underGraduate Homework In Mathematics

SetTheory 5

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 12 月 7 日

General fire extinguisher

PROBLEM I Prove: $F \subset \mathcal{N}$ is closed set $\iff$ $F = [T]$ for some $T \subset_{<\omega} \omega$.

SOLUTION.
- $\implies$: Let $T := T_F$, by the definition of T_F and $[T]$, we get $F \subset [T]$. For $f \in [T]$, $f \restriction n \in T$, so $\forall n \in \mathbb{N}, f \restriction n = g \restriction n$, $\exists g \in F$. So $d(f, F) \leq d(f, g) = \frac{1}{2^n} \to 0, n \to \infty$. Since F is closed, then $f \in F$.
- $\impliedby$: For any $[T] \in_{<\omega} \omega$, only need to prove $[T]$ is closed. Assume $f \in \overline{[T]}$, then $\forall n \in \mathbb{N}, \exists g \in [T], f \restriction n = g \restriction n$. Since $g \in [T]$, then $g \restriction n \in T$. So $f \in [T]$. So $[T]$ is closed.

$\square$

PROBLEM II Assume f is isolated point in closed set $F \subset \mathcal{N}$, then $\exists n \in \mathbb{N}, \forall g \in F, g \neq f \to g \restriction n \neq f \restriction n$.

SOLUTION. Since f is isolated, we get $\exists n \in \mathbb{N}, \forall g \in F \setminus \{f\}, d(f, g) > \frac{1}{2^n}$. Then $f \restriction n \neq g \restriction n$. $\square$

PROBLEM III A closed set $F \subset \mathcal{N}$ is perfect $\iff$ T_F is a perfect tree.

SOLUTION.
- $\implies$: For $t \in T_F$, $\exists f \in F, n \in \mathbb{N}, t = f \restriction n$. Since F is perfect, then F is not isolated, by PROBLEM II $\forall n, \exists g \in F, g \neq f$ such that $d(f, g) < \frac{1}{2^{n+1}}$. Then $t = f \restriction n \sqsubset g$. Since $f \neq g$, Then, $\exists m \in \mathbb{N}, m > n$ such that $f \restriction m \neq g \restriction m$. So $t \sqsubset f \restriction m, t \sqsubset g \restriction m$, and $f \restriction m, g \restriction m$ are incomparable. So T_F is perfect.
- $\impliedby$: For $f \in F$, only need to prove f is not isolated. Since T_F is perfect, then $\forall t := f \restriction n \in T_F$, where $f \in F, n \in \mathbb{N}$. $\exists s_1, s_2 \in T_F$ such that $t \sqsubset s_1, s_2$ and s_1, s_2 are incomparable. Then $s_1, s_2 \sqsubset f$ is impossible. Without loss of generality assume $s_1 \not\sqsubset f$. so $s_1 = g \restriction m$ for some $g \in F, m \in \mathbb{N}$. So $d(f, g) \leq \frac{1}{2^{n+1}}$. So f is not isolated.

$\square$

PROBLEM IV For $\alpha < \omega_1$, we let $\Sigma_0 = \{O \subset \mathbb{R} : O \text{ is open }\}$, and $\Pi_0 = = \{F \subset \mathbb{R} : F \text{ is closed }\}$. And $\Sigma_{\alpha+1} = \{\bigcup_{n \in \mathbb{N}} A(n) : A \in_{\mathbb{N}} \Pi_\alpha$. $\Pi_{\alpha+1} = \{\mathbb{R} \setminus A : A \in \Sigma_\alpha\}$. $\Sigma_\alpha = \bigcup_{\beta < \alpha} \Sigma_\beta, \Pi_\alpha = \bigcup_{\beta < \alpha} \Pi_\beta$ for limit ordinal α. Prove that $\mathcal{B}(\mathbb{R}) = \bigcup_{\alpha < \omega_1} \Sigma_\alpha$.

