PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

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These are notes I wrote up for my own comprehension while following the text of Hotta–Takeuchi–Tanisaki [HTT08]. No part of this work is original, save for any errors I may have introduced throughout my reading.

1. T-Structures

Definition 1.1. Let $\mathcal{D}$ be a triangulated category, and $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ its full subcategories. Set $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n]$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t-structure on $\mathcal{D}$ if

- $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$
- For all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.
- For any $X \in \mathcal{D}$ there exists an exact triangle

$$X_0 \to X \to X_1 \xrightarrow{+1} \cdots \tag{1.1}$$

  such that $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 1}$.

The full subcategory $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is called the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

The heart of a t-structure allows us to get a good grasp on the ambient derived category.

Theorem 1.2. *$\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an Abelian category, and any exact sequence*

$$0 \to X \to Y \to Z \to 0$$

gives rise to a distinguished triangle

$$X \to Y \to Z \xrightarrow{+1} \cdots$$

in $\mathcal{D}$.

This allows us to define the functor H^0, which sends our derived category to its heart.

Definition 1.3. The functor

$$H^0 : \mathcal{D} \to \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \qquad X \mapsto \tau^{\leq 0}\tau^{\geq 0}X.$$

Likewise, define $H^n(X) = H^0(X[n])$.

Now let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. Denote by $\mathcal{C}_1$, $\mathcal{C}_2$ the hearts of the respective categories.

Definition 1.4. The additive functor

$${}^pF : \mathcal{C}_1 \to \mathcal{C}_2, \qquad X \mapsto H^0(F(X)).$$

We say that ${}^pF$ is left t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$, and analogously for right t-exactness.

2. Perverse sheaves

Now we define a t-structure on the derived category of constructable sheaves. Recall that we have a Verdier duality functor $\mathbf{D}_X : D^b_{con}(X)^{op} \to D^b_{con}(X)$.

Definition 2.1. For $F^\bullet \in \mathcal{D}^b_{con}(X)$, we have $F^\bullet \in \mathcal{D}^{\leq 0}_{con}(X)$ if and only if

$$\dim \operatorname{supp}(H^j(F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$. Likewise, we have $F^\bullet \in \mathcal{D}^{\geq 0}_{con}(X)$ if and only if

$$\dim \operatorname{supp}(H^j(\mathbf{D}_X F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$.

It is not obvious that this is indeed a t-structure, so it must be stated seperately.

Theorem 2.2. *The pair $(\mathcal{D}^{\leq 0}_{con}, \mathcal{D}^{\geq 0}_{con})$ forms a t-structure.*

Definition 2.3. We call the heart of this t-structure $\mathrm{Perv}(\mathbb{C}_X)$ the perverse sheaves on X. The functor $\mathbf{D}_X$ induces a duality of $\mathrm{Perv}(\mathbb{C}_X)$. Now, for any functor $F : \mathcal{D}^b_{con}(X) \to \mathcal{D}^b_{con}(Y)$, denote by ${}^pF$ the composition ${}^pH^0 \circ F$, viewed as a functor $\mathrm{Perv}(\mathbb{C}_X) \to \mathrm{Perv}(\mathbb{C}_Y)$. In particular, we get the associated functors

$${}^pf^{-1}, {}^pf^! : \mathrm{Perv}(\mathbb{C}_Y) \to \mathrm{Perv}(\mathbb{C}_X),$$

and

$${}^pRf_*, {}^pRf_! : \mathrm{Perv}(\mathbb{C}_X) \to \mathrm{Perv}(\mathbb{C}_Y).$$

Next, we define minimal extensions of perverse sheaves. Let $X = Z \sqcup U$ with $i : Z \to X$ proper and $j : U \to X$ open. For $F^\bullet \in \mathcal{D}^b_{con}(X)$, we have natural maps

$$j_! F^\bullet \to Rf_* F^\bullet, \qquad s \mapsto s,$$

mapping a section to itself. If $F^\bullet$ is a perverse sheaf on U, we also get maps

$$ {}^pj_! F^\bullet \to {}^pf_* F^\bullet. \tag{2.1}$$

Definition 2.4. We denote by ${}^pf_{!*}F^\bullet$ the image of (2.1), and call it the minimal extension of $F^\bullet \in \mathrm{Perv}(\mathbb{C}_U)$.

