theta$.

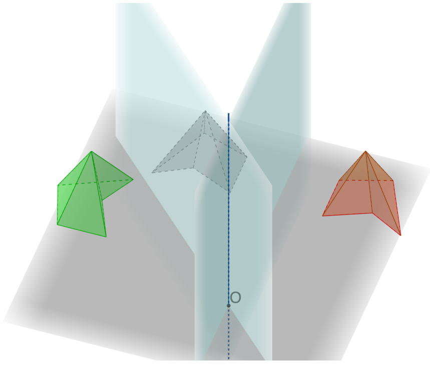

FIGURE 10. A rotation in 3-D can be carried out by applying reflections in two planes. The axis of rotation is the line of intersection of the planes. The angle of rotation is double the angle from the first plane to the second. Here, the red object is rotated by approximately a quarter of a turn about the line through O to give the green object. The grey object is intermediate, produced after the first reflection.

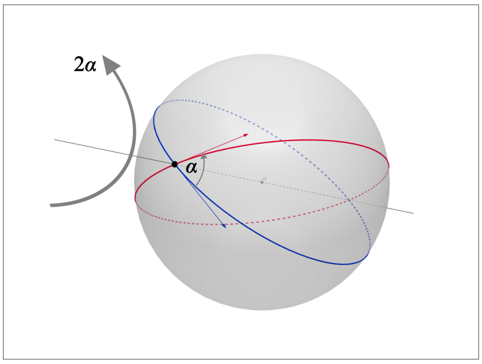

FIGURE 11. Another way of visualising a pair of reflections in 3-D in planes that intersect. Here the planes are represented by the great circles (one blue, one red) where they meet a sphere centred on the line of intersection of the planes. The axis of rotation is the line through the circle intersection points. The angle of rotation is double the angle, α, from the blue 'plane' to the red one (we reflect in the blue plane first).

1.4. Rotations about intersecting axes

This section focuses on 3-D rotations where the axis passes through the origin. For a pair of rotations with axes that intersect at some other point, we can always change the coordinate system with a shift to ensure the axes pass through O so we do not lose generality by treating the intersection point as the origin.

In Section 1.2.2, we looked at how a composition of rotations could be carried out by representing them as reflections such that they share a reflection which ends up being applied twice in succession and 'cancels' itself out. This method can be adapted for 3-D rotations with intersecting axes.

[2] en.wikipedia.org/wiki/Right-hand_rule

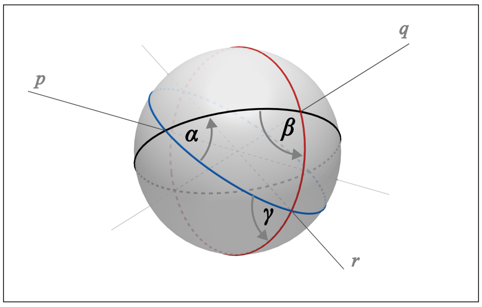

FIGURE 12. Choosing reflection planes for rotations so that one is shared. A rotation about p by 2α is achieved by reflecting in the blue and black 'planes'. A 2β rotation about q by reflections in black and red. The composition is done by reflecting in the blue and red planes. This is the 3-D version of Figure 7 (right).

Let R_p and R_q be rotations about origin-intersecting axes p and q. We want to characterise $R_q \circ R_p$. We can again choose planes of reflection such that one of the planes is shared[3] This is done so that the shared plane is the second for R_p and the first for R_q.

This is illustrated in Figure 12 where R_p is carried out by reflecting first in the blue plane then the black one. The reflections for R_q are first in the black, then the red. In the composition the black plane reflections cancel each other out, so that it can be carried out by reflecting in the blue then the red planes. This is equivalent to the derivation in Equation 1.

1.4.1. A method for 3-D rotations about axes through the origin. If we have a pair of rotations R_p and R_q about axes through the origin with direction vectors $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$. Let the angles of rotation for R_p and R_q be $\theta = 2\alpha$ and $\phi = 2\beta$ respectively and let the composed rotation be $R_c = R_q R_p$.

Case 1: $\hat{\boldsymbol{p}} = \hat{\boldsymbol{q}}$

The axes are identical. R_c has axis $\hat{\boldsymbol{p}}$ and angle $\theta + \phi$

Case 2: $\hat{\boldsymbol{p}} = -\hat{\boldsymbol{q}}$

The axes are anti-parallel. R_c has axis $\hat{\boldsymbol{p}}$ and angle $\theta - \phi$

Case 3: Neither of the above

In this case, the lines containing $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$ represent distinct directions so we can define a plane that contains O and is spanned by $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{q}}$.

Let this plane be s, this will be the shared plane (illustrated by the black circle in Figure 12).

Rotate plane s about axis $\hat{\boldsymbol{p}}$ by an angle of $-\alpha$ to give plane t.

Rotate plane s about axis $\hat{\boldsymbol{q}}$ by an angle of β to give plane x.

The intersection of planes t and x provides the axis of rotation for R_c.

[3]An algebraic method to show why it is possible to choose such planes is given in Section 2.8.

The angle of rotation for R_c, γ, can be calculated by finding and doubling the angle between planes t and x. (In the 2-D case we were able to use the relation $\gamma = \alpha + \beta$ but this will no longer work for the spherical triangle in Figure 12).

1.4.2. A method for 3-D rotations about intersecting axes. We can now summarise the results of this Section in the checkpoint below.

CHECKPOINT 3. *Composing 3-D rotations about intersecting axes:*

- *If the axes intersect at a point* $\mathbf{p}$ *other than the origin apply a shift in coordinates so that the intersection is the origin,* $\mathbf{x} \to \mathbf{x} - \mathbf{p}$.
- *Apply the method of Section 1.4.1 to characterise the composed rotation.*
- *If a shift was applied at the start, this can now be reversed to give the final result, i.e.,* $\mathbf{x} \to \mathbf{x} + \mathbf{p}$

1.5. 3-D rotations about parallel axes

This section covers one further case for co-planar axes before we later look at composing arbitrary rotations in 3-D.

If two 3-D rotations are about axes that are parallel and do not coincide, then we can treat this as effectively a 2-D problem. There is a single plane that is perpendicular to both axes and the projection onto this plane gives the 2-D representation. This is illustrated in Figure 13

The cases to be handled for 2-D rotations are listed in Section 1.2.3. We can assume the axes do not coincide, because this is covered by the intersecting axes cases 1 and 2 in Section 1.4.1.

So, the relevant 2-D cases are 2a and 2b in Section 1.2.3 which leads to either a rotation or to a translation, depending on whether the rotation angles add up to a multiple of 2π.

1.6. 3-D rotations about arbitrary axes

In Section 1.4 we found a method for composing rotations about axes that intersect at the origin. Here we start by considering a single rotation about an axis through an arbitrary point, not necessarily the origin. This will later be used to develop a method for composing a pair of such 'general' rotations with axes are not co-planar (not intersecting and not parallel).

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1.6.1. Decomposing a general rotation. When its axis goes through some arbitrary point, we can decompose a rotation into a sequence of simpler steps as follows:

- A translation to shift the axis onto the origin
- A rotation about an axis through the origin that is parallel to the required axis

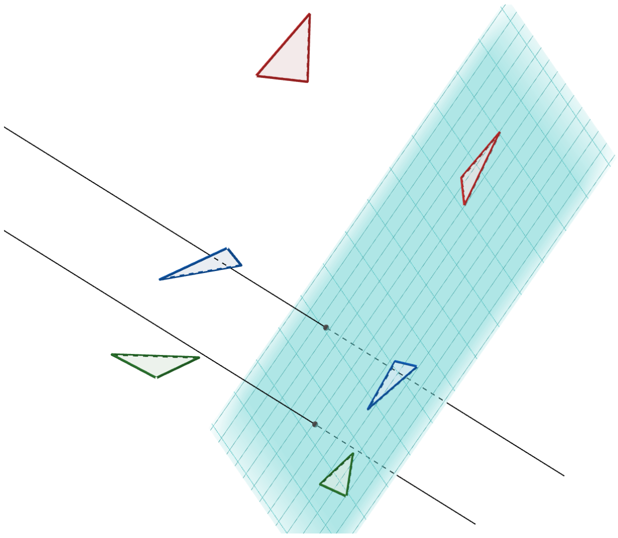


FIGURE 13. Two 3-D rotations about parallel axes. Projecting onto a plane perpendicular to both axes reduces this to a 2-D problem. The points of intersection of the axes with the plane define the 2-D centres of rotation.

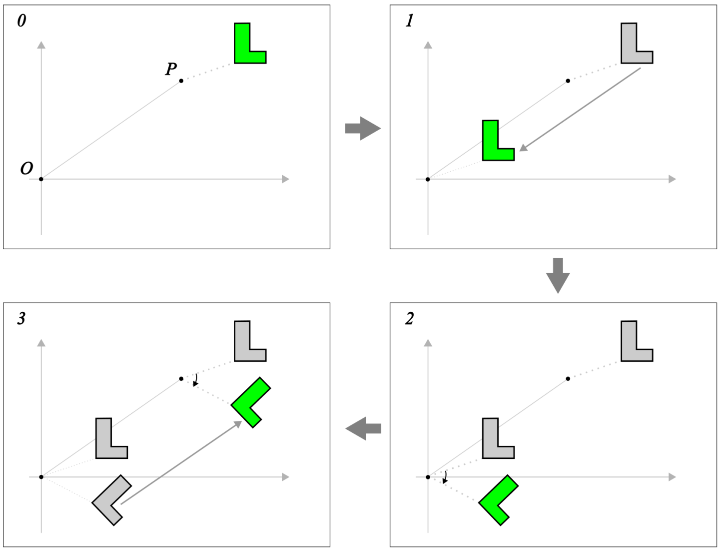


FIGURE 14. A rotation about an arbitrary point can be decomposed into three steps. Reading clockwise: 0) Start position, 1) a translation by $P \to O$, 2) a rotation about O, 3) a translation $O \to P$. This generalises to 3-D.

- A translation to push the axis back to its first location

This is illustrated for the 2-D case in Figure 14.

The following parts will use a notation taken from Artin [**1**].

CHECKPOINT 4. *Let $\mathcal{M}$ denote a rotation about an axis going through the origin O. Treat $\mathcal{M}$ as a function that maps points, i.e., we write $\mathbf{a} \to \mathcal{M}(\mathbf{a})$ for some point $\mathbf{a}$.*

If the direction of the rotation axis is $\hat{\boldsymbol{l}}$ and the angle of rotation is θ, we can choose to make these explicit by writing $\mathcal{M}_{\hat{\boldsymbol{l}},\theta}$, or we can simply write $\mathcal{M}$.

Denote a translation by a vector $\mathbf{p}$ as $t_{\mathbf{p}}$, i.e. $t_{\mathbf{p}}(\mathbf{x}) = \mathbf{x} + \mathbf{p}$.

1.6.2. Representing 'general' rotations: A three-step form. Let a 'general' rotation be one about an axis, with a direction vector $\hat{\boldsymbol{l}}$ that goes through some arbitrary point $\mathbf{p}$, i.e., not necessarily the origin O.

As discussed, such a rotation can be broken down into separate steps.

