

# PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

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These are notes I wrote up for my own comprehension while following the text of Hotta–Takeuchi–Tanisaki [HTT08]. No part of this work is original, save for any errors I may have introduced throughout my reading.

## 1. T-STRUCTURES

**Definition 1.1.** Let  $\mathcal{D}$  be a triangulated category, and  $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$  its full subcategories. Set  $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n]$ . The pair  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  defines a t-structure on  $\mathcal{D}$  if

- $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$
- For all  $X \in \mathcal{D}^{\leq 0}$  and  $Y \in \mathcal{D}^{\geq 1}$  we have  $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ .
- For any  $X \in \mathcal{D}$  there exists an exact triangle

$$(1.1) \quad X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1} \dots$$

such that  $X_0 \in \mathcal{D}^{\leq 0}$  and  $X_1 \in \mathcal{D}^{\geq 1}$ .

The full subcategory  $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$  is called the heart of the t-structure  $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ .

The heart of a t-structure allows us to get a good grasp on the ambient derived category.

**Theorem 1.2.**  $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$  is an Abelian category, and any exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

gives rise to a distinguished triangle

$$X \rightarrow Y \rightarrow Z \xrightarrow{+1} \dots$$

in  $\mathcal{D}$ .

This allows us to define the functor  $H^0$ , which sends our derived category to its heart.

**Definition 1.3.** The functor

$$H^0 : \mathcal{D} \rightarrow \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \quad X \mapsto \tau^{\leq 0} \tau^{\geq 0} X.$$

Likewise, define  $H^n(X) = H^0(X[n])$ .

Now let  $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$  be a functor of triangulated categories. Denote by  $\mathcal{C}_1, \mathcal{C}_2$  the hearts of the respective categories.

**Definition 1.4.** The additive functor

$${}^p F : \mathcal{C}_1 \rightarrow \mathcal{C}_2, \quad X \mapsto H^0(F(X)).$$

We say that  ${}^p F$  is left t-exact if  $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$ , and analogously for right t-exactness.

## 2. PERVERSE SHEAVES

Now we define a t-structure on the derived category of constructable sheaves. Recall that we have a Verdier duality functor  $\mathbf{D}_X : D_{\text{con}}^b(X)^{\text{op}} \rightarrow D_{\text{con}}^b(X)$ .

**Definition 2.1.** For  $F^\bullet \in \mathcal{D}_{\text{con}}^b(X)$ , we have  $F^\bullet \in \mathcal{D}_{\text{con}}^{\leq 0}(X)$  if and only if

$$\dim \text{supp}(H^j(F^\bullet)) \leq -j$$

for all  $j \in \mathbb{Z}$ . Likewise, we have  $F^\bullet \in \mathcal{D}_{\text{con}}^{\geq 0}(X)$  if and only if

$$\dim \text{supp}(H^j(\mathbf{D}_X F^\bullet)) \leq -j$$

for all  $j \in \mathbb{Z}$ .

It is not obvious that this is indeed a t-structure, so it must be stated separately.

**Theorem 2.2.** *The pair  $(\mathcal{D}_{\text{con}}^{\leq 0}, \mathcal{D}_{\text{con}}^{\geq 0})$  forms a t-structure.*

**Definition 2.3.** We call the heart of this t-structure  $\text{Perv}(\mathbb{C}_X)$  the perverse sheaves on  $X$ . The functor  $\mathbf{D}_X$  induces a duality of  $\text{Perv}(\mathbb{C}_X)$ . Now, for any functor  $F : \mathcal{D}_{\text{con}}^b(X) \rightarrow \mathcal{D}_{\text{con}}^b(Y)$ , denote by  ${}^p F$  the composition  ${}^p H^0 \circ F$ , viewed as a functor  $\text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y)$ . In particular, we get the associated functors

$${}^p f^{-1}, {}^p f^! : \text{Perv}(\mathbb{C}_Y) \rightarrow \text{Perv}(\mathbb{C}_X),$$

and

$${}^p Rf_*, {}^p Rf_! : \text{Perv}(\mathbb{C}_X) \rightarrow \text{Perv}(\mathbb{C}_Y).$$

