# PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

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These are notes I wrote up for my own comprehension while following the text of Hotta–Takeuchi–Tanisaki [HTT08]. No part of this work is original, save for any errors I may have introduced throughout my reading.

## 1. T-Structures

**Definition 1.1.** Let $\mathcal{D}$ be a triangulated category, and $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}$ its full subcategories. Set $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[n]$. The pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defines a t-structure on $\mathcal{D}$ if

- $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$
- For all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.
- For any $X \in \mathcal{D}$ there exists an exact triangle

$$X_0 \to X \to X_1 \xrightarrow{+1} \cdots \tag{1.1}$$

  such that $X_0 \in \mathcal{D}^{\leq 0}$ and $X_1 \in \mathcal{D}^{\geq 1}$.

The full subcategory $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is called the heart of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

The heart of a t-structure allows us to get a good grasp on the ambient derived category.

**Theorem 1.2.** *$\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an Abelian category, and any exact sequence*

$$0 \to X \to Y \to Z \to 0$$

*gives rise to a distinguished triangle*

$$X \to Y \to Z \xrightarrow{+1} \cdots$$

*in $\mathcal{D}$.*

This allows us to define the functor $H^0$, which sends our derived category to its heart.

**Definition 1.3.** The functor

$$H^0 : \mathcal{D} \to \mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}, \qquad X \mapsto \tau^{\leq 0}\tau^{\geq 0}X.$$

Likewise, define $H^n(X) = H^0(X[n])$.

Now let $F : \mathcal{D}_1 \to \mathcal{D}_2$ be a functor of triangulated categories. Denote by $\mathcal{C}_1$, $\mathcal{C}_2$ the hearts of the respective categories.

**Definition 1.4.** The additive functor

$${}^pF : \mathcal{C}_1 \to \mathcal{C}_2, \qquad X \mapsto H^0(F(X)).$$

We say that ${}^pF$ is left t-exact if $F(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$, and analogously for right t-exactness.

## 2. Perverse sheaves

Now we define a t-structure on the derived category of constructable sheaves. Recall that we have a Verdier duality functor $\mathbf{D}_X : D^b_{con}(X)^{op} \to D^b_{con}(X)$.

**Definition 2.1.** For $F^\bullet \in \mathcal{D}^b_{con}(X)$, we have $F^\bullet \in \mathcal{D}^{\leq 0}_{con}(X)$ if and only if

$$\dim \operatorname{supp}(H^j(F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$. Likewise, we have $F^\bullet \in \mathcal{D}^{\geq 0}_{con}(X)$ if and only if

$$\dim \operatorname{supp}(H^j(\mathbf{D}_X F^\bullet)) \leq -j$$

for all $j \in \mathbb{Z}$.

It is not obvious that this is indeed a t-structure, so it must be stated seperately.

**Theorem 2.2.** *The pair $(\mathcal{D}^{\leq 0}_{con}, \mathcal{D}^{\geq 0}_{con})$ forms a t-structure.*

**Definition 2.3.** We call the heart of this t-structure $\operatorname{Perv}(\mathbb{C}_X)$ the perverse sheaves on $X$. The functor $\mathbf{D}_X$ induces a duality of $\operatorname{Perv}(\mathbb{C}_X)$. Now, for any functor $F : \mathcal{D}^b_{con}(X) \to \mathcal{D}^b_{con}(Y)$, denote by ${}^pF$ the composition ${}^pH^0 \circ F$, viewed as a functor $\operatorname{Perv}(\mathbb{C}_X) \to \operatorname{Perv}(\mathbb{C}_Y)$. In particular, we get the associated functors

$${}^pf^{-1}, {}^pf^! : \operatorname{Perv}(\mathbb{C}_Y) \to \operatorname{Perv}(\mathbb{C}_X),$$

and

$${}^pRf_*, {}^pRf_! : \operatorname{Perv}(\mathbb{C}_X) \to \operatorname{Perv}(\mathbb{C}_Y).$$

Next, we define minimal extensions of perverse sheaves. Let $X = Z \sqcup U$ with $i : Z \to X$ proper and $j : U \to X$ open. For $F^\bullet \in \mathcal{D}^b_{con}(X)$, we have natural maps

$$j_! F^\bullet \to Rf_* F^\bullet, \qquad s \mapsto s,$$

mapping a section to itself. If $F^\bullet$ is a perverse sheaf on $U$, we also get maps

$$ {}^pj_! F^\bullet \to {}^pf_* F^\bullet. \tag{2.1}$$

**Definition 2.4.** We denote by ${}^pf_{!*}F^\bullet$ the image of (2.1), and call it the minimal extension of $F^\bullet \in \operatorname{Perv}(\mathbb{C}_U)$.

