underGraduate Homework In Mathematics

FunctionalAnalysis 13

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General fire extinguisher

PROBLEM I $\mathcal{X}, \mathcal{Y}$ are B space, T is linear operator from $\mathcal{X}$ to $\mathcal{Y}$, $\forall g \in \mathcal{Y}^*$, $\sup_{x \in \mathcal{X}, \|x\|=1} g(Tx) < \infty$, prove: T is bounded.

SOLUTION. Let $\mathcal{C} = \{x \in \mathcal{X} : \|x\| = 1\} \subset \mathcal{X}$, $\mathcal{E} = \{Tx : x \in \mathcal{C}\}$, then, $\mathcal{E} \subset \mathcal{Y}$, moreover, $\mathcal{E}$ is a subsapce of $\mathcal{Y}$. Since $\mathcal{Y} \subset \mathcal{Y}^{**}$, then $\mathcal{E} \subset \mathcal{Y}^{**}$. Besides, $\forall g \in \mathcal{Y}^*, \sup_{x \in \mathcal{C}} g(Tx) < \infty$, $\mathcal{Y}^*$ is B space, by Banach-Steinhaus theorem, $\sup_{x \in \mathcal{C}} \|Tx\| < \infty$. Therefore, T is bounded.

$\square$

PROBLEM II $\mathcal{H}$ is a Hilbert space, $x_n \to x, n \to \infty \iff \|x_n\| \to \|x\|, n \to \infty$ and $x_n \rightharpoonup x, n \to \infty$.

SOLUTION.

1. " $\Longrightarrow$ ": $|\,\|x_n\| - \|x\|\,| \le \|x_n - x\| \to 0, n \to \infty$, $\forall f \in H^*$, $|f(x_n) - f(x)| = |f(x_n - x)| \le \|f\| \|x_n - x\| \to 0, n \to \infty$.
2. " $\Longleftarrow$ ": Consider $f(y) := (y, x)$, then f is linear, obviously. $\forall y \in \mathcal{X}, \|y\| = 1$, $|(y, x)| \le \|y\| \|x\| = \|x\| < \infty$, then $f \in H^*$. So $(x_n, x) = f(x_n) \to f(x) = (x, x) = \|x\|^2$. And $(x, x_n) = \overline{(x_n, x)} \to \overline{\|x\|^2} = \|x\|^2$. Therefore, $\|x_n - x\| = \|x_n\|^2 - (x_n, x) - (x, x_n) + \|x\|^2 \to 0, n \to \infty$.

$\square$