Step	Transformation
1. Translate so that point $\mathbf{p}$ is shifted to O	$t_{-\mathbf{p}}$
2. Rotate about an axis through O with direction $\hat{\boldsymbol{l}}$	$\mathcal{M}$
3. Translate so that the O is shifted back to $\mathbf{p}$	$t_{\mathbf{p}}$

Such a rotation can therefore be written as $t_{\mathbf{p}}\ \mathcal{M}\ t_{-\mathbf{p}}$, where $\mathcal{M}$ represents the rotation about an axis through O with direction $\hat{\boldsymbol{l}}$.

CHECKPOINT 5. ***A three-step form for a general rotation***

Consider a rotation about an axis that goes through an arbitrary point $\mathbf{p}$ with a direction given by the unit vector $\hat{\boldsymbol{l}}$. The rotation can be written as

$$t_{\mathbf{p}}\ \mathcal{M}\ t_{-\mathbf{p}}$$

where $\mathcal{M} = \mathcal{M}_{\hat{\boldsymbol{l}},\theta}$ is a rotation about an axis through the origin with direction $\hat{\boldsymbol{l}}$.

1.6.3. Two-step transformations: Composing a translation and an origin rotation. In the previous section, we looked at a three-step form for a general rotation. Here, we consider a two-step transformation made by composing a translation and a rotation about O. This will be later used to develop a more compact form for general rotations.

We have a choice when composing the translation and rotation to generate the transformation, i.e., which to apply first. Let us look at the case where the translation is applied first so that the two step transformation is written $\mathcal{M}\ t_{\mathbf{a}}$ where the translation is through the vector $\mathbf{a}$.

We can show that the two-step transformation $\mathcal{M}\ t_{\mathbf{a}}$ is equivalent to another two-step transformation where the rotation $\mathcal{M}$ is applied first. In fact:

$$\mathcal{M}\ t_{\mathbf{a}} = t_{\mathbf{a}'}\ \mathcal{M} \qquad \text{where} \qquad \mathbf{a}' = \mathcal{M}(\mathbf{a})$$

The above is true because

$$\begin{aligned}
\mathcal{M}\ t_{\mathbf{a}}(\mathbf{x}) &= \mathcal{M}(\mathbf{x} + \mathbf{a}) && \text{apply the translation} \\
&= \mathcal{M}(\mathbf{x}) + \mathcal{M}(\mathbf{a}) && \text{a rotation about origin is linear} \\
&= \mathcal{M}(\mathbf{x}) + \mathbf{a}' \\
&= t_{\mathbf{a}'} \circ \mathcal{M}(\mathbf{x}) && \text{express as composition} \\
&= t_{\mathbf{a}'}\ \mathcal{M}(\mathbf{x}) && \text{dropping circle notation}
\end{aligned}$$

where the second equality applies because a rotation about the origin is linear (see Section 3.3).

The above shows that the order of applying a rotation and a transformation does not really matter apart from the difference in the translation that is applied. The same overall transformation can be obtained by carrying out the rotation $\mathcal{M}$ first or second[4].

CHECKPOINT 6. *A two-step transformation consisting of a rotation $\mathcal{M}$ about an axis through O and a translation can be represented in two ways: We can apply $\mathcal{M}$ first or second and the possible forms are*

$$\mathcal{M}\, t_{\mathbf{a}} = t_{\mathbf{a}'}\, \mathcal{M} \text{ where } \mathbf{a}' = \mathcal{M}(\mathbf{a}) \tag{2}$$

1.6.4. Converting a rotation between three- and two-step forms. Using Equation 2, we can show that a general rotation in three-step form can be re-written in two-step form:

$$\begin{aligned}
t_{\mathbf{p}}\, \mathcal{M}\, t_{-\mathbf{p}} &= t_{\mathbf{p}}\ [\mathcal{M}\, t_{-\mathbf{p}}] \\
&= t_{\mathbf{p}}\ \left[t_{\mathcal{M}(-\mathbf{p})}\, \mathcal{M}\right] \\
&= t_{\mathbf{p}}\ \left[t_{-\mathcal{M}(\mathbf{p})}\, \mathcal{M}\right] \qquad \mathcal{M} \text{ is linear} \\
&= t_{\mathbf{p}}\ \left[t_{-\mathbf{p}'}\, \mathcal{M}\right] \qquad \text{setting } \mathbf{p}' = \mathcal{M}(\mathbf{p}) \\
&= \left[t_{\mathbf{p}}\, t_{-\mathbf{p}'}\right]\ \mathcal{M} \\
&= t_{\mathbf{p}-\mathbf{p}'}\, \mathcal{M} \qquad \text{replace two translations with one} \\
&= t_{\mathbf{q}}\, \mathcal{M} \qquad \text{where } \mathbf{q} = \mathbf{p} - \mathbf{p}'
\end{aligned}$$

where we can see that both the two-step form and the three-step form contain the same rotation step ($\mathcal{M}$) but a translation is needed for the two-step form ($t_{\mathbf{p}-\mathbf{p}'}$) that is different to either of the translations in the three-step form.

Note that the vector $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ is the difference between $\mathbf{p}$ and its image $\mathbf{p}'$ under the rotation. Being related by the rotation, $\mathbf{p}$ and $\mathbf{p}'$ must both lie in a plane perpendicular to the axis of $\mathcal{M}$. This means that their difference, $\mathbf{q}$ is also perpendicular to the axis.

In other words, when written in two-step form, a rotation about an arbitrary axis can be viewed as a rotation about an axis through the origin, O, followed by a translation by a vector *perpendicular* to the rotation axis.

CHECKPOINT 7. *A general rotation in three-step form:*

$$t_{\mathbf{p}}\, \mathcal{M}\, t_{-\mathbf{p}}$$

is equivalent to a two-step transformation: a rotation about the origin, $\mathcal{M}$, followed by a translation $t_{\mathbf{q}}$

$$t_{\mathbf{p}}\, \mathcal{M}\, t_{-\mathbf{p}} = t_{\mathbf{q}}\, \mathcal{M} \tag{3}$$

where the vector $\mathbf{q} = \mathbf{p} - \mathcal{M}(\mathbf{p})$ is perpendicular to the axis of $\mathcal{M}$.

This is illustrated in Figure 15 which shows how an object is rotated about a general axis to give the brown shape. The same result can be obtained by applying a rotation

[4]It can be shown that every isometry can be expressed in the form $t_a M$, this includes the case where M is a reflection which we are not considering here.

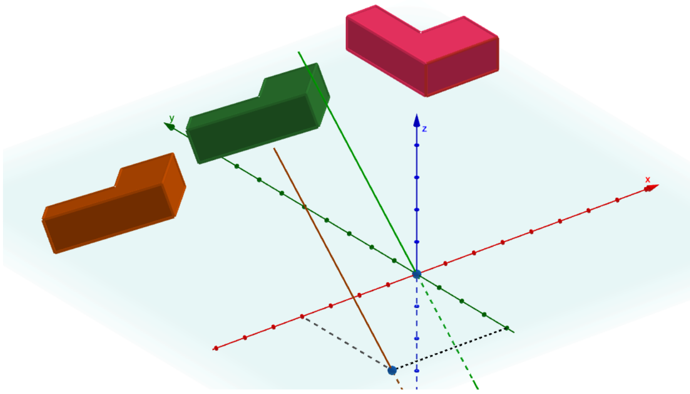


FIGURE 15. The red shape is rotated about an arbitrary axis (brown line). The same result can be obtained by rotating about a parallel axis through the origin (green) and then applying a translation perpendicular to the axis.

about a parallel axis through the origin (green line) and then applying a translation perpendicular to the axis.

1.6.5. Composing two general 3-D rotations. We can apply the two-step form of general rotations to the task of composing them.

Consider two general rotations, Φ and Θ, about axes through points $\mathbf{p}$ and $\mathbf{r}$. We will use the following three- and two-step representations:

Φ	Θ	Description
$t_{\mathbf{r}}\mathcal{K}t_{-\mathbf{r}}$	$t_{\mathbf{p}}\mathcal{L}t_{-\mathbf{p}}$	Three-step form, where $\mathcal{K} = \mathcal{K}_{\hat{\mathbf{a}},\phi}$ and $\mathcal{L} = \mathcal{L}_{\hat{\mathbf{b}},\theta}$
$t_{\mathbf{r}-\mathcal{K}(\mathbf{r})}\mathcal{K}$	$t_{\mathbf{p}-\mathcal{L}(\mathbf{p})}\mathcal{L}$	Re-writing in two-step form using Equation 3
$t_{\mathbf{s}}\mathcal{K}$	$t_{\mathbf{q}}\mathcal{L}$	Two-step form, writing $\mathbf{s} = \mathbf{r} - \mathcal{K}(\mathbf{r})$ and $\mathbf{q} = \mathbf{p} - \mathcal{L}(\mathbf{p})$

All the important properties of the rotations are known: the axis direction vectors $\{\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}$; the angles of rotation $\phi, \theta\}$; and the points through which the axes pass $\mathbf{p}, \mathbf{r}\}$. All the other variables can be derived from these properties.

Applying Θ then Φ gives the composition $\Phi\Theta$ where

$$\begin{aligned}
\Phi\Theta &= t_{\mathbf{s}}\mathcal{K}\, t_{\mathbf{q}}\mathcal{L} \\
&= t_{\mathbf{s}}\, t_{\mathcal{K}(\mathbf{q})}\, \mathcal{K}\mathcal{L} && \text{using Equation 2} \\
&= t_{\mathbf{s}+\mathcal{K}(\mathbf{q})}\mathcal{K}\mathcal{L} && \text{replace two translations with one} \\
&= t_{\mathbf{u}}\mathcal{K}\mathcal{L} && \text{writing } \mathbf{u} = \mathbf{s} + \mathcal{K}(\mathbf{q}) \\
&= t_{\mathbf{u}}\mathcal{M} && \text{writing } \mathcal{M} = \mathcal{K}\mathcal{L}
\end{aligned}$$

The transformation obtained by composing the rotations, $\Phi\Theta$, is characterised by the rotation $\mathcal{M}$ about the origin and the translation $t_{\mathbf{u}}$.

Rotations $\mathcal{K}$ and $\mathcal{L}$ are both around origin-intersecting axes and we denote their composition $\mathcal{K}\mathcal{L}$ as $\mathcal{M}$ (which is also about an axis through the origin). Making its axis and angle explicit, we can write $\mathcal{M} = \mathcal{M}_{\hat{\mathbf{c}},\psi}$. The direction of the axis through the origin, $\hat{\boldsymbol{c}}$, and the angle of rotation, ψ, can be found using the method described in Section 1.4.1 using the corresponding information from $\mathcal{K}$ and $\mathcal{L}$.

The translation in the composition above is through the vector $\mathbf{u}$ where

$$\begin{aligned}
\mathbf{u} &= \mathbf{s} + \mathcal{K}(\mathbf{q}) \\
&= \mathbf{r} - \mathcal{K}(\mathbf{r}) + \mathcal{K}(\mathbf{p} - \mathcal{L}(\mathbf{p})) \\
&= \mathbf{r} - \mathcal{K}(\mathbf{r}) + \mathcal{K}(\mathbf{p}) - \mathcal{K}\mathcal{L}(\mathbf{p})
\end{aligned}$$

which can be summarised as

$$\mathbf{u} = \mathbf{r} - \mathcal{K}(\mathbf{r} - \mathbf{p}) - \mathcal{K}\mathcal{L}(\mathbf{p}) \tag{4}$$

The above shows that $\mathbf{u}$ can be calculated directly from the properties of the original general rotations: the axis points, $\mathbf{p}$ and $\mathbf{r}$, and the origin-rotations $\mathcal{L}$ and $\mathcal{K}$.

