Instructions:

- Follow directions and answer questions with requested supporting work.
- Clearly indicate your answer in the allotted space or by putting a box around it.
- No cellphones, laptops, books, notes, supporting materials, or external aids are allowed on this exam.

Name: ____________________________

UID: ____________________________

Problem #	Score
1	
2	
3	
4	
5	
Total	

1. True of False. **(2 points each)**
(a) If $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ is a basis of $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$, we have

$$\begin{bmatrix} u_1 \ u_2 \ \cdots \ u_n \end{bmatrix} [x]_{\mathcal{B}} = x.$$

True

(b) If the columns of $A \in \mathbb{R}^{m \times n}$ are orthonormal, then $A^T A = I$.
False. This is only true if $m \geq n$.

(c) $im(A)^{\perp} = ker(A^T)$.
True

(d) Suppose $n \in \mathbb{R}^n$; $T(x) = nn^T x$ is the orthogonal projection onto $V = \{\alpha n : \alpha \in \mathbb{R}\}$.
False. This is only true if $\|n\| = 1$.

(e) An orthogonal matrix must have linearly independent columns.
True

(f) $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times k}$, then $im(AB) \subseteq im(A)$.
True

(g) **(3 points)** Write down the Rank-nullity theorem.
For $n \times m$ matrix A,

$$\dim(ker(A)) + \dim(im(A)) = m$$

(h) **(2 points)** If $\{u_1, u_2, \cdots, u_m\}$ is a set of orthonormal vectors of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ is an arbitrary vector. What is the relation between $p = (u_1 \cdot x)^2 + (u_2 \cdot x)^2 + \cdots + (u_m \cdot x)^2$ and $\|x\|^2$ (equal, greater than or equal to, less than or equal to)?
$(u_1 \cdot x)^2 + (u_2 \cdot x)^2 + \cdots + (u_m \cdot x)^2 \leq \|x\|^2$

(i) **(3 points)** Write down the definition of the least-squares solution of a linear system $Ax = b$ where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
x^* is a least-squares solution to $Ax = b$ if $\|b - Ax^*\| \leq \|b - Ax\|$ for all $x \in \mathbb{R}^n$.

2. Consider the linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T(x) = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Find an orthonormal bases for $ker(T)$ and $im(T)$.

This is the orthogonal projection onto the plane spanned by $(1, 1, -1, -1)$ and $(1, -1, 1, -1)$. An orthonormal basis for $im(T)$ is naturally $\frac{1}{2}(1, 1, -1, -1), \frac{1}{2}(1, -1, 1, -1)$. By inspection, an orthonormal basis for $ker(T)$ is $\frac{1}{\sqrt{2}}(1, 0, 0, 1), \frac{1}{\sqrt{2}}(0, 1, 1, 0)$ (or can be found by first finding a basis for $ker(T)$ then apply Gram-Schmidt process). Since $ker(T) = im(T)^{\perp}$, the combination of basis forms a orthonormal basis or $\mathbb{R}^4$.

3. Consider a subspace V spanned by the basis $\mathcal{B} = \{v_1, v_2\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Determine whether the following vectors are in V. If the vector is in V, compute its coordinates with respect to $\mathcal{B}$.

(a) **(3 points)** $x = \begin{bmatrix} 3 \\ -4 \\ -11 \end{bmatrix}$.

(b) **(3 points)** $y = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(c) **(4 points)** Find a vector v_3 such that $\mathcal{C} = \{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$. Compute the coordinates with respect to $\mathcal{C}$ for the vectors in (a) and (b).

(a) $x = 10v_1 - 7v_2$, and $[x]_{\mathcal{B}} = (10, -7)$.

(b) $y = 2v_1 - v_2 + (0, 1, 0)$, and $y \notin V$.

(c) Let $v_3 = (0, 1, 0)$, then $[x]_{\mathcal{C}} = (10, -7, 0)$, $[y]_{\mathcal{C}} = (2, -1, 1)$.

4. (a) **(7 points)** Compute the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 6 \\ 1 & 4 \end{bmatrix}$$

(b) **(3 points)** Show for the above matrix A and its QR factorization $A = QR$, $\|Ax\|^2 = \|Rx\|^2$ for all $x \in \mathbb{R}^2$. (Note the difference that $Ax \in \mathbb{R}^4$ and $Rx \in \mathbb{R}^2$.)

(a)

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 10 \\ 0 & 2 \end{bmatrix}$$

The bonus problem follows from $A^T A = R^T R$.

5. **(Bonus problem. 5 points.)** Let V be a subspace in $\mathbb{R}^m$. For the orthogonal projection $proj_V(x)$ for $x \in \mathbb{R}^m$, the following property always holds:

$$x - proj_V(x) \in V^\perp.$$

Use Pythagorean Theorem to show that

$$\|x - proj_V(x)\|^2 \leq \|x - y\|^2 \text{ for all } y \in V.$$

Hint: Since $y - proj_V(x) \in V$, we know that $x - proj_V(x)$ is perpendicular to $y - proj_V(x)$. Apply Pythagorean Theorem to the right triangle formed by these two vectors.