## under Graduate Homework In Mathematics

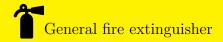
Functional Analysis 8

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2023年11月1日



ROBEM I Let  $A = \{e_k\}$  are orthonormal basis in inner product space X. Prove:  $\forall x, y \in X$ ,  $\sum_{k=1}^{\infty} |(x, e_k)(y, e_k)| \leq ||x|| ||y||$ .

SOLITION. Since  $A = \{e_k\}$  are orthonormal basis in inner product space X, then  $\forall x \in X$ ,  $||x|| = \sum_{k=1}^{\infty} |(x,e_k)|^2$ , so by Holder inequation, we get  $\sum_{k=1}^{\infty} |(x,e_k)(y,e_k)| \leq (\sum_{k=1}^{\infty} |(x,e_k)|^2)^{\frac{1}{2}} (\sum_{k=1}^{\infty} |(y,e_k)|^2)^{\frac{1}{2}} = ||x|| ||y||$ .

ROBEM II H is Hilbert space,  $\{e_k\}, \{e'_k\}$  are two kinds of orthonormal set in inner product space H,  $\sum_{k=1}^{\infty} \|e_k - e'_k\|^2 < 1$ . Prove: if one of  $\{e_k\}, \{e'_k\}$  is complete, then the other is complete.

SOLION. Let  $\{e_n\}_{n=1}^{\infty}$  is complete. If  $\{e'_n\}_{n=1}^{\infty}$  is not complete, then  $\exists x_0: \theta \neq x_0 \notin Span\{\{e'_n\}_{n=1}^{\infty}\}$  s.t.  $(x_0, e'_n) = 0 \forall n$ . So  $||x_0||^2 = \sum_{n=1}^{\infty} |(x_0, e_n)|^2 = \sum_{n=1}^{\infty} |(x_0, e_n - e'_n)|^2 \leq ||x_0||^2 \sum_{n=1}^{\infty} ||e_n - e'_n||^2 < ||x_0||$ . Contradiction!

 $\mathbb{R}^{OBEM}$  III H is an inner space, these propositions below are equal:

- 1.  $x \perp y$ ;
- 2.  $||x + \alpha y|| \ge ||x||$ ,  $\alpha \in \mathbb{C}$ ;
- 3.  $||x + \alpha y|| = ||x \alpha y||, \forall \alpha \in \mathbb{C}$ .

SOUTHOW. 1.  $x \perp y \Rightarrow \|x + \alpha y\| \ge \|x\|$ ,  $\alpha \in \mathbb{C} : \|x + \alpha y\| = \|x\| + |\alpha| \|y\| \ge \|x\|$ ,  $\forall \alpha \in \mathbb{C}$ .

- 2.  $x \perp y \Leftarrow \|x + \alpha y\| \ge \|x\|$ ,  $\alpha \in \mathbb{C}$ : If  $(x,y) \neq 0$ , let  $\alpha = r(x,y)$ ,  $r \in \mathbb{R}$ , then by  $\|x + \alpha y\| \ge \|x\|$ , we get  $|\alpha|^2 \|y\|^2 + 2\operatorname{Re}\overline{\alpha}(x,y) \ge 0$ , that is  $r^2|(x,y)|^2 \|y\|^2 + 2r|(x,y)|^2 \ge 0$ , so  $\Delta = 4 \le 0$ . Contradiction!
- 3.  $x \perp y \Rightarrow ||x + \alpha y|| = ||x \alpha y||, \forall \alpha \in \mathbb{C}: ||x + \alpha y|| = ||x|| + |\alpha| ||y|| = ||x \alpha y||.$
- 4.  $x \perp y \Leftarrow ||x + \alpha y|| = ||x \alpha y||, \forall \alpha \in \mathbb{C}$ : If  $(x, y) \neq 0$ , let  $\alpha = r(x, y), r \in \mathbb{R}$ , then by  $||x + \alpha y|| = ||x \alpha y||$ , we get  $\text{Re}\overline{\alpha}(x, y) = 0$ , that is  $r^2|(x, y)|^2 = 0, \forall r \in \mathbb{R}$ . Contradiction!