Checkpoint 8. *For a pair of general 3-D rotations, $\Phi = t_{\mathbf{r}}\mathcal{K}t_{-\mathbf{r}}$ and $\Theta = t_{\mathbf{p}}\mathcal{L}t_{-\mathbf{p}}$, the transformation resulting from the composition $\Phi \circ \Theta$ can be written in two step form:*

$$\Phi\ \Theta = t_{\mathbf{u}}\mathcal{M}$$

where $\mathcal{M} = \mathcal{K}\mathcal{L}$ and $\mathbf{u} = \mathbf{r} - \mathcal{K}(\mathbf{r} - \mathbf{p}) - \mathcal{K}\mathcal{L}(\mathbf{p})$. The axis and angle for the rotation $\mathcal{M} = \mathcal{M}_{\hat{\mathbf{c}},\psi}$, can be calculated using the method in Section 1.4.1.

The above shows that we are able to compose two rotations to get a two-step representation, as an origin-rotation and a translation, $t_{\mathbf{u}}\mathcal{M}$. When we do this, what can we say about the result?

We know from Equation 3, that if the translation vector $\mathbf{u}$ is perpendicular to the rotation axis of $\mathcal{M}$, then the result is a (general) rotation. But, the composition may not have such a translation vector (one that is perpendicular to the rotation axis), so what can we say about the transformation *in general*? We look at this question in Section 1.6.6.

1.6.6. What can we say about a two-step transform $t_{\mathbf{u}}\ \mathcal{M}$? In Section 1.6.5, we looked at how composing a pair of rotations can provide a two-step representation for the resulting transformation.

Assume we are given such a two-step transform, $t_{\mathbf{u}}\mathcal{M}$, where $\mathcal{M} = \mathcal{M}_{\hat{\mathbf{c}},\theta}$. To characterise this transformation, we look at the cases where:

- Case 1: $\mathbf{u}$ is perpendicular to $\hat{\boldsymbol{c}}$
- Case 2: $\mathbf{u}$ is parallel to $\hat{\boldsymbol{c}}$
- Case 3: There is an arbitrary angle between $\mathbf{u}$ and $\hat{\boldsymbol{c}}$.

Cases 1 and 2 can be treated as special cases of the more general Case 3. We consider the cases in turn in the following sections. Each time, we assume that the key properties of the two-step transformation are non-trivial, in particular:

$$\mathbf{u} \neq \mathbf{0} \text{ and } \theta \neq 0$$

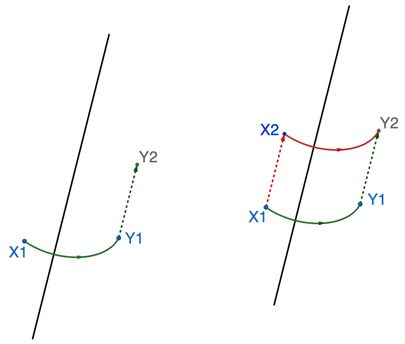


FIGURE 16. A twist or screw transformation. Left: A point $X1$ undergoing a rotation (to $Y1$) followed by a translation by a vector parallel to the rotation axis, ending with $Y2$. Right: The same result would be obtained if we had translated first and then rotated $X1 \to X2 \to Y2$.

1.6.6.1. *Case 1:* $\mathbf{u}$ *is perpendicular to* $\hat{\mathbf{c}}$*.* Here, we can characterise the composition result $t_{\mathbf{u}}\ \mathcal{M}$ as a general rotation (Equation 3), meaning that we can write it in the three-step form $t_{\mathbf{p}}\mathcal{M}t_{-\mathbf{p}}$ for some vector $\mathbf{p}$ that needs to be determined.

If we have $t_{\mathbf{p}}\mathcal{M}t_{-\mathbf{p}} = t_{\mathbf{p}-\mathbf{p}'}\mathcal{M}$ where $\mathbf{p}' = \mathcal{M}(\mathbf{p})$, then, to find $\mathbf{p}$, we need to solve for it in the equation

$$t_{\mathbf{u}}\mathcal{M} = t_{\mathbf{p}-\mathbf{p}'}\mathcal{M}$$

Recall that $\mathbf{u}$ is known, so we need to solve for $\mathbf{p}$ in the equation $\mathbf{p} - \mathcal{M}(\mathbf{p}) = \mathbf{u}$. When $\mathbf{u}$ is perpendicular to $\hat{\mathbf{c}}$ this can be done using basic geometry and the details are given in Section 3.5.

1.6.6.2. *Case 2:* $\mathbf{u}$ *is parallel to* $\hat{\mathbf{c}}$*.* In this case, we can show that the result of the composition $t_{\mathbf{u}}\mathcal{M}$ is neither a rotation nor a translation, it is a 'screw' or 'twist' transformation.

The effect of applying a rotation to a point, followed by a translation parallel to the rotation axis is illustrated in Figure 16 (Left).

As an aside, note that, because the translation vector and rotation axis are parallel, the same result is obtained whether the rotation is applied before or after the translation, as shown in Figure 16 (Right). We can write $\mathbf{u} = \alpha\hat{\mathbf{c}}$, for some scalar value α, so $\mathbf{u}$ lies on the axis of rotation $\mathcal{M}$ which means that $\mathcal{M}(\mathbf{u}) = \mathbf{u}$.

For some point $\mathbf{x}$,

$$\mathcal{M}t_{\mathbf{u}}(\mathbf{x}) = \mathcal{M}(\mathbf{u}+\mathbf{x}) = \mathcal{M}(\mathbf{u}) + \mathcal{M}(\mathbf{x}) = \mathbf{u} + \mathcal{M}(\mathbf{x}) = t_{\mathbf{u}}\mathcal{M}(\mathbf{x})$$

Therefore, when $\mathbf{u}$ is parallel to $\hat{\mathbf{c}}$, for rotation $\mathcal{M} = \mathcal{M}_{\mathbf{c},\theta}$ and translation $t_{\mathbf{u}}$, we have

$$t_{\mathbf{u}}\mathcal{M} = \mathcal{M}t_{\mathbf{u}}$$

The composition is not a rotation because any 3-D rotation must leave an entire line of points unchanged: the axis of rotation. Applying the transformation $t_{\mathbf{u}}\mathcal{M}$ leaves *no* points unchanged. After applying the first step, $\mathcal{M}_{\mathbf{c},\theta}$, only the axis is left unchanged (the line through O with direction $\mathbf{c}$. However, the next step applies a non-zero translation along the axis. This will modify all points on the axis. Points not on the axis, that were rotated by the previous step, cannot be returned to their

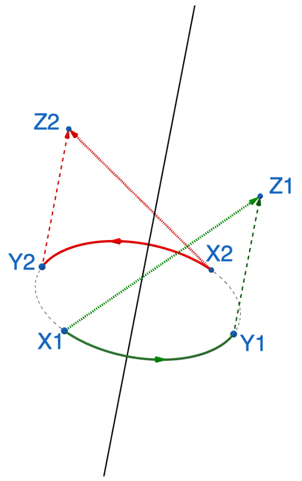


FIGURE 17. A two-step 'twist' transformation $t_{\mathbf{u}}\mathcal{M}$, where $\mathbf{u}$ parallel to the rotation axis of $\mathcal{M}$. Applying it to two points, $X1$ and $X2$, we can see that the displacements they undergo are different. This means that such a transformation cannot be a translation.

starting position because that would require a translation perpendicular to the axis. This means that all points are modified by the combined transformation.

The composition is not a translation because, for a translation, there is a fixed displacement between every point $\mathbf{x}$ and its image, $\mathcal{M}(\mathbf{x}) + \mathbf{u}$. This would require that for any point $\mathbf{x}$, the displacement $\mathcal{M}(\mathbf{x}) + \mathbf{u} - \mathbf{x}$ is fixed. But it is very easy to find points that undergo different displacements under this particular composite transformation $t_{\mathbf{u}}\mathcal{M}$ and a simple example is shown in Figure 17.

In two-step form, the composition $t_{\mathbf{u}}\mathcal{M}$ is in fact represented as simply as possible, and cannot be further decomposed. This kind of transformation is sometimes called a screw transformation (en.wikipedia.org/wiki/Screw_axis) in fields such as robotics. It has some similarities with a glide reflection in 2-D (en.wikipedia.org/wiki/Glide_reflection), which is obtained by composing a reflection with a translation parallel to the axis of reflection.

1.6.6.3. *Case 3: There is an arbitrary angle between* $\mathbf{u}$ *and* $\hat{\mathbf{c}}$. When the directions of the translation vector $\mathbf{u}$ and the axis $\hat{\mathbf{c}}$ for rotation $\mathcal{M}$ are free to vary at some arbitrary angle to each other, we can apply the results from the previous two cases to characterise the transformation.

We can separate the vector $\mathbf{u}$ into two components where one is parallel to $\hat{\mathbf{c}}$ and the other is perpendicular to $\hat{\mathbf{c}}$.

$$\mathbf{u} = \mathbf{w} + \mathbf{s} \qquad \text{where } \mathbf{w} \parallel \hat{\mathbf{c}} \text{ and } \mathbf{s} \perp \hat{\mathbf{c}}$$

So we can re-write the translation $t_{\mathbf{u}}$ as $t_{\mathbf{w}+\mathbf{s}} = t_{\mathbf{w}}\, t_{\mathbf{s}}$. This means we can re-write the two-step transformation:

$$t_{\mathbf{u}}\mathcal{M} = t_{\mathbf{w}}\, t_{\mathbf{s}}\mathcal{M}$$

Using Case 1 (Section 1.6.6.1), we see that the last two terms, $t_{\mathbf{s}}\mathcal{M}$ represent a general rotation. If the vector $\mathbf{s}$ is zero, then $t_{\mathbf{s}}\mathcal{M} = \mathcal{M}$, which is a rotation about the origin.

Either way, represent the general rotation $t_{\mathbf{s}}\mathcal{M}$ by Θ, so we can write

$$t_{\mathbf{u}}\mathcal{M} = t_{\mathbf{w}}\, t_{\mathbf{s}}\mathcal{M} = t_{\mathbf{w}}\, \Theta$$

If the vector $\mathbf{w}$ is non-zero, then the result is a screw transformation where the axis of Θ and $\mathbf{w}$ are parallel, using Case 2 (Section 1.6.6.2).

1.6.7. A method for general rotations in 3-D. We can now assemble these results to characterise the transformation obtained by composing two rotations.

CHECKPOINT 9. *For a pair of general 3-D rotations, $\Phi = t_{\mathbf{r}}\mathcal{K}t_{-\mathbf{r}}$ and $\Theta = t_{\mathbf{p}}\mathcal{L}t_{-\mathbf{p}}$, generate the two-step form $\Phi\Theta = t_{\mathbf{u}}\mathcal{M}$, where $\mathcal{M} = \mathcal{M}_{\hat{\mathbf{c}},\theta}$, using the method described in Checkpoint 8.*

Consider the vector $\mathbf{u}$ and the axis of rotation, $\hat{\mathbf{c}}$, for $\mathcal{M}$:

- *If $\mathbf{u}$ is zero, the composition $\Phi\Theta$ is a rotation about an axis through the origin.*
- *If $\mathbf{u}$ is perpendicular to $\hat{\mathbf{c}}$, the composition $\Phi\Theta$ is a general rotation about some axis not intersecting the origin.*
- *If $\mathbf{u}$ is parallel to $\hat{\mathbf{c}}$, then $\Phi\Theta$ is a screw transformation with a rotation about an axis through the origin.*
- *If $\mathbf{u}$ is at some arbitrary angle to $\hat{\mathbf{c}}$, then $\Phi\Theta$ is a screw transformation with a general rotation.*

1.7. Discussion

Further questions:

What are the conditions under which composing a pair of 3-D rotations leads to a another rotation (rather than a screw transform)?

