COMBINATION2

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ProblemI. Caculate integral between $1 - 1000$ which is neither square number nor cubic number.

Solution. $A := [0, 1000] \cap \mathbb{N}, B := \{a \in A : \exists b \in \mathbb{N}, b^2 = a\}, C = \{a \in A : \exists b \in \mathbb{N}, b^3 = a\}, D = B \cap C, E := \{a \in A : \exists b \in \mathbb{N}, b^6 = a\}$. $\forall a \in D$, $\exists b \in A, a = b^2, \exists c \in A, a = c^3$, then let $b = p_1^{r_1} \cdots p_n^{r_n}, q = q_1^{s_1} \cdots q_m^{s_m}$, where p_i, q_j, are prime $i = 1, \cdots, n, j = 1, \cdots, m$ and $p_i \neq p_j, i \neq j, q_i \neq q_j, i \neq j$. So $p_1^{2r_1} \cdots p_n^{2r_n} = q_1^{3s_1} \cdots q_m^{3s_m}$. Then by Prime factorization theorem, $n = m$, $\forall i, \exists j$ s.t. $p_i = q_j$ and $2r_i = 3s_j$. WLOG, let $\forall i = 1, \cdots, n, p_i = q_i$, so $2r_i = 3r_i$. Since $(2, 3) = 1$, so $3|r_i, 2|s_i$. Assume $r_i = 3k_i$, so $s_i = 2k_i$. Therefore $(\prod_{i=1}^n p_i^{k_i})^6 = b^2 = a$. So $a \in E$. $\forall a \in E$, $\exists e \in A$, $e^6 = a$, so $(e^2)^3 = a = (e^3)^2$, so $a \in B \cap C = D$. So $D = E$.
While $|B| = |[0, 1000^{1/2}] \cap \mathbb{N}| = 31, |C| = |[0, 1000^{1/3}] \cap \mathbb{N}| = 10, |D| = |E| = |[0, 1000^{1/6}] \cap \mathbb{N}| = 3$. By the Inclusion-Exclusion Principle, $F := \{a \in A : \forall b, b^2 \neq a, b^3 \neq a\} = (A \setminus (B \cup C)) \cup (B \cap C)$, so $F = |A| - (|B| + |C|) + |B \cap C| = 1000 - (31 + 10) + 3 = 962$. □

ProblemII. Caculate the permtation of $\{1, 2, 3, 4, 5, 6\}$ $i_1 i_2 i_3 i_4 i_5 i_6$, where $i_1 \neq 1, 5$, $i_2 \neq 2, 3, 5,$, $i_4 \neq 4$, $i_5 \neq 5, 6$.

Solution. $U := \{i_1 i_2 i_3 i_4 i_5 i_6 : \exists \sigma \in S_6, \sigma(k) = i_k, k = 1, \cdots, 6\}$ $A := \{i_1 i_2 i_3 i_4 i_5 i_6 : \exists \sigma \in S_6, \sigma(k) = i_k, k = 1, \cdots, 6, i_1 \neq 1, 5,\ i_2 \neq 2, 3, 5, i_4 \neq 4, i_5 \neq 5, 6.\}$. Since $i_1, i_2, i_5 \neq 5$, i_3, i_4 or $i_5 = 5$. $A_j := \{a \in A : i_j = 5\}, j = 3, 4, 5$. Since $\sigma_{35} : A_3 \to A_5, \sigma_{35} = (35) \in S_6$, $\sigma_{35}(i_1 i_2 i_3 i_4 i_5 i_6) = i_1 i_2 i_5 i_4 i_3 i_6$, it is obviously that σ_{35} is well-defined and injective, and $\sigma_{53} \circ \sigma_{35} = \mathrm{id}$. So σ_{35} is bijective, then $|A_3| = |A_5|$. So we only need to caculate A_3, A_4.