In other words, we have maps

$$ {}^pj_! F^\bullet \twoheadrightarrow {}^pj_{!*} F^\bullet \hookrightarrow {}^pj_* F^\bullet. \tag{2.2}$$

**Theorem 2.5.** *Let $F^\bullet \in Perv(\mathbb{C}_U)$ as above. Then*

*(i) ${}^pf_*F^\bullet$ has no non-trivial sub-objects whose support is contained in $Z$.*
*(ii) ${}^pj_!F^\bullet$ has no non-trivial quotient object whose support is contained in $Z$.*
*(iii) ${}^pj_{!*}F^\bullet$ has neither sub- or quotient objects whose support is contained in $Z$.*

Now suppose that $F^\bullet$ is a simple object in $\operatorname{Perv}(\mathbb{C}_U)$. We claim that ${}^pj_{!*}F^\bullet$ is also simple in $\operatorname{Perv}(\mathbb{C}_X)$. Indeed, for any subobject $G^\bullet \subseteq {}^pj_{!*}F^\bullet$ we may consideer the exact sequence

$$0 \to G^\bullet \to {}^pj_{!*}F^\bullet \to H^\bullet \to 0. \tag{2.3}$$

Since $j^! = j^{-1}$ is exact here, we may apply it to obtain

$$0 \to j^{-1}G^\bullet \to F^\bullet \to j^{-1}H^\bullet \to 0. \tag{2.4}$$

Since $F^\bullet$ is simple, $j^{-1}G^\bullet$ or $j^{-1}H^\bullet$ is zero, or in other words, one of $G^\bullet$ and $H^\bullet$ is supported in $Z$. By the previous theorem, this makes it zero.

**Theorem 2.6.** $G = {}^p j_{!*}F^\bullet$ *is the unique perverse sheaf satisfying the conditions*

*(i)* $G^\bullet|_U \simeq F^\bullet$,
*(ii)* $i^{-1}G^\bullet \in {}^p\mathcal{D}^{\leq -1}_{con}(Z)$,
*(iii)* $i^! G^\bullet \in {}^p\mathcal{D}^{\geq 1}_{con}(Z)$.

## 3. Intersection cohomology

Let $X$ be an irreducible complex projective variety (or an irreducible compact analytic space) of dimension $d$. We may start by defining the intersection cohomology complex in the following way.

**Definition 3.1.** Take $U \subseteq X^{\mathrm{reg}}$ to be a Zariski open subset of the smooth locus of $X$. For $\underline{\mathbb{C}}_U[d]$ the constant perverse sheaf on $U$, define $IC^\bullet_X$ to be a minimal extension to all of $X$.

We may explicitly construct $IC^\bullet_X$. Fix a Whitney stratification $X = \bigsqcup_\alpha X_\alpha$,

$$X_k = \coprod_{\dim X_\alpha \leq k} X_\alpha, \quad U_k := X \smallsetminus X_{k-1}, \tag{3.1}$$

with each $X_k \smallsetminus X_{k-1}$ a smooth $k$-dimensional complex manifold. We have a family of maps $j_k : U_k \hookrightarrow U_{k-1}$, for $k = 1, \ldots, d$. Altogether, this fits into

$$\emptyset \hookrightarrow U_d \hookrightarrow U_{d-1} \hookrightarrow \cdots \hookrightarrow U_1 \hookrightarrow X. \tag{3.2}$$

**Theorem 3.2.** $IC^\bullet_X$ *is quasi-isomorphic to the complex*

$${}^p j_{!*}(\mathbb{C}_U[d_X]) \simeq \left(\tau^{\leq -1}Rj_{1*}\right) \circ \left(\tau^{\leq -2}Rj_{2*}\right) \circ \cdots \circ \left(\tau^{\leq -d}Rj_{d*}\right)\left(\underline{\mathbb{C}}_U[d]\right). \tag{3.3}$$