Next, we define minimal extensions of perverse sheaves. Let  $X = Z \sqcup U$  with  $i : Z \rightarrow X$  proper and  $j : U \rightarrow X$  open. For  $F^\bullet \in \mathcal{D}_{\text{con}}^b(X)$ , we have natural maps

$$j_! F^\bullet \rightarrow Rf_* F^\bullet, \quad s \mapsto s,$$

mapping a section to itself. If  $F^\bullet$  is a perverse sheaf on  $U$ , we also get maps

$$(2.1) \quad {}^p j_! F^\bullet \rightarrow {}^p f_* F^\bullet.$$

**Definition 2.4.** We denote by  ${}^p f_{!*} F^\bullet$  the image of (2.1), and call it the minimal extension of  $F^\bullet \in \text{Perv}(\mathbb{C}_U)$ .

In other words, we have maps

$$(2.2) \quad {}^p j_! F^\bullet \twoheadrightarrow {}^p j_{!*} F^\bullet \hookrightarrow {}^p f_* F^\bullet.$$

**Theorem 2.5.** *Let  $F^\bullet \in \text{Perv}(\mathbb{C}_U)$  as above. Then*

- (i)  ${}^p f_* F^\bullet$  has no non-trivial sub-objects whose support is contained in  $Z$ .
- (ii)  ${}^p j_! F^\bullet$  has no non-trivial quotient object whose support is contained in  $Z$ .
- (iii)  ${}^p j_{!*} F^\bullet$  has neither sub- or quotient objects whose support is contained in  $Z$ .

Now suppose that  $F^\bullet$  is a simple object in  $\text{Perv}(\mathbb{C}_U)$ . We claim that  ${}^p j_{!*} F^\bullet$  is also simple in  $\text{Perv}(\mathbb{C}_X)$ . Indeed, for any subobject  $G^\bullet \subseteq {}^p j_{!*} F^\bullet$  we may consider the exact sequence

$$(2.3) \quad 0 \rightarrow G^\bullet \rightarrow {}^p j_{!*} F^\bullet \rightarrow H^\bullet \rightarrow 0.$$

Since  $j^! = j^{-1}$  is exact here, we may apply it to obtain

$$(2.4) \quad 0 \rightarrow j^{-1} G^\bullet \rightarrow F^\bullet \rightarrow j^{-1} H^\bullet \rightarrow 0.$$

Since  $F^\bullet$  is simple,  $j^{-1} G^\bullet$  or  $j^{-1} H^\bullet$  is zero, or in other words, one of  $G^\bullet$  and  $H^\bullet$  is supported in  $Z$ . By the previous theorem, this makes it zero.

**Theorem 2.6.**  $G = {}^p j_{!*} F^\bullet$  is the unique perverse sheaf satisfying the conditions

- (i)  $G^\bullet|_U \simeq F^\bullet$ ,
- (ii)  $i^{-1}G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\leq -1}(Z)$ ,
- (iii)  $i^!G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\geq 1}(Z)$ .

### 3. INTERSECTION COHOMOLOGY

Let  $X$  be an irreducible complex projective variety (or an irreducible compact analytic space) of dimension  $d$ . We may start by defining the intersection cohomology complex in the following way.

**Definition 3.1.** Take  $U \subseteq X^{\text{reg}}$  to be a Zariski open subset of the smooth locus of  $X$ . For  $\mathbb{C}_U[d]$  the constant perverse sheaf on  $U$ , define  $IC_X^\bullet$  to be a minimal extension to all of  $X$ .

