underGraduate Homework In Mathematics

FunctionalAnalysis 7

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General fire extinguisher

PROBLEM I Prove: There is no such inner product such that $\forall f \in C[a,b], (f,f)^{\frac{1}{2}} = \max_{x\in[a,b]} |f(x)|$.

SOLUTION . Since $x_1 = \frac{x-a}{b-a}, x_2 = \frac{x-b}{a-b}$, $x_1, x_2 \in C[a,b]$, then, $x_1 + x_2 = \frac{x-a+b-x}{b-a} = 1$, $x_1 - x_2 = \frac{2x-a-b}{a-b}$. Let $\|f\| = \max_{x\in[a,b]} |f(x)|$, then $(C[a,b], \|\cdot\|)$ is B^* space. Since $\|x_1 + x_2\| = 1, \|x_1 - x_2\| = \max_{x\in[a,b]} |\frac{2x-a-b}{a-b}| = 1, \|x_1\| = 1 = \|x_2\|$, then $\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 1^2 + 1^2 < 2(\|x_1\|^2 + \|x_2\|^2) = 2 \times (1+1) = 4$. Then, there is no such inner product such that $\forall f \in C[a,b], (f,f)^{\frac{1}{2}} = \max_{x\in[a,b]} |f(x)|$. □

PROBLEM II $f : L^2[0,T] \to [0,+\infty)$, $x \mapsto |\int_0^T \mathrm{e}^{-(T-\tau)} x(\tau)\,\mathrm{d}\tau|$.Prove: f can reach maximum in S^2, calculate the maximum and find the element of x which reaches the maximum.

SOLUTION . $\forall x \in S^2$, $f(x) = |\int_0^T \mathrm{e}^{-(T-\tau)} x(\tau)\,\mathrm{d}\tau| = \mathrm{e}^T \int_0^T |e^\tau x(\tau)|\,\mathrm{d}\tau \le \mathrm{e}^T (\int_0^T \mathrm{e}^{2\tau}\,\mathrm{d}\tau)^{\frac{1}{2}} (\int_0^T x(\tau)^2\,\mathrm{d}\tau)^{\frac{1}{2}} = \mathrm{e}^T (\int_0^T \mathrm{e}^{2\tau}\,\mathrm{d}\tau)^{\frac{1}{2}} = (\frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2T})^{\frac{1}{2}}$.By the equation condition of Cauchy Schwartz inequation, when $x = t\mathrm{e}^u$, the inequation can be an equation. So let $x = \pm(\frac{2}{\mathrm{e}^{2T}-1})^{\frac{1}{2}} \mathrm{e}^u$, $f(x) = (\frac{1}{2} - \frac{1}{2}\mathrm{e}^{-2T})^{\frac{1}{2}} = \max_{t\in S^2} f(t)$. □

PROBLEM III $(X, (\cdot,\cdot))$ is an inner product space, $M \subset N \subset X$. Prove $N^\perp \subset M^\perp$.

SOLUTION . Since $\forall x \in N^\perp := \{x \in X : \forall y \in N, (x,y) = 0\}$, $\forall y \in M \subset N$, $y \in N$, $(x,y) = 0$, then $x \in M^\perp := \{x \in X : \forall y \in M, (x,y) = 0\}$. So $N^\perp \subset M^\perp$. □