Graduate Homework In Mathematics

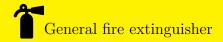
Functional Analysis 13

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ROBEM I \mathcal{X}, \mathcal{Y} are B space, T is linear operator from \mathcal{X} to $\mathcal{Y}, \forall g \in \mathcal{Y}^*, \sup_{x \in \mathcal{X}, ||x|| = 1} g(Tx) < \infty$, prove: T is bounded.

SOLION. Let $\mathcal{C} = \{x \in \mathcal{X} : ||x|| = 1\} \subset \mathcal{X}, \ \mathcal{E} = \{Tx : x \in \mathcal{C}\}, \text{ then, } \mathcal{E} \subset \mathcal{Y}, \text{ moreover, } \mathcal{E} \text{ is a subsapce of } \mathcal{Y}. \text{ Since } \mathcal{Y} \subset \mathcal{Y}^{**}, \text{ then } \mathcal{E} \subset \mathcal{Y}^{**}. \text{ Besides, } \forall g \in \mathcal{Y}^*, \sup_{x \in \mathcal{C}} g(Tx) < \infty, \mathcal{Y}^* \text{ is } B \text{ space, by Banach-Steinhaus theorem, } \sup_{x \in \mathcal{C}} ||Tx|| < \infty. \text{ Therefore, } T \text{ is bounded.}$

ROBEM II \mathcal{H} is a Hilbert space, $x_n \to x, n \to \infty \iff ||x_n|| \to ||x||, n \to \infty \text{ and } x_n \rightharpoonup x, n \to \infty.$

- SOLUTION. 1. " \Longrightarrow ": $|\|x_n\| \|x\|| \le \|x_n x\| \to 0, n \to \infty, \forall f \in H^*, |f(x_n) f(x)| = |f(x_n x)| \le \|f\| \|x_n x\| \to 0, n \to \infty.$
 - 2. " \Leftarrow ": Consider f(y) := (y, x), then f is linear, obviously. $\forall y \in \mathcal{X}, ||y|| = 1, |(y, x)| \le ||y|| ||x|| = ||x|| < \infty$, then $f \in H^*$. So $(x_n, x) = f(x_n) \to f(x) = (x, x) = ||x||^2$. And $(x, x_n) = \overline{(x_n, x)} \to \overline{||x||^2} = ||x||^2$. Therefore, $||x_n x|| = ||x_n||^2 (x_n, x) (x, x_n) + ||x||^2 \to 0, n \to \infty$.