1. Consider $A_3 := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_1 \neq 1, i_2 \neq 2, 3, i_3 = 5, i_4 \neq 4, i_6 \neq 6\}$. Let $B := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3\}, B_j := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_j \neq j\}, B_{jk} := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_t \neq t, t = j, k\}, B_{jkl} := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_t \neq t, t = j, k, l\}$. By the Inclusion-Exclusion Principle, $|B_1 \cup B_4 \cup B_6| = |B_1| + |B_4| + |B_6| - |B_{14}| - |B_{16}| - |B_{46}| + |B_{146}|$.
   Besides, $\forall l, k \in \{1, 4, 6\}, l \neq k, \varphi_{lk} : B \setminus B_l \to B \setminus B_k$, $j_1 j_2 j_3 j_4 j_5 j_6 := \varphi_{lk}(i_1 i_2 i_3 i_4 i_5 i_6)$ s. t. $i_s = k, s \neq l, j_k = k, j_l = i_k, j_s = l$. So φ_{lk} is well defined.$t_1 t_2 t_3 t_4 t_5 t_6 := \varphi_{kl} \circ \varphi_{lk}(i_1 i_2 i_3 i_4 i_5 i_6)$ s.t. $i_s = k, s \neq l, j_k = k, j_l = i_k, j_s = l$, so $t_l = l, t_s = k, t_k = j_l = i_k$, so $t_1 t_2 t_3 t_4 t_5 t_6 = i_1 i_2 i_3 i_4 i_5 i_6$. So $\varphi_{kl} \circ \varphi_{lk}$ is id. It is the same for $\varphi_{lk} \circ \varphi_{kl}$. So φ_{lk} is bijective. So $|B \setminus B_l| = |B \setminus B_k|$.
   Moreover, $\forall l, k, r \in \{1, 4, 6\}, l \neq k, l \neq r, k \neq r$, let $C_{rl} := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_r \neq r, i_l = l\}$. $\psi_{lk} : C_{rl} \to C_{rk}$, $j_1 j_2 j_3 j_4 j_5 j_6 := \psi_{lk}(i_1 i_2 i_3 i_4 i_5 i_6)$ s. t. $i_s = k, s \neq l, j_k = k, j_l = i_k, j_s = l$. So ψ_{lk} is well defined. It is the same to proof ψ_{lk} is bijective as before we have proved. Noticing that $B_r \setminus C_{rk} = B_{rk}$, so $|B_{rk}| = |B_{pl}|, r, k, p, l \in \{1, 4, 6\}, r \neq k, p \neq k$
   By caculating, we get $|B| = A_4^2 \times A_3^3 = 4 \times 3 \times 3 \times 2 \times 1 = 72$, $|B \setminus (B_1 \cup B_4 \cup B_6)| = |\{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_t = t, t = 1, 4, 6\}| = 0$, $|B \setminus B_1| = |\{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_3 = 5, i_2 \neq 2, 3, i_1 = 1\}| = A_3^2 \times A_2^2 = 3 \times 2 \times 2 \times 1 = 12$. $|C_{14}| = A_2^2 + C_2^1 \times C_2^1 \times A_2^2 = 2 + 2 \times 2 \times 2 \times 1 = 10$. $|B_{14}| = |B_1 \setminus C_{14}| = (72 - 12) - 10 = 50$.
   Therefore, $|B_1 \cup B_4 \cup B_6| = |B| - |B \setminus (B_1 \cup B_4 \cup B_6)| = 72 - 0 = |B_1| + |B_4| + |B_6| - |B_{14}| - |B_{16}| - |B_{46}| + |B_{146}| = 3 \times (72 - 12) - 3 \times 50 + |B_{146}|$, so $|A_3| = |B_{146}| = 42$.

2. Consider $A_4 := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_1 \neq 1, i_2 \neq 2, 3, i_4 = 5, i_6 \neq 6\}$. Let $D := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_4 = 5, i_2 \neq 2, 3\}, D_{jk} := \{i_1 i_2 i_3 i_4 i_5 i_6 \in U : i_4 = 5, i_2 \neq 2, 3, i_t \neq t, t = j, k\}$. By the Inclusion-Exclusion Principle,

$|D_1 \cup D_6| = |D_1| + |D_6| - |D_{16}|$.

