Sec 1.1 Intro to Linear Systems

Recall algebra, e.g. $x + 5 = 3$. Generalize to two variables:

$$\begin{cases} x + y = 5 \\ 3x - y = -1 \end{cases}.$$

Solving intuitively, $x = 1, y = 4$. The problem on Page 1:

$$\begin{cases} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{cases}. \tag{1}$$

Answer is $x = 2.75, y = 4.25, z = 9.25$.

Some systems are not (uniquely) solvable.

$$\begin{cases} 2x + 4y + 6z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 6 \end{cases}. \tag{2}$$

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{cases}. \tag{3}$$

Geometric interpretation: find points that lie on all three planes.

"Degrees of freedom" (from Sec 1.3)

$$\begin{cases} x + z = -7 \\ x + 3z = 3 \\ x + 5z = 13 \end{cases} \qquad \begin{cases} x + y + z = 1 \\ y + 3z = 3 \end{cases} \qquad \begin{cases} x + y + 4z = 1 \\ x - y + z = 1 \\ 3x + y - z = 5 \\ x + 4y - 6z = 0 \end{cases}$$

First: $x = -12, z = 5$ but no constraint on y.

Quick check, doesn't prove solvability.

Geometric interpretation (from Sec 1.3)

- $ax + by + cz = 0$ defines a plane perpendicular to (a, b, c) passing origin. Translate it to get $ax + by + cz = d$.
- Intersection of planes, either unique or infinitely many solutions. (Houdini demo)

Solvability (from Sec 1.3)

Not solvable: contradiction after some reduction. See (3).
Infinite solutions: parametrization. See (2).

Sec 1.2 Matrices, Vectors, and Gauss-Jordan Elimination

- matrix dimension; row, column, index notation
- identity, zero, square, upper/lower triangular, symmetric matrices
- vector, vector spaces $\mathbb{R}^n$ (column vectors!)
- solve (1) using extended matrix
- Gaussian reduction: three operations
- RREF: definition, solve (1), show (2) is not full rank

Sec 1.3 On the Solutions of Linear Systems; Matrix Algebra

Rank

Matrices from (1) and (2) have rank 3 and 2. Full rank matrix has identity in RREF.

Matrix Algebra

- from linear system to matrix-vector equation
- matrix addition, matrix-vector multiplication, matrix-matrix multiplication
- distribution law, commutative law etc.
- linear combination
- interpret matrix-vector multiplication as linear combination with columns

Sec 2.1 Linear Transformations

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

- linearity
- $Ae_i = T(e_i)$
- Finding the corresponding matrix
- ~~Markov chain: EXAMPLE 9 on p.5, distribution vectors and transition matrices~~ (skipped)

Sec 2.2 Linear Transformation in Geometry

- Geometric meaning of the four entries of a 2-by-2 matrix (scaling, shearing)
- ~~orthogonal projection, reflection in 2D~~ take home
- orthogonal projection, reflection w.r.t. a plane in 3D
- rotation in 2D

Sec 2.3 Matrix Products

- function composition
- non-commutativity
- ~~distributivity~~ in homework
- ~~block matrix multiplication~~ skip

Sec 2.4 The Inverse of a Linear Transformation

- injective, surjective, invertible functions and their composition
- invertible matrices: RREF, rank, row operations
- invertible linear systems: solvability
- $AA^{-1} = A^{-1}A = I$
- prove $(AB)^{-1} = B^{-1}A^{-1}$
- 2-by-2 matrix inverse formula

Sec 3.1 Image and Kernel of a Linear Transformation

- definition of image, kernel, and span
- finding image and kernel of matrix $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Sec 3.2 Subspaces of $\mathbb{R}^n$; Bases and Linear Independence

Subspaces

- subspace: closed under linear combination
- image and kernel are subspaces
- geometric interpretation

Bases

- linear independence; link to rank of a matrix
- nontrivial kernel = not linearly independent columns of a matrix, e.g. (page 129)

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

- basis = linearly independent spanning set
- find basis for subspaces $im(A), ker(A)$
- basis and unique representation

Sec 3.3 Dimension of a Subspace of $\mathbb{R}^n$

- dimension = # of vectors in a basis
- e.g. $\mathbb{R}^n = span(\{e_1, \cdots, e_n\})$
- dimension is unique
- dimension = maximal # of linearly independent vectors = minimal # of spanning vectors (Theorem 3.3.4 page 136)
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- $v_1, \cdots, v_k$: linear independent $\Rightarrow Sv_1, \cdots, Sv_k$: linear independent where $S \in \mathbb{R}^{n\times n}$ is invertible
- example 1 page 136
- row reduction messes up the columns
- $rank(A) = rank(SA)$ for any $A \in \mathbb{R}^{m\times n}$ and $S \in GL(m, \mathbb{R})$
- rank-nullity theorem: from RREF, # pivot is rank, those columns without pivot is free variable (nullity), summing to n

Midterm 1 Review

- orthogonal projection to a one dimensional subspace $\{\alpha n : \alpha \in \mathbb{R}\}$
- Finding the inverse of a matrix by applying row reduction to the matrix A and I simultaneously
- meaning of rank (# nonzero rows in RREF, dimension of image)

- examples of linear systems with unique solution, infinitely many solutions, no solution
- link between rank, RREF, solvability of a linear system, linear independence of the columns, injectivity, surjectivity of the linear map; practice this with square matrices and rectangular matrices
- meaning of a subspace
- meaning of linear independence
- how to find basis of the image and kernel of a given linear map $T(x) = Ax$