The answer to this is that a rotation can only be produced by composing two rotations with axes that are co-planar. A proof using basic geometric methods is not given here but a later proof will be given after a presentation of algebraic methods for characterising isometries and orthogonal transformations (Section 2.16).

Can composing rotations about non-coplanar axes lead to a translation? My guess is that the answer is now but this is still TODO.

CHAPTER 2

Algebraic methods

2.1. Orthogonal transformations

An *orthogonal transformation* T (Wikipedia page) is

- *Linear*: For points $\mathbf{x}$, $\mathbf{y}$, and a scalar c
  - $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
  - $T(c\mathbf{x}) = cT(\mathbf{x})$.
- *Preserves inner products*: Writing the inner product of $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{x} \cdot \mathbf{y}$, we have $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

These properties can be used to show that an orthogonal transformation also

- *Preserves distances between points*: For points $\mathbf{x}$, $\mathbf{y}$, we have $|T(\mathbf{x}-\mathbf{y})| = |\mathbf{x}-\mathbf{y}|$
- *Preserves angles between lines*: This a consequence of preserving distances, three points in a triangle will get mapped to points with the same pairwise distances, i.e., to a congruent triangle so that the angles between the edges are the same.
- *Maps the zero vector to itself*: for any orthogonal transformation, T, $T(\mathbf{0}) = \mathbf{0}$.

2.2. Isometries

An *isometry* is any transformation that

- preserves distances
- preserves angles (implied by the preservation of distances)

An isometry does not necessarily satisfy the other properties listed earlier for orthogonal transformations (linearity, inner-product preservation, mapping the zero vector to itself).

All orthogonal transformations are isometries but not all isometries are orthogonal.

An example of a 2-D isometry that is *not* orthogonal is a translation $t_v(x) = x + v$ where the vector $v = (1, 0)$. The origin is mapped to $(1, 0)$ so it is not mapped to itself. If we consider the points $x = (1, 0)$ and $y = (1, 1)$ then we have $x + y = (2, 1)$ and

$$t_v(x) = (2,0) \qquad t_v(y) = (2,1) \qquad t_v(x+y) = (3,1) \qquad t_v(x)+t_v(y) = (4,1)$$

So the linearity property fails because $t_v(x+y) \neq t_v(x) + t_v(y)$.

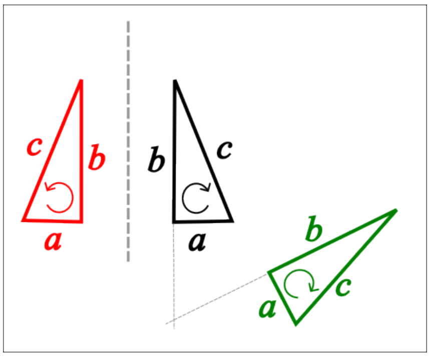


FIGURE 1. *Transformations and orientation.* The black triangle is reflected to give the red one and this is an example of an orientation-changing transformation: tracing a path along the edges, in order of size, gives a clockwise path in the original black triangle and an anti-clockwise path in the reflected red triangle. If the black triangle is rotated, to give the green one, this is orientation-preserving: tracing the edges in order of size gives clockwise paths in both cases. Both the transformations, the reflection and the rotation, are isometries and all the triangles are congruent.

Another example of a 2-D isometry that is not orthogonal is any rotation about a point that *not* the origin. An illustration of why a non-origin rotation is not linear is given in Section 3.3.

2.3. Classifying orthogonal transformations

2.3.1. Orientation-preserving vs orientation-changing. One way of distinguishing orthogonal transformations is by how they affect the orientation of objects. A simple 2-D example is shown in Figure 1 where a black triangle is reflected to give the red triangle. The triangles are both congruent, having identical length sides. If the side lengths are all distinct and are, in order of increasing size, a, b, c, then the edges in order of size trace a clockwise path on the black triangle but trace an anti-clockwise path on the red reflected version - this represents the *change of orientation*. If, on the other hand, we rotate the original triangle, to give the green one, then tracing the edges in order of size still results in a clockwise path which means the rotation *preserves orientation*.

In 3-D, one way to think about this is by considering three unit vectors that make up a coordinate frame, for example, the $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ vectors along the coordinate axes. Such a triplet of vectors can be described as having a 'handedness', i.e., it can form a right-handed or a left-handed system.

We can draw the x and y axes on a page so that the x axis (direction $\hat{\boldsymbol{\imath}}$) goes to the right and the y axis (direction $\hat{\boldsymbol{\jmath}}$) goes upwards. For a left-handed coordinate frame, the direction of the z-vector ($\hat{\boldsymbol{k}}$) would then be going into the page. In a right-handed frame, the opposite is true, the $\hat{\boldsymbol{k}}$ direction would be coming out of the page towards the viewer. These are illustrated in Figure 2.

One kind of orthogonal transformation (linear, preserving lengths and angles) is a reflection in a plane through the origin. If we apply a reflection to a coordinate

FIGURE 2. Left: A left-handed coordinate frame. Right: A right-handed coordinate frame. This particular pair of frames are related by a reflection in a plane containing the $\hat{\imath}$ and $\hat{\jmath}$ vectors confirming that a reflection changes handedness (orientation).

frame, then its handedness will be changed: a left-handed frame will be reflected to generate a right-handed frame and vice-versa.

Rotations are another kind of orthogonal transformation and the handedness of a frame is not changed under rotation. To see why, we recall that a rotation can be represented by two successive reflections. The first reflection will change the orientation while the second will also change orientation, thus undoing the effect of the first and giving a composed transformation (rotation) that preserves orientation.

In this context, we say that orthogonal transformations can be

- *Orientation-preserving*
- *Orientation-reversing*

depending on whether they keep the handedness of objects (like coordinate frames) or modify them.

If we successively apply two transformations S and T then the composition that results will preserve or change orientation depending on the nature of S and T:

Preserves orientation?

S	T	$S \circ T$
yes	yes	yes
yes	no	no
no	yes	no
no	no	yes

For example, in the second row, S could be a rotation and T could be a reflection with their composition will be orientation-changing, being either a reflection or an 'improper rotation' (see Section 2.3.2 for more detail).

2.3.2. Classifying orthogonal transformations by their fixed points.

The set of points fixed by an orthogonal transformation gives another way to categorise them.

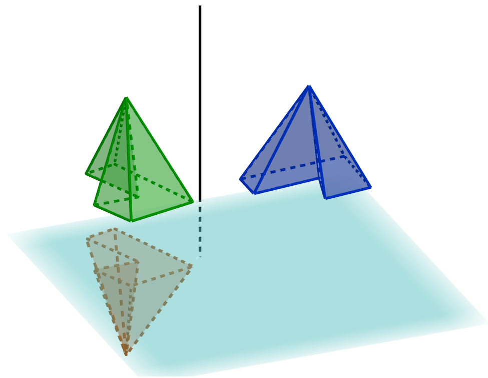


FIGURE 3. An illustration of an improper rotation. The blue pyramid is rotated about the axis (black line) to give the green pyramid which is then reflected in the plane to give the brown pyramid. The plane is perpendicular to the rotation axis.

Let T be some orthogonal transformation, we say $\mathbf{x}$ is a fixed point if $T(\mathbf{x}) = \mathbf{x}$.

The zero vector at the origin is mapped to itself by an orthogonal transformation so $\mathbf{0}$ is always a fixed point.

If $\mathbf{x}$ and $\mathbf{y}$ are two fixed points of an orthogonal transformation T, then $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{x} + \mathbf{y}$ where the first equality uses the linearity of T. Also, for a fixed point $\mathbf{x}$ of T and for any scalar value c, $T(c\mathbf{x}) = cT(\mathbf{x}) = c\mathbf{x}$.

So the fixed points of an orthogonal transformation form a *linear subspace*, adding pairs of fixed points gives a fixed point, and scaling them also gives fixed points.

Focusing on three dimensions, there are only a limited number of linear sub-spaces of $\mathbb{R}^3$. These are: all of $\mathbb{R}^3$, planes through the origin, lines through the origin and the space consisting of the single zero vector at the origin: $\{\mathbf{0}\}$.

So, we can classify an orthogonal transformation T according to the kind of subspace that it fixes:

(1) *All of* $\mathbb{R}^3$: The entire space is fixed, therefore T is the identity transformation $T(\mathbf{x}) = \mathbf{x}$ for any point $\mathbf{x}$.

(2) *A plane through the origin*: In this case, T is a reflection in the plane.

(3) *A line through the origin*: In this case, T is a rotation where the given fixed line is the rotation axis.

(4) *The single point at the origin*: In this case, T is an 'improper rotation' which can be obtained by applying a (non-trivial) rotation about an axis followed by a reflection in a plane perpendicular to the axis (Wikipedia page). Both the axis and the plane contain the origin. Figure 3 gives an illustration of an improper rotation.

We can consider the different cases according to how the resulting transformations affect orientation. The identity transformation (case 1) and rotations (case 3) are orientation-preserving orthogonal transformations. A single reflection (case 2) and an improper rotation[1] (case 4), are orientation-reversing.

[1] which can be represented by three reflections

2.4. Classifying isometries in 3-D

We can bring together some of the ideas of the previous sections to categorise isometries which include, but are not limited to, orthogonal transformations.

The first way we will divide them is based on whether they have any fixed points. If a transformation does have a fixed point, we will further sub-divide according to the type of space formed by their fixed points.

2.4.0.1. *3-D isometries with at least one fixed point.* The following table considers isometries with at least one fixed point. They are divided according to whether the origin, O, is contained in the set of fixed points or not.

Fixed point(s)	Contain O	Transformation
All of $\mathbb{R}^3$	Yes	Identity
Plane	Yes	Orthogonal reflection in a plane through O
Line	Yes	Orthogonal rotation about an axis through O
Point	Yes	Orthogonal improper rotation for an axis and plane through O
Plane	No	Reflection in an arbitrary plane
Line	No	Rotation about an arbitrary axis
Point	No	Improper rotation for an arbitrary axis and plane.

The cases in the last three rows of the table, when the origin is not a fixed point of the transformation, are general versions of a rotation, reflection, or improper rotation - the do not fix the origin, are not linear and are therefore not orthogonal transformations.

2.4.1. 3-D isometries with no fixed points.

To list the types of 3-D isometry, T, that have no fixed points, we will represent them in the form $T = t_u M$, where M is a transformation that does have fixed points and is followed by a translation through a vector u.

We list three kinds of isometries that do not have fixed points[2]:

(1) T is a simple translation, in this case $M = I$, the identity transformation and $T = t_u$ for a non-zero vector u.

(2) T is a screw transformation[3] , this occurs when M is a rotation and u is a non-zero vector in the direction of the axis of rotation.

(3) T is a glide reflection[4] , this occurs when M is a reflection and u is a non-zero vector contained in the plane of reflection.

