

## Sec 1.1 Intro to Linear Systems

Recall algebra, e.g.  $x + 5 = 3$ . Generalize to two variables:

$$\begin{cases} x + y = 5 \\ 3x - y = -1 \end{cases}.$$

Solving intuitively,  $x = 1, y = 4$ . The problem on Page 1:

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases} \quad (1)$$

Answer is  $x = 2.75, y = 4.25, z = 9.25$ .

Some systems are not (uniquely) solvable.

$$\begin{cases} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{cases} \quad (2)$$

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{cases} \quad (3)$$

Geometric interpretation: find points that lie on all three planes.

### “Degrees of freedom” (from Sec 1.3)

$$\begin{cases} x + z = -7 \\ x + 3z = 3 \\ x + 5z = 13 \end{cases} \quad \begin{cases} x + y + z = 1 \\ y + 3z = 3 \end{cases} \quad \begin{cases} x + y + 4z = 1 \\ x - y + z = 1 \\ 3x + y - z = 5 \\ x + 4y - 6z = 0 \end{cases}$$

First:  $x = -12, z = 5$  but no constraint on  $y$ .

Quick check, doesn't prove solvability.

### Geometric interpretation (from Sec 1.3)

- $ax + by + cz = 0$  defines a plane perpendicular to  $(a, b, c)$  passing origin. Translate it to get  $ax + by + cz = d$ .
- Intersection of planes, either unique or infinitely many solutions. (Houdini demo)

### Solvability (from Sec 1.3)

Not solvable: contradiction after some reduction. See (3).

Infinite solutions: parametrization. See (2).

## Sec 1.2 Matrices, Vectors, and Gauss-Jordan Elimination

- matrix dimension; row, column, index notation
- identity, zero, square, upper/lower triangular, symmetric matrices
- vector, vector spaces  $\mathbb{R}^n$  (column vectors!)
- solve (1) using extended matrix
- Gaussian reduction: three operations
- RREF: definition, solve (1), show (2) is not full rank

## Sec 1.3 On the Solutions of Linear Systems; Matrix Algebra

### Rank

Matrices from (1) and (2) have rank 3 and 2. Full rank matrix has identity in RREF.

### Matrix Algebra

- from linear system to matrix-vector equation
- matrix addition, matrix-vector multiplication, matrix-matrix multiplication
- distribution law, commutative law etc.
- linear combination
- interpret matrix-vector multiplication as linear combination with columns

## Sec 2.1 Linear Transformations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (6)$$

- linearity
- $Ae_i = T(e_i)$
- Finding the corresponding matrix
- ~~Markov chain: EXAMPLE 9 on p.5, distribution vectors and transition matrices~~  
(skipped)

## Sec 2.2 Linear Transformation in Geometry

- Geometric meaning of the four entries of a 2-by-2 matrix (scaling, shearing)
- ~~orthogonal projection, reflection in 2D~~ take home
- orthogonal projection, reflection w.r.t. a plane in 3D
- rotation in 2D

## Sec 2.3 Matrix Products

- function composition
- non-commutativity
- ~~distributivity~~ in homework
- ~~block matrix multiplication~~ skip

## Sec 2.4 The Inverse of a Linear Transformation

- injective, surjective, invertible functions and their composition
- invertible matrices: RREF, rank, row operations
- invertible linear systems: solvability
- $AA^{-1} = A^{-1}A = I$
- prove  $(AB)^{-1} = B^{-1}A^{-1}$
- 2-by-2 matrix inverse formula

## Sec 3.1 Image and Kernel of a Linear Transformation

- definition of image, kernel, and span
- finding image and kernel of matrix  $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ ,  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Sec 3.2 Subspaces of $\mathbb{R}^n$ ; Bases and Linear Independence

### Subspaces

- subspace: closed under linear combination
- image and kernel are subspaces
- geometric interpretation

**Bases**

- linear independence; link to rank of a matrix
- nontrivial kernel = not linearly independent columns of a matrix, e.g. (page 129)

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

- basis = linearly independent spanning set
- find basis for subspaces  $\text{im}(A), \text{ker}(A)$
- basis and unique representation