Noticing, $\exists f$ is bijection between B, D just like before. So do $D \setminus D_l = \{i_1i_2i_3i_4i_5i_6 \in U : i_4 = 5, i_2 \neq 2, 3, i_l = l\}$ and $B \setminus B_l, l \in \{1, 6\}$

By caculating, $|D| = |B| = 72$, $|D \setminus (D_1 \cup D_6)| = |\{i_1i_2i_3i_4i_5i_6 \in U : i_4 = 5, i_2 \neq 2, 3, i_1 = 1, i_6 = 6\}| = 2$, $|D_1 \cup D_6| = |D| - |D \setminus (D_1 \cup D_6)| = 72 - 2 = |D_1| + |D_6| - |D_{16}| = 2 \times (72 - 12) - |D_{16}|$, then $|D_{16}| = 50 = |A_4|$

Therefore, the total number is $|A| = |A_3| + |A_5| + |A_4| = 50 \times 2 + 50 = 150$. $\square$

ProblemIII. Put n different balls into different k boxes, none of boxes is empty. Caculate the different ways.

Solution. Let every different balls have a number, and so do those boxes. Let's say balls names $\{1, \cdots, n\}$, and boxes named $\{1, \cdots, k\}$. Consider $C := \{f \in \{1, \cdots, n\}^{\{1, \cdots, k\}}\}$, $C_i = \{f \in C : i \notin f[n]\}$, where $f[n] := \{f(i) : i \in \{1, \cdots, n\}\}$. Since $\forall g \leq k$, $\forall t_1, \cdots, t_g \in \{1, \cdots, k\}$, $|\cap_{i=1}^{g} C_{t_i}| = |\{f \in C : t_1, \cdots, t_g \notin f[n]\}| = (k - g)^n$. By the Inclusion-Exclusion Principle, we get $|\cap_{i=1}^{k} (C_i)^c| = |C| + \sum_{s=1}^{k} (-1)^s \sum_{1 \leq i_1 < \cdots < i_s \leq k} |\cap_{t=1}^{g} C_{i_t}| = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n$. $\square$

ProblemIV. Caculate the prime between $1 - 120$.

Solution. If $1 \leq i \leq 120$ is not a prime, then $\exists 1 < d \leq \sqrt{120}$ s.t. $d|i$. So we only need to exclude those number i, which can be divided by $1 < d \leq \sqrt{120}$. $\forall B \subset [1, 120] \cap \mathbb{N}, A_B := \{1 \leq a \leq 120 : \forall d \in B, d|a\}$ So $A_{\{2\}} := \{2k : 1 \leq k \leq 60\}$, $A_{\{3\}} := \{3k : 1 \leq k \leq 40\}$, $A_{\{5\}} := \{5k : 1 \leq k \leq 24\}$, $A_{\{7\}} := \{7k : 1 \leq k \leq 17\}$, $A_{\{2,3\}} := \{6k : 1 \leq k \leq 20\}$, $A_{\{2,5\}} := \{10k : 1 \leq k \leq 12\}$, $A_{\{2,7\}} := \{14k : 1 \leq k \leq 8\}$, $A_{\{3,5\}} := \{15k : 1 \leq k \leq 8\}$, $A_{\{3,7\}} := \{21k : 1 \leq k \leq 5\}$, $A_{\{5,7\}} := \{35k : 1 \leq k \leq 3\}$, $A_{\{2,3,5\}} := \{30k : 1 \leq k \leq 4\}$, $A_{\{2,3,7\}} := \{42k : 1 \leq k \leq 2\}$, $A_{\{2,5,7\}} := \{70\}$, $A_{\{3,5,7\}} := \{105\}$, $A_{\{2,3,5,7\}} = \emptyset$.

By the Inclusion-Exclusion Principle, $|\{1, \cdots, 120\} \setminus \cup_{d \in \{2,3,5,7\}} A_{\{d\}}| = |\{1 \leq a \leq 120 : \forall d \in \{2, 3, 5, 7\}, d \nmid a\}| = 120 - (60 + 40 + 24 + 17) + (20 + 12 + 8 + 8 + 5 + 3) - (4 + 2 + 1 + 1) + 0 = 120 - 141 + 56 - 8 = 27$. But 1 is not prime, and $\{2, 3, 5, 7\} \cap \{1 \leq a \leq 120 : \forall d \in \{2, 3, 5, 7\}, d \nmid a\} = \emptyset$, so the total number is $27 - 1 + 4 = 30$. $\square$

ProblemV. There are n kinds different balls, each kind of ball has 2. Arrange these $2n$ balls into a circle, same balls are not adjacent. Caculate the different arrangement.

