1. True of False. **(2 points each)**
(a) There exists a 3-by-2 matrix $A \in \mathbb{R}^{3\times 2}$ with $\operatorname{rank}(A) = 3$.

(b) If a square matrix is full rank, then it's row Reduced Echelon Form (RREF) must be the identity matrix.

(c) If $T(x) = Ax$ where $A \in \mathbb{R}^{4\times 2}$ is a 4-by-2 matrix, then T must be injective (one-to-one).

(d) If $T(x) = Ax$ where $A \in \mathbb{R}^{4\times 2}$ is a 4-by-2 matrix, then T cannot be surjective (onto).

(e) A linear system has a unique solution if the columns of the corresponding matrix are linearly independent.

(f) The set $S = \{x \in \mathbb{R}^n \mid Ax = b \text{ for some } x \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^n$ where $A \in \mathbb{R}^{n\times n}$ is a nonzero square matrix and $b \in \mathbb{R}^n$ is a nonzero vector.

(g) The set $S = \{(x, y, z) \mid ax + by + cz = 0\}$ is a plane parallel to the vector $n = (a, b, c)$.

(h) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation where $n \neq m$, then T is not invertible.

(i) **(4 points)** Write down the definition of linear dependence.

2. Solve the following system (or explain why it does not have any solution). **(6 points)** Characterize the geometry of the associated linear map (e.g. scaling, orthogonal projection, reflection, rotation, or shear). **(4 points)**

$$\begin{cases} x + 2y + 3z - u = 0 \\ 2x + 4y + 6z - 2u = 0 \\ 3x + 6y + 9z - 3u = 0 \\ -x - 2y - 3z + u = 1 \end{cases}$$

Hint: Note that the system is not much but

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3. (a) **(5 points)** Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) **(5 points)** Derive the necessary and sufficient condition such that

$$B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

is invertible. Compute its inverse and verify that the inverse of upper triangular is upper triangular as well.

4. (a) **(3 points)** Give an example of a system of 3 linear equations with 3 variables that has infinitely many solutions.

(b) **(3 points)** Give an example of a system of 3 linear equations with 3 variables that has no solution.

(c) **(2 points)** Give an example of a 4-by-4 matrix that has rank 2.

(d) **(2 points)** Give an example of $A, B \in \mathbb{R}^{2\times 2}$ such that $AB \neq BA$.

5. **(10 points)** Describe the geometry of the image and kernel of the linear map $T(x) = Ax$ (as point, line, plane, or $\mathbb{R}^3$) where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ -1 & -2 & -1 \end{bmatrix}.$$

Find a basis for $im(A)$ and $ker(A)$ and verify that $\dim(im(A)) + \dim(ker(A)) = 3$.