In Mathematics

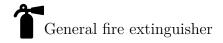
SetTheory 4

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

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problem Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a,b) \sim (c,d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and verifty that your definitions don't depend on the choice of representatives.

SPETON. Define $[(a,b)] +_{\mathbb{Q}} [(c,d)] = [(ad+bc,bd)], [(a,b)] \cdot_{\mathbb{Q}} [(c,d)] = [(ac,bd)],$ and $[(a,b)] <_{\mathbb{Q}} [(c,d)] \iff abd^2 < cdb^2$. Next to prove these definitions don't depend on the choice of representatives.

- 1. $+_{\mathbb{Q}}$: Let $(a,b) \sim (e,f), (c,d) \in \mathbb{Q}$), so af = be. Thus, $(ad + bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed + fc)bd$. So $(ad + bc, bd) \sim (ed + fc, df)$. So $+_{\mathbb{Q}}$ is well defined.
- 2. $\cdot_{\mathbb{Q}}$: Let $(a,b) \sim (e,f), (c,d) \in \mathbb{Q}$), so af = be. Then, acfd = bced = bdec, i.e. $(ac,bd) \sim (ec,fd)$.
- 3. $<_{\mathbb{Q}}$: Let $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. So $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ and $a_1b_1d_1^2 < c_1d_1b_1^2$. Thus, $a_1b_1d_1^2b_2^2d_2^2 < c_1d_1b_1^2b_2^2d_2^2$. Then, $a_2b_1^2d_1^2b_2d_2^2 < c_2d_1^2b_1^2b_2^2d_2$. So $a_2d_2^2b_2 < c_2b_2^2d_2$. Therefore, we prove $(a_2, b_2) < (c_2, d_2)$.

 \mathbb{R}^{OBEM} I The set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$ has cardinality \mathfrak{c} (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on \mathbb{R} is determined by its values at rational points.]

SPETION. Let $S := \{ f \in \mathbb{RR} : fiscontinous \}$. Consider $\theta : S \to 2^{\mathbb{Q}}, f \mapsto \{(a,b) \in \mathbb{Q} \times \mathbb{Q} : f(a) < b \}$.

- 1. f is a injection: Assume $\theta(f) = \theta(g)$,
 - (a) $\forall x \in \mathbb{Q}$, so $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(g)\} \sup y \in \mathbb{Q} : y < g(x) = g(x).$
 - (b) $\forall x \in \mathbb{R}, \ \exists \{x_n\}_{n=1}^{\infty} \subset \mathbb{Q} \text{ such that } x_n \to x, \text{ then } f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(x).$

So we get f = g. So $\operatorname{card}\mathbb{R}\mathbb{R} \leq \operatorname{card}2^{\mathbb{Q}} = 2^{\aleph_0}$.

2. Obviously card $\mathbb{RR} \geq 2^{\aleph_0}$, so we get they are equal.

 $\mathbb{R}^{OB}\mathbb{E}M$ II There are at least \mathfrak{c} countable order-types of linearly ordered sets.

SPINON. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x, y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \land 0 < x < a_{\frac{y}{2}}\}$$

And for $(x, y), (z, w) \in \tau_a$ we define $(x, y) < (z, w) \iff y < w \land y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove a = b. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know (x,0) can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \land_{1 \leq i < j \leq x-1} t_i \neq t_j, \forall k = 1, \dots x-1, t_k < p$. And θ is isomorphism. So $\theta(x,0) = (x,0)$. For (x,1), we let b_0 satisfy $\theta(0,1) = (b_0,m)$. Since the set $\{(x,y): y=1\}$ can be defined by $\psi(p) = \forall r, s(r,s , where <math>\tau(r) := \{s: s < r\}$ and $[r,s] = \{y: r < y < s\}$. we get $\theta[\{(x,y): y=1\}] = \{(x,y): y=1\}$. So we can delete the element whose second coordinary is 0,1, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x,2n+1) = (x,2n+1)$. So $a_n = \operatorname{card}\{(x,2n+1) \in \tau_a\} = \operatorname{card}\{(x,2n+1) \in \tau_b\} = b_n$ and thus a = b.