Instructions:

- Follow directions and answer questions with requested supporting work.
- Clearly indicate your answer in the allotted space or by putting a box around it.
- No cellphones, laptops, books, notes, supporting materials, or external aids are allowed on this exam.

Name: ______________________________

UID: ______________________________

Problem #	Score
1	
2	
3	
4	
5	
Total	

1. True of False. **(2 points each)**
(a) There exists a 3-by-2 matrix $A \in \mathbb{R}^{3\times 2}$ with $\text{rank}(A) = 3$.
false

(b) If a square matrix is full rank, then it's row Reduced Echelon Form (RREF) must be the identity matrix.
true

(c) If $T(x) = Ax$ where $A \in \mathbb{R}^{4\times 2}$ is a 4-by-2 matrix, then T must be injective (one-to-one).
false

(d) If $T(x) = Ax$ where $A \in \mathbb{R}^{4\times 2}$ is a 4-by-2 matrix, then T cannot be surjective (onto).
true

(e) A linear system has a unique solution if the columns of the corresponding matrix are linearly independent.
false. Consider 3-by-2 linear system

(f) The set $S = \{x \in \mathbb{R}^n \mid Ax = b \text{ for some } x \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^n$ where $A \in \mathbb{R}^{n\times n}$ is a nonzero square matrix and $b \in \mathbb{R}^n$ is a nonzero vector.
false

(g) The set $S = \{(x, y, z) \mid ax + by + cz = 0\}$ is a plane parallel to the vector $n = (a, b, c)$.
false

(h) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation where $n \neq m$, then T is not invertible.
true

(i) **(4 points)** Write down the definition of linear dependence.
A set of vector $S = \{v_1, \cdots, v_k\}$ is linear dependent if there exist nontrivial scalars $\alpha_1, \cdots, \alpha_k \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0.$$

2. Solve the following system (or explain why it does not have any solution). **(6 points)** Characterize the geometry of the associated linear map (e.g. scaling, orthogonal projection, reflection, rotation, or shear). **(4 points)**

$$\begin{cases} x + 2y + 3z - u = 0 \\ 2x + 4y + 6z - 2u = 0 \\ 3x + 6y + 9z - 3u = 0 \\ -x - 2y - 3z + u = 1 \end{cases}$$

Hint: Note that the system is not much but

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has no solution as $(0, 0, 0, 1) \notin L = \{\alpha(1, 2, 3, -1) : \alpha \in \mathbb{R}\}$, where the linear map is not much but 15proj_L (note that the vector $n = (1, 2, 3, -1)$ is not a unit vector).

3. (a) **(5 points)** Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 2 \end{bmatrix}.$$

(b) **(5 points)** Derive the necessary and sufficient condition such that

$$B = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

is invertible. Compute its inverse and verify that the inverse of upper triangular is upper triangular as well.

(a)

$$A^{-1} = \begin{bmatrix} 1 & -4 & -6 \\ 0 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

(b) The condition is $adg \neq 0$ and the inverse is

$$B^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} & \frac{bc-cd}{adf} \\ 0 & \frac{1}{d} & \frac{-c}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$$

4. (a) **(3 points)** Give an example of a system of 3 linear equations with 3 variables that has infinitely many solutions.

(b) **(3 points)** Give an example of a system of 3 linear equations with 3 variables that has no solution.

(c) **(2 points)** Give an example of a 4-by-4 matrix that has rank 2.

(d) **(2 points)** Give an example of $A, B \in \mathbb{R}^{2\times 2}$ such that $AB \neq BA$.

(a)

$$\begin{cases} x+y+z=1 \\ 2x+2y+2z=2 \\ z=1 \end{cases}$$

(b)

$$\begin{cases} x+y+z=1 \\ 2x+2y+2z=3 \\ z=1 \end{cases}$$

(c)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d)

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. **(10 points)** Describe the geometry of the image and kernel of the linear map $T(x) = Ax$ (as point, line, plane, or $\mathbb{R}^3$) where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ -1 & -2 & -1 \end{bmatrix}.$$

Find a basis for $im(A)$ and $ker(A)$ and verify that $\dim(im(A)) + \dim(ker(A)) = 3$.
Note that $3a_1 - a_2 - a_3 = 0$ where a_i's denote the columns of A. $im(T) = span(\{a_1, a_2\})$ is a plane, $ker(T) = span(\{(3, -1, -1)^T\})$ is a line.