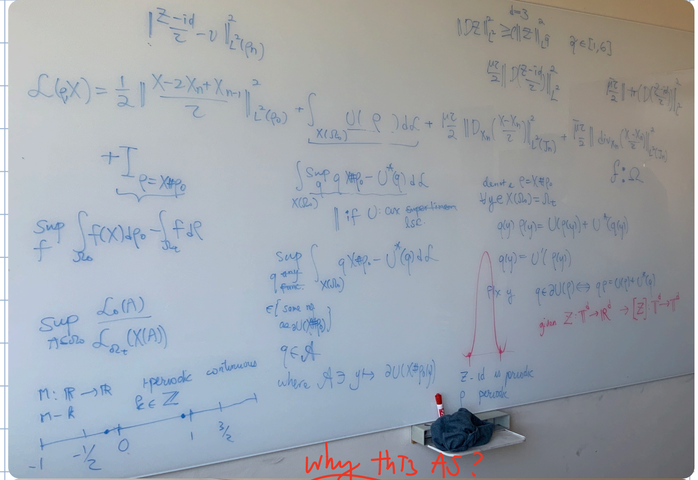
$$\mathcal{L}(\rho, X) = I_{\rho = X\#\rho_0} + \frac{1}{2}\left\| \frac{X - X_n}{\tau} - v_n \circ X_n \right\|^2_{L^2(\rho_0)} + \int_{X(\Omega_0)} U(\rho)\, d\mathcal{L}$$
$$+ \frac{\mu\tau}{2}\left\| D_{X_n}\left(\frac{X-X_n}{\tau}\right)\right\|^2_{L^2} + \frac{\bar{\mu}\tau}{2}\left\|\mathrm{div}_{X_n}\left(\frac{X-X_n}{\tau}\right)\right\|^2_{L^2}$$

$$\sup_{f\in L^1(\Omega_t)} \int_{\Omega_0} f(X)\, d\rho_0 - \int_{\Omega_t} f\, d\rho$$

✱ to come back to discuss limit passing on this



Prop. functions: $\mathbb{T}^d \to \mathbb{R}^d$ are just periodic $\mathbb{R}^d\to\mathbb{R}^d$.

functions: $\mathbb{T}^d\to\mathbb{T}^d$ are equiv. classes in $\mathbb{T}^d\to\mathbb{R}^d$.

$$f:\mathbb{T}^d\to\mathbb{R}^d \Rightarrow [f]: \mathbb{T}^d\to\mathbb{T}^d$$
$$x\mapsto [x]$$

Continuous function $[f]\in C^0(\mathbb{T}^d\to\mathbb{T}^d)$ satisfies

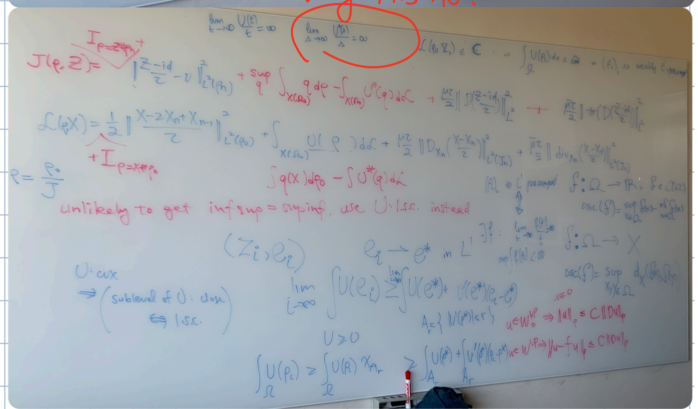
$\forall \varepsilon>0\ \exists\delta>0\ \forall x,y,\ |x-y|_{\mathbb{T}^d}<\delta \Rightarrow |f(x)-f(y)|_{\mathbb{T}^d}<\varepsilon$

($\because \mathbb{T}^d$: cpt $\therefore$ conti. ⇔ unif. conti.) Take $\varepsilon<1$

$\therefore \exists$ a version $f\in[f]$, $N > \lceil \varepsilon\cdot(\#\ \delta\text{-ball to cover } \mathbb{T}^d)\rceil$

$f:[0,1]^d \to [0,2N]^d$ continuous w.r.t. $\mathrm{dist}_{\mathbb{R}^d}$.

why this A3 ?



Lemma $\{A_i\}_{i\in\mathbb{N}}$: L^1 seq. cpt. ⇔ $\exists f$: superlinear ($\lim_{t\to\infty} f(t)/t = \infty$) s.t. $\sup_{i\in\mathbb{N}} \int f(A_i) < \infty$

(Keyword: L^1 precompact, equiintegrability) ⇝ this gives ρ_i precompactness by adding $\lim_{t\to\infty} \frac{U(t)}{t} = \infty$

Prop. $\int U(\rho)\, d\mathcal{L}$ is l.s.c. w.r.t. $\rho_i\rightharpoonup\rho^*$ in L^1 if ① U: convex ② U' exists

pf) convexity ⇒ $U(\rho_i)\ge U(\rho^*) + U'(\rho^*)(\rho_i-\rho^*)$

$$\int U(\rho_i) \ge \int_{A_r} U(\rho_i) \ge \int_{A_r} U(\rho^*) + U'(\rho^*)(\rho_i-\rho^*)$$

