underGraduate Homework In Mathematics

FunctionalAnalysis 10

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General fire extinguisher

PROBLEM I Let $f \in \mathcal{X}^*$, $f \neq 0$, let d $:= \inf\{\|x\| : f(x) = 1, x \in \mathcal{X}\}$, prove: $\|f\| = \frac{1}{d}$.

SOLUTION. First of all $d > 0$, that is because f is continue, $\exists \delta > 0$, $\forall \|x\| < \delta$, $|f(x)| \leq 1$. So $d \geq \delta$. Besides, $\exists x \neq 0$, such that $f(x) \neq 0$, then $\{x \in \mathcal{X} : f(x) = 1\}$ is not empty.

1. $\forall \|x\| = 1$, $|f(x)| \leq \frac{1}{d}$: if not, $\exists \|x\| = 1, |f(x)| > \frac{1}{d}$, let $x = \frac{x}{f(x)}$, so $f(\frac{x}{f(x)}) = 1$, $\left\|\frac{x}{f(x)}\right\| = \frac{\|x\|}{|f(x)|} = \frac{1}{|f(x)|} < d$. So $\inf\{\|x\| : f(x) = 1\} < d$.
2. $\|f\| \geq \frac{1}{d}$: Since $\exists \{x_n\}_{n=1}^{\infty}$, such that $f(x_n) = 1$, $\lim_{n\to\infty} \|x_n\| = d$. Then, $y_n := \frac{x_n}{\|x_n\|}$, so $\|y_n\| = 1$, $|f(y_n)| = \frac{|f(x_n)|}{\|x_n\|} = \frac{1}{\|x_n\|} \to \frac{1}{d}$.

□

PROBLEM II Let $f \in \mathcal{X}^*$, prove: $\forall \varepsilon > 0$, $\exists x_0 \in \mathcal{X}$, such that $f(x_0) = \|f\|$, and $\|x_0\| < 1 + \varepsilon$.

SOLUTION. $\forall \varepsilon > 0$, $n = \left[\frac{\|f\|}{1+\varepsilon}\right] + 1$, so $\exists \|x\| = 1$, $|f(x)| \geq \|f\| - \frac{\varepsilon}{n} > \frac{1}{n}$, Let $y := x\mathrm{e}^{-\mathrm{i}\theta}$, where $\theta := \arg f(x)$, then $f(y) = \mathrm{e}^{-\mathrm{i}\theta} f(x) \geq 0$, $f(y) = |f(x)|$. So $z = y + \frac{\varepsilon}{nf(y)} y$, $f(z) = f(y) + \frac{\varepsilon}{nf(y)} f(y) = f(y) + \frac{\varepsilon}{n} \geq \|f\|$, $\|z\| \leq |(k+1)| \, \|x\| = \frac{\varepsilon}{nf(y)} + 1 < 1 + \varepsilon$. Therefore, $f(z) = \|f\|$ and $\|z\| < 1 + \varepsilon$. □

PROBLEM III Let $T : \mathcal{X} \to \mathcal{Y}$ is linear, let $N(T) := \{x \in \mathcal{X} : Tx = 0\}$.

1. If $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$, prove: $N(T)$ is closed subspace of $\mathcal{X}$.
2. Can we infer $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$ throught that $N(T)$ is closed subspace in $\mathcal{X}$.
3. If f is a linear functional, prove: $f \in \mathcal{X}^* \iff N(f)$ is closed subspace in $\mathcal{X}$.

SOLUTION.