In case 1, applying a non-zero translation, will modify every point x, $T(x) = t_u(x) = x + u$, it is not possible for any point to remain fixed.

In case 2, the first step applies a rotation which affects every point except those on the axis. Each non-axis point is displaced by a vector contained in a plane perpendicular to the rotation axis. After that every point is translated in a direction parallel to the axis. None of the non-axis points can be returned to their starting position with this translation and all axis points are shifted along the axis. Thus all points are moved and there are no fixed points.

Case 3 is very similar, the first step is a reflection which affects all points not in the reflection plane and the second step is a translation by a vector parallel to the

[2] No proof is presented that the listing is complete.
[3] en.wikipedia.org/wiki/Screw_axis, Section 1.6.6.2
[4] en.wikipedia.org/wiki/Glide_reflection

reflection plane which will affect all points in the plane and cannot return any of the points moved in the first step to their starting position. Again, there are no fixed points.

2.5. Reflections in planes that are linked by a rotation

Here we show a relationship between reflections in planes that are linked by a rotation. We will use this later in Section 2.8.

Let P be a plane through the origin and let T be the transformation that reflects in this plane. Let R be some rotation about an axis through O. Let $P' = R(P)$ be result of applying the rotation to the plane P. Because the rotation axis for R and the plane P both go through the origin, the second plane, P', will also go through the origin. Let T' be the (linear) reflection in plane P'.

We will show that $T' = R\,T\,R^{-1}$.

For a point $\mathbf{x}$, let its image under the reflection in P be $\mathbf{y} = T(\mathbf{x})$. The line segment from $\mathbf{x}$ to $\mathbf{y}$ intersects the plane P at some point $\mathbf{q}$, say, which will be the mid-point of the line segment.

Rotation R is orthogonal so it will preserve distances and angles. Consider the three points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{q}$ The distances between them are preserved by R so the corresponding distances between $R(\mathbf{x})$, $R(\mathbf{y})$, and $R(\mathbf{q})$ will be the same. In particular, $R(\mathbf{q})$ will be the mid-point of $R(\mathbf{x})$ and $R(\mathbf{y})$. The angle preserving property can be used to show that the segment from $R(\mathbf{x})$ to $R(\mathbf{y})$ will also be perpendicular to the rotated plane P'. This shows that $R(\mathbf{x})$ and $R(\mathbf{y})$ are related to each other by a reflection in plane P'.

Writing this out, we can derive

$$\begin{aligned}
T'(R(\mathbf{x})) &= R(\mathbf{y}) \\
T' \circ R(\mathbf{x}) &= R(\mathbf{y}) \\
R^{-1} \circ T' \circ R(\mathbf{x}) &= \mathbf{y} \\
R^{-1} \circ T' \circ R(\mathbf{x}) &= T(\mathbf{x}) \\
R^{-1}\,T'\,R(\mathbf{x}) &= T(\mathbf{x})
\end{aligned}$$

The above shows the relationship between T (reflection in plane P) and T' (reflection in a rotated version of P). Because it is true for all points $\mathbf{x}$, we can simply write

$$R^{-1}\,T'\,R = T$$

which is equivalent to

$$T' = R\,T\,R^{-1} \tag{5}$$

2.6. Conjugation of transformations

In Section 2.5, we saw that one reflection T' was linked to another reflection T by the relation $T' = R\,T\,R^{-1}$ where R is a rotation that takes one plane of reflection to the other.

Preceding a transformation by the inverse of a second transformation then following it by the second is a common pattern and is called *conjugation*[5] where we say that T is conjugated by R to give T'.

We have already seen an example of conjugation (Checkpoint 5) to represent a general rotation in the form

$$t_{\mathbf{p}}\mathcal{M}t_{-\mathbf{p}}$$

Where $t_{\mathbf{p}}$ is a translation and $\mathcal{M}$ is a rotation about an axis through the origin. If we write the translation $t_{\mathbf{p}}$ as T, then we have $t_{-\mathbf{p}} = T^{-1}$ and the above can be re-written more clearly as a conjugation of $\mathcal{M}$ by T:

$$T\,\mathcal{M}\,T^{-1}$$

2.7. Two linear reflections make a rotation

Let P and Q be two planes that contain O, and let S_P and S_Q be the (linear) reflections in P and Q respectively. What can we say about the result of composing the reflections: $S_Q\,S_P$?

If the planes P and Q are identical then the reflections are identical and each will undo the other, so, in this case, $S_Q\,S_P = S_P\,S_Q = I$, the identity transformation (which can be viewed as a rotation by an angle of zero).

So, assume P and Q are distinct planes. The composition of reflections, $T = S_Q\,S_P$ represents the successive application of two orientation-reversing transformations. Therefore T must be orientation-preserving (See Section 2.3.1). The only orientation preserving orthogonal transformations are the identity and rotations. The reflection planes are distinct so T is not the identity meaning it must be a rotation.

There is also a geometric argument for why the composition of reflections in distinct planes through O, $T = S_Q\,S_P$, must be a rotation. This is illustrated for the 2-D case in Section 3.4, Figure 2.

For the 3-D case, with planes P and Q distinct, if we consider a point $\mathbf{x}$ with $\mathbf{y} = S_P(\mathbf{x})$ and $\mathbf{z} = S_Q(\mathbf{y}) = S_Q\,S_P(\mathbf{x})$, then note that the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are all contained in a single plane that is perpendicular to the line of intersection of P and Q (recall P and Q both must go through O, so they must intersect).

If we consider the points $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in this plane, the argument for the 2-D case can be applied. The line of intersection of the planes is fixed by both reflections, therefore it is fixed by their composition and this makes it the axis of rotation.

2.8. We have choice in the reflections to generate a rotation

This has already been illustrated for the 2-D case in Figure 5. We have also used this property when finding a method to compose two 3-D rotations about intersecting axes in Section 1.4.

Here, we can show that there is such a choice algebraically using the result in Equation 5.

Let P and Q be planes through O, and let S_P and S_Q be the reflections in P and Q respectively.

We have shown that $S_Q\,S_P$ is a rotation with the line of intersection, l, of the planes as its axis, let's call the rotation R.

[5] See Wikipedia entry

Let M be some other rotation about l and let $P' = M(P)$ and $Q' = M(Q)$ be the result of applying the rotation to each of the planes P and Q.

The reflections in the rotated planes P' and Q' are $S_{P'}$ and $S_{Q'}$.

With Equation 5, we can write the reflection in each rotated plane in terms of the rotation M and the original reflections:

$$S_{P'} = M\ S_P\ M^{-1} \qquad\qquad S_{Q'} = M\ S_Q\ M^{-1}$$

Now we can write the composition of reflections in the rotated planes $S_{Q'}S_{P'}$:

$$S_{Q'}S_{P'} = M\ S_Q\ M^{-1}M\ S_P\ M^{-1} = M\ S_Q\ S_P\ M^{-1} = M\ R\ M^{-1}$$

The rotation M, its inverse M^{-1}, R are all about the same axis so the order in which they are applied does not affect the result. This means we can write

$$S_{Q'}S_{P'} = M\ R\ M^{-1} = R\ M\ M^{-1} = R = S_Q S_P$$

which shows that $S_{Q'}S_{P'} = S_Q S_P$, i.e., we obtain the same rotation, whether we reflect successively in the original planes or in the planes after rotating them by M. This shows that the rotation we obtain only depends on the planes intersecting in the same axis and that the angle between them is the same. Their orientation about the axis does not matter.

2.9. A 3-D orthogonal transformation can be represented by at most three reflections

This section presents an non-constructive argument for why a 3-D orthogonal transformation can be represented by at most three reflections. See Section 2.10 for a proof that is constructive and generates the reflections needed.

To be specific, a 3-D orthogonal transformation can be represented by the composition of one, two, or three reflections in planes that pass through the origin, O.

There are only four kinds of 3-D orthogonal transformation: The identity, a reflection, a rotation, and an improper rotation (See Section 2.3.2).

Let T be a 3-D orthogonal transformation, if T is the identity, we can represent it by two successive reflections in a single plane. If T is a reflection, then, by definition, it can be represented by one reflection. If T is a rotation, we can represent by two successive reflections in planes that intersect along the rotation axis. The angle between the planes needs to be half the angle of rotation. If T is an improper rotation, then it is the result of applying a rotation (two reflections in planes through the axis) followed by a reflection in a plane perpendicular to the axis - i.e., three rotations altogether.

2.10. A 3-D orthogonal transformation can be represented by at most three reflections – Constructive proof

For an alternative proof, let M be an orthogonal 3-D transformation and consider an orthonormal[6] set of basis vectors $\{i, j, k\}$ for $\mathbb{R}^3$ (for example $i = (1, 0, 0)$, $j = (0, 1, 0)$, and $k = (0, 0, 1)$). Every point x in $\mathbb{R}^3$ can be expressed as a linear combination of $\{i, j, k\}$: $x = \alpha i + \beta j + \gamma k$, for some scalar values α, β, and γ.

Let the basis vectors be mapped to $\{u, v, w\}$ under the transformation, i.e., $M(i) = u$, $M(j) = v$, $M(k) = w$. M preserves angles and distances, meaning that $\{u, v, w\}$ is also an orthonormal set of vectors that can act as a basis of $\mathbb{R}^3$.

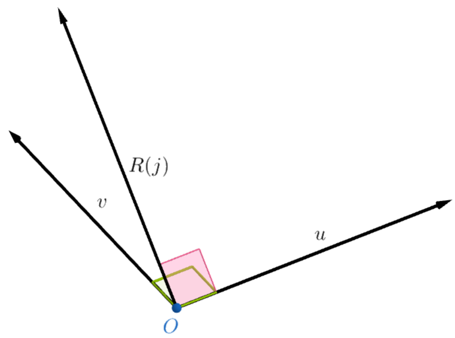
[6]Perpendicular to each other in pairs and each having unit length

Construct R, a reflection in a plane through O that takes i to u: $R(i) = u$. Note that R is an orthogonal transformation. If we find that R takes j to v and k to w already, then we can stop because a single reflection was needed. Otherwise we will continue.

We have:

$$0 = i \cdot j = R(i) \cdot R(j) = u \cdot R(j)$$

where the first equality uses the orthonormal property and the second is true because R preserves inner products. The above shows that $R(j) \perp u$



Regarding lengths, all the vectors we are considering have unit length:

$$|v| = |T(j)| = |j| = 1 \qquad \text{and} \qquad |R(j)| = |j| = 1$$

Because both T and R preserve length.

So we have $v \perp u$ and $R(j) \perp u$ and $|v| = |R(j)| = 1$. v and $R(j)$ form two sides of an equilateral parallelogram (with side length 1). The vector $v + R(j)$ forms the main diagonal of this parallelogram and bisects the angle between v and $R(j)$.

Consider a reflection S in the plane spanned by u and the vector $v + R(j)$, a plane that will also contain the origin. The reflection S will

- Take $R(j)$ to v
- Leave u unchanged.

We can summarise the effect of the composite transformation SR on i and j, the first two vectors in the original orthonormal basis:

- $SR(i) = S(u) = u$
- $SR(j) = v$

Now we can look at the effect of SR on k. If $SR(k) = w$, then we are done and two reflections were sufficient.

The case where $SR(k) \neq w$: We know that w is perpendicular to both u and v and therefore is perpendicular to the plane spanned by u and v.

