COMBINATION2

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Problem I. 确定数 $3^4 \times 5^2 \times 11^7 \times 13^8$ 的正整数因数的个数.

证明. Let $V_p(n) = \sup\{k \in \mathbb{N} : p^k | n\}$, p is prime. Let $p_1 = 3, p_2 = 5, p_3 = 11, p_4 = 13, a_1 = 4, a_2 = 2, a_3 = 7, a_4 = 8, a = 3^4 \times 5^2 \times 11^7 \times 13^8$, $A := \{n \in N : V_{p_i}(n) \leq a_i, i = 1, \cdots, 4, V_p(n) = 0, p$ is prime, and $p \neq p_i, 1 \leq i \leq 4\}$, $F := \{n \in \mathbb{N} : n | a\}$.

1. $\forall n \in A$, then $n = 3^{V_3(n)} \times 5^{V_5(n)} \times 11^{V_{11}(n)} \times 13^{V_{13}(n)}$, then $n|a$, since $p_i^{n_i} | p_i^{a_i}, 1 \leq i \leq 4$, so $n \in F$.
2. $\forall n \notin A$, if $\exists$ prime $p \neq p_i, 1 \leq i \leq 4$ s.t. then $V_p(n) > 0$, then$n \nmid a$, then $n \notin F$. If $\forall$ prime $p \neq p_i, 1 \leq i \leq 4$ s.t. $V_p(n) = 0$ and $\exists p_i | n, 1 \leq i \leq 4$ s.t. $V_{p_i}(n) > a_i$, then $p_i^{V_p(n)} | n$ but $p_i^{V_p(n)} \nmid a$, so $n \notin F$.

Then $A = F$. So $|F| = |A| = 5 * 3 * 8 * 9 = 1080$ □

Problem II. 在 $0 - 9999$ 之间有多少个整数只有一位数字是 *5* ?

证明. The number between $0 - 9999$ can be written as $a = a_4 * 10^3 + a_3 * 10^2 + a_2 * 10^1 + a_1 * 10^0$. $\varphi : [0, 9999] \cap \mathbb{N} \to A := \{(a_1, a_2, a_3, a_4) : 0 \leq a_j \leq 9, j = 1, \cdots, 4\}$, $\varphi(a) := (a_1, a_2, a_3, a_4)$. Obviously, φ is bijection. Let $A_i := \{(a_1, a_2, a_3, a_4) : a_i = 5, 0 \leq a_j \leq 9, j = 1, \cdots, 4\}$. Consider $\theta_{ij} : A_i \to A_j$, $(a_1, a_2, a_3, a_4) \mapsto (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)})$, where $\sigma \in S_4, \sigma = (i\ j)$ Obviously, θ_{ij} is bijection. So the amount is $4|A_1|$. Obviously, $|A_1| = 9^3 = 729$. □

Problem III. 比 *5400* 大的四位数中, 数字 *2* 和 *7* 不出现, 且各位数字不同的整数有多少个?

证明. The number between $0-9999$ can be written as $a = a_4 * 10^3 + a_3 * 10^2 + a_2 * 10^1 + a_1 * 10^0$. $\varphi : [0, 9999] \cap \mathbb{N} \to A := \{(a_1, a_2, a_3, a_4) : 0 \leq a_j \leq 9, j = 1, \cdots, 4\}$, $\varphi(a) := (a_1, a_2, a_3, a_4)$. Obviously, φ is bijection. Since $2, 7$ can't appear in any digit, then $B := \{(a_1, a_2, a_3, a_4) : a_i \in \{0, 1, 3, 4, 6, 8, 9\}, 1 \leq i \leq 4, a_i \neq a_j, i \neq j, 1 \leq i, j \leq 4, \varphi^{-1}(a_1, a_2, a_3, a_4) \in [5400, \infty) \cap \mathbb{N}\}$. Let $A_i := \{(a_1, a_2, a_3, a_4) \in B : a_4 = i, a > 5400\}$, $A_{i,j} := \{(a_1, a_2, a_3, a_4) \in B : a_4 = i, a_3 = j, a > 5400\}$