1. $\forall x, y \in \mathcal{X}, a, b \in \mathbb{K}$, $f(ax + by) = af(x) + bf(y) = 0$. So $ax + by \in N(T)$. $\{x_n\}_{n=1}^{\infty} \subset N(T), \lim_{n\to\infty} x_n = x$. Since $T \in \mathscr{L}(\mathcal{X}, \mathcal{Y})$, then $f(x) = \lim_{n\to\infty} f(x_n) = 0$. Therefore, $N(T)$ is closed.
2. No. Consider $\mathcal{X} := l^1$, where the norm on $\mathcal{X}$ is $\|x\| := \sup_{n\geq\infty} |x(n)|$, $x(n)$ is the $n-$th number of x. a such that $a(k) = 1, k = 1, a(k) = -1, k = 2, a(k) = 0, k > 2$. $f : \mathcal{X} \to \mathbb{K}$, $f(x) = \sum_{n=1} x(n)$. Let $T : \mathcal{X} \to \mathcal{X}$, $T(x) = x - af(x)$. Since $x \in l^1$, then $|f(x)| = |\sum_{n\in\mathbb{N}_+} x(n)| \leq \sum_{n\in\mathbb{N}_+} |x(n)| < \infty$ So $\sum_{n\in\mathbb{N}_+} |x(n) - f(x)a(n)| \leq \sum_{n\in\mathbb{N}_+} |x(n)| + |f(x)| < \infty$. Terefore, T is well-defined. Besides, T is linear obviously. And $\forall x \in N(T)$, $x = af(x) \iff x(n) = f(x)a(n), n \in \mathbb{N}_+$, and $f(x) = \sum_{n\in\mathbb{N}_+} x(n) = 0$. Therefore, $N(T) = \{\theta\}$. Besides, $\mathcal{X}$ can be a disstance space, then $N(T)$ is closed. However, $\|f\| = \infty$, that is because $f(x_n) = n$, where $x_n(k) = \mathbb{1}_{k\leq n}$. So $\|x_n\| = 1$, $\|f(x_n)\| = n \to \infty, n \to \infty$. And $\forall x : \|x\| = 1, \|af(x)\| = \|x - T(x)\| \leq \|x\| + \|T(x)\| = 1 + \|T(x)\|$, thus, $\|T\| = \infty$.
3. By Item 1, we only need to prove $N(T)$ is closed $\implies T \in \mathcal{X}^*$.
   (a) If $N(T) = \mathcal{X}$, then $\|T\| = 0$, so $T \in \mathcal{X}^*$.

(b) f $N(T) \subsetneq \mathcal{X}$, $\exists x \in \mathcal{X} \setminus N(T)$, such that $T(x) \neq 0$. So $x_0 := \frac{x}{T(x)} \in \mathcal{A} := \{x : T(x) = 1\}$. Obviously, $x_0 + N(T) \subset \mathcal{A}$, $\forall y \in \mathcal{A}$, $T(y - x_0) = T(y) - T(x_0) = 1 - 1 = 0$. Therefore, $\mathcal{A} \subset x_0 + N(T)$. Let $d := \inf\{\|x\| : x \in \mathcal{A}\}$. So $d \geq 0$. If $d = 0$, then $\{x_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $\|x_n\| \to 0, n \to \infty$. Consider $y_n = x_n - x_0 \in N(T)$, then $\|y_n\| = \|x_n - x_0\| \leq \|x_n\| + \|x_0\| \to \|x_0\|, n \to \infty$. Then $\{y_n\}_{n=1}^{\infty} \subset N(T)$ is bounded. Besides, $N(T)$ is closed, then $\exists \{y_{n_k}\}_{k=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty}$ such that $\exists y_0 \in N(T), y_{n_k} \to y_0, k \to \infty$. For convenience's sake, assume $\lim_{n\to\infty} y_n = y_0$. So $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|x_0 + y_n\| = \lim_{n\to\infty} \|x_0 + y_0\| = 0$. Therefore, $x_0 + y_0 = 0$, then $x_0 \in N(T)$, i.e. $T(x_0) = 0$. Contradiction! Thus $d > 0$. Same as PROBLEM I, then $\|T\| = \frac{1}{d} < \infty$. Therefore, $T \in \mathscr{L}(\mathcal{X}, \mathbb{K})$.

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