

PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

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These are notes I wrote up for my own comprehension while reading the text of Hotta–Takeuchi–Tanisaki [HTT08].

1. T-STRUCTURES

Definition 1.1. Let \mathcal{D} be a triangulated category, and $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ its full subcategories. Set $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n]$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t-structure on \mathcal{D} if

- $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$
- For all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.
- For any $X \in \mathcal{D}$ there exists an exact triangle

$$(1.1) \quad X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1} \dots$$

such that $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 1}$.

The full subcategory $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is called the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

The heart of a t-structure allows us to get a good grasp on the ambient derived category.

Theorem 1.2. $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an Abelian category, and any exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

gives rise to a distinguished triangle

$$X \rightarrow Y \rightarrow Z \xrightarrow{+1} \dots$$

in \mathcal{D} .

This allows us to define the functor H^0 , which sends our derived category to its heart.

Definition 1.3. The functor

$$H^0 : \mathcal{D} \rightarrow \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \quad X \mapsto \tau^{\leq 0} \tau^{\geq 0} X.$$

Likewise, define $H^n(X) = H^0(X[n])$.

Now let $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ be a functor of triangulated categories. Denote by $\mathcal{C}_1, \mathcal{C}_2$ the hearts of the respective categories.

Definition 1.4. The additive functor

$${}^p F : \mathcal{C}_1 \rightarrow \mathcal{C}_2, \quad X \mapsto H^0(F(X)).$$

We say that ${}^p F$ is left t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$, and analogously for right t-exactness.

2. PERVERSE SHEAVES

Now we define a t-structure on the derived category of constructable sheaves. Recall that we have a Verdier duality functor $\mathbf{D}_X : D_{\text{con}}^b(X)^{\text{op}} \rightarrow D_{\text{con}}^b(X)$.

Definition 2.1. For $F^\bullet \in \mathcal{D}_{\text{con}}^b(X)$, we have $F^\bullet \in \mathcal{D}_{\text{con}}^{\leq 0}(X)$ if and only if

$$\dim \text{supp}(H^j(F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$. Likewise, we have $F^\bullet \in \mathcal{D}_{\text{con}}^{\geq 0}(X)$ if and only if

$$\dim \text{supp}(H^j(\mathbf{D}_X F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$.

It is not obvious that this is indeed a t-structure, so it must be stated separately.

Theorem 2.2. *The pair $(\mathcal{D}_{\text{con}}^{\leq 0}, \mathcal{D}_{\text{con}}^{\geq 0})$ forms a t-structure.*

Definition 2.3. We call the heart of this t-structure $\text{Perv}(\mathbb{C}_X)$ the perverse sheaves on X . The functor \mathbf{D}_X induces a duality of $\text{Perv}(\mathbb{C}_X)$. Now, for any functor $F : \mathcal{D}_{\text{con}}^b(X) \rightarrow \mathcal{D}_{\text{con}}^b(Y)$, denote by ${}^p F$ the composition ${}^p H^0 \circ F$, viewed as a functor $\text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y)$. In particular, we get the associated functors

$${}^p f^{-1}, {}^p f^! : \text{Perv}(\mathbb{C}_Y) \rightarrow \text{Perv}(\mathbb{C}_X),$$

and

$${}^p Rf_*, {}^p Rf_! : \text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y).$$

Next, we define minimal extensions of perverse sheaves. Let $X = Z \sqcup U$ with $i : Z \rightarrow X$ proper and $j : U \rightarrow X$ open. For $F^\bullet \in \mathcal{D}_{\text{con}}^b(X)$, we have natural maps

$$j_! F^\bullet \rightarrow Rf_* F^\bullet, \quad s \mapsto s,$$

mapping a section to itself. If F^\bullet is a perverse sheaf on U , we also get maps

$$(2.1) \quad {}^p j_! F^\bullet \rightarrow {}^p f_* F^\bullet.$$

Definition 2.4. We denote by ${}^p f_{!*} F^\bullet$ the image of (2.1), and call it the minimal extension of $F^\bullet \in \text{Perv}(\mathbb{C}_U)$.

In other words, we have maps

$$(2.2) \quad {}^p j_! F^\bullet \twoheadrightarrow {}^p j_{!*} F^\bullet \hookrightarrow {}^p j_* F^\bullet.$$

Theorem 2.5. *Let $F^\bullet \in \text{Perv}(\mathbb{C}_U)$ as above. Then*

- (i) ${}^p f_* F^\bullet$ has no non-trivial sub-objects whose support is contained in Z .
- (ii) ${}^p j_! F^\bullet$ has no non-trivial quotient object whose support is contained in Z .
- (iii) ${}^p j_{!*} F^\bullet$ has neither sub- or quotient objects whose support is contained in Z .