Also, because S and R are both orthogonal transformations, their composition SR is also orthogonal and therefore preserves inner products:

$$\begin{aligned} 0 = i \cdot k \quad &= SR(i) \cdot SR(k) = u \cdot SR(k) \\ 0 = j \cdot k \quad &= SR(j) \cdot SR(k) = v \cdot SR(k) \end{aligned}$$

so that the vector $SR(k)$ is also perpendicular to u and v.

The length of $SR(k)$ is one because $|k| = 1$ and both R and S preserve lengths.

The above shows that we have two distinct unit vectors w and $SR(k)$ that are both perpendicular to the plane spanned by u and v. This can only happen if w and $SR(k)$ are related by a reflection in the plane spanned by u and v.

Let T be a reflection in the plane spanned by u and v, so that T fixes u and v. Consider the composite transformation TSR:

- $TSR(i) = TS(u) = T(u) = u$
- $TSR(j) = T\ S(\ R(j)\ ) = T(v) = v$
- $TSR(k) = T(\ SR(k)\ ) = w$ by construction above.

so that the three reflections take the original basis vectors $\{i, j, k\}$ to the new basis vectors $\{u, v, w\}$.

As discussed earlier, any vector x can be written as a combination of the original basis vectors $x = \alpha i + \beta j + \gamma k$ and and we can use this to represent its image under the original orthogonal transformation M:

$$\begin{aligned} M(x) &= M(\alpha i + \beta j + \gamma k) \\ &= M(\alpha i) + M(\beta j) + M(\gamma k) \\ &= \alpha M(i) + \beta M(j) + \gamma M(k) \\ &= \alpha u + \beta v + \gamma w \end{aligned}$$

So the effect of the transformation M on any vector is determined by its effect on the basis vectors and we have shown earlier that the transformation that maps the basis vectors can be represented by at most three reflections.

2.11. Can three orthogonal reflections be 'reduced'?

In the earlier sections, we showed that any orthogonal transformation can be represented by at most three reflections. We can turn things around by asking whether, when we are given three successive reflections, the resulting transformation can be represented by fewer reflections.

Let M be an orthogonal transformation represented by three reflections S_U, S_V, and S_W, in planes U, V, W that all contain the origin.

First note that each reflection changes the orientation. Applying three reflections can only lead to another orientation changing transformation. This means that the resulting transformation will either be a reflection or an improper rotation.

2.11.1. Case 1: Successive planes coincide.

Writing the composition as $M = S_W S_V S_U$, we can consider the simple cases where either $U = V$ or $V = W$.

If $U = V$, then the reflections S_U and S_V cancel each other out and we have $M = S_W$. Similarly, if $V = W$, then we have $M = S_U$. In either case, M can be represented by a single reflection.

In the following, we assume that $U \neq V$ and $V \neq W$.

2.11.2. Case 2: The reflections planes all intersect in a single line. Consider the case where the lines of intersection all coincide, i.e., $U \cap V = V \cap W$.

The line $U \cap V$ represents the axis of the rotation R defined by the first pair of reflections:

$$R := S_V S_U$$

We have seen that the same rotation can be represented by two reflections in different planes (Section 2.8):

$$R = S_V S_U = S_{V'} S_{U'}$$

as long as V' and U' intersect in the same axis $(U \cap V)$ and have the same angle of separation.

In particular, we can rotate the pair V', U' around the common line of intersection (of the planes U, V, W) until the plane V' coincides with the plane W.

Now we can re-write M

$$\begin{aligned} M &= S_W S_V S_U \\ &= S_W S_{V'} S_{U'} && \text{equivalence of rotation} \\ &= S_W S_W S_{U'} && \text{we made } V' \text{ coincide with } W \\ &= S_{U'} \end{aligned}$$

So, when the three planes share a common line of intersection, the composed reflections can be reduced to a single reflection, $S_{U'}$. This case is illustrated schematically in Figure 4.

2.11.3. Case 3: The three planes do not intersect in a common line. In this case, the three planes which all contain the origin, intersect pair-wise in three distinct lines.

The transformation $M = S_W S_V S_U$ represents an improper rotation and cannot be reduced (stated without proof). The representation, however, is not unique, and it is possible to represent the same transformation by reflections in planes U', V' and W' where W' is perpendicular to the line of intersection of U' and V'. See Section 2.10 for details. In this form, we can say that the improper rotation has been represented 'canonically'.

2.12. Reducing longer sequences of reflections (more than three)

Given a sequence of n orthogonal reflections, that are composed to give a single transformation:

$$M = S_n S_{n-1} \cdots S_1$$

where $n > 3$ and P_k denotes the plane for reflection S_k, we can ask how to reduce the sequence to one for a single orthogonal transformation (with three or fewer reflections).

We can assume that no successive planes of reflections are the same so that no adjacent pairs of reflections cancel each other out. If such pairs exist, we can always remove them and start with the sequence that remains.

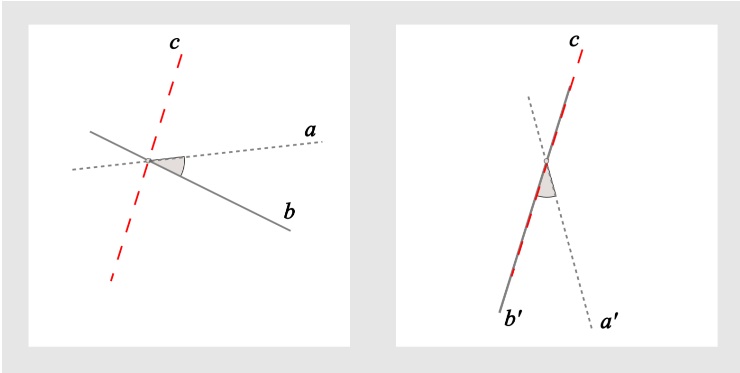


FIGURE 4. Applying three reflections in planes that intersect in a single line. This is shown schematically above with lines representing planes viewed along their 'edge' and the intersection line going through the page. Reflecting in a then b gives the same rotation as reflecting in a' then b' which are rotated versions of a and b through the same angle. The combination of all three reflections is the same in both cases $abc = a'b'c$ and because the planes for b' and c coincide, the reflections cancel and we are left with a single reflection: a'.

One approach to reducing them would be to group the reflections in pairs to give rotations. This would give

$$\begin{aligned} M &= R_m R_{m-1} \cdots R_1 \qquad n \text{ even and } m = n/2 \text{ or} \\ M &= R_m R_{m-1} \cdots R_1 S_1 \qquad n \text{ odd and } m = (n-1)/2 \end{aligned}$$

Where each rotation is an orthogonal rotation about an axis through the origin (the intersection of the planes of the corresponding pair of reflections).

In either of the above sequences each successive adjacent pair of rotations can be replaced with a single rotation that is their composition. This step would reduce the number of rotations and can be carried on until only one rotation remains (even n) or a single rotation and reflection remain (odd n).

2.13. Two-step forms for general versions of orthogonal transformations

As previously discussed Section 2.4.0.1), there are three types of general isometry with at least one fixed point: reflection, rotation, and improper rotation. The fixed points for these three types of transformation form a plane (reflection), a line (rotation), a single point (improper rotation).

In the next three parts, we show that any of these can be written in a two-step form consisting of orthogonal transformation followed by a translation.

More specifically, if we write the two-step form as $t_u M$ for some orthogonal transformation M and translation t_u by a vector u, we will see that the vector u is always perpendicular to the fixed point set of the orthogonal transformation M. For example, if M is a rotation, then u is perpendicular to the axis of rotation (Lemma 2.13.2), or, if M is a reflection, then u is perpendicular to the plane of reflection (Lemma 2.13.1).

2.13.1. A general reflection as a two-step form. A general reflection in a plane (that need not contain the origin) can be represented by a translation to the origin, followed by a linear reflection in a plane through O, followed by a translation back to reverse the first translation.

Let P be an arbitrary plane and p some point in the plane. The translation t_{-p} takes p to the origin. Let Q be the plane through O that is parallel to P. Let S_P and S_Q be reflections in the planes P and Q. Because the plane Q contains the origin, we know that the reflection S_Q is an orthogonal linear transformation. The reflection S_P can then be given by $t_p\, S_Q\, t_{-p}$.

LEMMA 2.13.1. *A general reflection in a plane P can be written as $t_v S$ where S is a reflection in a plane through O parallel to P and vector v is perpendicular to P (and S).*

Proof: Let Q be a plane through O parallel to P. The reflection in plane P, S_P can be written $t_p\, S_Q\, t_{-p}$ for some point $p \in P$. Applying this reflection to a general point x:

$$\begin{aligned} t_p\, S_Q\, t_{-p}(x) &= t_p\,(S_Q(x-p)) && \text{apply } t_{-p} \\ &= t_p\,(S_Q(x) - S_Q(p)) && S_Q \text{ is linear} \\ &= S_Q(x) - S_Q(p) + p && \text{apply } t_p \\ &= S_Q(x) + v \\ &= t_v(S_Q(x)) \end{aligned}$$

where $v = p - S_Q(p)$ is the displacement vector between p and its image under reflection in the plane Q, meaning that v is perpendicular to Q, and hence also to plane P. Dropping the x in the above, we have shown that

$$S_P = t_p\, S_Q\, t_{-p} = t_v S_Q \qquad \text{where } v \perp Q$$

■

2.13.2. A general rotation as a two-step form. Lemma 2.13.1 shows the two-step form that a general reflection can take. If we compose two reflections we can use this form to show that the resulting transformation can also be written in a similar form.

LEMMA 2.13.2. *A general rotation about an arbitrary axis can be written in the form $t_z R$ where R is a linear rotation about an axis through the origin and the translation is by a vector z perpendicular to the rotation axis.*

Proof: We have already shown this in Checkpoint 7 but here we will show it by using the fact that a general rotation is achieved by carrying out two reflections in a pair of planes that intersect in the axis.

Let the two reflections be written $t_u S_P$ and $t_v S_Q$, using the form from Lemma 2.13.1, where S_P and S_Q are reflections in planes P and Q through O. We know that $u \perp P$ and $v \perp Q$.

The composition is

$$t_v S_Q\, t_u S_P$$

Recall that S_Q is linear so that, for any point x

$$S_Q t_u(x) = S_Q(x+u) = S_Q(x) + S_Q(u) = t_{S_Q(u)}(S_Q(x)) = t_{u'} S_Q(x)$$

where $u' = S_Q(u)$. The point x is general so we can write $S_Q\, t_u = t_{u'}\, S_Q$.

This means we can re-write the composition

$$t_v S_Q \, t_u S_P = t_v t_{u'} \, S_Q S_P = t_{v+u'} \, S_Q S_P$$

The last two terms are $S_Q S_P$, successive reflections in planes through the origin. If Q and P are identical, they cancel each other out and $S_Q S_P = I$, the identity, otherwise $S_Q S_P$ is a rotation about an axis through the origin that is the line of intersection of P and Q. Let the line l be the axis of the rotation.

We have

$$u \perp P \Rightarrow u \perp l \text{ and } v \perp Q \Rightarrow v \perp l$$

Now, l is contained in Q and u' is the reflection of u in plane Q. This means that $u' \perp l$ [7]. $v + u' \perp l$ because each of v and u' are separately perpendicular to l.

Putting all this together, we can say that the composition of reflections can be written

$$t_v S_Q \, t_u S_P = t_{v+u'} \, S_Q S_P = t_z R$$

where R is a rotation about l and $z = v + u'$ is a vector perpendicular to l.