SOLUTION. Use MI easily we get $\bigcup_{\alpha < \omega_1} \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$. Now we prove $\mathcal{B}(\mathbb{R}) \subset \bigcup_{\alpha < \omega_1} \Sigma_\alpha$. Since open sets is subset of $\bigcup_{\alpha < \omega_1} \Sigma_\alpha$, we only need to prove $\bigcup_{\alpha < \omega_1} \Sigma_\alpha =: \mathcal{A}$ is σ-field. Easily we get $\Sigma_\alpha \subset \Sigma_{\alpha+2}$. Obviously $\mathbb{R} \in \mathcal{A}$. For $A \in \mathcal{A}$, assume $A \in \Sigma_\alpha$. Then $\mathbb{R} \setminus A \in \Pi_{\alpha+1} \subset \Sigma_{\alpha+1} \subset \mathcal{A}$. Assume $A \in_{\mathbb{N}} \mathcal{A}$, let $f \in_{\mathbb{N}} \omega_1, f(n) = \min\{\alpha \in \omega_1 : A(n) \in \Sigma_\alpha\}$. Consider $\sup \operatorname{ran} f =: \gamma$. Since $\forall \alpha \in \operatorname{ran} f, \alpha$ is countable. And $\operatorname{ran} f$ is countable. So $\sup \operatorname{ran} f$ is countable, thus $\sup \operatorname{ran} f < \omega_1$. Then $\operatorname{ran} A \subset \Pi_{\gamma+1}$. So we get $\bigcup_{n \in \mathbb{N}} A(n) \subset \Sigma_{\gamma+2} \subset \mathcal{A}$. So we get $\mathcal{A}$ is σ-field. So $\mathcal{B}(\mathbb{R}) \subset \mathcal{A}$, thus $\mathcal{A} = \mathcal{B}(\mathbb{R})$. $\square$

PROBLEM V Show that $\mathcal{M} := \{A \subset \mathbb{R} : A \text{ is measurable}\}$ is a σ-field.

Lemma 1. For $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$, $|\mathcal{A}| = alpha_0$, then $\mu^*(\bigcup_{A \in \mathcal{A}} A) \leq \sum_{A \in \mathcal{A}} \mu^*(A)$.

证明. Since $|\mathcal{A}| = \alpha_0$, let $\mathcal{A} = \{A_1, A_2, \cdots, A_n, \cdots\}$. $\forall n \in \mathbb{N}$, $\varepsilon > 0, \exists O_n \in \mathcal{O}, A_n \subset O_n$ and $\mu^*(A_n) \leq |O_n| + \frac{\varepsilon}{2^{n+1}}$. Let $U := \bigcup_{n \in \mathbb{N}} O_n$, then $\bigcup_{n \in \mathbb{N}} A_n \subset U$. So $\mu^*(\bigcup_{n \in \mathbb{N}} A_n \leq |U| \leq \sum_{n \in \mathbb{N}} |O_n| \leq \sum_{n \in \mathbb{N}} \mu^*(A_n) + \varepsilon$. Since ε is arbitry, then $\mu^*(\bigcup_{n \in \mathbb{N}} A_n = \sum_{n \in \mathbb{N}} \mu^*(A_n)$. $\square$

Lemma 2. If $G \in G_\delta$, then $\forall \varepsilon > 0, \exists O \in \mathcal{O}, G \subset O \wedge \mu^*(O \setminus G) \leq \varepsilon$.

证明. 1. G is bonded: Assume $G \subset [-M, M], M > 0$, and $G = \bigcap_{n\in\mathbb{N}} O_n$, where $O_n \in \mathcal{O}$. Since $G = \bigcap_{n\in\mathbb{N}} \bigcap_{k=0}^{m} O_m$, then without loss of generality, we can assume $O_n \supset O_{n+1}, n \in \mathbb{N}$. Besides, since $G = \bigcap_{n\in\mathbb{N}}(O_n \cap (-M-1, M+1))$. So, we can assume $O_n \subset (-M-1, M+1)$. So $|O_n|$ is declining and bounded. Thus, $\lim_{n\to\infty} |O_n| = a$. Therefore, if $m_k, 0 \leq k < n$ have define, let we define m_n, $\forall \varepsilon > 0, \exists N, \forall l, m \geq N$, $|O_l| - |O_m| < \frac{\varepsilon}{2^{n-1}}$. Let $m_n = N$, then $\{O_{m_n}\}_{n=0}^{\infty} \subset \{O_n\}_{n=0}^{\infty}$ is a sub sequence, and $\lim_{n\to\infty} |O_{m_n}| = a$, $G = \bigcap_{n\in\mathbb{N}} O_{m_n}$, $|O_{m_n}| - |O_{m_{n+1}}| < \frac{\varepsilon}{2^{n-1}}$. Thus, we can assume $\{O_n\}_{n=0}^{\infty}$ such that $\forall n, |O_n| - |O_{n+1}| < \frac{\varepsilon}{2^n}$ By Lemma 1, so

2. G is not bounded: Let $G_n = G \cap B(0, n)$, then $G = \bigcup_{n\in\mathbb{N}} G_n$. So $\forall \varepsilon > 0$, $\exists O_n \supset G_n$ such that $\mu^*(O_n \setminus G_n) \leq \frac{\varepsilon}{2^n}$. Then $O = \bigcup_{n\in\mathbb{N}} O_n \in \mathcal{O}$, $O \setminus G \subset \bigcup_{n\in\mathbb{N}} O_n \setminus G_n$, so by Lemma 1, $\mu^*(O \setminus G) \leq \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^n} < \varepsilon$.

