

$$\begin{aligned} \mathcal{L}(\rho, X) &= I_{\rho=X \# \rho_0} + \frac{1}{2} \left\| \frac{X-X_n}{\varepsilon} - U_n^* X_n \right\|_{L^2(\rho_0)}^2 + \int_{X(\Omega_\varepsilon)} U(\rho) d\mathcal{L} \\ &\quad + \frac{\mu^2}{2} \left\| D_{X_n} \left( \frac{X-X_n}{\varepsilon} \right) \right\|_{L^2}^2 + \frac{\bar{\mu}^2}{2} \left\| \operatorname{div}_{X_n} \left( \frac{X-X_n}{\varepsilon} \right) \right\|_{L^2}^2 \end{aligned}$$

Prop. functions:  $\mathbb{T}^d \rightarrow \mathbb{R}^d$  are just periodic  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ .

functions:  $\mathbb{T}^d \rightarrow \mathbb{R}^d$  are equiv. classes in  $\mathbb{T}^d \rightarrow \mathbb{R}^d$ .

$$f: \mathbb{T}^d \rightarrow \mathbb{R} \Rightarrow [f]: \mathbb{T}^d \rightarrow \mathbb{T}^d \\ x \mapsto [x]$$

Continuous function  $[f] \in C^0(\Pi^d \rightarrow \Pi^d)$  satisfies

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \|x - y\|_T^d < \delta \Rightarrow |f(x) - f(y)|_T^d < \epsilon$$

( $\because \Pi^d$  is cpt  $\therefore$  conti.  $\Leftrightarrow$  unif. conti.) Take  $\varepsilon < 1$

$\therefore \exists$  a version  $f \in [f]$ ,  $N > \lceil \varepsilon \cdot (\# \delta\text{-ball to cover } T^d) \rceil$

$$f: [0,1]^d \rightarrow [0,2N]^d \text{ continuous w.r.t. dist}_{\mathbb{R}^d}.$$

Lemma  $\{A_i\}_{i \in \mathbb{N}} : L'$  sep. cpt.  $\Leftrightarrow \exists f: \text{superlinear} \left( \lim_{t \rightarrow \infty} \frac{f(t)}{t} = \infty \right) \text{ s.t. } \sup_{i \in \mathbb{N}} \int f(A_i) < \infty$

(Keyword: L' precompact, equiintegrability)  $\rightsquigarrow$  this gives  $\pi$  precompactness by adding  $\lim_{t \rightarrow \infty} \frac{U(t)}{t} = \infty$

Prop.  $\int U(p) dL$  is l.s.c. w.r.t.  $p_i \rightarrow p^*$  in  $L'$  if ①  $U$ : convex ②  $U'$  exists

$$\text{Pf) convexity} \Rightarrow U(p_i) \geq U(p^*) + U'(p^*)(p_i - p^*)$$

$$\int_U U(\rho_i) \geq \int_{A_r} U(\rho_i) \geq \int_{A_r} U(\rho^*) + \underbrace{U'(\rho^*)}_{\text{Convexity}} \underbrace{(\rho_i - \rho^*)}_{\text{Distance}}$$

$\boxed{A_r := \{ U'(\rho^*) \leq r \}}$

$$\liminf_{i \rightarrow \infty} \int U(p_i) \geq \int_{A_r} U(p^*) \quad \forall r \in \mathbb{R}$$

$$\therefore \liminf_{i \rightarrow \infty} \int_U U(p_i) \geq \int_U U(p^*) = \sup_r \int_{A_r} U(p^*)$$

Take minimising seq.  $(\rho_i, z_i)$  for  $J(\rho, z)$

$$\mathcal{L}(e, z^0 x_n)$$

$$I_{\ell} = \mathcal{Z} \# p_n + \left\| \frac{\bar{z} - id}{\bar{z}} - v_n \right\|_{L^2(p_n)}^2 + \int_{\mathcal{Z}(\Omega_n)} \cup$$

$$\sup_{\Omega} \|Z_i\|_{L^\infty} < \infty. \text{ since } \text{dist}_T < \text{const.}$$

Lemma:  $\sup_i \|z_i\|_\infty < \infty$ ,  $z_i \rightarrow z^*$  in  $L^1 \Rightarrow z_i \rightarrow z^*$  in  $L^q$  for  $q \in [1, \infty)$

② "  $|\sum| < \infty$  "  $\sup_i \|z_i\|_p < \text{const.}$

③ BAT: if  $\sup_i \|z_i\|_{W^{1,p}} < \infty$  then up to subseq.  $z_i \rightarrow z^*$  in  $W^{1,\frac{p}{p-1}}$  (in particular  
 $p=2$  works!)

↓ Consider  $p=2$ ,  $DZ_i \rightarrow DZ^*$  in  $L^2$

$\bar{z}_i \rightarrow \bar{z}^*$  in  $L^p$  for any  $p \in [1, \infty)$

due to Sobolev embedding see Evans Ch 5 S 5

We want  $\mathcal{Z}^i \rightarrow \mathcal{Z}^*$  in

$$\begin{aligned}
 & \text{Integrability of } f(z) \text{ on } \mathbb{R} \text{ implies } \int_{\mathbb{R}} |f(z)| dz < \infty. \\
 & \text{Let } Z = \int_{\mathbb{R}} z f(z) dz = \int_{\mathbb{R}} z \sum_{n=1}^{\infty} \frac{c_n}{n!} z^n dz = \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}} z^{n+1} dz. \\
 & \text{If } n \neq -1, \int_{\mathbb{R}} z^{n+1} dz \text{ exists. If } n = -1, \int_{\mathbb{R}} z^{-1} dz \text{ does not exist.} \\
 & \text{Therefore, } Z \text{ is well-defined if } c_{-1} = 0. \\
 & \text{If } c_{-1} \neq 0, \int_{\mathbb{R}} z^{-1} dz \text{ does not exist, so } Z \text{ is not well-defined.} \\
 & \text{Thus, } Z \text{ is well-defined if and only if } c_{-1} = 0. \\
 & \text{This shows that } f(z) \text{ is analytic at } z=0 \text{ if and only if } c_{-1} = 0. \\
 & \text{Therefore, } f(z) \text{ is analytic at } z=0 \text{ if and only if } \int_{\mathbb{R}} z f(z) dz \text{ exists.} \\
 & \text{This proves that } \int_{\mathbb{R}} z f(z) dz \text{ exists if and only if } f(z) \text{ is analytic at } z=0. \\
 & \text{Therefore, } \int_{\mathbb{R}} z f(z) dz \text{ exists if and only if } f(z) \text{ is analytic at } z=0. \\
 & \text{This completes the proof.}
 \end{aligned}$$

**DEFINITION.** Let  $X$  and  $Y$  be Banach spaces,  $X \subset Y$ . We say that  $X$  is compactly embedded in  $Y$ , written

$$X \subset\subset Y,$$

- (i)  $\|x\|_Y \leq C\|x\|_X$  ( $x \in X$ ) for some constant  $C$ ,

272

## 5. SOBOLEV SPACES

*and*

- (ii) each bounded sequence in  $X$  is precompact in  $Y$ .

**THEOREM 1** (Rellich-Kondrachov Compactness Theorem). *Assume  $U$  is a bounded open subset of  $\mathbb{R}^n$ , and  $\partial U$  is  $C^1$ . Suppose  $1 \leq p < n$ . Then*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each  $1 \leq q < p^*$ .