1. Find the orthogonal projection of $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace V of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Hint: First observe that v_1, v_2, v_3 are mutually orthogonal, hence by normalizing these vectors, we obtain an orthonormal basis $\{u_1, u_2, u_3\}$ of V. Suppose we can complete this basis by finding a unit vector u_4 such that $\{u_1, u_2, u_3, u_4\}$ makes an orthonormal basis of $\mathbb{R}^4$. Recall that the orthogonal projection on V can be computed by

$$\begin{aligned} proj_V(x) &= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \\ u_4^T \end{bmatrix} x \\ &= \left(u_1 u_1^T + u_2 u_2^T + u_3 u_3^T \right) x \\ &= proj_{u_1}(x) + proj_{u_2}(x) + proj_{u_3}(x). \end{aligned}$$

Note that in the last expression it doesn't require knowledge of u_4.

2. Find u_4 for the above problem.