underGraduate Homework In Mathematics

SetTheory 4

王胤雅

王胤雅是傻逼

201911010205

201911010205@mail.bnu.edu.cn

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General fire extinguisher

problem Consider $\mathbb{Q} = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim$, where $(a, b) \sim (c, d) \iff ad = bc$. Define $+_{\mathbb{Q}}, \cdot_{\mathbb{Q}}$ and $<_{\mathbb{Q}}$ and verifty that your definitions don't depend on the choice of representatives.

SOLUTION . Define $[(a, b)] +_{\mathbb{Q}} [(c, d)] = [(ad + bc, bd)], [(a, b)] \cdot_{\mathbb{Q}} [(c, d)] = [(ac, bd)]$, and $[(a, b)] <_{\mathbb{Q}} [(c, d)] \iff abd^2 < cdb^2$. Next to prove these definitions don't depend on the choice of representatives.

1. $+_{\mathbb{Q}}$: Let $(a, b) \sim (e, f), (c, d) \in \mathbb{Q})$, so $af = be$. Thus, $(ad + bc)bf = ad^2f + bdcf = bed^2 + bdcf = (ed + fc)bd$. So $(ad + bc, bd) \sim (ed + fc, df)$. So $+_{\mathbb{Q}}$ is well defined.
2. $\cdot_{\mathbb{Q}}$: Let $(a, b) \sim (e, f), (c, d) \in \mathbb{Q})$, so $af = be$. Then, $acfd = bced = bdec$, i.e. $(ac, bd) \sim (ec, fd)$.
3. $<_{\mathbb{Q}}$: Let $(a_1, b_1) \sim (a_2, b_2), (c_1, d_1) \sim (c_2, d_2)$ and $(a_1, b_1) < (c_1, d_1)$. So $a_1b_2 = a_2b_1, c_1d_2 = c_2d_1$ and $a_1b_1d_1^2 < c_1d_1b_1^2$. Thus, $a_1b_1d_1^2b_2^2d_2^2 < c_1d_1b_1^2b_2^2d_2^2$. Then, $a_2b_1^2d_1^2b_2d_2^2 < c_2d_1^2b_1^2b_2^2d_2$. So $a_2d_2^2b_2 < c_2b_2^2d_2$. Therefore, we prove $(a_2, b_2) < (c_2, d_2)$.

$\square$

PROBLEM I The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$). [A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

SOLUTION . Let $S := \{f \in \mathbb{R}\mathbb{R} : fiscontinous\}$. Consider $\theta : S \to 2^{\mathbb{Q}}, f \mapsto \{(a, b) \in \mathbb{Q} \times \mathbb{Q} : f(a) < b\}$.

1. f is a injection: Assume $\theta(f) = \theta(g)$,
   (a) $\forall x \in \mathbb{Q}$, so $f(x) = \sup\{y \in \mathbb{Q} : y < f(x)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(f)\} = \sup\{y \in \mathbb{Q} : (x, y) \in \theta(g)\} \sup y \in \mathbb{Q} : y < g(x) = g(x)$.
   (b) $\forall x \in \mathbb{R}$, $\exists\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ such that $x_n \to x$, then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$.

   So we get $f = g$. So $\mathrm{card}\mathbb{R}\mathbb{R} \le \mathrm{card}2^{\mathbb{Q}} = 2^{\aleph_0}$.
2. Obviously $\mathrm{card}\mathbb{R}\mathbb{R} \ge 2^{\aleph_0}$, so we get they are equal.

□

PROBLEM II There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

SOLUTION. For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the ordertype

$$\tau_a = \{(x,y) \in \mathbb{Z} \times \mathbb{N} : 2 \nmid y \wedge 0 < x < a_{\frac{y}{2}}\}$$

And for $(x,y),(z,w) \in \tau_a$ we define $(x,y) < (z,w) \iff y < w \wedge y = w, x < z$. Now we will show that if $a \neq b$, then $\tau_a \neq \tau_b$. Assume $\tau_a \cong \tau_b$, we need to prove $a = b$. assume $\theta : \tau_a \to \tau_b$ is the isomorfism.

We know $(x,0)$ can be defined as $\phi(p) = \exists_{k=1}^{x-1} t_k, \wedge_{1 \leq i < j \leq x-1} t_i \neq t_j, \forall k = 1, \cdots x-1, t_k < p$. And θ is isomorphism. So $\theta(x,0) = (x,0)$. For $(x,1)$, we let b_0 satisfy $\theta(0,1) = (b_0, m)$. Since the set $\{(x,y) : y = 1\}$ can be defined by $\psi(p) = \forall r, s(r, s < p \wedge \tau(r) \wedge \tau(s) \to \text{card}[r,s] < \infty)$, where $\tau(r) := \{s : s < r\}$ and $[r,s] = \{y : r < y < s\}$. we get $\theta[\{(x,y) : y = 1\}] = \{(x,y) : y = 1\}$. So we can delete the element whose second coordinary is $0, 1$, and θ is isomorphism, too. Do this repeatedly, we get $\theta(x, 2n+1) = (x, 2n+1)$. So $a_n = \text{card}\{(x, 2n+1) \in \tau_a\} = \text{card}\{(x, 2n+1) \in \tau_b\} = b_n$ and thus $a = b$. □