We may explicitly construct  $IC_X^\bullet$ . Fix a Whitney stratification  $X = \bigsqcup_\alpha X_\alpha$ ,

$$(3.1) \quad X_k = \coprod_{\dim X_\alpha \leq k} X_\alpha, \quad U_k := X \setminus X_{k-1},$$

with each  $X_k \setminus X_{k-1}$  a smooth  $k$ -dimensional complex manifold. We have a family of maps  $j_k : U_k \hookrightarrow U_{k-1}$ , for  $k = 1, \dots, d$ . Altogether, this fits into

$$(3.2) \quad \emptyset \hookrightarrow U_d \hookrightarrow U_{d-1} \hookrightarrow \dots \hookrightarrow U_1 \hookrightarrow X.$$

**Theorem 3.2.**  $IC_X^\bullet$  is quasi-isomorphic to the complex

$$(3.3) \quad {}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq (\tau^{\leq -1} Rj_{1*}) \circ (\tau^{\leq -2} Rj_{2*}) \circ \dots \circ (\tau^{\leq -d} Rj_{d*})(\mathbb{C}_U[d]).$$

*Proof.* Suppose that  $F^\bullet$  is a perverse sheaf on  $U_k$  whose restriction to any strata  $X_\alpha \subseteq U_k$  has locally constant  $\mathcal{H}^i$ . We show that  $G^\bullet := \tau^{\leq -k} Rj_{k*}(F^\bullet)$  satisfies the unique characterization of a minimal extension from Theorem 2.6. First, note that  $U_k$  consists of strata of dimension  $\geq k$ , so that  $\mathcal{H}^r(F^\bullet) = 0$  for  $r > -k$ . In particular,

$$[\tau^{\leq -k} Rj_{k*} F^\bullet]|_{U_k} \simeq F^\bullet,$$

so that what we have is indeed an extension of  $F^\bullet$ , satisfying (i). Next, set  $Z := U_{k-1} \setminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$ . Denote by  $i : Z \rightarrow U_k$  the associated close embedding. Then  $i^{-1}G^\bullet$  has locally constant cohomology sheaves on each  $X_\alpha \subseteq Z$ , so that  $\mathcal{H}^r(i^{-1}G^\bullet) = 0$  for  $r > -k$ . This implies that  $i^{-1}G^\bullet \in {}^p\mathcal{D}_{\text{con}}^{\leq -1}$ , thus satisfying condition (ii). Finally, consider the triangle

$$(3.4) \quad G \rightarrow Rj_{k*} F^\bullet \rightarrow \tau^{\geq -k+1} Rj_{k*} F^\bullet \xrightarrow{+1} \dots$$

Applying  $i^!$ , we note that the middle vanishes. This gives us isomorphisms

$$i^! G^\bullet \simeq i^! (\tau^{\geq -k+1} Rj_{k*} F^\bullet)[-1]$$

so that  $\mathcal{H}^r(i^! G^\bullet) = 0$  for  $r < -k$ . But since  $i^! G$  has locally constant cohomology on  $X_\alpha \subseteq Z$ , we get  $i^! G \in {}^p\mathcal{D}_{\text{con}}^{\geq 1}(Z)$ , proving (iii).  $\square$

Importantly, the intersection cohomology complex is self-dual in the way we expect of perverse sheaves.

**Theorem 3.3.**  $IC_X^\bullet \simeq \mathbb{D}_X(IC_X^\bullet)$ . Furthermore, there exist canonical morphisms

$$(3.5) \quad \mathbb{C}_X \rightarrow IC_X^\bullet[-d] \rightarrow \omega_X^\bullet[-2d].$$

*Proof.* We have an isomorphism in the derived category of constructible sheaves from above

$$(3.6) \quad \tau^{\leq -d} {}^p j_{1*} \mathbb{C}_X[d] \simeq Rj_{1*} \circ \cdots \circ Rj_{d*} (\mathbb{C}_X[d]) \simeq (j_* \mathbb{C})[d].$$

This is none other than the intersection complex of  $X$ , and we see that it admits a canonical map from  $\mathbb{C}_X$ . The second map in the composition comes from taking the Verdier dual,  $IC_X^\bullet[d] \rightarrow \omega_X^\bullet$ .  $\square$

With the intersection complex in place, we are ready to define intersection cohomology.