Now suppose that F^\bullet is a simple object in $\text{Perv}(\mathbb{C}_U)$. We claim that ${}^p j_{!*} F^\bullet$ is also simple in $\text{Perv}(\mathbb{C}_X)$. Indeed, for any subobject $G^\bullet \subseteq {}^p j_{!*} F^\bullet$ we may consider the exact sequence

$$(2.3) \quad 0 \rightarrow G^\bullet \rightarrow {}^p j_{!*} F^\bullet \rightarrow H^\bullet \rightarrow 0.$$

Since $j^! = j^{-1}$ is exact here, we may apply it to obtain

$$(2.4) \quad 0 \rightarrow j^{-1} G^\bullet \rightarrow F^\bullet \rightarrow j^{-1} H^\bullet \rightarrow 0.$$

Since F^\bullet is simple, $j^{-1} G^\bullet$ or $j^{-1} H^\bullet$ is zero, or in other words, one of G^\bullet and H^\bullet is supported in Z . By the previous theorem, this makes it zero.

Theorem 2.6. $G = {}^p j_{!*} F^\bullet$ is the unique perverse sheaf satisfying the conditions

- (i) $G^\bullet|_U \simeq F^\bullet$,
- (ii) $i^{-1}G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\leq -1}(Z)$,
- (iii) $i^!G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\geq 1}(Z)$.

3. INTERSECTION COHOMOLOGY

Let X be an irreducible complex projective variety (or an irreducible compact analytic space) of dimension d . We may start by defining the intersection cohomology complex in the following way.

Definition 3.1. Take $U \subseteq X^{\text{reg}}$ to be a Zariski open subset of the smooth locus of X . For $\mathbb{C}_U[d]$ the constant perverse sheaf on U , define IC_X^\bullet to be a minimal extension to all of X .

We may explicitly construct IC_X^\bullet . Fix a Whitney stratification $X = \bigsqcup_\alpha X_\alpha$,

$$(3.1) \quad X_k = \coprod_{\dim X_\alpha \leq k} X_\alpha, \quad U_k := X \setminus X_{k-1},$$

with each $X_k \setminus X_{k-1}$ a smooth k -dimensional complex manifold. We have a family of maps $j_k : U_k \hookrightarrow U_{k-1}$, for $k = 1, \dots, d$. Altogether, this fits into

$$(3.2) \quad \emptyset \hookrightarrow U_d \hookrightarrow U_{d-1} \hookrightarrow \dots \hookrightarrow U_1 \hookrightarrow X.$$

Theorem 3.2. IC_X^\bullet is quasi-isomorphic to the complex

$$(3.3) \quad {}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq (\tau^{\leq -1} Rj_{1*}) \circ (\tau^{\leq -2} Rj_{2*}) \circ \dots \circ (\tau^{\leq -d} Rj_{d*})(\mathbb{C}_U[d]).$$

Proof. Suppose that F^\bullet is a perverse sheaf on U_k whose restriction to any strata $X_\alpha \subseteq U_k$ has locally constant \mathcal{H}^i . We show that $G^\bullet := \tau^{\leq -k} Rj_{k*}(F^\bullet)$ satisfies the unique characterization of a minimal extension from Theorem 2.6. First, note that U_k consists of strata of dimension $\geq k$, so that $\mathcal{H}^r(F^\bullet) = 0$ for $r > -k$. In particular,

$$[\tau^{\leq -k} Rj_{k*} F^\bullet]|_{U_k} \simeq F^\bullet,$$

so that what we have is indeed an extension of F^\bullet , satisfying (i). Next, set $Z := U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$. Denote by $i : Z \rightarrow U_k$ the associated close embedding. Then $i^{-1}G^\bullet$ has locally constant cohomology sheaves on each $X_\alpha \subseteq Z$, so that $\mathcal{H}^r(i^{-1}G^\bullet) = 0$ for $r > -k$. This implies that $i^{-1}G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\leq -1}$, thus satisfying condition (ii). Finally, consider the triangle

$$(3.4) \quad G \rightarrow Rj_{k*} F^\bullet \rightarrow \tau^{\geq -k+1} Rj_{k*} F^\bullet \xrightarrow{+1} \dots$$

Applying $i^!$, we note that the middle vanishes. This gives us isomorphisms

$$i^! G^\bullet \simeq i^! (\tau^{\geq -k+1} Rj_{k*} F^\bullet)[-1]$$

so that $\mathcal{H}^r(i^! G^\bullet) = 0$ for $r < -k$. But since $i^! G$ has locally constant cohomology on $X_\alpha \subseteq Z$, we get $i^! G \in {}^p\mathcal{D}_{\text{con}}^{\geq 1}(Z)$, proving (iii). \square

Importantly, the intersection cohomology complex is self-dual in the way we expect of perverse sheaves.

