underGraduate Homework In Mathematics

FunctionalAnalysis 8

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PROBLEM I Let $A = \{e_k\}$ are orthonormal basis in inner product space X. Prove: $\forall x, y \in X$, $\sum_{k=1}^{\infty} |(x, e_k)(y, e_k)| \le \|x\| \, \|y\|$.

SOLUTION . Since $A = \{e_k\}$ are orthonormal basis in inner product space X, then $\forall x \in X$, $\|x\| = \sum_{k=1}^{\infty} |(x, e_k)|^2$, so by Holder inequation, we get $\sum_{k=1}^{\infty} |(x, e_k)(y, e_k)| \le (\sum_{k=1}^{\infty} |(x, e_k)|^2)^{\frac{1}{2}} (\sum_{k=1}^{\infty} |(y, e_k)|^2)^{\frac{1}{2}} = \|x\| \, \|y\|$. □

PROBLEM II H is Hilbert space, $\{e_k\}, \{e'_k\}$ are two kinds of orthonormal set in inner product space H, $\sum_{k=1}^{\infty} \|e_k - e'_k\|^2 < 1$. Prove: if one of $\{e_k\}, \{e'_k\}$ is complete, then the other is complete.

SOLUTION . Let $\{e_n\}_{n=1}^{\infty}$ is complete. If $\{e'_n\}_{n=1}^{\infty}$ is not complete, then $\exists x_0 : \ \theta \ne x_0 \notin Span\{\{e'_n\}_{n=1}^{\infty}\}$ s.t. $(x_0, e'_n) = 0 \forall n$. So $\|x_0\|^2 = \sum_{n=1}^{\infty} |(x_0, e_n)|^2 = \sum_{n=1}^{\infty} |(x_0, e_n - e'_n)|^2 \le \|x_0\|^2 \sum_{n=1}^{\infty} \|e_n - e'_n\|^2 < \|x_0\|$. Contradiction! □

PROBLEM III H is an inner space, these propositions below are equal:

1. $x \perp y$;
2. $\|x + \alpha y\| \ge \|x\|$, $\alpha \in \mathbb{C}$;
3. $\|x + \alpha y\| = \|x - \alpha y\| , \forall \alpha \in \mathbb{C}$.

SOLUTION . 1. $x \perp y \Rightarrow \|x + \alpha y\| \ge \|x\|$, $\alpha \in \mathbb{C} : \|x + \alpha y\| = \|x\| + |\alpha| \, \|y\| \ge \|x\| , \forall \alpha \in \mathbb{C}$.

2. $x \perp y \Leftarrow \|x + \alpha y\| \ge \|x\|$, $\alpha \in \mathbb{C}$: If $(x, y) \ne 0$, let $\alpha = r(x, y), r \in \mathbb{R}$, then by $\|x + \alpha y\| \ge \|x\|$, we get $|\alpha|^2 \|y\|^2 + 2\mathrm{Re}\overline{\alpha}(x, y) \ge 0$, that is $r^2 |(x, y)|^2 \|y\|^2 + 2r|(x, y)|^2 \ge 0$, so $\Delta = 4 \le 0$. Contradiction!
3. $x \perp y \Rightarrow \|x + \alpha y\| = \|x - \alpha y\| , \forall \alpha \in \mathbb{C}$: $\|x + \alpha y\| = \|x\| + |\alpha| \, \|y\| = \|x - \alpha y\|$.
4. $x \perp y \Leftarrow \|x + \alpha y\| = \|x - \alpha y\| , \forall \alpha \in \mathbb{C}$: If $(x, y) \ne 0$, let $\alpha = r(x, y), r \in \mathbb{R}$, then by $\|x + \alpha y\| = \|x - \alpha y\|$, we get $\mathrm{Re}\overline{\alpha}(x, y) = 0$, that is $r^2 |(x, y)|^2 = 0, \forall r \in \mathbb{R}$. Contradiction!

□