FUNCTIONAL 1

王胤雅
SID:201911010205
201911010205@mail.bnu.edu.cn

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Problem I. 设 (X, d) 是距离空间, 令 $\rho(x, y) = \frac{\mathrm{d}(x,y)}{1+\mathrm{d}(x,y)}$. 求证 (X, ρ) 也是距离空间.

Solution.

- Since $\forall x, y \in X, \mathrm{d}(x, y) \geq 0$, then $\rho(x, y) = \frac{\mathrm{d}(x,y)}{1+\mathrm{d}(x,y)} \geq 0$. If $\rho(x, y) = 0 \Leftrightarrow \mathrm{d}(x, y) = 0 \Leftrightarrow x = y$.
- Since $\forall x, y \in X$, $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, then $\rho(x, y) = \frac{\mathrm{d}(x,y)}{1+\mathrm{d}(x,y)} = \frac{\mathrm{d}(y,x)}{1+\mathrm{d}(y,x)} = \rho(y, x)$.
- $\forall x, y, z \in X$, $\rho(x, y) = \frac{\mathrm{d}(x,y)}{1+\mathrm{d}(x,y)}$.
  If $\mathrm{d}(x, y) \leq \max\{\mathrm{d}(x, z), \mathrm{d}(z, y)\}$, by the increasing of function $\frac{x}{1+x}$ on $[0, +\infty)$, then $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\} \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$.
  If $\mathrm{d}(x, y) > \max\{\mathrm{d}(x, z), \mathrm{d}(z, y)\}$, then $\rho(x, y) = \frac{\mathrm{d}(x,y)}{1+\mathrm{d}(x,y)} \leq \frac{\mathrm{d}(x,z)}{1+\mathrm{d}(x,y)} + \frac{\mathrm{d}(z,y)}{1+\mathrm{d}(x,y)} \leq \frac{\mathrm{d}(x,z)}{1+\mathrm{d}(x,z)} + \frac{\mathrm{d}(z,y)}{1+\mathrm{d}(z,y)} = \rho(x, z) + \rho(z, y)$.

□

Problem II. $[0, 1]$ 上的全体多项式记为 $P[0, 1]$, 定义距离

$$\mathrm{d}(p, q) = \int_0^1 |p(x) - q(x)| \mathrm{d}x \tag{1}$$

其中 p, q 是多项式. 证明 $(P[0, 1], \mathrm{d})$ 是不完备的, 并指出它的完备化空间.

Solution. First of all, $(P[0, 1], \mathrm{d})$ is not complete.
Consider $f_n(x) = \sum_{k=1}^{n} \frac{1}{k+1} x^k, x \in [0, 1]$, so $\{f_n : n \in \mathbb{N}\} \subset P[0, 1]$. $\forall n \geq m$, $\mathrm{d}(f_m, f_n) = \int_0^1 |f_m - f_n| \mathrm{d}x \leq \sum_{k=m+1}^{n} \frac{1}{(1+k)^2} \to 0$, as $m, n \to \infty$, $\{f_n\}$ is a cauchy series. While $f(x) = \sum_{k=1}^{\infty} \frac{1}{1+k} x, x \in [0, 1]$, $\mathrm{d}(f_n, f) = \int_0^1 |f_n - f| \mathrm{d}x = \sum_{k=n+1}^{\infty} \frac{1}{(1+k)^2} \to 0$. By the uniqueness of limit, f is the limit of $\{f_n\}$. $\forall n$, $f^{(n)}(0) = \frac{n!}{n+1} \neq 0$, so $f \notin P[0, 1]$.
Secondly, proof $L^1[0, 1]$ is the completeness of $P[0, 1]$.

Lemma 1 (Stone–Weierstrass theorem)**.** $\forall f \in C[0, 1], \exists\{f_n \in P[0, 1] : n \in \mathbb{N}\}$ *satisfies*

$$\max_{0 \leq x \leq 1} |f_n(x) - f(x)| \to 0, n \to \infty$$

.

证明. $\forall x \in [0,1]$, $\{X_n : n \in \mathbb{N}_+\} \overset{i.i.d.}{\sim} B(1,x), S_n := \sum_{k=1}^n X_k$. Consider $b_n(x) = \sum_{k=1}^n f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = \mathbb{E}(\frac{S_n}{n})$ Since f is uniformly continuous on $[0,1]$, then $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x, y \in [0,1] : |x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Then,

$$\begin{aligned}
&|\mathbb{E}(f(\frac{S_n}{n})) - f(x)| \\
=&|\int_0^1 f(\frac{S_n}{n}) - f(x)\mathrm{d}x| \\
\leq&\int_0^1 |f(\frac{S_n}{n}) - f(x)|\mathrm{d}x \\
\leq&\int_{|\frac{S_n}{n}-f(x)|<\delta} |f(\frac{S_n}{n}) - f(x)|\mathrm{d}x + \int_{|\frac{S_n}{n}-f(x)|\geq\delta} |f(\frac{S_n}{n}) - f(x)|\mathrm{d}x \\
\leq&\varepsilon + 2\sup_{t\in[0,1]} |f(x)|\mathbb{E}\mathbb{1}_{\{|\frac{S_n}{n}-f(x)|\geq\delta\}}
\end{aligned} \tag{2}$$

By Chebyshev's inequality,

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{|\frac{S_n}{n}-f(x)|\geq\delta} \\
\leq&\frac{\mathbb{E}|\frac{S_n}{n} - f(x)|^2}{\delta^2} \\
=&\frac{x(1-x)}{n\delta^2} \\
\leq&\frac{1}{4n\delta^2}
\end{aligned} \tag{3}$$

