underGraduate Homework In Mathematics

FunctionalAnalysis 12

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

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PROBLEM I $\mathcal{X}$ is a linear space on $\mathbb{C}$. p is a seminorm on $\mathcal{X}$. $p(x_0) \neq 0, x_0 \in \mathcal{X}$. Prove: $\exists f$ is a linear funcional on $\mathcal{X}$ such that

1. $f(x_0) = 1$
2. $|f(x_0)| \leq \frac{p(x)}{p(x_0)}$, $\forall x \in \mathcal{X}$.

SOLUTION. Consider $p^* : \mathcal{X} \to \mathbb{C}$, $x \mapsto \frac{p(x)}{p(x_0)}$. Obviously p^* is a seminorm on $\mathcal{X}$. Let $\mathcal{X}_0 := \text{Span}\{x_0\} \subset \mathcal{X}$ is a subspace of $\mathcal{X}$. $f : \mathcal{X}_0 \to \mathbb{C}$, $\alpha x_0 \mapsto \alpha f(x_0)$, where $f(x_0) = 1$. So f is a linear funcional on $\mathcal{X}_0$. And $\forall x \in \mathcal{X}_0, x = \alpha x_0, |f(x)| = |\alpha||f(x_0)| \leq |\alpha| = \frac{p(\alpha x_0)}{p(x_0)} = p^*(x)$. Thus, by Hahn-Banach theorem, $\exists \tilde{f} : \mathcal{X} \to \mathbb{R}$ is a linear funcional on $\mathcal{X}$ such that

1. $\tilde{f}(x) = f(x), \forall x \in \mathcal{X}_0$.
2. $|\tilde{f}(x)| \leq p^*(x) = \frac{p(x)}{p(x_0)}, \forall x \in \mathcal{X}$.

So $\tilde{f}(x_0) = f(x_0)$. □

PROBLEM II $\mathcal{X}$ is a B^* space, $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ such that $\forall f \in \mathcal{X}^*, \{f(x_n)\}_{n=1}^{\infty}$ is bounded. Prove that $\{x_n\}_{n=1}^{\infty}$ is bounded.

SOLUTION. Since there is an embedding map from $\mathcal{X} \to \mathcal{X}^{**}$, which keeps norm. Regard $\{x_n\}_{n=1}^{\infty}$ as subset of $\mathcal{X}^{**}$. And $\mathcal{X}^{**} = \mathcal{L}(\mathcal{X}^*, \mathbb{K})$, $\mathcal{X}^* = \mathcal{L}(X, \mathbb{K})$. $\mathbb{K}$ is complete, so $\mathcal{X}^*$ is a B space. Besides, $\forall f \in \mathcal{X}^*, \sup_{n\in\mathbb{N}_+} |x_n(f)| = \sup_{n\in\mathbb{N}_+} |f(x_n)| < \infty$. By Banach-Steinhaus theorem, $\sup_{n\in\mathbb{N}_+} \|x_n\| < \infty$ □

PROBLEM III $\mathcal{X}$ is a B^* space, $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$. Prove that $\forall x \in \mathcal{X}, \inf_{y\in\mathcal{X}_0} \|x - y\| = \sup\{|f(x)| : f \in \mathcal{X}\ \|f\| = 1, f|_{\mathcal{X}_0} = 1\}$.

Lemma 1. $\mathcal{X}$ is a B^* space, let $H_f^{\lambda} := \{x \in \mathcal{Z} : f(x) = \lambda\}$ where is a linear functional on $\mathcal{X}$. If $\|f\| = 1$, then $|f(x)| = d(x, H_f^0), \forall x \in \mathcal{X}$, where $\mathrm{d}(x, H_f^0) := \inf_{z\in H_f^0} \|x - z\|$.

证明. Since $\|f\| = 1$, then $\exists z \notin H_f^0$. And $x \in H_f^0$, $f(x) = 0$, let $z = x$, then $\|x - z\| = 0$. Next consider $x \notin H_f^0$:

1. $|f(x)| \leq \mathrm{d}(x, H_f^0)$: Since $\forall \varepsilon > 0$, $\exists y \in H_f^0$ such that $\|x - y\| \leq \mathrm{d}(x, H_f^0) + \varepsilon$. And $|f(x)| = |f(x - y)| \leq \|f\|\,\|x - y\| \leq \mathrm{d}(x, H_f^0) + \varepsilon \to \mathrm{d}(x, H_f^0), \varepsilon \to 0$.
2. $|f(x) \geq d(x, H_f^0)|$: $\forall \varepsilon > 0$, $\exists y \in x$

□

SOLUTION. By Lemma 1, we have that $\inf_{y\in\mathcal{X}_0} \|x - y\| \geq \sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$. Consider $\mathcal{X}_0 = \mathcal{X}$ is possible, we define: $\sup\varnothing = 0$.

1. $\mathcal{X}_0 = \mathcal{X}$: $f(x) = 0, \forall x \in \mathcal{X}$, so $\nexists f : \|f\| = 1$, so $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = \sup\varnothing$. Obviously, the conclusion is true.
2. $\mathcal{X}_0 \subsetneq \mathcal{X}$:

(a) If $x \notin \mathcal{X}_0$, since $\mathcal{X}_0$ is closed, then $d := \inf_{y \in \mathcal{X}_0} \|x - y\| > 0$, by Hahn-Banach theorem, $\exists f \in \mathcal{X}^*$ such that $\|f\| = 1, f|_{\mathcal{X}_0} = 0, f(x) = d$. Thus, $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = |f(x)|$. So $\inf_{y \in \mathcal{X}_0 \|x-y\|} = |f(x)| = \sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$.

(b) If $y \in \mathcal{X}_0$, take $x \notin \mathcal{X}_0$,f such that $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = |f(x)|$. So $f(y) = 0$, then $\{|f(y)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$ is not empty. And $\forall f \in \mathcal{X}^*$ such that $\|f\| = 1, f|_{\mathcal{X}_0} = 0$, then $f(y) = 0$. Thus, $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = 0 = \inf_{y \in \mathcal{X}_0} \|x - y\|$.

$\square$