Sec 1.1 Intro to Linear Systems

Recall algebra, e.g. $x + 5 = 3$. Generalize to two variables:

$$\begin{cases} x + y = 5 \\ 3x - y = -1 \end{cases}.$$

Solving intuitively, $x = 1, y = 4$. The problem on Page 1:

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases}. \tag{1}$$

Answer is $x = 2.75, y = 4.25, z = 9.25$.

Some systems are not (uniquely) solvable.

$$\begin{cases} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{cases}. \tag{2}$$

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{cases}. \tag{3}$$

Geometric interpretation: find points that lie on all three planes.

"Degrees of freedom" (from Sec 1.3)

$$\begin{cases} x + z = -7 \\ x + 3z = 3 \\ x + 5z = 13 \end{cases} \qquad \begin{cases} x + y + z = 1 \\ y + 3z = 3 \end{cases} \qquad \begin{cases} x + y + 4z = 1 \\ x - y + z = 1 \\ 3x + y - z = 5 \\ x + 4y - 6z = 0 \end{cases}$$

First: $x = -12, z = 5$ but no constraint on y.

Quick check, doesn't prove solvability.

Geometric interpretation (from Sec 1.3)

- $ax + by + cz = 0$ defines a plane perpendicular to (a, b, c) passing origin. Translate it to get $ax + by + cz = d$.
- Intersection of planes, either unique or infinitely many solutions. (Houdini demo)

Solvability (from Sec 1.3)

Not solvable: contradiction after some reduction. See (3).
Infinite solutions: parametrization. See (2).

Sec 1.2 Matrices, Vectors, and Gauss-Jordan Elimination

- matrix dimension; row, column, index notation
- identity, zero, square, upper/lower triangular, symmetric matrices
- vector, vector spaces $\mathbb{R}^n$ (column vectors!)
- solve (1) using extended matrix
- Gaussian reduction: three operations
- RREF: definition, solve (1), show (2) is not full rank

Sec 1.3 On the Solutions of Linear Systems; Matrix Algebra

Rank

Matrices from (1) and (2) have rank 3 and 2. Full rank matrix has identity in RREF.

Matrix Algebra

- from linear system to matrix-vector equation
- matrix addition, matrix-vector multiplication, matrix-matrix multiplication
- distribution law, commutative law etc.
- linear combination
- interpret matrix-vector multiplication as linear combination with columns

Sec 2.1 Linear Transformations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

- linearity
- $Ae_i = T(e_i)$
- Finding the corresponding matrix
- ~~Markov chain: EXAMPLE 9 on p.5, distribution vectors and transition matrices~~ (skipped)

Sec 2.2 Linear Transformation in Geometry

- Geometric meaning of the four entries of a 2-by-2 matrix (scaling, shearing)
- ~~orthogonal projection, reflection in 2D~~ take home
- orthogonal projection, reflection w.r.t. a plane in 3D
- rotation in 2D

Sec 2.3 Matrix Products

- function composition
- non-commutativity
- ~~distributivity~~ in homework
- ~~block matrix multiplication~~ skip

Sec 2.4 The Inverse of a Linear Transformation

- injective, surjective, invertible functions and their composition
- invertible matrices: RREF, rank, row operations
- invertible linear systems: solvability
- $AA^{-1} = A^{-1}A = I$
- prove $(AB)^{-1} = B^{-1}A^{-1}$
- 2-by-2 matrix inverse formula

Sec 3.1 Image and Kernel of a Linear Transformation

- definition of image, kernel, and span
- finding image and kernel of matrix $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Sec 3.2 Subspaces of $\mathbb{R}^n$; Bases and Linear Independence

Subspaces

- subspace: closed under linear combination
- image and kernel are subspaces
- geometric interpretation

Bases

- linear independence; link to rank of a matrix
- nontrivial kernel = not linearly independent columns of a matrix, e.g. (page 129)

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

- basis = linearly independent spanning set
- find basis for subspaces $im(A), ker(A)$
- basis and unique representation

