## under Graduate Homework In Mathematics

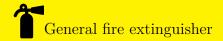
SetTheory 2

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1 QUESTION 2

## 1 Question

ROBEM I Let  $(U, \leq), (V, \prec)$  be two well-orderings. Consider  $f := \{(x, y) : x \in U \land y \in V \land (U_x, \leq) \}$  prove f is isomorphism from some initial segment of U to some initial segment of V.

ROBEM II The relation " $(P, \leq) \cong (Q, \leq)$ " is an equivalence relation (on the class of all partially ordered sets).

ROBEM III Let  $\mathcal{A}$  denote the class of all well orderings. For any  $a, b \in \mathcal{A}$ , define  $a \prec b \iff a$  is isomorphic to an initial segment of b. Show that  $\prec$  is a well ordering on  $\mathcal{A}/\cong$ , where  $\cong$  is the equivalence relation given in ROBEM II.

## BOBEM IV

- 1. If (W, <) is a well ordering and  $U \subset W$ , then  $(U, < \cap (U \times U))$  is a well ordering.
- 2. If  $(W_1, <_1)$  and  $(W_2, <_2)$  are two well orderings and  $W_1 \cap W_2 = \emptyset$ , then  $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$  is a well ordering, where

$$\prec = <_1 \cup <_2 \cup \{(a,b) \mid a \in W_1 \land b \in W_2\}$$

3. If  $(W_1, <_1)$  and  $(W_2, <_2)$  are two well orderings, then  $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$  is a well ordering, where

$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \lor (b_1 = b_2 \land a_1 <_1 a_2)$$

**BOBEM** V Show that the following are equivalent:

- 1. T is transitive;
- 2.  $\bigcup T \subseteq T$ ;
- 3.  $T \subseteq \mathscr{P}(T)$ .

ROBEM VI Let  $\alpha, \beta, \gamma \in \text{Ord}$  and let  $\alpha < \beta$ . Then

- a  $\alpha + \gamma < \beta + \gamma$ .
- b  $\alpha \cdot \gamma < \beta \cdot \gamma$ .
- c  $\alpha^{\gamma} \leq \beta^{\gamma}$ .

Given examples to show that < cannot be replaced by < in either inequality.

Example 1. a Let  $\alpha = 0, \beta = 1, \gamma = \omega$ , then  $\alpha < \beta$  but  $\alpha + \gamma = \omega = 1 + \omega = \beta + \gamma$ .

- b Let  $\alpha = 1, \beta = 2, \gamma = \omega$ , then  $\alpha \cdot \gamma = \omega = 2 \cdot \omega = \omega$ .
- c Let  $\alpha = 2, \beta = 3, \gamma = \omega$ , then  $\alpha^{\gamma} = \beta^{\gamma}$ .

**POBLEM** VII Show that the following rules do not hold for all  $\alpha, \beta, \gamma \in \text{Ord}$ :

- a If  $\alpha + \gamma = \beta + \gamma$  then  $\alpha = \beta$ .
- b If  $\gamma > 0$  and  $\alpha \cdot \gamma = \beta \cdot \gamma$  then  $\alpha = \beta$ .

1 QUESTION 3

c 
$$(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$$
.

**POBEM** VIII Find a set  $A \subset \mathbb{Q}$  such that  $(A, <_{\mathbb{Q}}) \cong (\alpha, \in)$ , where

a 
$$\alpha = \omega + 1$$
,

b 
$$\alpha = \omega \cdot 2$$
,

$$c \alpha = \omega \cdot \omega$$

$$d \alpha = \omega^{\omega}$$
,

e 
$$\alpha = \varepsilon_0$$
.

f  $\alpha$  is any ordinal  $< \omega_1$ .

ROBEM IX An ordinal  $\alpha$  is a limit ordinal iff  $\alpha = \omega \cdot \beta$  for some  $\beta \in \text{Ord}$ .

ROBEM X Find the first three  $\alpha > 0$  s.t.  $\xi + \alpha = \alpha$  for all  $\xi < \alpha$ .

 $\mathbb{R}^{OBEM}$  XI Find the least  $\xi$  such that

a 
$$\omega + \xi = \xi$$
.

b 
$$\omega \cdot \xi = \xi, \xi \neq 0$$
.

c 
$$\omega^{\xi} = \xi$$
.

(Hint for (1): Consider a sequence  $\langle \xi_n \rangle$  s.t.  $\xi_{n+1} = \omega + \xi_n$ .)

Lemma 1. If  $f: \operatorname{Ord} \to \operatorname{Ord}$  and  $a \leq b \to f(a) \leq f(b)$  and  $f(\sup B) = \sup f(B)$  for any B is subset of Ord, let  $a_0 = 0, a_{n+1} = f(a_n)$ , then  $\xi = \sup\{a_n : n \in \mathbb{N}\}$  is the least  $\xi$  such that  $f(\xi) = \xi$ .

证明. First we prove  $a_{n+1} \ge a_n$ . Use MI it's obvious.

Second we prove  $f(\xi) = \xi$ . Obviously  $f(\xi) = f(\sup\{a_n\}) = \sup\{f(a_n)\} = \sup\{a_{n+1}\} = \lim a_{n+1} = \lim a_n = \xi$ .

Finally we prove  $\xi$  is the least. Assume  $f(\alpha) = \alpha$ , then use MI we can easily prove  $\alpha \ge a_n \forall n < \omega$ . So  $\alpha \ge \sup\{a_n\} = \xi$ .