

# Group Representation 5

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 10 月 27 日

**PROBLEM I**  $K$  is a field,  $A$  is algebra on  $K$ ,  $\emptyset \neq A_1 \subset A$ , we call  $A_1$  is a subalgebra of  $A$ , if  $A_1$  is a subring of  $A$  which contains 1 of  $A$  and  $A_1$  is a subspace of  $A$  on  $K$  and is also an algebra on  $K$ . Let  $Z(A) := \{c \in A : ca = ac, \forall a \in A\}$ . Prove:  $Z(A)$  is a subalgebra of  $A$ , we call  $Z(A)$  is the center of algebra  $A$ .

**SOLUTION.** 1.  $Z(A)$  is a subring of  $A$  which contains 1 of  $A$ : Since  $\forall a \in A$ ,  $A$  is a ring, then  $1a = a1$ . So  $1 \in Z(A)$ .  $\forall c_1, c_2 \in Z(A)$ ,  $\forall a \in A$ ,  $(c_1 - c_2)a = (c_1 + (-c_2))a = c_1a + (-1)c_2a = ac_1 + (-1)ac_2 = ac_1 + a(-1)c_2 = a(c_1 + (-1)c_2) = a(c_1 - c_2)$ , then  $c_1 - c_2 \in Z(A)$ .  $(c_1c_2)a = c_1(c_2a) = c_1(ac_2) = (c_1a)c_2 = (ac_1)c_2 = a(c_1c_2)$ , then  $c_1c_2 \in A$ .

2.  $Z(A)$  is a subspace of  $A$  on  $K$ :  $\forall k \in K$ ,  $\forall c, c_1, c_2 \in Z(A)$ ,  $\forall a \in A$ , since  $A$  is an algebra on  $K$ , then  $(kc)a = k(ca) = k(ac) = a(kc)$ , then  $kc \in Z(A)$ . And by Item 1, we get  $c_1 + c_2 \in Z(A)$ .

3. By Item 1, Item 2, we get  $Z(A)$  is also an algebra on  $K$ .  
So  $Z(A)$  is a subalgebra of  $A$ . □

**PROBLEM II** Let  $G$  is infinite group,  $K$  is a field. Prove:

1.  $\sum_{g \in G} g \in Z(K[G])$ ;
2.  $C_a := \{gag^{-1} : g \in G\}$ ,  $\sum_{x \in C_a} x \in Z(K[G])$ .

**SOLUTION.** 1.  $\forall \sum_{h \in G} a_h h \in K([G])$ , then  $\sum_{g \in G} g \sum_{h \in G} a_h h = \sum_{g \in G} \sum_{x \in G} x a_{x^{-1}g} x^{-1}g = \sum_{g \in G} \sum_{x \in G} a_{x^{-1}g} g = \sum_{g \in G} \sum_{x \in G} a_x g = \sum_{g \in G} \sum_{x \in G} a_x x x^{-1}g = \sum_{x \in G} \sum_{g \in G} a_x x x^{-1}g = \sum_{h \in G} a_h h \sum_{g \in G} g$ .

2.  $\forall \sum_{h \in G} a_h h \in K([G])$ , then  $\sum_{x \in C_a} x \sum_{g \in G} a_g g = \sum_{x \in C_a} \sum_{g \in G} x a_g g = \sum_{x \in C_a} \sum_{g \in G} a_g x g = \sum_{g \in G} a_g \sum_{x \in C_a} x g, \sum_{g \in G} a_g g \sum_{x \in C_a} x = \sum_{g \in G} \sum_{x \in C_a} a_g g x = \sum_{g \in G} \sum_{x \in C_a} a_g g x = \sum_{g \in G} a_g \sum_{x \in C_a} g x$ . Since  $\forall g \in G$ , then  $\{hah^{-1} : h \in G\} = \{g(g^{-1}h)a(h^{-1}g) : h \in G\} = \{ghah^{-1} : h \in G\}$ , then,  $\sum_{x \in C_a} x \sum_{g \in G} a_g g = \sum_{g \in G} a_g g \sum_{x \in C_a} x$ . □