In other words, we have maps

$$ {}^pj_! F^\bullet \twoheadrightarrow {}^pj_{!*} F^\bullet \hookrightarrow {}^pj_* F^\bullet. \tag{2.2}$$

Theorem 2.5. *Let $F^\bullet \in Perv(\mathbb{C}_U)$ as above. Then*

(i) ${}^pf_*F^\bullet$ has no non-trivial sub-objects whose support is contained in Z.
(ii) ${}^pj_!F^\bullet$ has no non-trivial quotient object whose support is contained in Z.
(iii) ${}^pj_{!*}F^\bullet$ has neither sub- or quotient objects whose support is contained in Z.

Now suppose that $F^\bullet$ is a simple object in $\mathrm{Perv}(\mathbb{C}_U)$. We claim that ${}^pj_{!*}F^\bullet$ is also simple in $\mathrm{Perv}(\mathbb{C}_X)$. Indeed, for any subobject $G^\bullet \subseteq {}^pj_{!*}F^\bullet$ we may consideer the exact sequence

$$0 \to G^\bullet \to {}^pj_{!*}F^\bullet \to H^\bullet \to 0. \tag{2.3}$$

Since $j^! = j^{-1}$ is exact here, we may apply it to obtain

$$0 \to j^{-1}G^\bullet \to F^\bullet \to j^{-1}H^\bullet \to 0. \tag{2.4}$$

Since $F^\bullet$ is simple, $j^{-1}G^\bullet$ or $j^{-1}H^\bullet$ is zero, or in other words, one of $G^\bullet$ and $H^\bullet$ is supported in Z. By the previous theorem, this makes it zero.

Theorem 2.6. *$G = {}^p j_{!*} F^\bullet$ is the unique perverse sheaf satisfying the conditions*

(i) $G^\bullet|_U \simeq F^\bullet$,
(ii) $i^{-1}G^\bullet \in {}^p\mathcal{D}^{\leq -1}_{con}(Z)$,
(iii) $i^! G^\bullet \in {}^p\mathcal{D}^{\geq 1}_{con}(Z)$.

3. Intersection cohomology

Let X be an irreducible complex projective variety (or an irreducible compact analytic space) of dimension d. We may start by defining the intersection cohomology complex in the following way.

Definition 3.1. Take $U \subseteq X^{\text{reg}}$ to be a Zariski open subset of the smooth locus of X. For $\underline{\mathbb{C}}_U[d]$ the constant perverse sheaf on U, define $IC_X^\bullet$ to be a minimal extension to all of X.

We may explicitly construct $IC_X^\bullet$. Fix a Whitney stratification $X = \bigsqcup_\alpha X_\alpha$,

$$X_k = \coprod_{\dim X_\alpha \leq k} X_\alpha, \quad U_k := X \smallsetminus X_{k-1}, \tag{3.1}$$

with each $X_k \smallsetminus X_{k-1}$ a smooth k-dimensional complex manifold. We have a family of maps $j_k : U_k \hookrightarrow U_{k-1}$, for $k = 1, \dots, d$. Altogether, this fits into

$$\emptyset \hookrightarrow U_d \hookrightarrow U_{d-1} \hookrightarrow \cdots \hookrightarrow U_1 \hookrightarrow X. \tag{3.2}$$

Theorem 3.2. *$IC_X^\bullet$ is quasi-isomorphic to the complex*

$${}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq \left(\tau^{\leq -1} Rj_{1*}\right) \circ \left(\tau^{\leq -2} Rj_{2*}\right) \circ \cdots \circ \left(\tau^{\leq -d} Rj_{d*}\right) \left(\underline{\mathbb{C}}_U[d]\right). \tag{3.3}$$