**Sec 3.3 Dimension of a Subspace of  $\mathbb{R}^n$** 

- dimension = # of vectors in a basis
- e.g.  $\mathbb{R}^n = \text{span}(\{e_1, \dots, e_n\})$
- dimension is unique
- dimension = maximal # of linearly independent vectors = minimal # of spanning vectors (Theorem 3.3.4 page 136)
- $v_1, \dots, v_k$ : linear independent  $\Rightarrow Sv_1, \dots, Sv_k$ : linear independent where  $S \in \mathbb{R}^{n \times n}$  is invertible
- example 1 page 136
- row reduction messes up the columns
- $\text{rank}(A) = \text{rank}(SA)$  for any  $A \in \mathbb{R}^{m \times n}$  and  $S \in GL(m, \mathbb{R})$
- rank-nullity theorem: from RREF, # pivot is rank, those columns without pivot is free variable (nullity), summing to  $n$

**Midterm 1 Review**

- linear systems with a unique solution, infinitely many solutions, no solution (examples and such)
- orthogonal projection to a one dimensional subspace  $\{\alpha n : \alpha \in \mathbb{R}\}$
- row reduction, RREF, solving linear system, extended matrix, inverting a matrix by applying row reduction simultaneously on  $A$  and  $I$

- meaning of rank (# nonzero rows in RREF, dimension of image)
- links between rank, RREF, solvability of the linear system, invertibility of the matrix, linear independence of the columns, injectivity, surjectivity of the linear map; practice this with square matrices and rectangular matrices
- meaning of a subspace, linear combination, linear dependence, spanning set, basis, dimension
- how to find basis of the image and kernel of a given linear map  $T(x) = Ax$

### Sec 3.3 Dimension of a Subspace of $\mathbb{R}^n$ (Cont'd)

- rank nullity theorem: rank is number of pivots in RREF

### Sec 3.4 Coordinates

- Notation of coordinates (page 149)
- example 2 (page 150)
- $[x]_{\mathcal{B}} = S^{-1}x$
- example 3 on page 151
- diagram on page 155
- $x = [x]_{\mathcal{E}}, [T]_{\mathcal{B}}$  using example 3
- rotation on a plane spanned by  $(2, 1, -1)$  and  $(0, 1, 1)$
- similar matrices (definition)
- diagonal  $[T]_{\mathcal{B}}$  (page 157)

### Sec 5.1 Orthogonal Projections and Orthonormal Bases

- orthonormal basis (definition)
- $S^{-1} = S^T$  for orthonormal basis
- orthogonal projection done by inner product with orthonormal basis,  $\text{diag}(1, \dots, 1, 0, \dots, 0)Sx$

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- theorem 5.1.6 (page 207)
- ~~orthogonal complement (definition)~~ skip

- ~~theorem 5.1.8 (page 208)~~ skip
- ~~Cauchy-Schwartz inequality~~ skip

## Sec 5.2 Gram-Schmidt Process and QR Factorization

- Gram-Schmidt process: Example 1 (page 219)
- QR factorization: Example 2 (page 222)

## Sec 5.3 Orthogonal Transformations and Orthogonal Matrices

- Definition 5.3.1 (page 225)
- $\|Ax\|^2 = x^T A^T A x = x^T x \Leftrightarrow A^T A = I$ , i.e. columns of  $A$  are orthonormal! (Theorem 5.3.3 on page 227)
- Theorem 5.3.10 on page 232)

## Sec 5.4 Least Squares and Data Fitting

- Definition 5.4.4 Least-squares solutions  $\|b - Ax^*\| \leq \|b - Ax\| \forall x \in \mathbb{R}^m$ .
- Thm 5.4.1, 5.4.2, 5.4.3, 5.4.5, or alternatively, multi-variable calculus to show  $A^T(Ax^* - b) = 0$ .
- Theorem 5.4.6, 5.4.7
- (maybe skip theorems) Example 4 (page 242), Example 6 (page 244) (linear regression)

## Intro to Determinants

- 2 by 2 and 3 by 3

## Midterm 2 Review

- rank-nullity theorem (statement and usage)
- coordinates (computation with  $S = [v_1 \cdots v_n]$ ,  $S[x]_{\mathcal{B}} = x$ ), both for subspaces or  $\mathbb{R}^n$
- orthonormal basis, both for subspaces or  $\mathbb{R}^n$
- $S^{-1} = S^T$ ,  $\therefore [x]_{\mathcal{B}}$  can be found by  $u_i^T x$
- orthogonal projection using orthonormal basis
- orthogonal complement (definition)

- Gram-Schmidt process, QR factorization
- orthogonal matrix (definition)
- Least-squares solutions:  $Ax^* - b \in \text{im}(A)^\perp$ ,  $A^T Ax^* = A^T b$