*Proof.* Suppose that $F^\bullet$ is a perverse sheaf on $U_k$ whose restriction to any strata $X_\alpha \subseteq U_k$ has locally constant $\mathscr{H}^i$. We show that $G^\bullet := \tau^{\leq -k}Rj_{k*}(F^\bullet)$ satisfies the unique characterization of a minimal extension from Theorem Theorem 2.6. First, note that $U_k$ consists of strata of dimension $\geq k$, so that $\mathscr{H}^r(F^\bullet) = 0$ for $r > -k$. In particular,

$$\left[\tau^{\leq -k}Rj_{k*}F^\bullet\right]|_{U_k} \simeq F^\bullet,$$

so that what we have is indeed an extension of $F^\bullet$, satisfying (i). Next, set $Z := U_{k-1} \smallsetminus U_k = \bigsqcup_{\dim X_\alpha = k-1} X_\alpha$. Denote by $i : Z \to U_k$ the associated closde embedding. Then $i^{-1}G^\bullet$ has locally constant cohomology sheaves on each $X_\alpha \subseteq Z$, so that $\mathscr{H}^r(i^{-1}G^\bullet) = 0$ for $r > -k$. This implies that $i^{-1}G^\bullet \in {}^p\mathcal{D}^{\leq -1}_{con}$, thus satisfying condition (ii). Finally, consider the triangle

$$G \to Rj_{k*}F^\bullet \to \tau^{\geq -k+1}Rj_{k*}F^\bullet \xrightarrow{+1} \cdots \tag{3.4}$$

Applying $i^!$, we note that the middle vanishes. This gives us isomorphisms

$$i^!G^\bullet \simeq i^!(\tau^{\geq -k+1}Rj_{k*}F^\bullet)[-1]$$

so that $\mathscr{H}^r(i^!G^\bullet) = 0$ for $r < -k$. But since $i^!G$ has locally constant cohomology on $X_\alpha \subseteq Z$, we get $i^!G \in {}^p\mathcal{D}^{\geq 1}_{con}(Z)$, proving (iii). $\square$

Importantly, the intersection cohomology complex is self-dual in the way we expect of perverse sheaves.

**Theorem 3.3.** $IC^\bullet_X \simeq \mathbb{D}_X(IC^\bullet_X)$. *Furthermore, there exist canonical morphisms*

$$\underline{\mathbb{C}}_X \to IC^\bullet_X[-d] \to \omega^\bullet_X[-2d]. \tag{3.5}$$

*Proof.* We have an isomorphism in the derived category of constructible sheaves from above

$$\tau^{\leq -d}\ {}^{p}j_{!*}\underline{\mathbb{C}}_X[d] \simeq Rj_{1*} \circ \cdots \circ Rj_{d*}\left(\underline{\mathbb{C}}_X[d]\right) \simeq (j_* \mathbb{C})[d]. \tag{3.6}$$

This is none other than the intersection complex of $X$, and we see that it admits a canonical map from $\underline{\mathbb{C}}_X$. The second map in the composition comes from taking the Verdier dual, $IC^\bullet_X[d] \to \omega^\bullet_X$. □

With the intersection complex in place, we are ready to define intersection cohomology.

**Definition 3.4.** For $i \in \mathbb{Z}$, we define

$$IH^i(X) = H^i\left(R\Gamma\left(X, IC^\bullet_X[-d]\right)\right), \tag{3.7}$$

$$IH^i_c(X) = H^i\left(R\Gamma_c\left(X, IC^\bullet_X[-d]\right)\right). \tag{3.8}$$

Importantly, the intersection cohomology of $X$ satisfies Poincare duality.

**Theorem 3.5.** *Let $X$ be irreducible of dimension $d$. Then*

$$IH^i(X) \simeq \left[IH^{2d-i}_c(X)\right]^*. \tag{3.9}$$

*Proof.* Let $a_X : X \to \{p\}$ be the unique map to a point. We have

$$R\mathrm{Hom}_{\mathbb{C}}(Ra_{X!}IC^\bullet, \mathbb{C}) \simeq Ra_{X*}R\mathrm{Hom}_{\mathbb{C}_X}(IC^\bullet_X, \omega^\bullet_X). \tag{3.10}$$

by Verdier duality. But $IC^\bullet_X$ is self-dual, so that

$$R\mathrm{Hom}_{\mathbb{C}_X}(IC^\bullet_X, \omega^\bullet_X) = \mathbf{D}_X(IC^\bullet_X) = IC^\bullet_X.$$