1. A_5 :
   (a) $a \in A_{54}$, if $a_1 = 1$, then $a_2 \in \{0, 3, 4, 6, 8, 9\}$; if $a_1 = 0$, then $a_2 \in \{1, 3, 4, 6, 8, 9\}$. Then $|A_{54}| = 6 + 6 = 12$.
   (b) $a \in A_{5j}, j \leq 6$, $\theta_{jk} : A_{5j} \to A_{5k}, j \neq k$, $(a_1, a_2, a_3, a_4) \mapsto (\sigma(a_1), \sigma(a_2), \sigma(a_3), \sigma(a_4))$, where $\sigma \in S_9, \sigma = (jk)$. When $a_1, a_2 \neq k$, then $\theta_{jk}(a_1, a_2, 5, j) = (a_1, a_2, 5, k) \in A_{5k}$; when $a_1 = k$, then $a_2 \notin \{k, j\}$, then $\theta_{jk}(k, a_2, 5, j) = (j, a_2, 5, k) \in A_{5k}$; it is the same for $a_2 = k$. So θ_{jk} is well-defined. It is trivial that θ_{jk} is injection. And $\theta_{kj} \circ \theta_{jk} = \mathrm{id}$, so θ_{jk} is bijection. $\forall a \in A_{56}$, $a_i \in \{0, 1, 3, 4, 8, 9\}, i = 1, 2$ so $|A_{56}| = A_6^2 = 30$.

   So $|A_5| = |A_{54} \cup (\cup_{j \in \{6,8,9\}} A_{5j})| = 12 + 30 \times 3 = 102$.

2. As for $A_i, i \in \{6, 8, 9\}$, $\theta_{ij} : A_i \to A_j, i \neq j$, $(a_1, a_2, a_3, a_4) \mapsto (\sigma(a_1), \sigma(a_2), \sigma(a_3), \sigma(a_4))$, where $\sigma \in S_9, \sigma = (ij), i, j \in \{6, 8, 9\}$.When $a_1, a_2, a_3 \neq j$, then $\theta_{ij}(a_1, a_2, a_3, i) = (a_1, a_2, a_3, j) \in A_j$; when $a_1 = j$, then $a_2, a_3 \notin \{i, j\}$, then $\theta_{ij}(i, a_2, a_3, j) = (j, a_2, a_3, i) \in A_j$; it is the same for $a_2, a_3 = j$. So θ_{ij} is well-defined. It is trivial that θ_{ij} is injection. And $\theta_{ji} \circ \theta_{ij} = \mathrm{id}$, so θ_{jk} is bijection. $\forall a \in A_6$, then $a_i = \{0, 1, 3, 4, 5, 8, 9\} i = 1, 2, 3$, then $|A_6| = A_7^3 = 7 \times 6 \times 5 = 210$.

So the total number is $|\cup_{i\geq 5} A_i| = 102 + 210 * 3 = 732$. □

Problem IV. *10 个字母的字符串中 (由 26 个英文小写字母中的一些字母组成, 可以有重复字母), 两个相邻字母都不相同的字符串有多少个.*

证明. $A := \{Allthecharacter\}$, $E := \{a \in A^26 : a_i \neq a_{i+1} 1 \leq i \leq 25\}$ □

Problem V. *在 26 个英文大写字母的全排列中, 使得任两个元音字母 (A, E, I, O, U) 都不相邻的排列共有多少个.*

证明. $21! * A_{22}^5 = \frac{21! * 22!}{17!}$ □

Problem VI. *把 18 人分成 4 个小组, 使各组人数分别为 5 5 4 4 人, 有多少种分法.*

证明. $\frac{C_{18}^5 C_{13}^5 C_8^4}{2! * 2!} = \frac{18!13!8!}{5!13!5!8!4!4!2!2!} = 306306$ □

Problem VII. *将 a, b, c, d, e, f, g, h 排成一行, 要求 a 在 b 的左侧, b 在 c 的左侧, 问有多少种排法?*

证明. $5! * C_6^3 = 600$ □

Problem VIII. *3 个男生和 7 个女生聚餐, 围坐在圆桌旁, 任意两个男生不相邻的坐法有多少种?*

证明. $C_3^1 * \frac{8!}{8} * C_2^1 * C_6^1 * C_5^1 = \frac{3*8!6!2!5!}{8*5!*4!} = 907200$ □

Problem IX. *设 $k, k_1, k_2, \ldots, k_n$ 为正整数, 且满足 $k_1 + k_2 + \cdots + k_n = k$, 将 k 个不同的物品放入 n 个不同的盒子 $B_1, B_2, \ldots, B_n$ 中, 使得 B_j 中放入 $k_j (1 \leq j \leq n)$ 个物品, 问不同的放法有多少种?*