Proof. Suppose that $F^\bullet$ is a perverse sheaf on U_k whose restriction to any strata $X_\alpha \subseteq U_k$ has locally constant $\mathscr{H}^i$. We show that $G^\bullet := \tau^{\leq -k} Rj_{k*}(F^\bullet)$ satisfies the unique characterization of a minimal extension from Theorem Theorem 2.6. First, note that U_k consists of strata of dimension $\geq k$, so that $\mathscr{H}^r(F^\bullet) = 0$ for $r > -k$. In particular,

$$\left[\tau^{\leq -k} Rj_{k*} F^\bullet\right]|_{U_k} \simeq F^\bullet,$$

so that what we have is indeed an extension of $F^\bullet$, satisfying (i). Next, set $Z := U_{k-1} \smallsetminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$. Denote by $i : Z \to U_k$ the associated closde embedding. Then $i^{-1}G^\bullet$ has locally constant cohomology sheaves on each $X_\alpha \subseteq Z$, so that $\mathscr{H}^r(i^{-1}G^\bullet) = 0$ for $r > -k$. This implies that $i^{-1}G^\bullet \in {}^p\mathcal{D}^{\leq -1}_{con}$, thus satisfying condition (ii). Finally, consider the triangle

$$G \to Rj_{k*}F^\bullet \to \tau^{\geq -k+1} Rj_{k*} F^\bullet \xrightarrow{+1} \cdots \tag{3.4}$$

Applying $i^!$, we note that the middle vanishes. This gives us isomorphisms

$$i^! G^\bullet \simeq i^!(\tau^{\geq -k+1} Rj_{k*} F^\bullet)[-1]$$

so that $\mathscr{H}^r(i^! G^\bullet) = 0$ for $r < -k$. But since $i^! G$ has locally constant cohomology on $X_\alpha \subseteq Z$, we get $i^! G \in {}^p\mathcal{D}^{\geq 1}_{con}(Z)$, proving (iii). ☐

Importantly, the intersection cohomology complex is self-dual in the way we expect of perverse sheaves.

Theorem 3.3. *$IC_X^\bullet \simeq \mathbb{D}_X(IC_X^\bullet)$. Furthermore, there exist canonical morphisms*

$$\underline{\mathbb{C}}_X \to IC_X^\bullet[-d] \to \omega_X^\bullet[-2d]. \tag{3.5}$$

Proof. We have an isomorphism in the derived category of constructible sheaves from above

$$\tau^{\leq -d}\ {}^{p}j_{!*}\underline{\mathbb{C}}_X[d] \simeq Rj_{1*} \circ \cdots \circ Rj_{d*}\left(\underline{\mathbb{C}}_X[d]\right) \simeq (j_*\mathbb{C})[d]. \tag{3.6}$$

This is none other than the intersection complex of X, and we see that it admits a canonical map from $\underline{\mathbb{C}}_X$. The second map in the composition comes from taking the Verdier dual, $IC^\bullet_X[d] \to \omega^\bullet_X$. □

With the intersection complex in place, we are ready to define intersection cohomology.

Definition 3.4. For $i \in \mathbb{Z}$, we define

$$IH^i(X) = H^i\left(R\Gamma\left(X, IC^\bullet_X[-d]\right)\right), \tag{3.7}$$

$$IH^i_c(X) = H^i\left(R\Gamma_c\left(X, IC^\bullet_X[-d]\right)\right). \tag{3.8}$$

Importantly, the intersection cohomology of X satisfies Poincare duality.

Theorem 3.5. *Let X be irreducible of dimension d. Then*

$$IH^i(X) \simeq \left[IH^{2d-i}_c(X)\right]^*. \tag{3.9}$$

Proof. Let $a_X : X \to \{p\}$ be the unique map to a point. We have

$$R\mathrm{Hom}_{\mathbb{C}}(Ra_{X!}IC^\bullet, \mathbb{C}) \simeq Ra_{X*}R\mathrm{Hom}_{\mathbb{C}_X}(IC^\bullet_X, \omega^\bullet_X). \tag{3.10}$$

by Verdier duality. But $IC^\bullet_X$ is self-dual, so that

$$R\mathrm{Hom}_{\mathbb{C}_X}(IC^\bullet_X, \omega^\bullet_X) = \mathbf{D}_X(IC^\bullet_X) = IC^\bullet_X.$$