**Definition 3.4.** For  $i \in \mathbb{Z}$ , we define

$$(3.7) \quad IH^i(X) = H^i(R\Gamma(X, IC_X^\bullet[-d])),$$

$$(3.8) \quad IH_c^i(X) = H^i(R\Gamma_c(X, IC_X^\bullet[-d])).$$

Importantly, the intersection cohomology of  $X$  satisfies Poincare duality.

**Theorem 3.5.** Let  $X$  be irreducible of dimension  $d$ . Then

$$(3.9) \quad IH^i(X) \simeq [IH_c^{2d-i}(X)]^*.$$

*Proof.* Let  $a_X : X \rightarrow \{p\}$  be the unique map to a point. We have

$$(3.10) \quad R\mathrm{Hom}_{\mathbb{C}}(Ra_{X!} IC_X^\bullet, \mathbb{C}) \simeq Ra_{X*} R\mathrm{Hom}_{\mathbb{C}_X}(IC_X^\bullet, \omega_X^\bullet).$$

by Verdier duality. But  $IC_X^\bullet$  is self-dual, so that

$$R\mathrm{Hom}_{\mathbb{C}_X}(IC_X^\bullet, \omega_X^\bullet) = \mathbf{D}_X(IC_X^\bullet) = IC_X^\bullet.$$

Thus we get an isomorphism

$$(3.11) \quad [R\Gamma_c(X, IC_X^\bullet)]^* \simeq R\Gamma(X, IC_X^\bullet).$$

$\square$

**Example 3.6.** Let  $X$  be an irreducible complex projective variety with isolated singular points  $p_1, \dots, p_k$ . Then it suffices to consider the stratification

$$X = \{p_1, \dots, p_k\} \sqcup X^{\mathrm{reg}}.$$

Then we have  $X_0 = \cdots = X_{d-1} = \{p_1, \dots, p_k\}$ , and  $X_d = X$ . Thus, we get the inclusions of complex manifolds

$$\emptyset \rightarrow U_d = X^{\mathrm{reg}} = U_1 \xrightarrow{j_1} X.$$

The only interesting map in this stratification is  $j_1$ , the inclusion of the smooth locus. In particular, by Theorem 3.2,

$$IC_X^\bullet \simeq \tau^{\leq -1} (Rj_{1*} \mathbb{C}_{U_1}).$$

Now, associated to truncation is the exact couple

$$(3.12) \quad IC_X^\bullet[-d] \rightarrow Rj_{1*} \mathbb{C}_{U_1} \rightarrow \tau^{\geq d} (Rj_{1*} \mathbb{C}_{U_1}) \xrightarrow{+1} \cdots$$

But now we can apply  $R\Gamma$  and take cohomology. For  $0 \leq i < d$ , we have  $IH^i(X) = H^i(X, Rj_{1*} \overline{\mathbb{C}}_{U_1}) = H^i(X^{\mathrm{reg}}, \mathbb{C})$ . For  $i = d$ , we do not get a clear vanishing on the right, but the map  $IH^d(X) \rightarrow H^d(X^{\mathrm{reg}}, \mathbb{C})$  is injective. To analyze this case more carefully, consider the canonical morphism  $\mathbb{C}_X \rightarrow IC_X^\bullet[-d]$  guaranteed by Theorem Theorem 3.3.

We may associate to this natural map a new exact couple

$$(3.13) \quad \mathbb{C}_X \rightarrow IC_X^\bullet[-d] \rightarrow F^\bullet \xrightarrow{+1} \cdots$$

Here  $F^\bullet$  is a constructible sheaf supported only on the zero-dimensional closed subset  $\{p_1, \dots, p_k\} = X_0$ . In particular,  $H^i(F^\bullet) = 0$  for all  $i \geq d$ . Thus, we get  $IH^i(X) = H^i(X, \mathbb{C})$  for  $d < i \leq 2d$ . After  $i = 2d$ , the cohomology of  $\underline{\mathbb{C}}_X$  vanishes, and with it  $IH^i(X)$ . Summarizing,

$$(3.14) \quad IH^i(X) = \begin{cases} H^i(X^{\text{reg}}, \mathbb{C}) & : 0 \leq i < d \\ \text{im}(H^i(X, \mathbb{C}) \rightarrow H^i(X^{\text{reg}}, \mathbb{C})) & : i = d \\ H^i(X, \mathbb{C}) & : d < i \leq 2d \\ 0 & : \text{otherwise} \end{cases}$$

□

#### 4. COMPUTATIONS VIA SAITO'S MIXED HODGE MODULES

To compute intersection complexes of standard perverse sheaves, we use Saito's mixed Hodge modules ([Sai88], [Sai90]). For a quick summary of this theory, refer to [dCRS21, §2.1]. Throughout these computations, we loosely follow [dCM09, §2].