Theorem 3.3. $IC_X^\bullet \simeq \mathbb{D}_X(IC_X^\bullet)$. Furthermore, there exist canonical morphisms

$$(3.5) \quad \mathbb{C}_X \rightarrow IC_X^\bullet[-d] \rightarrow \omega_X^\bullet[-2d].$$

Proof. We have an isomorphism in the derived category of constructible sheaves from above

$$(3.6) \quad \tau^{\leq -d} {}^p j_{1*} \mathbb{C}_X[d] \simeq Rj_{1*} \circ \cdots \circ Rj_{d*} (\mathbb{C}_X[d]) \simeq (j_* \mathbb{C})[d].$$

This is none other than the intersection complex of X , and we see that it admits a canonical map from \mathbb{C}_X . The second map in the composition comes from taking the Verdier dual, $IC_X^\bullet[d] \rightarrow \omega_X^\bullet$. \square

With the intersection complex in place, we are ready to define intersection cohomology.

Definition 3.4. For $i \in \mathbb{Z}$, we define

$$(3.7) \quad IH^i(X) = H^i(R\Gamma(X, IC_X^\bullet[-d])),$$

$$(3.8) \quad IH_c^i(X) = H^i(R\Gamma_c(X, IC_X^\bullet[-d])).$$

Importantly, the intersection cohomology of X satisfies Poincare duality.

Theorem 3.5. Let X be irreducible of dimension d . Then

$$(3.9) \quad IH^i(X) \simeq [IH_c^{2d-i}(X)]^*.$$

Proof. Let $a_X : X \rightarrow \{p\}$ be the unique map to a point. We have

$$(3.10) \quad R\mathrm{Hom}_{\mathbb{C}}(Ra_{X!} IC_X^\bullet, \mathbb{C}) \simeq Ra_{X*} R\mathrm{Hom}_{\mathbb{C}_X}(IC_X^\bullet, \omega_X^\bullet).$$

by Verdier duality. But IC_X^\bullet is self-dual, so that

$$R\mathrm{Hom}_{\mathbb{C}_X}(IC_X^\bullet, \omega_X^\bullet) = \mathbf{D}_X(IC_X^\bullet) = IC_X^\bullet.$$

Thus we get an isomorphism

$$(3.11) \quad [R\Gamma_c(X, IC_X^\bullet)]^* \simeq R\Gamma(X, IC_X^\bullet).$$

\square

Example 3.6. Let X be an irreducible complex projective variety with isolated singular points p_1, \dots, p_k . Then it suffices to consider the stratification

$$X = \{p_1, \dots, p_k\} \sqcup X^{\mathrm{reg}}.$$

Then we have $X_0 = \cdots = X_{d-1} = \{p_1, \dots, p_k\}$, and $X_d = X$. Thus, we get the inclusions of complex manifolds

$$\emptyset \rightarrow U_d = X^{\mathrm{reg}} = U_1 \xrightarrow{j_1} X.$$

The only interesting map in this stratification is j_1 , the inclusion of the smooth locus. In particular, by Theorem 3.2,

$$IC_X^\bullet \simeq \tau^{\leq -1} (Rj_{1*} \mathbb{C}_{U_1}).$$

Now, associated to truncation is the exact couple

$$(3.12) \quad IC_X^\bullet[-d] \rightarrow Rj_{1*} \mathbb{C}_{U_1} \rightarrow \tau^{\geq d} (Rj_{1*} \mathbb{C}_{U_1}) \xrightarrow{+1} \cdots$$

But now we can apply $R\Gamma$ and take cohomology. For $0 \leq i < d$, we have $IH^i(X) = H^i(X, Rj_{1*} \overline{\mathbb{C}}_{U_1}) = H^i(X^{\mathrm{reg}}, \mathbb{C})$. For $i = d$, we do not get a clear vanishing on the right, but the map $IH^d(X) \rightarrow H^d(X^{\mathrm{reg}}, \mathbb{C})$ is injective. To analyze this case more carefully, consider the canonical morphism $\mathbb{C}_X \rightarrow IC_X^\bullet[-d]$ guaranteed by Theorem 3.3.

We may associate to this natural map a new exact couple

$$(3.13) \quad \mathbb{C}_X \rightarrow IC_X^\bullet[-d] \rightarrow F^\bullet \xrightarrow{+1} \cdots$$

Here F^\bullet is a constructible sheaf supported only on the zero-dimensional closed subset $\{p_1, \dots, p_k\} = X_0$. In particular, $H^i(F^\bullet) = 0$ for all $i \geq d$. Thus, we get $IH^i(X) = H^i(X, \mathbb{C})$ for $d < i \leq 2d$. After $i = 2d$, the cohomology of $\underline{\mathbb{C}}_X$ vanishes, and with it $IH^i(X)$. Summarizing,

$$(3.14) \quad IH^i(X) = \begin{cases} H^i(X^{\text{reg}}, \mathbb{C}) & : 0 \leq i < d \\ \text{im}(H^i(X, \mathbb{C}) \rightarrow H^i(X^{\text{reg}}, \mathbb{C})) & : i = d \\ H^i(X, \mathbb{C}) & : d < i \leq 2d \\ 0 & : \text{otherwise} \end{cases}$$

□

REFERENCES

- [HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. Regular holonomic d-modules. In *D-Modules, Perverse Sheaves, and Representation Theory*. Springer, 2008.