Thus,

$$\sup_{t\in[0,1]} |\mathbb{E}(f(\frac{S_n}{n})) - f(x)| \leq \varepsilon + \frac{1}{2n\delta^2}\sup_{t\in[0,1]} |f(x)| \tag{4}$$

Last, let $n \to \infty$, and then let $\varepsilon \to 0$, we get $\sup_{t\in[0,1]} |\mathbb{E}(f(\frac{S_n}{n})) - f(x)| \to 0, n \to \infty$. That means $b_n \to f$ uniformly. □

Lemma 2 (Lusin Theorem). *f is a measurable function on E, and f is finite a.s., $\forall \delta > 0$, $\exists F \subset E$ is closed, F satisfies $m(E \setminus F) < \delta$, f is continuous on F.*

证明. Since f is finite a.s.

- When f is simple measurable functions. Let $f = \sum_{k=1}^n a_k E_k$, $a_k \in \mathbb{R} \forall k = 1, \cdots, n$, $E_i \cap E_j = \emptyset, i \neq j, E = \cup_{k=1}^n E_k$. $\forall \delta > 0, k = 1, \cdots, n$, $\exists F_k \subset E_k$, $m(E_k \setminus F_k) < \frac{\delta}{n}$. $f = a_k, \forall x \in F_k$. Let $F = \cup_{k=1}^n F_k$, so F is closed, and f is continuous on F. Besides, $m(E \setminus F) = m((\cup_{k=1}^n E_k) \setminus (\cup_{k=1}^n F_k)) \leq m(\cup_{k=1}^n (E_k \setminus F_k)) \leq \sum_{k=1}^n m(E_k \setminus F_k) \leq \delta$
- When f is a measurable function. Let $g = \frac{f}{1+|f|}$, then $f = \frac{g}{1-|g|}$, that means $f \in C[0,1] \Leftrightarrow g \in C[0,1]$, and g is bounded. W.L.O.G., f is bounded. So $\exists\{\varphi_n$ is simply measurable function: $n \in N\}, \varphi_n \to f$ uniformly. $\forall \delta > 0, \varphi_n$, $\exists F_n \subset E$ is closed, and φ_n is continuous on F, $m(E \setminus F_n) < \frac{\delta}{2^n}$. $F = \cap_{n=1}^\infty F_n$, F is closed, and $\forall \varphi_n$, φ_n is continuous on F, $m(E \setminus F) \leq m(E \setminus \cap_{n=1}^\infty F_n) = m(\cup_{n=1}^\infty (E \setminus F_n)) \leq \sum_{n=1}^\infty m(E \setminus F_n) < \sum_{n=1}^\infty \frac{\delta}{2^n} = \delta$.

□

Lemma 3. *$\forall f \in L^1(E)$, $\forall \varepsilon > 0, \exists g \in C(E)$, $\text{supp}(g)$ is compact, satisfies $\int_E |f(x) - g(x)|\mathrm{d}x < 0$.*

证明. It is easy to find φ is measurable on E, which satisfies $\text{supp}\varphi$ is compact and $\int_E |f(x) - \varphi(x)|\mathrm{d}x < \frac{\varepsilon}{2}$.
Let $|\varphi| \leq M$. By lemma2, we can find $g \in C(E)$ that satisfies $m(|\varphi(x) - g(x)| > 0)\frac{\varepsilon}{4M}$ and $g(x) \leq M$.

$$\begin{aligned}
&\int_E |f(x) - g(x)|\mathrm{d}x \\
\leq &\int_E |f(x) - \varphi(x)|\mathrm{d}x + \int_E |\varphi(x) - g(x)|\mathrm{d}x \\
\leq &\frac{\varepsilon}{2} + 2Mm(|\varphi(x) - g(x)| > 0) \\
\leq &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned} \tag{5}$$

□

$\forall f \in L^1[0,1]$, by lemma3, $\exists \{g_n \in C[0,1] : n \in \mathbb{N}\}$ satisfies $\int_{[0,1]} |f - g_n|\mathrm{d}x \to 0, n \to \infty$. Then by lemma1, $\forall g_n$, $\exists \{g_{n,m} \in P[0,1] : m \in \mathbb{N}\}$ satisfies $g_{n,m} \to g_n$ uniformly.
$\forall \varepsilon > 0$, $\exists N$, $\forall n > N$, $\int_{[0,1]} |g_n(x) - f(x)|\mathrm{d}x < \frac{\varepsilon}{2}$, $\exists M_n$, $m > M_n$, $\max_{t \in [0,1]} |g_{n,m}(t) - g_n(t)| < \frac{\varepsilon}{2}$. Let $m_n = M_n + 1$, $\{g_{n,m_n} \in P[0,1] : n \in \mathbb{N}\}$,

$$\begin{aligned}
&\int_{[0,1]} |g_{n,m_n} - f|\mathrm{d}x \\
\leq &\int_{[0,1]} |g_{n,m_n} - g_n|\mathrm{d}x + \int_{[0,1]} |g_n - f|\mathrm{d}x \\
\leq &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned} \tag{6}$$

Therefore, $P[0,1]$ is dense in $L^1[0,1]$. Let $\theta : P[0,1] \to L^1[0,1]$, which is an embed mapping. It is obvious that θ is isometry, $L^1[0,1]$ is complete, $\theta(P[0,1]) = P[0,1]$ is dense in $L^1[0,1]$.

□