Let  $X$  be a variety over  $\mathbb{C}$ . Saito [Sai90] defines a category  $\mathcal{D}^b \text{MHM}(X)$  of complexes of mixed Hodge modules, equipped with Hodge-theoretic weight formalism and a natural morphism  $\text{rat} : \mathcal{D}^b \text{MHM}(X) \rightarrow \mathcal{D}_{\text{con}}^b(X, \mathbb{Q})$ . This category has two t-structures: the one corresponding to the standard t-structure on  $\mathcal{D}_{\text{con}}^b(X, \mathbb{Q})$ , and the perverse t-structure. This sets a heart: if  $K \in \text{MHM}(X)$ , then  $\text{rat}(K) \in \text{Perv}(X, \mathbb{Q})$ . When restricted to a suitable dense open set, the objects in  $\text{MHM}(X)$  become admissible variations of polarizable mixed Hodge structures.

More importantly, for  $f : X \rightarrow B$  a proper morphism of varieties and  $K \in \text{MHM}(X)$ , if the derived pushforwards (with respect to the standard t-structures)  $R^i f_* K$ ,  $R^i f_! K$  are perverse sheaves, they may be endowed with the structure of an object in  $\mathcal{D}^b \text{MHM}(X)$ . More concretely, we have the following theorem.

**Theorem 4.1** (Decomposition Theorem). *Let  $K \in \text{MHM}(X)$  be of pure weight  $w$ . Then there is a splitting of  $Rf_* K$  with respect to the perverse t-structure*

$$(4.1) \quad Rf_* K = \bigoplus_k {}^p \mathcal{H}^k(Rf_* K)[-k],$$

where each summand  ${}^p \mathcal{H}^k(Rf_* K)[-k]$  is pure and semisimple in  $\mathcal{D}^b \text{MHM}(X)$  of weight  $w - k$ . (Should there be no  $[-k]$  shift?) Moreover, if  $\eta$  is an ample class on  $X$ , there is a relative Hard Lefschetz isomorphism

$$(4.2) \quad \eta^i : {}^p \mathcal{H}^{-i}(Rf_* K) \simeq {}^p \mathcal{H}^i(Rf_* K).$$

**4.1. Fibrations of smooth algebraic varieties.** In this section, we focus on the case of  $f : X \rightarrow B$ , where  $f$  is proper with positive-dimensional fibers, and  $X$  and  $B$  are smooth.

**Example 4.2.** Let  $X$  be a projective surface and  $B$  a curve. Denote by  $B^\circ$  the locus of regular values of  $f$ . Then  $R^1 f_* \mathbb{Q}_X|_{B^\circ}$  is a weight one variation of Hodge structures.

As a Hodge module,  $\mathbb{Q}_X(0)$  is pure of weight 0. Over  $B^\circ$ , its pushforward decomposes as

$$(4.3) \quad \mathbb{Q}_{B^\circ}(0) \oplus R^1 f_* \mathbb{Q}_X|_{B^\circ}[-1] \oplus \mathbb{Q}_{B^\circ}(-1)[-2].$$

Now let  $\Delta = B \setminus B^\circ$ . Suppose further that the singular fibers of  $f$  are irreducible. Over  $p \in \Delta$ , we get

$$(4.4) \quad Rf_* \mathbb{Q}_X|_p = H^0(X_p, \mathbb{Q})[2] \oplus H^1(X_p, \mathbb{Q})[1] \oplus H^2(X_p, \mathbb{Q})[0].$$

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