FUNCTIONAL2

王胤雅

SID:201911010205

201911010205@mail.bnu.edu.cn

2023 年 9 月 25 日

Problem I. 证明存在闭区间 $[0,1]$ 上的连续函数 $x(t)$, 使得

$$x(t) = \frac{1}{2}\sin x(t) - a(t).$$

其中 $a(t)$ 是给定的 $[0,1]$ 上的连续函数

证明. $\theta : C[0,1] \to C[0,1]$, $x(t) \mapsto 1/2\sin x(t) - a(t)$, $\rho(x(t), y(t)) := \max_{0 \le t \le 1}|x(t) - y(t)|$ is the distance on $C[0,1]$.

$$
\begin{aligned}
&|\theta(x(t)) - \theta(y(t))| \\
=&|1/2\sin x(t) - a(t) - (1/2\sin y(t) - a(t))| \\
=&|1/2\sin x(t) - 1/2\sin y(t)| \\
=&|\sin\frac{x(t) - y(t)}{2}\cos\frac{x(t) + y(t)}{2}| \\
\le&|\frac{x(t) - y(t)}{2}|
\end{aligned}
\tag{1}
$$

Therefore, $\rho(\theta(x(t)), \theta(y(t))) \le \frac{1}{2}\rho(x(t), y(t))$. Thus, θ is a contraction mapping on $(C[0,1], \rho)$. By contraction mapping principle, $\exists x(t) \in C[0,1]$ statisfies $x(t) = \frac{1}{2}\sin x(t) - a(t)$.

$\square$

Problem II.
$$x(t) - \lambda \int_0^1 e^{t-s}x(s)\mathrm{d}s = y(t),$$

其中 $y(t) \in C[0,1]$, λ 为常数, $|\lambda| < 1$. 证明存在唯一解 $x(t) \in C[0,1]$.

证明. $x(t) - \lambda\int_0^1 e^{t-s}x(s)\mathrm{d}s = y(t), \Leftrightarrow e^{-t}x(t) - \lambda\int_0^1 e^{-s}x(s)\mathrm{d}s = e^{-t}y(t)$.
Let $z(t) = e^{-t}x(t), w(t) = e^{-t}y(t)$, $\theta : C[0,1] \to C[0,1]$, $z(t) \mapsto \lambda\int_0^1 z(s)\mathrm{d}s - w(t)$,

$\rho(x(t), y(t)) := \max_{0 \le t \le 1} |x(t) - y(t)|$ is the distance on $C[0,1]$.

$$|\theta(u(t)) - \theta(v(t))|$$

$$= |(\lambda \int_0^1 u(s)\mathrm{d}s - w(t)) - (\lambda \int_0^1 v(s)\mathrm{d}s - w(t))| \tag{2}$$

$$= |\lambda \int_0^1 u(s)\mathrm{d}s - \lambda \int_0^1 v(s)\mathrm{d}s|$$

$$\le |\lambda| \int_0^1 |u(s) - v(s)|\mathrm{d}s$$

$$\le |\lambda|\rho(u(s), v(s))$$

Therefore, $\rho(\theta(x(t)), \theta(y(t))) \le |\lambda|\rho(x(t), y(t))$, where $|\lambda| < 1$. Thus, θ is a contraction mapping on $(C[0,1], \rho)$. By contraction mapping principle, $\exists |z(t) \in C[0,1]$ statisfies $z(t) = \lambda \int_0^1 z(s)\mathrm{d}s - w(t)$. $x(t) = \mathrm{e}^t z(t)$ is what we need. $\qquad\square$