Sec 3.3 Dimension of a Subspace of $\mathbb{R}^n$

- dimension = # of vectors in a basis
- e.g. $\mathbb{R}^n = span(\{e_1, \cdots, e_n\})$
- dimension is unique
- dimension = maximal # of linearly independent vectors = minimal # of spanning vectors (Theorem 3.3.4 page 136)
- $v_1, \cdots, v_k$: linear independent $\Rightarrow Sv_1, \cdots, Sv_k$: linear independent where $S \in \mathbb{R}^{n \times n}$ is invertible
- example 1 page 136
- row reduction messes up the columns
- $rank(A) = rank(SA)$ for any $A \in \mathbb{R}^{m \times n}$ and $S \in GL(m, \mathbb{R})$
- rank-nullity theorem: from RREF, # pivot is rank, those columns without pivot is free variable (nullity), summing to n

Midterm 1 Review

- linear systems with a unique solution, infinitely many solutions, no solution (examples and such)
- orthogonal projection to a one dimensional subspace $\{\alpha n : \alpha \in \mathbb{R}\}$
- row reduction, RREF, solving linear system, extended matrix, inverting a matrix by applying row reduction simultaneously on A and I

- meaning of rank (# nonzero rows in RREF, dimension of image)
- links between rank, RREF, solvability of the linear system, invertibility of the matrix, linear independence of the columns, injectivity, surjectivity of the linear map; practice this with square matrices and rectangular matrices
- meaning of a subspace, linear combination, linear dependence, spanning set, basis, dimension
- how to find basis of the image and kernel of a given linear map $T(x) = Ax$

Sec 3.3 Dimension of a Subspace of $\mathbb{R}^n$ (Cont'd)

- rank nullity theorem: rank is number of pivots in RREF

Sec 3.4 Coordinates

- Notation of coordinates (page 149)
- example 2 (page 150)
- $[x]_{\mathcal{B}} = S^{-1}x$
- example 3 on page 151
- diagram on page 155
- $x = [x]_{\mathcal{E}}$, $[T]_{\mathcal{B}}$ using example 3
- rotation on a plane spanned by $(2, 1, -1)$ and $(0, 1, 1)$
- similar matrices (definition)
- diagonal $[T]_{\mathcal{B}}$ (page 157)

Sec 5.1 Orthogonal Projections and Orthonormal Bases

- orthonormal basis (definition)
- $S^{-1} = S^T$ for orthonormal basis
- orthogonal projection done by inner product with orthonormal basis, $\text{diag}(1, \cdots, 1, 0, \cdots, 0)Sx$
- theorem 5.1.6 (page 207)
- ~~orthogonal complement (definition)~~ skip
- ~~theorem 5.1.8 (page 208)~~ skip
- ~~Cauchy-Schwartz inequality~~ skip

Sec 5.2 Gram-Schmidt Process and QR Factorization

- Gram-Schmidt process: Example 1 (page 219)
- QR factorization: Example 2 (page 222)

Sec 5.3 Orthogonal Transformations and Orthogonal Matrices

- Definition 5.3.1 (page 225)
- $\|Ax\|^2 = x^T A^T A x = x^T x \Leftrightarrow A^T A = I$, i.e. columns of A are orthonormal! (Theorem 5.3.3 on page 227)
- Theorem 5.3.10 on page 232)

Sec 5.4 Least Squares and Data Fitting

- Definition 5.4.4 Least-squares solutions $\|b - Ax^*\| \leq \|b - Ax\| \forall x \in \mathbb{R}^m$.
- Thm 5.4.1, 5.4.2, 5.4.3, 5.4.5, or alternatively, multi-variable calculus to show $A^T(Ax^* - b) = 0$.
- Theorem 5.4.6, 5.4.7
- (maybe skip theorems) Example 4 (page 242), Example 6 (page 244) (linear regression)

Sec 6.1 Intro to Determinants

- 2 by 2 and 3 by 3
- $\det(A) = u \cdot (v \times w)$ for 3 by 3 matrices
- $\det(A) = 0$ then u, v, w are linear dependent
- Similarly, $\det(A) \neq 0$ then A is invertible for 2 by 2 matrices

Midterm 2 Review

- rank-nullity theorem (statement and usage)
- coordinates (computation with $S = [v_1 \cdots v_n]$, $S[x]_{\mathcal{B}} = x$), both for subspaces or $\mathbb{R}^n$
- orthonormal basis, both for subspaces or $\mathbb{R}^n$
- $S^{-1} = S^T$, $\therefore$[x]$_{\mathcal{B}}$ can be found by $u_i^T x$
- orthogonal projection using orthonormal basis

