Instructions:

- Follow directions and answer questions with requested supporting work.
- Clearly indicate your answer in the allotted space or by putting a box around it.
- No cellphones, laptops, books, notes, supporting materials, or external aids are allowed on this exam.

Name: ____________________________

UID: ____________________________

Problem #	Score
1	
2	
3	
4	
5	
Total	

1. True of False. **(2 points each)**
(a) If $\mathcal{B} = \{u_1, u_2, \cdots, u_n\}$ is a basis of $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$, we have

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} [x]_{\mathcal{B}} = x.$$

(b) If the columns of $A \in \mathbb{R}^{m\times n}$ are orthonormal, then $A^T A = I$.

(c) $im(A)^\perp = ker(A^T)$.

(d) Suppose $n \in \mathbb{R}^n$; $T(x) = nn^T x$ is the orthogonal projection onto $V = \{\alpha n : \alpha \in \mathbb{R}\}$.

(e) An orthogonal matrix must have linearly independent columns.

(f) $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times k}$, then $im(AB) \subseteq im(A)$.

(g) **(3 points)** Write down the Rank-nullity theorem.

(h) **(2 points)** If $\{u_1, u_2, \cdots, u_m\}$ is a set of orthonormal vectors of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ is an arbitrary vector. What is the relation between $p = (u_1 \cdot x)^2 + (u_2 \cdot x)^2 + \cdots + (u_m \cdot x)^2$ and $\|x\|^2$ (equal, greater than or equal to, less than or equal to)?

(i) **(3 points)** Write down the definition of the least-squares solution of a linear system $Ax = b$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

2. Consider the linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T(x) = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Find an orthonormal bases for $ker(T)$ and $im(T)$.

3. Consider a subspace V spanned by the basis $\mathcal{B} = \{v_1, v_2\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Determine whether the following vectors are in V. If the vector is in V, compute its coordinates with respect to $\mathcal{B}$.

(a) **(3 points)** $x = \begin{bmatrix} 3 \\ -4 \\ -11 \end{bmatrix}$.

(b) **(3 points)** $y = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(c) **(4 points)** Find a vector v_3 such that $\mathcal{C} = \{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$. Compute the coordinates with respect to $\mathcal{C}$ for the vectors in (a) and (b).

4. (a) **(7 points)** Compute the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \\ 1 & 6 \\ 1 & 4 \end{bmatrix}$$

(b) **(3 points)** Show for the above matrix A and its QR factorization $A = QR$, $\|Ax\|^2 = \|Rx\|^2$ for all $x \in \mathbb{R}^2$. (Note the difference that $Ax \in \mathbb{R}^4$ and $Rx \in \mathbb{R}^2$.)

5. **(Bonus problem. 5 points.)** Let V be a subspace in $\mathbb{R}^m$. For the orthogonal projection $proj_V(x)$ for $x \in \mathbb{R}^m$, the following property always holds:

$$x - proj_V(x) \in V^{\perp}.$$

Use Pythagorean Theorem to show that

$$\|x - proj_V(x)\|^2 \leq \|x - y\|^2 \text{ for all } y \in V.$$

Hint: Since $y - proj_V(x) \in V$, we know that $x - proj_V(x)$ is perpendicular to $y - proj_V(x)$. Apply Pythagorean Theorem to the right triangle formed by these two vectors.