证明. The positive solution of equation $k_1 + k_2 + \cdots + k_n = k$ equal to the non-negtive solution $x_1 + x_2 + \cdots + x_n = k - n$ which is $C_{k-n+n-1}^{n-1}$. So the different way to deposite different items is $\frac{n!(k-1)!}{(n-1)!(k-n)!} = \frac{n(k-1)!}{(k-n)!}$ □

Problem X. *将 r 个相同的球放入 k 个不同的盒子中, 有多少种不同的放法?*

证明. $\frac{(r+k-1)!}{r!(k-1)!}$ □

Problem XI. *将 6 个蓝球, 5 个红球, 4 个白球, 3 个黄球排成一排, 要求黄球不挨着, 问有多少种排列方式.*

证明. First we arrange blue, red and white balls , the amount of arrangement is $\frac{(6+5+4)!}{6!5!4!} = 630630$. Then we arrange the yellow ones, the amount of arrangement is $\frac{(6+5+4)!}{6!5!4!} * C_{16}^3 = 2118916800$ □

Problem XII. *不等式 $x_1 + x_2 + \cdots + x_9 < 2000$ 的正整数解有多少个?*

证明. Equal to amount of the non-negtive solution of $x_1 + x_2 + \cdots + x_9 < 1991$, that is $\sum_{k=0}^{1990} \frac{(k+8)!}{k!8!}$ □

Problem XIII. *证明 $(1+\sqrt{3})^{2m+1} + (1-\sqrt{3})^{2m+1}$ 是一个整数.*

证明.

$$\begin{aligned}
&(1+\sqrt{3})^{2m+1} + (1-\sqrt{3})^{2m+1} \\
=&\sum_{k=0}^{2m+1} \sqrt{3}^k + \sum_{k=0}^{2m+1} (-\sqrt{3})^k \\
=&\sum_{l=0}^{m} (\sqrt{3}^{2l} + \sqrt{3}^{2l}) + (\sqrt{3}^{2l+1} - \sqrt{3}^{2l+1}) \\
=&\sum_{l=0}^{m} 2 * 3^l
\end{aligned} \tag{1}$$

□

Problem XIV. *用多项式定理展开 $(x_1 + x_2 + x_3)^4$.*

证明.

$$
\begin{aligned}
&(x_1+x_2+x_3)^4\\
=&\sum_{n_1+n_2+n_3=4}\frac{4!}{n_1!n_2!n_3!}x_1^{n_1}x_2^{n_2}x_3^{n_3}\\
=&x_1^4+x_2^4+x_3^4+4x_1x_2^3+4x_1x_3^3+4x_2x_1^3+4x_2x_3^3+4x_3x_1^3+4x_3x_2^3+6x_1^2x_2^2+6x_1^2x_3^2\\
&+6x_2^2x_3^2+12x_1x_2x_3^2+12x_1x_2^2x_3+12x_1^2x_2x_3
\end{aligned}
\tag{2}
$$

□

Problem XV. 用牛顿二项式定理近似计算 $10^{\frac{1}{3}}$.

证明.

$$
\begin{aligned}
&10^{\frac{1}{3}}\\
=&(1+9)^{\frac{1}{3}}\\
=&\sum_{k=0}^{\infty}\frac{\frac{1}{3}\cdots(\frac{1}{3}-k+1)}{k!}9^k\\
=&\frac{1}{3}*\frac{4}{3}+\frac{1}{3}*9-\frac{\frac{1}{3}*\frac{2}{3}*9^2}{2*1}+\sum_{k=3}^{\infty}\frac{\frac{1}{3}*\cdots*(\frac{1}{3}-k+1)}{k!}3^{2k}\\
=&\frac{10}{3}+\sum_{m=0}^{\infty}\frac{1*\cdots*(1-3(m+2))}{(m+3)!}3^{m+3}\\
=&\frac{10}{3}+\sum_{m=0}^{\infty}\frac{2*\cdots*(-3m-5)}{(m+3)!}3^{m+3}
\end{aligned}
\tag{3}
$$

□

Problem XVI. 运用数学归纳法证明

$$\frac{1}{(1-z)^n}=\sum_{k=0}^{\infty}\begin{pmatrix} n+k-1\\ k\end{pmatrix}z^k,\quad |z|<1.$$

