GroupRepresentation 3

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 10 月 27 日

1 homework

PROBLEM I Let ϕ is representation of $\mathrm{GL}_n(K)$ over K^n. And $\phi(A)\alpha := A\alpha$. Prove:ϕ is faithful and irreducible and $n-$dimentional.

SOLUTION . It is obvious that ϕ is $n-$dimentional. $\forall A, B \in GL_n(K)$, $A \neq B$, $\exists \alpha \in K$, $A\alpha \neq B\alpha$, so $\phi(A)\alpha = A\alpha \neq B\alpha = \phi(B)\alpha$. So ϕ is injective, so it is faithful. $\forall \alpha, \beta \in K^n \setminus \{0\}$, $\exists A \in GL_n(K)$ s.t. $A(\alpha) = \beta$, so there is no invariant subspace of K^n. □

PROBLEM II For $A \in \mathrm{GL}_n(K)$, let $\psi(A)X = AX, \forall X \in M_n(K)$. Then:

1. ψ is n^2-dimentional representation of $\mathrm{GL}_n(K)$ over K.
2. For $j : 1 \leq j \leq n$, let $M_n^{(j)}(K) := \{(a_{ik})_{n\times n} : a_{ik} \neq 0 \to k = j\}$. Prove $M_n^{(j)}$ is invariant subspace of $\mathrm{GL}_n(K)$. Let ψ is subrepresentation of ψ in $M_n^{(j)}$, prove ψ_j is irreducible and $\psi = \bigoplus_{j=1}^n \psi_j$.
3. Prove $\psi_j \cong \phi$, where $\phi = (\text{ PROBLEM } I).\phi$

SOLUTION .
1. Since $M_n(K)$ is n^2-dimentional on K and $\forall A, B \in GL_n(K), \forall X \in M_n(K)$, $\psi(AB)X = ABX = \psi(A)BX = \psi(A)(B)X$, so ψ is a homomorphism. So ψ is a n^2-dimentional representation.
2. $\forall A \in GL_n(K), \forall X \in M_n^{(j)}(K)$, let $X = (x_{ik})_{n\times n}$, $A = (a_{ik})_{n\times n}$, $\phi(A)X = AX =: (b_{ik})_{n\times n}$, $b_{ik} = \sum_{l=1}^n a_{il}x_{lk} \neq 0$, then $k = j$, so $AX \in M_n^{(j)}(K)$, so $M_n^{(j)}(K)$ is invariant subspace. Since $M_n(K) = \oplus_{j=1}^n M_n^{(j)}(K)$, so $\psi = \oplus_{j=1}^n \psi_j$. Consider $\tau : M_n^{(j)}(K) \to K^n$, $(\tau(X))_k = x_{kj}$, so τ is a isomophism. Obviously, ψ is a isomophism between ψ_j and ϕ, $\forall j = 1, \cdots, n$. While ϕ is irreducible, so ψ_j is irreducible.
3. As Item 2 has prooved.

□

PROBLEM III Let $K=\mathbb{C}$ and $n=2$ in (Group representation second homework).(Problem 3), prove the subrepresentation of ϕ over $M_2^0(\mathbb{C})$ is irreducible.

SOLUTION . Since $\forall X\in M_2(\mathbb{C})$, X can be diagonalized on $\mathbb{C}$. $\forall X\in M_n^0(\mathbb{C})$, $\exists A\in M_n(\mathbb{C})$, s.t. $\phi(A)(X)=\begin{pmatrix}\lambda & 0\\ 0 & -\lambda\end{pmatrix}=\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. So $\forall X\in E$, where E is the invariant subspace of $M_2^0(\mathbb{C})$. so $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\in V$. So $M_2^0(\mathbb{C})\subset E$, so $E=M_2^0(\mathbb{C})$. □

PROBLEM IV Assume $n\geq 3$ and $n\nmid \operatorname{char} K$, proof: then $n-$ dimentional permutate representation of S_n can be decomposed as the direct sum of a main representation and a $n-1-$ dimentional irreducible subrepresentation

