GroupRepresentation 12

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PROBLEM I Assume $G = \langle a \rangle$ is the n-ranked cyclic group. Prove that

$$f(a^r) = \sum_{j=0}^{n-1} k_j \xi^{rj}$$

where $f : G \to \mathbb{C}$ is a function and $\xi = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$ and

$$k_j = \frac{1}{n} \sum_{r=0}^{n-1} f(a^r) \xi^{-rj}$$

SOLUTION. Easily we have $\hat{G} = \{\varphi^j : j = 0, \cdots, n-1\}$, where $\varphi(a^r) = \xi^r$. So we have $f(a^r) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \hat{f}(\varphi^j)\varphi^j(a^r) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \hat{f}(\varphi^j)\xi^{rj}$. And $\hat{f}(\varphi^j) = \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} f(a^r)\overline{\varphi^j(a^r)} = \frac{1}{\sqrt{n}} \sum_{r=0}^{n-1} f(a^r)\xi^{-rj}$. So we finally get the given equation. $\square$

PROBLEM II Assume $f : G \to \mathbb{C}$ and G is Abel group. Prove that f is const $\iff$ $\hat{f}(\varphi) = 0, \forall \varphi \in \hat{G} \setminus \{1\}$.

SOLUTION. " $\implies$ ": Easily we have $\hat{f}(\varphi) = \frac{1}{\sqrt{n}} \sum_{g \in G} f(g)\varphi(g)$. Assume $\varphi \neq 1$, since f is const, we only need to prove $\sum_{g \in G} \varphi(g) = 0$. Since $\varphi \perp 1$, we get $\sum_{g \in G} \varphi(g) = 0$. " $\impliedby$ ": Easily we have $f(g) = \sum_{\varphi \in \hat{G}} \frac{1}{\sqrt{n}} \sum_{\varphi \in \hat{G}} \hat{f}(\varphi)\varphi(g)$. Since $\varphi \neq 1 \to \hat{f}(\varphi) = 0$, we get $f(g) = \frac{1}{\sqrt{n}} \hat{f}(1)$ is a const. $\square$

PROBLEM III Find a 3-dim irriducible complex character of S_4.

SOLUTION. Consider $S_4 \times \Omega := \{1, 2, 3, 4\} \to \Omega, \sigma x = \sigma(x)$. Easily this group action is double transitive. Let φ is the reperesentation obtained by this group action, then $\varphi = \varphi_0 \oplus \varphi_1$, where φ_0 is main reperesentation and $\dim \varphi_1 = 3$ and φ_1 is irriducible. Easily S_4 has 5 conjugate classes and $\{(1), (12), (123), (12)(34), (1234)\}$ is reperesentation element. Let χ, χ_0, χ_1 are character of $\varphi, \varphi_0, \varphi_1$, then $\chi = \chi_0 + \chi_1 = 1 + \chi_1$. So $\chi_1((1)) = \chi((1)) - 1 = 3, \chi_1((12)) = \chi((12)) - 1 = 1, \chi_1((123)) = \chi((123)) - 1 = 0, \chi_1((12)(34)) = \chi((12)(34)) - 1 = -1, \chi_1((1234)) = \chi((1234)) - 1 = -1$. $\square$

PROBLEM IV Find a 4-dim irriducible complex character of A_5.

SOLUTION. Consider $A_5 \times \Omega := \{1, 2, 3, 4, 5\} \to \Omega, \sigma x := \sigma(x)$, easily it's double transitive. Let φ is the reperesentation obtained by this group action, then $\varphi = \varphi_0 \oplus \varphi_1$, where φ_0 is main reperesentation and φ_1 is 4-dim irriducible. Let χ, χ_0, χ_1 are character of $\varphi, \varphi_0, \varphi_1$, then $\chi = \chi_0 + \chi_1 = 1 + \chi_1$. Easily A_5 has 5 conjugate classes and $\{(1), (123), (12)(34), (12345), (12543)\}$ is a group of reperesentation elements. Finally we get $\chi_1((1)) = 4, \chi_1((123)) = 1, \chi_1((12)(34)) = 0, \chi_1((12345)) = \chi_1((12543)) = -1$. $\square$