■

2.13.3. A general improper rotation as a two-step form. A general improper rotation is a rotation about some arbitrary axis followed by a reflection in a plane perpendicular to the axis. In Sections 2.13.2 and 2.13.1, we saw that a general rotation and reflection can each be represented in a two-step form.

Let l be a line through O parallel to the axis for the improper rotation. The rotation part of the improper rotation can be expressed as $t_z R$ where R is a rotation about l and t_z is a translation by vector z perpendicular to l (Section 2.13.2).

The reflection part of the improper rotation will be in a plane perpendicular to l. Let S_Q be a reflection in a plane Q through the origin perpendicular to l. Then the reflection part of the improper rotation can be expressed as $t_v S_Q$ where v is perpendicular to Q and parallel to l (Section 2.13.1).

So the improper rotation can be written as

$$t_v S_Q \, t_z R$$

We have, for any point x

$$S_Q t_z(x) = S_Q(x + z) = S_Q(x) + S_Q(z) = S_Q(x) + z' = t_{z'} S_Q(x)$$

where $z' = S_Q(z)$, the image of z in the reflection S_Q.

Note that vector z is perpendicular to the axis of R and, since plane Q is also perpendicular to this axis, we have $z' = S_Q(z) = z$.

So the improper rotation becomes

$$\begin{aligned} t_v S_Q \, t_z R &= t_v t_{z'} S_Q R \\ &= t_v t_z S_Q R \\ &= t_u S_Q R \qquad \text{where } u = v + z \end{aligned}$$

The transformation $S_Q R$ is a rotation about l through O followed by a reflection in plane Q which also contains O and is perpendicular to l – this means that $S_Q R$

[7] Let w be the point on l closest to u, the line from w to u is perpendicular to l. The line from w to u is reflected to a line from w to u' under S_Q. Both lines are in a plane perpendicular to Q. So the line from w to u' will remain at a right angle to l.

is an orthogonal (linear) transformation, specifically an improper rotation. Denote $Z = S_Q R$ for this improper rotation and we have the desired two-step form as $t_u Z$.

The orthogonal transformation Z, as an improper rotation, has only a single fixed point, $O = (0,0,0)$. For any vector u, we have $u \cdot (0,0,0) = 0$, so for the two step form $t_u Z$ we continue to have a translation vector u that is perpendicular to the fixed point set of Z.

2.14. If an isometry has any fixed points, then they are perpendicular to the translation in its two-step form.

In Section 2.4.0.1, we listed the isometries that have at least one fixed point. These consisted of the identity transformation and of reflections, rotations, or improper rotations which either fix the origin (are linear) or are 'general' (for example a rotation about an axis not intersecting the origin).

In the previous sections, we also expressed general versions of orthogonal transformations in a two-step form $t_u M$ where M is an orthogonal and t_u is a translation.

When M was a reflection, we saw that u is perpendicular to the plane of reflection (Section 2.13.1). When M was a rotation, we saw that u is perpendicular to the axis of rotation (Section 2.13.2). When M was an improper rotation, we saw that u was (trivially) perpendicular to the single fixed point of the transformation (Section 2.13.3).

In other words, in every case, the vector of translation in the two-step form was perpendicular to the space of fixed points for the orthogonal transformation part.

To start to see why this is the case, we consider the general *three-step* form of an isometry based on an orthogonal transformation. It can always be expressed as a translation to bring a particular point to the origin, followed by an orthogonal transformation, followed by a translation to take the origin back to the starting point. The starting point can be a point in the plane of reflection if the isometry is a reflection, a point on the axis if it is a rotation, or the central point in the case of an improper rotation.

We have seen that the three step form can be written as $t_p M t_{-p}$ and that this can be re-written in two-step form as follows

$$\begin{aligned} t_p M t_{-p} &= t_p \, t_{M(-p)} \, M \\ &= t_p \, t_{-M(p)} \, M \\ &= t_{p-M(p)} \, M \\ &= t_u \, M \qquad \text{where } u = p - M(p) \end{aligned}$$

The vector $u = p - M(p)$ represents the displacement between p and its image under the orthogonal transformation M. For example, if M is a rotation, then $p - M(p)$ is contained in a plane perpendicular to the rotation axis. If M is a reflection, then $p - M(p)$ is contained the line from p to its reflection, so is by definition perpendicular to the plane of reflection.

Now consider some point v in the set of fixed points for the orthogonal transformation M. As it is a fixed point, we have $M(v) = v$.

Let us calculate the inner product of $u = p - M(p)$ and the fixed point of M, v:

$$\begin{aligned} u \cdot v &= [p - M(p)] \cdot v \\ &= p \cdot v - M(p) \cdot v \qquad \text{because inner product is linear} \\ &= p \cdot v - M(p) \cdot M(v) \qquad v \text{ is a fixed point of } M \\ &= p \cdot v - p \cdot v \qquad M \text{ preserves inner products} \\ &= 0 \Rightarrow u \perp v \end{aligned}$$

The above gives a general proof that the translation vector in the two-step form of a general rotation, reflection, or improper rotation, will be perpendicular to its set of fixed points.

2.15. Identifying whether an isometry has a fixed point

In Section 2.14, we showed that, if an isometry has any fixed point(s), then any of them is perpendicular to the translation in the two-step form. This can be used to build a method for characterising whether an isometry has any fixed points.

Let an isometry be written in two-step form as $t_u M$ where M is an orthogonal transformation. Consider the transformation $I - M$, where I is the identity transformation. $I - M$ gives the displacement vector from the image of a point under M to its starting point:

$$x \to I(x) - M(x) = x - M(x)$$

Viewed as a function, we can consider the *image* of $I - M$, $im(I - M)$, the space of all vectors v that are produced when applying $I - M$ to any point. If v is contained in $im(I - M)$, the $v = x - M(x)$ for some point x.

We can consider the space of vectors $im(I - M)$ for each type of orthogonal transformation M:

- If M is a rotation about an origin-containing axis l, then $im(I - M)$ is the set of all possible displacement vectors between a point and its rotation about l. These vectors form a plane through the origin perpendicular to l.
- If M is a reflection in a plane P containing O, then $im(I - M)$ is the set of all possible displacement vectors between a point and its reflection in P. These vectors form a line perpendicular to P going through the origin.
- If M is an improper rotation, the $im(I - M)$ is the set of all possible displacements between a point and its image - these vectors can have any magnitude and directions and form all of $\mathbb{R}^3$.

In each case above, the vectors in $im(I - M)$ form a sub-space that is orthogonal to the space of fixed points for M.

Let us denote the fixed points of a transformation T by F_T. For example, if T is a reflection, the F_T is the plane of reflection. If T is a translation by a non-zero vector, then there are no fixed points and $F_T = \varnothing$, the empty set.

Notation for the fixed points of a transformation.

With the above in mind, we can prove the following lemmas. The first characterises the space $im(I - M)$ as the orthogonal complement[8] of the fixed points of M and the second gives a way to test whether an isometry has a fixed point.

LEMMA 2.15.1. *Let M be an orthogonal transformation, then space of points $im(I - M)$ is the orthogonal complement of F_M*

[8] en.wikipedia.org/wiki/Orthogonal_complement

Proof: Because M is orthogonal, it preserves inner products, we have $x \cdot y = M(x) \cdot M(y)$ for any pair of vectors x, y.

All orthogonal transformations fix the origin, so there is at least one fixed point M. Let u be a fixed point of M so that $M(u) = u$.

Let $w \in im(I - M)$ so that $w = v - M(v)$ for some v.

$$\begin{aligned}
u \cdot w &= u \cdot (v - M(v)) \\
&= u \cdot v - u \cdot M(v) \qquad \text{because the inner product is linear} \\
&= u \cdot v - M(u) \cdot M(v) \qquad u = M(u) \\
&= u \cdot v - u \cdot v \qquad M \text{ preserves inner products} \\
&= 0
\end{aligned}$$

So, for any fixed point u of M and any point w in the image of $I - M$, we have $u \cdot w = 0$ so that $u \perp w$, which proves that the space of points $im(I - M)$ is the orthogonal complement of the space of fixed points, F_M.

■

LEMMA 2.15.2. *An isometry T, written in two-step form as $T = t_u M$, has a fixed point if and only if $u \in im(I - M)$.*

Proof: Assume T has some fixed point v. Then

$$\begin{aligned}
v = T(v) &= t_u(M(v)) \\
&= M(v) + u \\
\Rightarrow u &= v - M(v) \\
&= (I - M)\,(v)
\end{aligned}$$

which shows that u is the image of v under the transformation $I - M$, so that $u \in im(I - M)$.

Going the other way, assume that $u \in im(I - M)$. This implies that there exists v such that $u = (I - M)(v) = v - M(v)$. From this we can derive:

$$\begin{aligned}
u &= v - M(v) \\
\Rightarrow v &= M(v) + u \\
\Rightarrow v &= T(v)
\end{aligned}$$

Where the last equality follows because $T(v) = t_u(M(v)) = M(v) + u$. In summary, we have shown that $v = T(v)$ so that v is a fixed point of T.

■

2.16. When can the composition of two general 3-D rotations give another rotation

In Section 1.7, we raised the question of when the composition of two 3-D rotations leads to a rotation. Earlier, we had observed that composing two rotations can lead to a screw (or twist) transformation (Section 1.6.6.2). Note that a screw transformation has no fixed points. We will use arguments about fixed points to answer the question above.

Specifically, in the following, we show that if the composition of two rotations produces another rotation then the axes of the initial pair must be co-planar. This means that if the axes of the two rotations are skew (non-intersecting, non-co-planar) then their composition *cannot* produce a rotation.

THEOREM 2.16.1. *Let R_1 and R_2 be two non-trivial 3-D rotations. If the composition $R = R_2R_1$ is a rotation, then the axes of R_1 and R_2 are co-planar.*

Proof: While R_1 and R_2 are considered to be 'general' rotations, we can always apply a shift to the coordinates so that the axis F_{R_1} of R_1 passes through the origin. This means that R_1 can be treated as an orthogonal (linear) transformation without any loss of generality.

We can write each rotation as a pair of reflections:

$$R_1 = T_1S_1 \qquad R_2 = T_2S_2$$

so that the composition becomes $R = R_2 \circ R_1 = T_2S_2\, T_1S_1$.

As discussed, we can treat R_1 as a linear orthogonal transformation. T_1 and S_1 are therefore also linear orthogonal transformations, their planes of reflection intersect the origin.

We have choice in the pair of reflections that generate R_2 (See Section 2.8 or Figure 5). In particular, we can choose the reflections for T_2S_2 such that the plane of reflection for S_2 contains the origin. This means we can assume that S_2 is also a linear orthogonal reflection (as well as T_1 and S_1).

We are assuming that the rotations are non-trivial so that $T_1 \neq S_1$ and $T_2 \neq S_2$. If it happens to be the case that $S_2 = T_1$, then these reflections have the same plane of reflection, and this plane contains both F_{R_1} and F_{R_2} proving that the axes are co-planar.

In the following, we assume $S_2 \neq T_1$

The first three reflections applied in the composition, $S_2T_1S_1$, are all orthogonal (linear) transformations. After they have been applied, one more reflection, T_2 is needed to generate the full composition R_2R_1.

We distinguish two cases for the transformation given by the first three reflections:

(1) $S_2T_1S_1$ can be reduced to a single reflection.