Thus we get an isomorphism

$$[R\Gamma_c(X, IC^\bullet_X)]^* \simeq R\Gamma(X, IC^\bullet_X). \tag{3.11}$$

□

Example 3.6. Let X be an irreducible complex projective variety with isolated singular points $p_1, \ldots, p_k$. Then it suffices to consider the stratification

$$X = \{p_1, \ldots, p_k\} \sqcup X^{\mathrm{reg}}.$$

Then we have $X_0 = \cdots = X_{d-1} = \{p_1, \ldots, p_k\}$, and $X_d = X$. Thus, we get the inclusions of complex manifolds

$$\emptyset \to U_d = X^{\mathrm{reg}} = U_1 \xrightarrow{j_1} X.$$

The only interesting map in this stratification is j_1, the inclusion of the smooth locus. In particular, by Theorem 3.2,

$$IC^\bullet_X \simeq \tau^{\leq -1}\left(Rj_{1*}\underline{\mathbb{C}}_{U_1}\right).$$

Now, associated to truncation is the exact couple

$$IC^\bullet_X[-d] \to Rj_{1*}\underline{\mathbb{C}}_{U_1} \to \tau^{\geq d}\left(Rj_{1*}\underline{\mathbb{C}}_{U_1}\right) \xrightarrow{+1} \cdots \tag{3.12}$$

But now we can apply $R\Gamma$ and take cohomology. For $0 \leq i < d$, we have $IH^i(X) = H^i(X, Rj_{1*}\overline{\mathbb{C}}_{U_1}) = H^i(X^{\mathrm{reg}}, \mathbb{C})$. For $i = d$, we do not get a clear vanishing on the right, but the map $IH^d(X) \to H^d(X^{\mathrm{reg}}, \mathbb{C})$ is injective. To analyze this case more carefully, consider the canonical morphism $\underline{\mathbb{C}}_X \to IC^\bullet_X[-d]$ guarenteed by Theorem Theorem 3.3.

We may associate to this natural map a new exact couple

$$\underline{\mathbb{C}}_X \to IC^\bullet_X[-d] \to F^\bullet \xrightarrow{+1} \cdots \tag{3.13}$$

Here $F^\bullet$ is a constructable sheaf supported only on the zero-dimensional closed subset $\{p_1, \ldots, p_k\} = X_0$. In particular, $H^i(F^\bullet) = 0$ for all $i \geq d$. Thus, we get $IH^i(X) = H^i(X, \mathbb{C})$ for $d < i \leq 2d$. After $i = 2d$, the cohomology of $\underline{\mathbb{C}}_X$ vanishes, and with it $IH^i(X)$. Summarizing,

$$IH^i(X) = \begin{cases} H^i(X^{\text{reg}}, \mathbb{C}) & : 0 \leq i < d \\ \operatorname{im}(H^i(X, \mathbb{C}) \to H^i(X^{\text{reg}}, \mathbb{C})) & : i = d \\ H^i(X, \mathbb{C}) & : d < i \leq 2d \\ 0 & : \text{otherwise} \end{cases} \tag{3.14}$$

□

4. Computations via Saito's mixed Hodge modules

To compute intersection complexes of standard perverse sheaves, we use Saito's mixed Hodge modules ([Sai88], [Sai90]). For a quick summary of this theory, refer to [dCRS21, §2.1]. Throughout these computations, we loosely follow [dCM09, §2].

References

[dCM09] Mark de Cataldo and Luca Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. *Bulletin of the American Mathematical Society*, 46(4):535–633, 2009.

[dCRS21] Mark Andrea A de Cataldo, Antonio Rapagnetta, and Giulia Saccà. The Hodge numbers of O'Grady 10 via Ngô strings. *Journal de Mathématiques Pures et Appliquées*, 156:125–178, 2021.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki. Regular holonomic D-modules. In *D-Modules, Perverse Sheaves, and Representation Theory*. Springer, 2008.

[Sai88] Morihiko Saito. Modules de Hodge polarisables. *Publications of the Research Institute for Mathematical Sciences*, 24(6):849–995, 1988.

[Sai90] Morihiko Saito. Mixed Hodge modules. *Publications of the Research Institute for Mathematical Sciences*, 26(2):221–333, 1990.