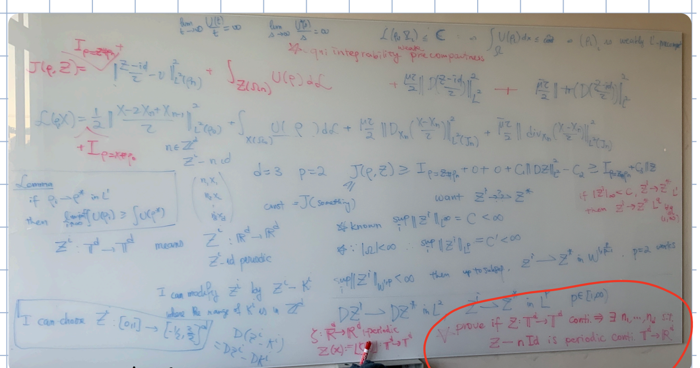
$\because \rho_i\rightharpoonup\rho^*$ in L^1, $U'(\rho^*)\chi_{A_r}\in L^\infty$

$$\boxed{A_r := \{U'(\rho^*)\le r\}}$$

$\therefore \int_{A_r} U'(\rho^*)(\rho_i-\rho^*) \xrightarrow{i\uparrow\infty} 0$

$$\liminf_{i\to\infty}\int U(\rho_i) \ge \int_{A_r} U(\rho^*)\qquad \forall r\in\mathbb{R}$$

$$\therefore \liminf_{i\to\infty} \int U(\rho_i) \ge \int U(\rho^*) = \sup_r \int_{A_r} U(\rho^*) \quad ※$$



Take minimizing seq. (ρ_i, Z_i) for $J(\rho,Z)$

$$= \mathcal{L}(\rho, Z\circ X_n)$$
$$= I_{\rho=Z\#\rho_n} + \left\|\frac{Z-id}{\tau} - v_n\right\|^2_{L^2(\rho_n)} + \int_{Z(\Omega_n)} U(\rho)\, d\mathcal{L} + \frac{\mu\tau}{2}\left\|\frac{Z-id}{\tau}\right\|^2_{L^2}+\cdots$$

do this

① $\sup_i \|Z_i\|_{L^\infty} < \infty$ ~~const~~. since $\mathrm{dist}_{\mathbb{T}^d} <$ const.

Lemma $\sup_i\|Z_i\|_\infty<\infty$ ~~const~~, $Z_i\to Z^*$ in L^1 ⇒ $Z^i\to Z^*$ in L^q $\forall q\in[1,\infty)$

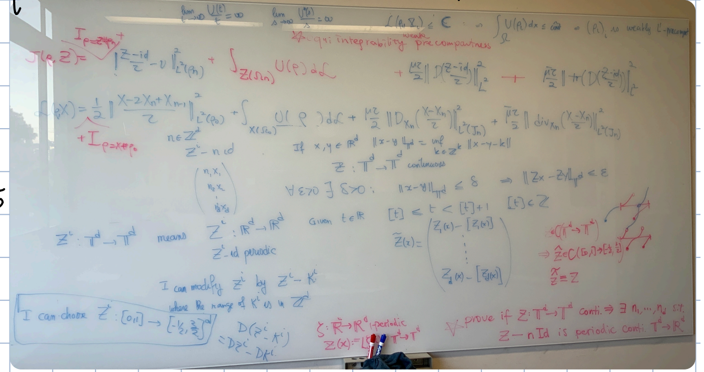
② $\because |\Omega|<\infty$ $\therefore \sup_i\|Z_i\|_{L^p} < \infty$ ~~const~~.

③ BAT: if $\sup_i\|Z_i\|_{W^{1,p}}<\infty$ then up to subseq. $Z_i \rightharpoonup Z^*$ in $W^{1,\frac{p}{p-1}}$ (in particular $p=2$ works!)

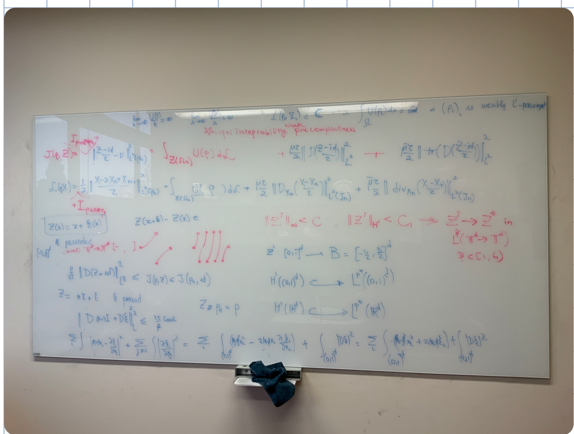
↓ Consider $p=2$, $DZ_i \rightharpoonup DZ^*$ in L^2

$Z_i \to Z^*$ in L^p for any $p \in [1,\infty)$

due to Sobolev embedding see Evans Ch5 S5



We want $Z^i \to Z^*$ in



DEFINITION. *Let X and Y be Banach spaces, $X \subset Y$. We say that X is* compactly embedded *in Y, written*

$$X \subset\subset Y,$$

provided

(i) $\|x\|_Y \le C\|x\|_X$ $(x \in X)$ *for some constant C,*

and

(ii) *each bounded sequence in X is precompact in Y.*

THEOREM 1 (Rellich-Kondrachov Compactness Theorem). *Assume U is a bounded open subset of $\mathbb{R}^n$, and ∂U is C^1. Suppose $1 \le p < n$. Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \le q < p^*$.