Thus we get an isomorphism

$$[R\Gamma_c(X, IC^\bullet_X)]^* \simeq R\Gamma(X, IC^\bullet_X). \tag{3.11}$$

□

**Example 3.6.** Let $X$ be an irreducible complex projective variety with isolated singular points $p_1, \ldots, p_k$. Then it suffices to consider the stratification

$$X = \{p_1, \ldots, p_k\} \sqcup X^{\mathrm{reg}}.$$

Then we have $X_0 = \cdots = X_{d-1} = \{p_1, \ldots, p_k\}$, and $X_d = X$. Thus, we get the inclusions of complex manifolds

$$\emptyset \to U_d = X^{\mathrm{reg}} = U_1 \xrightarrow{j_1} X.$$

The only interesting map in this stratification is $j_1$, the inclusion of the smooth locus. In particular, by Theorem 3.2,

$$IC^\bullet_X \simeq \tau^{\leq -1}\left(Rj_{1*}\underline{\mathbb{C}}_{U_1}\right).$$

Now, associated to truncation is the exact couple

$$IC^\bullet_X[-d] \to Rj_{1*}\underline{\mathbb{C}}_{U_1} \to \tau^{\geq d}\left(Rj_{1*}\underline{\mathbb{C}}_{U_1}\right) \xrightarrow{+1} \cdots \tag{3.12}$$

But now we can apply $R\Gamma$ and take cohomology. For $0 \leq i < d$, we have $IH^i(X) = H^i(X, Rj_{1*}\overline{\mathbb{C}}_{U_1}) = H^i(X^{\mathrm{reg}}, \mathbb{C})$. For $i = d$, we do not get a clear vanishing on the right, but the map $IH^d(X) \to H^d(X^{\mathrm{reg}}, \mathbb{C})$ is injective. To analyze this case more carefully, consider the canonical morphism $\underline{\mathbb{C}}_X \to IC^\bullet_X[-d]$ guarenteed by Theorem Theorem 3.3.

We may associate to this natural map a new exact couple

$$\underline{\mathbb{C}}_X \to IC^\bullet_X[-d] \to F^\bullet \xrightarrow{+1} \cdots \tag{3.13}$$

Here $F^\bullet$ is a constructable sheaf supported only on the zero-dimensional closed subset $\{p_1, \dots, p_k\} = X_0$. In particular, $H^i(F^\bullet) = 0$ for all $i \geq d$. Thus, we get $IH^i(X) = H^i(X, \mathbb{C})$ for $d < i \leq 2d$. After $i = 2d$, the cohomology of $\underline{\mathbb{C}}_X$ vanishes, and with it $IH^i(X)$. Summarizing,

$$IH^i(X) = \begin{cases} H^i(X^{\text{reg}}, \mathbb{C}) & : 0 \leq i < d \\ \text{im}(H^i(X, \mathbb{C}) \to H^i(X^{\text{reg}}, \mathbb{C})) & : i = d \\ H^i(X, \mathbb{C}) & : d < i \leq 2d \\ 0 & : \text{otherwise} \end{cases} \tag{3.14}$$

□

## 4. Computations via Saito's mixed Hodge modules

To compute intersection complexes of standard perverse sheaves, we use Saito's mixed Hodge modules ([Sai88], [Sai90]). For a quick summary of this theory, refer to [dCRS21, §2.1]. Throughout these computations, we loosely follow [dCM09, §2].

Let $X$ be a variety over $\mathbb{C}$. Saito [Sai90] defines a category $\mathcal{D}^b\,\text{MHM}(X)$ of complexes of mixed Hodge modules, equipped with Hodge-theoretic weight formalism and a natural morphism $\text{rat} : \mathcal{D}^b\,\text{MHM}(X) \to \mathcal{D}^b_{con}(X, \mathbb{Q})$. This category has two t-structures: the one corresponding to the standard t-structure on $\mathcal{D}^b_{con}(X, \mathbb{Q})$, and the perverse t-structure. This sets a heart: if $K \in \text{MHM}(X)$, then $\text{rat}(K) \in \text{Perv}(X, \mathbb{Q})$. When restricted to a suitable dense open set, the objects in $\text{MHM}(X)$ become admissable variations of polarizable mixed Hodge structures.