Solution. Let the n different balls be $\{a_1, a_2, \cdots, a_n\}$, then the different arrangement of $2n$ balls equal to the circle arrangement of set $\{2 \cdot a_1, \cdots, 2 \cdot a_n\}$. $A :=$ { all of the circle arrangement of $\{2 \cdot a_1, \cdots, 2 \cdot a_n\}$ }. $\forall 1 \leq l \leq n$, $\forall i_1, \cdots, i_l \in \{1, \cdots, n\}$, $A_{i_1, i_2, \cdots, i_l} :=$ { all of the circle arrangement of $\{2 \cdot a_1, \cdots, 2 \cdot a_n\}$ that appears $a_k a_k, k \in \{i_1, \cdots, i_l\}$}. $A_{i_1, i_2, \cdots, i_l}$ equals to the circle arrangement of $\{a_{i_1}, \cdots, a_{i_l}, 2 \cdot a_j, 1 \leq j \leq n, j \notin \{i_1, i_2, \cdots, i_l\}\}$.

By caculating, $|A| = \frac{1}{2n} \frac{2n!}{(2!)^n}$, $|A_{i_1, i_2, \cdots, i_l}| = \frac{1}{l + 2(n-l)} \frac{(l+2(n-l))!}{2!^{n-l}}$. So by the Inclusion-Exclusion Principle, we get $|A \setminus \cup_{k=1}^{n} A_k| = |\{$ all of the circle arrangement of $\{2 \cdot a_1, \cdots, 2 \cdots a_n\}$ that does't appear $a_k a_k, k = 1, \cdots, n\}| = |A| + \sum_{l=1}^{n} \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^l |A_{i_1, i_2, \cdots, i_l}| = \frac{1}{2n} \frac{2n!}{(2!)^n} + \sum_{l=1}^{n} (-1)^l \binom{n}{l} \frac{1}{l+2(n-l)} \frac{(l+2(n-l))!}{2!^{n-l}} = \frac{(2n-1)!}{(2!)^n} + \sum_{l=1}^{n} (-1)^l \binom{n}{l} \frac{(2n-l-1)!}{2!^{n-l}}$. $\square$

ProblemVI. Arrange $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$, none of $xxxx, yyy, zz$ appears. How many of these arrangement?

Solution. $A :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$}, $B_1 :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears $xxxx$}, $B_2 :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears yyy}, $B_3 :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears zz}, $B_{12} :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears $xxxx, yyy$}, $B_{13} :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears $xxxx, zz$}, $B_{23} :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears yyy, zz}, $B_{123} :=$ { all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that appears $xxxx, yyy, zz$}.

By the Inclusion-Exclusion Principle, $|A \setminus (\cup_{i=1}^{3} B_i)| = |\{$ all of the arrangement of $\{4 \cdot x, 3 \cdot y, 2 \cdot z\}$ that none of $xxxx, yyy, zz$ appears $\}| = |A| - (|B_1| + |B_2| + |B_3|) + (|B_{12}| + |B_{13}| + |B_{23}|) - |B_{123}| = \frac{9!}{4!3!2!} - (6\frac{5!}{3!2!} + 7\frac{6!}{4!2!} + 8\frac{7!}{4!3!}) + (\binom{3}{2} + \binom{4}{2} + \binom{5}{2}) - A_3^3 = 1260 - (60 + 105 + 280) + (3 + 6 + 10) - 6 = 828$. $\square$

ProblemVII. Pick 10 number from $\{\infty \cdot a, 3 \cdot b, 5 \cdot c, 7 \cdot d\}$, how many ways can you find?

Solution. Let $a_1 = a, a_2 = b, a_3 = c, a_4 = d, k_1 = 10, k_2 = 3, k_3 = 5, k_4 = 7$, $T :=$ {Pick 10 number from $\{\infty \cdot a, 3 \cdot b, 5 \cdot c, 7 \cdot d\}\}$. $S_\infty := \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$, $\mathcal{A}$ represents the combination of picking 10 number from S_∞. So $|\mathcal{A}| = \binom{10+4-1}{10} = \binom{13}{10} = 286$. $\forall 1 \le i \le 4$, $\mathcal{A}_i := \{x \in \mathcal{A} :$ the number of $a_i > k_i\}$. So $T = \cap_{i=1}^4 \mathcal{A}_i^c$. $\forall 1 \le l \le 4$, $i_1, \cdots, i_l \in 1, 2, 3, 4$, $\cap_{j=1}^l \mathcal{A}_{i_j} := \{x \in \mathcal{A} :$ the number of $a_{i_j} > k_{i_j}, j = 1, \cdots, l\}$. So $|\cap_{j=1}^l \mathcal{A}_{i_j}| = \binom{10-\sum_{j=1}^l (k_{i_j}+1)+3}{10-\sum_{k=1}^l (k_{i_j}+1)}$. Therefore, by the Inclusion-Exclusion Principle, $|\cup_{k=1}^4 A_k| = \sum_{l=1}^4 \sum_{1 \le i_1, \cdots, i_l \le 4} (-1)^{l-1} |\cap_{t=1}^l \mathcal{A}_{i_t}|$.
By caculating,