(2) $S_2T_1S_1$ cannot be reduced and is an improper rotation.

and these are the only two cases (See Section 2.11).

Case 1: $S_2T_1S_1$ can be reduced to a single reflection.

We know that $S_2 \neq T_1$ and $T_1 \neq S_1$. The only way that the three reflections can be reduced to a single reflection is if the three reflection planes all intersect in a single line (Section 2.11) and this line must also coincide with the axis of rotation F_{R_1}.

In particular, F_{R_1} is therefore contained in the plane of reflection for S_2 which also, by definition, contains F_{R_2}. So the two axes are co-planar as required.

Case 2: $S_2T_1S_1$ cannot be reduced to a single reflection and is an improper rotation.

Denote the partial composition of the linear reflections by M:

$$M = S_2 T_1 S_1$$

The full composition is given by $R = R_2 R_1 = T_2 S_2\, T_1 S_1 = T_2 M$.

If the plane of reflection for T_2 contains the origin, then T_2 is also orthogonal and the axis F_{R_2} for R_2 also goes through O. Since the reflections T_1 and S_1 for rotation R_1 also contain the origin, then F_{R_1} also intersects the origin so that both axes are co-planar.

Assume then that the plane for reflection T_2 does not contain the origin, i.e., that reflection T_2 is 'general'.

Express T_2 in two-step form $T_2 = t_v U$ where U is an orthogonal linear reflection and v is a vector perpendicular to the plane of reflection for U (See Lemma 2.13.1). We have $v \neq 0$ because the plane of reflection for T_2 does not intersect the origin.

By the assumption, we know that the composition $R = R_2 R_1$ is a rotation. Writing this using the two step form and the partial composition gives:

$$R = R_2 R_1 = T_2 M = t_v\; U\; M$$

Both U and M are orthogonal transformations so that the composition UM is also an orthogonal transformation. This orthogonal transformation is a rotation that we can denote as $W = UM$ to give $R = t_v W$ as a two-step form for the composed rotation.

So, the non-zero vector v is perpendicular to F_U, the plane of reflection for U, and to F_W, the axis (a line) for the orthogonal part of the final rotation, W (Lemma 2.13.2).

F_U represents a 2-D plane and v is perpendicular to this plane. The only way for v to also be perpendicular to the line F_W is for F_W to be contained in the plane F_U.

All points on the line F_W are fixed by rotation W (by definition). They are also all fixed by reflection F_U (because $F_W \in F_U$).

It follows that F_W is a line of fixed points for the composition $U \circ W$. This composition can be written

$$U \circ W = U \circ U \circ M = M = S_2\; T_1\; S_1$$

But, in this case, the three reflections composed $S_2\; T_1\; S_1$, form the orthogonal improper rotation M, which can only a single fixed point, the origin.

We therefore have a contradiction which means this case cannot lead to a rotation and, in all the earlier cases, we found that the axes of the two rotations are parallel.

■

CHAPTER 3

Other techniques

3.1. Quaternions

TODO

For two rotations about O, use quaternions to show how axis and angle of composition can be derived from those of the two rotations.

3.2. Rotation matrices

TODO

3.2.1. Finding the axis and angle from a 3-D rotation matrix. TODO

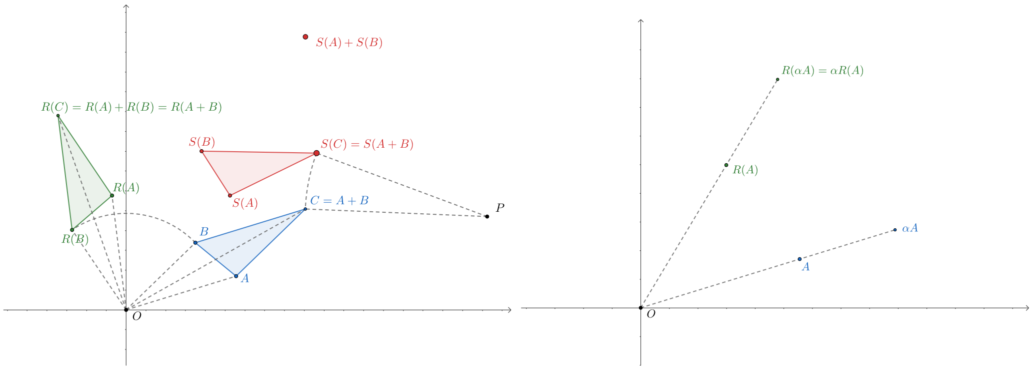


FIGURE 1. Left: Points A, B are rotated around the origin under rotation R to give $R(A), R(B)$. Their vector sum $C = A + B$ is rotated to $R(C) = R(A + B) = R(A) + R(B)$. S is a rotation about a different point P. For rotation S we don't have the same equality: $S(C) = S(A+B) \neq S(A)+S(B)$. Right: Scaling a point and rotating about the origin is the same as rotating and then scaling, $R(\alpha A) = \alpha R(A)$.

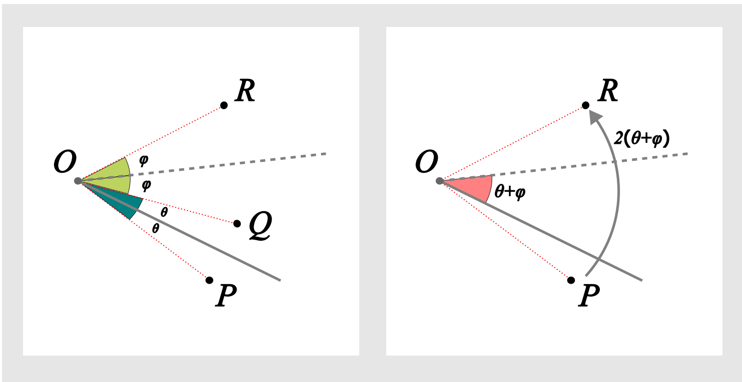


FIGURE 2. Two reflections in intersecting axes, the solid line and the dashed line, send $P \to Q \to R$. The angles on either side of the solid and dashed reflection lines are equal (left). The angle of rotation about O from P to R is double the angle between the lines (right).

3.3. Rotations and linearity

When a rotation R is about the origin, it is linear: for a pair of points $\mathbf{A}, \mathbf{B}$, we have $R(\mathbf{A} + \mathbf{B}) = R(\mathbf{A}) + R(\mathbf{B})$. This is shown for the blue and green sets of points in Figure 1.

Also, it is straightforward to show for a rotation R about the origin, $R(\alpha\mathbf{A}) = \alpha R(\mathbf{A})$.

If a rotation is *not* about the origin, then it is not linear. A simple example is shown in Figure 1 for rotation S about the non-origin point P. In this case, $S(\mathbf{A} + \mathbf{B})$ is a different point from $S(\mathbf{A}) + S(\mathbf{B})$

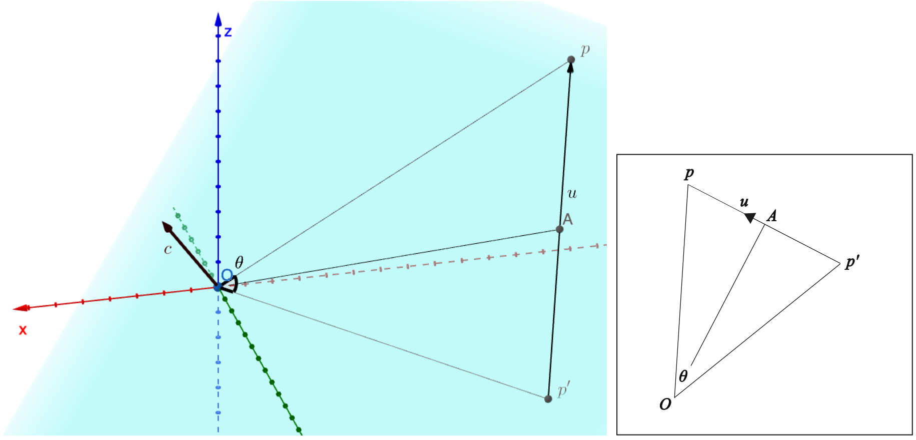


FIGURE 3. A point p and its image under a rotation $\mathcal{M}(p) = p'$ are both contained in a plane through O that is perpendicular to the rotation axis $\hat{c}$ of $\mathcal{M}$. The angle subtended by p and p' at the origin equals the angle of rotation. The vector $\mathbf{u}$ goes from p' to p. A flattened 2-D view is shown on the right. See Section 3.5.

3.4. Two 2-D reflections in intersecting lines gives a rotation

This section illustrates why two successive reflections in a pair of intersecting lines gives a rotation.

Consider the diagram on the left of Figure 2, starting with the point O, we can reflect it in the solid line to give Q and then reflect Q in the dashed line to give R.

Let the signed angle (e.g., anti-clockwise is positive) from OP to the solid line be θ. This means the angle from the solid line to OQ is also θ.

Let the angle from OQ to the dashed line be φ, then the angle from the dashed line to OR is also φ.

This means that the angle between the lines is $\theta + \phi$ (remember these can be signed).

The angle $\angle POR$ equals $2(\theta + \phi)$. Because the $\angle POR$ is equal to twice the angle between the two fixed lines, it is also fixed.

The distances $|OP|$, $|OQ|$, and $|OR|$ are all equal, so R is the result of rotating P about O by $2(\theta+\phi)$. The point P is arbitrary, we could have started with any point and, after the two prescribed reflections, the result can be reached with the same rotation.

3.5. Solving for $\mathbf{p}$ in $t_{\mathbf{u}}\mathcal{M} = t_{\mathbf{p}-\mathbf{p}'}\mathcal{M}$

The assumption here is that $\mathbf{p} \perp \mathbf{c}$, where $\mathcal{M} = \mathcal{M}_{\mathbf{c},\theta}$, i.e., the translation vector of the two-step form is perpendicular to the axis of the origin rotation $\mathcal{M}$.

This means that the vector $\mathbf{u}$ can be wholly contained in a plane through O and perpendicular to $\mathbf{c}$.

The equation $t_{\mathbf{u}}\mathcal{M} = t_{\mathbf{p}-\mathbf{p}'}\mathcal{M}$ gives $\mathbf{u} = \mathbf{p} - \mathbf{p}'$. So $\mathbf{u}$ is the difference between the point $\mathbf{p}$ (which we want to find) and its image under the rotation. The set up is illustrated in Figure 3.

The points $\mathbf{p}$ and $\mathbf{p}'$ must be equidistant from O and, together with u, form an isosceles triangle.

The line from O to A, the midpoint of $\mathbf{p}$ and $\mathbf{p}'$, is perpendicular to $\mathbf{c}$ and also to $\mathbf{u}$.

The direction of the vector $\overrightarrow{OA}$ can be found by taking the cross product of $\mathbf{c}$ and $\mathbf{u}$ and normalising.

The length OA can be found by noting it is the height of the isosceles triangle Opp' and that

$$\tan\left(\frac{\theta}{2}\right) = \frac{|\mathbf{u}|}{2OA}$$

where $|\mathbf{u}|$ and θ are known so that OA can be found. A flat view of the triangle within the plane is shown on the right of Figure 3.

After finding the length and direction of $\overrightarrow{OA}$, $\mathbf{p}$ can be found by using $\mathbf{p} = \overrightarrow{OA} + \mathbf{u}/2$.

Bibliography

[1] M. Artin. *Algebra*. Pearson Education, 2011.