□

SOLUTION. First, for $A = \mathbb{R}$, easily we can let $F = G = \mathbb{R}$. Then F is F_σ and G is G_δ. Second, assume $A \in \mathcal{M}$, consider $B = \mathbb{R} \setminus A$. Assume $F \subset A \subset G$ and $\mu^*(G \setminus F) = 0$. Then $G^c \subset B \subset F^c$. And G^c is F_σ, F^c is G_δ. And $\mu^*(F^c \setminus G^c) = \mu^*(G \setminus F) = 0$. So $B \in \mathcal{M}$. Finally, assume $A \in_{\mathbb{N}} \mathcal{M}$, we need to prove $\bigcup_{n\in\mathbb{N}} A_n =: A \in \mathcal{M}$. Use AC we can find $F \in_{\mathbb{N}} F_\sigma, G \in_{\mathbb{N}} G_\delta$ such that $F(n) \subset A_n \subset G(n), \mu^*(G(n) - F(n)) = 0$. Let $T = \bigcup_{n\in\mathbb{N}} F(n)$. Since $F(n)$ is F_σ, we get $T \in F_\sigma$. And easily $T = \bigcup_{n\in\mathbb{N}} F(n) \subset \bigcup_{n\in\mathbb{N}} A(n) = A$. □

PROBLEM VI Show that $\mathcal{A} := \{A \subset \mathbb{R} : A \text{ has property of Baire}\}$ is σ-field.

SOLUTION. Easily $\mathbb{R}\Delta\mathbb{R}$ is meager, so $\mathbb{R} \in \mathcal{A}$.

If $A \in \mathcal{A}$, we need to prove $\mathbb{R} \setminus A \in \mathcal{A}$. Assume $G \in \mathcal{O}$ and $A\Delta G$ is meager, write $B = \mathbb{R} \setminus A$, only need to prove $\exists U \in \mathcal{O}$, such that $B \setminus U, U \setminus B$ are meager. Let $U = \mathbb{R} \setminus \overline{G}$. Then $B \setminus U = A \setminus \overline{G}$ is meager. Now only need to prove $U \setminus B = \overline{G} \setminus A$ is meager. Since $G \setminus A$ is meager, we only need to prove $\overline{G} \setminus G$ is meager. In fact, we can prove $\overline{G} \setminus G$ is nowhere dense. Consider $I \in \mathcal{O}$, we need to prove $\exists J \subset I, J \in \mathcal{O}, J \cap \partial G = \varnothing$. If $I \cap \partial G = \varnothing$, we can let $J = I$. Else, assume $a \in I \cap \partial G$. Form the defination of ∂G, we get $\exists b \in I \cap G$. Let $J = I \cap G \neq \varnothing$ is OK. So $B\Delta U$ is meager.

Assume $A \in_{\mathbb{N}} \mathcal{P}(\mathcal{A})$, we need to prove $\bigcup_{n\in\mathbb{N}} A(n) =: A \in \mathcal{A}$. Assume $G(n) \in \mathcal{O}$ and $A(n)\Delta G(n)$ is meager. Consider $G := \bigcup_{n\in\mathbb{N}} G(n)$. We only need to prove $G\Delta A$ is meager. Only need $G \setminus A, A \setminus G$ is meager. Since $G \setminus A \subset \bigcup_{n\in\mathbb{N}} G(n) \setminus A(n)$ and $G(n) \setminus A(n)$ is meager, we get $G \setminus A$ is meager. For the same reason, we get $A \setminus G \subset \bigcup_{n\in\mathbb{N}} A(n) \setminus G(n)$ is meager.

So finally we get $\mathcal{A}$ is σ-field. □

PROBLEM VII Assume $A \subset_\omega \omega$ has the property of Baire, prove A is nonmerger $\iff \exists O \in \mathcal{O}({}_\omega\omega), O \neq \varnothing \wedge O \setminus A$ is meager.