underGraduate Homework In Mathematics

FunctionalAnalysis 12

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General fire extinguisher

PROBLEM I $\mathcal{X}$ is a linear space on $\mathbb{C}$. p is a seminorm on $\mathcal{X}$. $p(x_0) \neq 0, x_0 \in \mathcal{X}$. Prove: $\exists f$ is a linear funcional on $\mathcal{X}$ such that

1. $f(x_0) = 1$
2. $|f(x_0)| \leq \frac{p(x)}{p(x_0)}$, $\forall x \in \mathcal{X}$.

SOLUTION. Consider $p^* : \mathcal{X} \to \mathbb{C}$, $x \mapsto \frac{p(x)}{p(x_0)}$. Obviously p^* is a seminorm on $\mathcal{X}$. Let $\mathcal{X}_0 :=$ Span$\{x_0\} \subset \mathcal{X}$ is a subspace of $\mathcal{X}$. $f : \mathcal{X}_0 \to \mathbb{C}$, $\alpha x_0 \mapsto \alpha f(x_0)$, where $f(x_0) = 1$. So f is a linear funcional on $\mathcal{X}_0$. And $\forall x \in \mathcal{X}_0, x = \alpha x_0, |f(x)| = |\alpha||f(x_0)| \leq |\alpha| = \frac{p(\alpha x_0)}{p(x_0 v} = p^*(x)$. Thus, by Hahn-Banach theorem, $\exists \tilde{f} : \mathcal{X} \to \mathbb{R}$ is a linear funcional on $\mathcal{X}$ such that

1. $\tilde{f}(x) = f(x), \forall x \in \mathcal{X}_0$.
2. $|\tilde{f}(x)| \leq p^*(x) = \frac{p(x)}{p(x_0)}, \forall x \in \mathcal{X}$.

So $\tilde{f}(x_0) = f(x_0)$. $\square$