- orthogonal complement (definition)
- Gram-Schmidt process, QR factorization
- orthogonal matrix (definition)
- Least-squares solutions: $Ax^* - b \in im(A)^{\perp}$, $A^T A x^* = A^T b$

Sec 6.1 Intro to Determinants (Cont'd)

- n by n matrix (page 269)
- S_n permutation groups, transitions, decomposition of permutations
- determinant of triangular matrix
- Sec.6.1. #37 on page 276

Sec 6.2 Properties of the Determinant

- thm 6.2.1, transpose
- thm 6.2.2, linear in columns
- thm 6.2.3, row reductions
- thm 6.2.4, invertibility
- thm 6.2.6, product rule
- thm 6.2.8, inverse
- thm 6.2.10, Laplace expansion

Sec 6.3 Geometrical Interpretations of the Determinant; Cramer's Rule

- $Q \in \mathcal{U}(n) \Rightarrow \det(Q) = \pm 1$
- rotation matrices, reflection
- thm 6.3.3 with QR factorization
- interpretation of thm 6.3.3. with 2D and 3D geometry
- thm 6.3.4
- thm 6.3.6 with QR factorization
- Jacobian ("expansion factor")

Sec 7.1 Diagonalization

- diagonalizable matrices: $A = PDP^{-1}$
- interpret $AP = PD$ with Eigenvectors, eigenvalues, and eigenbases
- Example 4 (importance to determine the field)
- Example 5
- Example 7

Sec 7.2 Finding the Eigenvalues of a Matrix

- characteristic equation and its zeros
- Example 2 (then thm.7.2.2)
- characteristic polynomial (thm 7.2.5) $f_A(\lambda)$
- def 7.2.6 (algebraic multiplicity of an eigenvalue)
- Newton's method to find rational roots
- polynomial division
- decomposition of polynomials

Note: maybe mention the definition of trace and show its cyclic property.
May 25, 2019

Sec 7.3 Finding the Eigenvectors of a Matrix

- def 7.3.1 $E_\lambda = ker(A - \lambda I)$, Example 1, 2
- geometric multiplicity
- thm 7.3.3 (eigenbases and geometric multiplicities), 7.3.4 (all eigenvalues are distinct)
- thm 7.3.5 (similar matrices)
- thm 7.3.6 (geometric multiplicity $\leq$ algebraic multiplicity)

Sec 7.4 More on Dynaimcal Systems

- Example 1 (page 347)
- thm 7.4.1 (equilibria)
- Example 2 (page 351)

Sec 8.1 Symmetric Matrices

- thm 8.1.1 spectral theorem
- eigenspace are linear independent. For symmetric matrices, they are orthogonal
- thm 8.1.3
- thm 8.1.4 how to find orthogonal diagonalization

Sec 7.2 Quadratic Forms

- example 1 (turning quadratic forms into symmetric matrices)
- the benefits of diagonalizing the matrix associated with quadratic form (completing the squares)
- Definition 8.2.3 (positive/negative definite)
- Thm 8.2.4 (link of definiteness with EWs)
- Thm 8.2.7 (Ellipse or hyperbola)

Sec 8.3 Singular Values

- Review all decomposition we went over in the past: $A = QR, A = SDS^{-1}, A = SDS^T$
- Singular value decomposition $A = U\Sigma V^T$
- image of the unit circle
- Example 4
- Example 3
- Thm 8.3.4 #(nonzero singular values) = rank
- polar decomposition $A = (UV^T)(V\Sigma V^T) = RS$
- matrix norm $\|A\| = \max_{\|x\|=1} \|Ax\|$

Final review

- 2 by 2 singular value decomposition
- matrix norm
- quadratic form, its symmetric matrix, completion of squares, definiteness, maximum and minimum, principal axes
- dynamical system, limit behavior
- eigenvalue decomposition, links with rank, kernel, similar matrices, invertibility, etc
- determinant, geometric meaning, volume of parallelepiped
- QR factorization: GS, computing R, orthogonal projection, lease-squares solutions
- orthogonal projections
- rank-nullity theorem, finding bases of rank and kernel
- conditions of invertibility of a matrix