More importantly, for $f : X \to B$ a proper morphism of varieties and $K \in MHM(X)$, if the derived pushforwards (with respect to the standard t-structures) $R^i f_* K$, $R^i f_! K$ are perverse sheaves, they may be endowed with the structure of an object in $\mathcal{D}^b\,\text{MHM}(X)$. More concretely, we have the following theorem.

**Theorem 4.1** (Decomposition Theorem)**.** *Let $K \in \text{MHM}(X)$ be of pure weight $w$. Then there is a splitting of $Rf_*K$ with respect to the perverse t-structure*

$$Rf_*K = \bigoplus_k {}^p\mathcal{H}^k(Rf_*K)[-k], \tag{4.1}$$

*where each summand ${}^p\mathcal{H}^k(Rf_*K)[-k]$ is pure and semisimple in $\mathcal{D}^b\,\text{MHM}(X)$ of weight $w - k$. (Should there be no $[-k]$ shift?) Moreover, if $\eta$ is an ample class on $X$, there is a relative Hard Lefschetz isomorphism*

$$\eta^i : {}^p\mathcal{H}^{-i}(Rf_*K) \simeq {}^p\mathcal{H}^i(Rf_*K). \tag{4.2}$$

4.1. **Fibrations of smooth algebraic varieties.** In this section, we focus on the case of $f : X \to B$, where $f$ is proper with positive-dimensional fibers, and $X$ and $B$ are smooth.

**Example 4.2.** Let $X$ be a projective surface and $B$ a curve. Denote by $B^\circ$ the locus of regular values of $f$, $\Delta = B \setminus B^\circ$. Then $R^1 f_* \mathbb{Q}_X|_{B^\circ}$ is a weight one variation of Hodge structures. Denote the underlying local system by $\mathbb{V}$.

In order for $\mathbb{V}$ to appear in middle perversity in the decomposition theorem, we must shift the pushforward by 2: set $W = Rf_*\mathbb{Q}_X[2]$.

$$\begin{aligned} W|_{B^\circ} &= {}^p\mathcal{H}^{-2}(W|_{B^\circ})[2] \oplus {}^p\mathcal{H}^{-1}(W|_{B^\circ})[1] \oplus {}^p\mathcal{H}^{0}(W|_{B^\circ})[0] \\ &= \mathbb{Q}_{B^\circ}[2] \oplus \mathbb{V}[1] \oplus \mathbb{Q}_{B^\circ}. \end{aligned}$$

Next, we set our attention to the singular fibers of the pushforward. Let $p \in \Delta$ be a critical value. Then

$$\begin{aligned}
{}^p\mathcal{H}^{-2}(W)_p[2] &= H^0(X_p, \mathbb{Q})_p[2] = \mathbb{Q}_p[2],\\
{}^p\mathcal{H}^{-1}(W)_p[1] &= H^1(X_p, \mathbb{Q})_p[1],\\
{}^p\mathcal{H}^{-2}(W)_p[0] &= H^2(X_p, \mathbb{Q})_p[0] = \bigoplus_{X_p^{(i)} \subseteq X_p} \left\langle [X_p^{(i)}] \right\rangle_p [0],
\end{aligned}$$

where the last sum is taken over the irreducible components of $X_p$.

Altogether, the decomposition theorem yields

$$W = \mathbb{Q}_B[2] \oplus \overline{\mathbb{V}}[1] \oplus \mathbb{Q}_B[0] \oplus \bigoplus_{p \in \Delta} \left( \bigoplus_{X_p^{(i)} \subseteq X_p} \frac{\left\langle [X_p^{(i)}] \right\rangle}{\langle [X_p] \rangle} \right)_p [0]. \tag{4.3}$$

Here $\overline{\mathbb{V}} = R^1 f_* \mathbb{Q}_X$. We may also compute the intersection complex $IC_B(\mathbb{V})$ valued in the local system $\mathbb{V}$. It appears in this decomposition, and indeed, it is clear from the perverse degrees that

$$IC_B(\mathbb{V}) = \overline{\mathbb{V}}[1] \oplus \bigoplus_{p \in \Delta} \left( \bigoplus_{X_p^{(i)} \subseteq X_p} \frac{\left\langle [X_p^{(i)}] \right\rangle_p}{\langle [X_p] \rangle_p} \right) [0]. \tag{4.4}$$