SOLUTION . As we have prooved in the second homework in PROBLEM I. ϕ_{V_1} is the main representation, and V_2 is $n-1$dimentional, so we only need to proof V_2 is irreducible. $\forall\{0\}\neq V\subset V_2$ is a invariant subspace, $\forall x\in V\setminus\{0\}$, $x=\sum_{i=1}^n a_ix_i, \sum_{i=1}^n a_i=0$, if $a_i=k, i=1,\cdots,n$, so $\sum_{i=1}^n a_n=nk=0$, while $n\nmid \operatorname{char} K$, $k=0$, so $x=0$. W.L.O.G. Let $a_1\neq a_2$, so $\phi((12))x=a_2x_1+a_1x_2+\sum_{k=3}^n a_kx_k\in V$, then $x-\phi((12))x=(a_1-a_2)(x_1-x_2)\in V$, then $x_1-x_2\in V$, so $\phi((2j))(x_1-x_2)=x_1-x_j\in V$. While $\{x_1-x_2,\cdots,x_1-x_n\}\subset V$ and they are linear independent. Then $\dim(V)\geq n-1$, so $V=V_2$. Thus, $\phi|_{V_2}$ is irreducible. □

PROBLEM V Caculate the $1-$ dimentional $\mathbb{C}$ representation:

1. $(2,4)-$type of $8-$ order elementary Abel group.
2. the addition group of $\mathbb{Z}_p^n$

SOLUTION .

1. $G=\mathbb{Z}_2\times\mathbb{Z}_4$, $\phi(x,y)=\mathrm{e}^{\frac{(2x+y)\pi\mathrm{i}}{2}}$.
2. $\phi(a_1,\cdots,a_n)=\mathrm{e}^{\frac{2\sum_{k=1}^n a_k\pi\mathrm{i}}{p}}$

□

2 The second homework

PROBLEM I Group G has an action on set $\Omega=\{x_1,x_2,\cdots,x_n\}$, let (ϕ,V) be the $n-$ dimensional K permutation representation of G, where K is the field of vector space V, and

$$V=\left\{\sum_{i=1}^n a_ix_i \mid a_i\in K, i=1,2,\cdots,n\right\}.$$

Let $V_1=\left\langle\sum_{i=1}^n x_i\right\rangle, V_2=\left\{\sum_{i=1}^n a_ix_i \mid \sum_{i=1}^n a_i=0, a_i\in K\right\}$. Proof: (1) V_1 and V_2 are invariant subspaces of G ; (2) If $\operatorname{char} K\nmid n$, then $\varphi=\varphi_{V_1}\oplus\varphi_{V_2}$.

PROBLEM III $\mathcal{O}(n) := \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ is the set of all n-dimensional orthogonal matrix over $\mathbb{R}$. Let:

$$\begin{aligned} \varphi : \mathcal{O}(n) &\to \mathrm{GL}\left(M_n(\mathbb{R})\right) \\ A &\mapsto \varphi(A), \end{aligned} \tag{1}$$

$$\varphi(A)X := AXA^{-1} : \quad \forall X \in M_n(\mathbb{R}) \tag{2}$$

$M_n^+(\mathbb{R}) := \{A \in M_n^0(\mathbb{R}) : A = A^T\}$, $M_n^-(\mathrm{R}) := \{A \in M_n^0(\mathbb{R}) : A^T = -A\}$. (1) Proof: $M_n^+(\mathrm{R})$ and $M_n^-(\mathrm{R})$ are invariant spaces of φ; (2) Let the subrepresentation of φ on $\langle I \rangle$,$M_n^+(\mathbb{R})$, $M_n^-(\mathbb{R})$ be $\varphi_0, \varphi_1, \varphi_2$. Proof: $\varphi = \varphi_0 + \varphi_1 + \varphi_2$. (3) calculate a $\frac{1}{2}n(n-1)-$ dimensional $\mathbb{R}$ representation of $\mathcal{O}(n)$.