$$\begin{aligned}
&|\mathcal{A}| = 286, \\
&|\mathcal{A}_1| = \binom{10-(k_1+1)+3}{10-(k_1+1)} = \binom{2}{-1} = 0, \\
&|\mathcal{A}_2| = \binom{10-(k_2+1)+3}{10-(k_2+1)} = \binom{9}{6} = 84, \\
&|\mathcal{A}_3| = \binom{10-(k_3+1)+3}{10-(k_3+1)} = \binom{7}{4} = 35, \\
&|\mathcal{A}_4| = \binom{10-(k_4+1)+3}{10-(k_4+1)} = \binom{5}{2} = 10, \\
&|\mathcal{A}_{1l}| = \binom{10-(k_1+1)-(k_l+1)+3}{10-(k_1+1)-(k_l+1)} = \binom{-2}{-5} = 0, l \in \{2, 3, 4\}, \\
&|\mathcal{A}_{23}| = \binom{10-(k_2+1)-(k_3+1)+3}{10-(k_2+1)-(k_3+1)} = \binom{3}{0} = 1, \\
&|\mathcal{A}_{24}| = \binom{10-(k_2+1)-(k_4+1)+3}{10-(k_2+1)-(k_4+1)} = \binom{1}{-1} = 0, \\
&|\mathcal{A}_{34}| = \binom{10-(k_3+1)-(k_4+1)+3}{10-(k_3+1)-(k_4+1)} = \binom{-1}{-4} = 0, \\
&|\mathcal{A}_{1lt}| = \binom{10-(k_1+1)-(k_l+1)-(k_t+1)+3}{10-(k_1+1)-(k_l+1)-(k_t+1)} = 0, l, t \in \{2, 3, 4\}, \\
&|\mathcal{A}_{234}| = \binom{10-(k_2+1)-(k_3+1)-(k_4+1)+3}{10-(k_2+1)-(k_3+1)-(k_4+1)} = 0, \\
&|\mathcal{A}_{1234}| = \binom{10-(k_1+1)-(k_2+1)-(k_3+1)-(k_4+1)+3}{10-(k_1+1)-(k_2+1)-(k_3+1)-(k_4+1)} = 0,
\end{aligned} \tag{1}$$

so $|E| = 286 - (84 + 35 + 10) + 1 = 158$. □

ProblemVIII. Caculate the postive integral solution of equation $x_1 + x_2 + x_3 = 14$, where $x_i \le 8, i = 1, 2, 3$.

Solution. $y_i = x_i - 1, i = 1, \cdots, 3$, so $0 \le y_i \le 7$, $x_1 + x_2 + x_3 = 14 \Leftrightarrow y_1 + y_2 + y_3 = 11$. $A = \{$ all of the non-negtive solution of $y_1 + y_2 + y_3 = 11\}$. $A_i := \{$ all of the non-negtive solution of $y_1 + y_2 + y_3 = 11$ and $y_i > 7\}, i = 1, 2, 3$.
So $|A| = \binom{11+3-1}{11} = \binom{13}{2} = 78$. $|A_i| = \binom{11-(7+1)+3-1}{11-(7+1)} = \binom{5}{3} = 10, i = 1, 2, 3$, $|A_i \cap A_j| = 0, i, j \in 1, 2, 3, i \ne j$, $|A_1 \cap A_2 \cap A_3| = 0$. Therefore, $|A_1^c \cap A_2^c \cap A_3^c| = |A| - \sum_{i=1}^3 |A_i| + \sum_{1 \le i < j \le 3} |A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| = 78 - 3 \times 10 + 0 - 0 = 48$. □