证明.
- When $n=0$, trivial
- When $n=1$, it turns to

$$
\begin{aligned}
&\frac{1}{1-z}=\sum_{k=0}^{\infty}z^k\\
&\Leftrightarrow\\
&1=\sum_{k=0}^{\infty}z^k(1-z)\\
=&\sum_{k=0}^{\infty}z^k-z^{k+1}\\
=&1+\sum_{k=1}^{\infty}z^k-\sum_{k=0}^{\infty}z^{k+1}\\
=&1
\end{aligned}
\tag{4}
$$

- If n the equation is right, then we goes to $n+1$.

$$\begin{aligned}
&\frac{1}{(1-z)^{n+1}} \\
=&\frac{1}{1-z}\sum_{k=0}^{\infty}\left(\begin{array}{c} n+k \\ k \end{array}\right) z^k \\
=&\sum_{k=0}^{\infty} z^k \sum_{k=0}^{\infty}\left(\begin{array}{c} n+k \\ k \end{array}\right) z^k \\
=&\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left(\begin{array}{c} n+k \\ k \end{array}\right) z^m \\
=&\sum_{m=0}^{\infty}\left(\begin{array}{c} n+m \\ m \end{array}\right) z^m
\end{aligned} \tag{5}$$

□

Problem XVII. *By applying integral to binomial theorem, proof: $\forall n$, we have*

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}.$$

Solution. Since$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, apply integral on $[0,1]$to both side, we get $\frac{1}{n+1}(x+1)^{n+1}|_0^1 = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}x^{k+1}|_0^1$, that means $\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$. □

Problem XVIII. *Proof:*

$$\sum_{k=0}^{n}\binom{m}{k}\binom{m-k}{n-k} = 2^n\binom{m}{n}$$

Solution. $\sum_{k=0}^{n}\binom{m}{k}\binom{m-k}{n-k} = \sum_{k=0}^{n}\frac{m!}{k!(m-k)!}\frac{(m-k)!}{(n-k)!(m-n)!} = \sum_{k=0}^{n}\frac{m!}{n!(m-n)!}\frac{n!}{(n-k)!k!} = 2^n\binom{m}{n}$ □

Problem XIX. *Apply $m^2 = 2\binom{m}{2}+\binom{m}{1}$, calculate the value of $1^2+2^2+\cdots+n^2$.*

Solution. $\sum_{m=1}^{n} m^2 = \sum_{m=1}^{n}\left(2\binom{m}{2}+\binom{m}{1}\right) = 2\left(\binom{n+1}{3}-\binom{0}{2}\right)+\left(\binom{n+1}{2}-\binom{0}{1}\right) = \frac{(n+1)n(n-1)}{3}+\frac{(n+1)n}{2} = \frac{(n+1)n(2n+1)}{6}$.

□

Problem XX. *Let $q = \lceil \frac{n}{2} \rceil$, then*

$$\sum_{k=0}^{q}\binom{n}{2k}2^{n-2k} = \frac{3^n+1}{2}.$$

Solution. $(2+1)^n+(2-1)^n = 2\left(\sum_{k=0}^{q}\binom{n}{2k}2^{n-2k}\right) = 2\left(\sum_{k=0}^{q}\binom{n}{2k}2^{n-2k}\right)$ □

Problem XXI. *Proof:*

$$\sum_{k=0}^{n}\frac{k+2}{k+1}\binom{n}{k} = \frac{(n+3)2^n-1}{n+1}.$$

Solution. $\sum_{k=0}^{n}\binom{n}{k}x^k = (x+1)^n$, so $\int_{[0,1]}(x+1)^n = \int_{[0,1]}\sum_{k=0}^{n}\binom{n}{k}x^k$, so $\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$. Therefore, $\sum_{k=0}^{n}\frac{k+2}{k+1}\binom{n}{k} = \sum_{k=0}^{n}\binom{n}{k}+\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}+2^n = \frac{(n+3)2^n-1}{n+1}$ □

Problem XXII. *Using $m(m \geq 2)$ colors to paint a chess broad of $1 \times n$, every cell has one color. Let $h(m,n)$ be the amount of different painting methods in which every neibouring cell has different color and every color is used, calculate $h(m,n)$.*

Solution.

$$h(m,n) = \sum_{1=t_1<t_2<\cdots<t_m<t_{m+1}=n+1} m!\prod_{k=1}^{m}(k-1)^{t_k-t_{k-1}-1}$$

□