This is identified with $R^1 f_* \mathbb{Q}_X[1]$ if and only if the fibers of $f$ are all irreducible. $\square$

**Example 4.3.** We now consider the same example, but on the level of mixed Hodge modules. As a Hodge module, $\mathbb{Q}_X(0)$ is pure of weight 0. Over $B^\circ$, its pushforward decomposes as

$$\mathbb{Q}_{B^\circ}(0)[0] \oplus \mathcal{V}[-1] \oplus \mathbb{Q}_{B^\circ}(-1)[-2], \tag{4.5}$$

where $\mathcal{V}$ is the weight one variation of Hodge structures associated to the local system $\mathbb{V}$ above. Over $p \in \Delta$, we get

$$Rf_* \mathbb{Q}_X|_p = H^0(X_p, \mathbb{Q})[0] \oplus H^1(X_p, \mathbb{Q})[-1] \oplus H^2(X_p, \mathbb{Q})[-2]. \tag{4.6}$$

Hence, we get the same decomposition as in (4.3):

$$Rf_* \mathbb{Q}_X = \mathbb{Q}_B(0)[0] \oplus \overline{\mathcal{V}}[-1] \oplus \mathbb{Q}_B(-1)[-2] \oplus \bigoplus_{p \in \Delta} \left( \frac{\left\langle [X_p^{(1)}], \dots, [X_p^{(n)}] \right\rangle}{\left\langle [X_p^{(1)} + \dots + X_p^{(n)}] \right\rangle} \right)_p [-2]. \tag{4.7}$$

Notice that the relative Hard Lefschetz theorem sends $\mathbb{Q}_B(0)$ to $\mathbb{Q}_B(-1)[-2]$ isomorphically, and fixes all other primative cohomology. We get a mixed Hodge module associated to the intersection complex,

$$\mathscr{IC}_B(\mathcal{V}) = \overline{\mathcal{V}}[-1] \oplus \bigoplus_{p \in \Delta} \left( \frac{\left\langle [X_p^{(1)}], \dots, [X_p^{(n)}] \right\rangle}{\left\langle [X_p^{(1)} + \dots + X_p^{(n)}] \right\rangle} \right)_p [-2]. \tag{4.8}$$

Here $\overline{\mathcal{V}} = R^1 f_* \mathbb{Q}_X$. From this description, we see that it is a simple weight one mixed Hodge module if and only if the fibers of $f$ are irreducible.

We may compute $IH^k(B, \mathcal{V}) = H^k(R\Gamma(\mathscr{IC}_B(\mathcal{V})))$. By Saito, these carry mixed Hodge structures. In this case, we have $IH^1(B, \mathcal{V}) = H^0(\overline{\mathcal{V}})$ the non-vanishing invariant cycles, and $IH^2(B, \mathcal{V}) = \bigoplus_{p\in\Delta} H^2(X_p, \mathbb{Q})^{\text{prim}}$ the primative weight two cohomogy of the singular fibers. □

4.2. **Resolutions of normal isolated singularities.** We next consider another important class of morphisms: Let $X$ be a projective variety singular at $x$, and let $f : \widetilde{X} \to X$ be a resolution. We will use $f$ compute $IC_X$. In particular, we may recover $IH^*(X, \mathbb{Q})$, which satisfies Poincare duality.

**Example 4.4.** Let $C \subseteq \mathbb{P}^2$ be a plane curve, and consider $X$ the cone over $C$. We have a natural resolution $f : \widetilde{X} \to X$ given by the blow-up morphism, $\widetilde{X} = \text{Bl}_x X$. Consider $K = Rf_*\mathbb{Q}_{\widetilde{X}}$.

$\mathbb{Q}_{\widetilde{X}}(0)$ is a weight zero Hodge module on $\widetilde{X}$, so its pushforward $K \in \mathcal{D}^b\,\text{MHM}(X)$ is a complex of pure weight zero. The restriction to the smooth locus of $X$ is $K|_{X^\circ} = \mathbb{Q}_{X^\circ}(0)$, a local system. Over the singular point, the fiber is $\widetilde{X}_x \simeq S$, so its cohomology is

$$K_x = \mathbb{Q}_x(0)[0] \oplus H^1(C, \mathbb{Q})_x[-1] \oplus \mathbb{Q}_x(-1)[-2]. \tag{4.9}$$

Thus, the decomposition theorem yields

$$K = \mathbb{Q}_X(0)[0] \oplus H^1(C, \mathbb{Q})_x[-1] \oplus \mathbb{Q}_x(-1)[-2]. \tag{4.10}$$

Now, the first summand is supported in dimension one, while the second is supported in dimension zero. We therefore see that both of them of the same perverse degree, so that

$$\mathscr{IC}_X = \mathbb{Q}_X(0)[0] \oplus H^1(C, \mathbb{Q})_x[-1]. \tag{4.11}$$

In particular, this implies that

$$IH^i(X, \mathbb{Q}) = \begin{cases} H^1(C, \mathbb{Q}) & : i = 1 \\ H^i(X, \mathbb{Q}) & : i \neq 1 \end{cases} \tag{4.12}$$

Intersection cohomology satisfies Poincare duality, so we can mostly recover $H^*(X, \mathbb{Q})$ from this description:

$$H^0(X, \mathbb{Q}) = \mathbb{Q},\ H^1(X, \mathbb{Q}) = 0,\ H^2(X, \mathbb{Q}) =?,\ H^3(X, \mathbb{Q}) = \mathbb{Q}^{2g},\ H^4(X, \mathbb{Q}) = \mathbb{Q}. \tag{4.13}$$

We can check that $H^2(X, \mathbb{Q}) = \mathbb{Q}$ by noting that all non-ruling classes in $H^2(X, \mathbb{Q})$ can be contracted to zero at the cone point, and are therefore trivial. □

**Example 4.5.** This time, set $S \subseteq \mathbb{P}^2$ to be a quadric surface, and consider the cone $X \subseteq \mathbb{P}^3$ over $S$. We have two resolutions to choose from: the blow-up of the cone point $x \in X$, or the blow-up of a line $L \ni x$ in $X$.

First, let us consider the former: $\widetilde{X} = \text{Bl}_x X$, $f : \widetilde{X} \to X$. As before, let $K = Rf_*\mathbb{Q}_X$. We once again have $K|_{X^\circ} = \mathbb{Q}_{X^\circ}(0)$. Anything interesting will again have to happen over the cone point. The fiber $\widetilde{X}_x \simeq S$, which happens to be simply connected. We see that

$$K = \mathbb{Q}_X(0)[0] \oplus H^2(S, \mathbb{Q})_x[-2] \oplus \mathbb{Q}_x(-2)[-4]. \tag{4.14}$$

The first summand is supported in dimension two, while the primitive part of the second is supported in dimension zero. Thus, they have the same perversity, and

$$\mathscr{IC}_X = \mathbb{Q}_X(0)[0] \oplus H^2(S, \mathbb{Q})_x^{\text{prim}}[-2]. \tag{4.15}$$

Now we can again compute the intersection cohomology of $X$. We get

$$IH^i(X,\mathbb{Q}) = \begin{cases} H^2(X,\mathbb{Q}) \oplus \langle L_1 - L_2\rangle & : i = 2 \\ H^i(X,\mathbb{Q}) & : i \neq 2. \end{cases} \tag{4.16}$$

In particular, we can conclude that

$$H^0(X,\mathbb{Q}) = \mathbb{Q},\ H^2(X,\mathbb{Q}) = \mathbb{Q},\ H^4(X,\mathbb{Q}) = \mathbb{Q}^2,\ H^6(X,\mathbb{Q}) = \mathbb{Q}, \tag{4.17}$$

and all odd-degree cohomology vanishes from standard rationality arguments. □

**Example 4.6.** Had we instead taken $\widetilde{X} = \mathrm{Bl}_L X$, $f : \widetilde{X} \to X$, we would have $\widetilde{X}_x = \mathbb{P}^1$. Now if we consider $K = Rf_*\mathbb{Q}_{\widetilde{X}}$, the decomposition theorem yields

$$K = \mathbb{Q}_X(0)[0] \oplus \mathbb{Q}_x(-1)[-2] \simeq \mathscr{I}\mathscr{C}_X. \tag{4.18}$$

Hence, we see that some resolutions are better for computing intersection cohomology than others. Note that this map $f$ had the property that

$$Rf_*\mathscr{I}\mathscr{C}_{\widetilde{X}} = \mathscr{I}\mathscr{C}_X. \tag{4.19}$$

□

## References

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