Graduate Homework In Mathematics

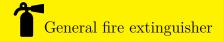
Functional Analysis 7

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ROBEM I Prove: There is no such inner product such that $\forall f \in C[a,b], (f,f)^{\frac{1}{2}} = \max_{x \in [a,b]} |f(x)|.$

SOUTION. Since $x_1 = \frac{x-a}{b-a}$, $x_2 = \frac{x-b}{a-b}$, $x_1, x_2 \in C[a, b]$, then, $x_1 + x_2 = \frac{x-a+b-x}{b-a} = 1$, $x_1 - x_2 = \frac{2x-a-b}{a-b}$. Let $||f|| = \max_{x \in [a,b]} |f(x)|$, then $(C[a,b], ||\cdot||)$ is B^* space. Since $||x_1 + x_2|| = 1$, $||x_1 - x_2|| = \max_{x \in [a,b]} |\frac{2x-a-b}{a-b}| = 1$, $||x_1|| = 1 = ||x_2||$, then $||x_1 + x_2||^2 + ||x_1 - x_2||^2 = 1^2 + 1^2 < 2(||x_1||^2 + ||x_2||^2) = 2 \times (1+1) = 4$. Then, there is no such inner product such that $\forall f \in C[a,b], (f,f)^{\frac{1}{2}} = \max_{x \in [a,b]} |f(x)|$.

ROBEM II $f: L^2[0,T] \to [0,+\infty), x \mapsto |\int_0^T e^{-(T-\tau)}x(\tau) d\tau|$. Prove: f can reach maximum in S^2 , calculate the maximum and find the element of x which reaches the maximum.

SOLITION. $\forall x \in S^2, f(x) = |\int_0^T e^{-(T-\tau)} x(\tau) d\tau| = e^T \int_0^T |e^\tau x(\tau)| d\tau \le e^T (\int_0^T e^{2\tau} d\tau)^{\frac{1}{2}} (\int_0^T x(\tau)^2 d\tau)^{\frac{1}{2}} = e^T (\int_0^T e^{2\tau} d\tau)^{\frac{1}{2}} = (\frac{1}{2} - \frac{1}{2}e^{-2T})^{\frac{1}{2}}.$ By the equation condition of Cauchy Schwartz inequation, when $x = te^u$, the inequation can be an equation. So let $x = \pm (\frac{2}{e^{2T}-1})^{\frac{1}{2}}e^u$, $f(x) = (\frac{1}{2} - \frac{1}{2}e^{-2T})^{\frac{1}{2}} = \max_{t \in S^2} f(t)$.

 \mathbb{R}^{OBEM} III $(X, (\cdot, \cdot))$ is an inner product space, $M \subset N \subset X$. Prove $N^{\perp} \subset M^{\perp}$.

SOLION. Since $\forall x \in N^{\perp} := \{x \in X : \forall y \in N, (x,y) = 0\}, \forall y \in M \subset N, y \in N, (x,y) = 0, \text{ then } x \in M^{\perp} := \{x \in X : \forall y \in M, (x,y) = 0\}. \text{ So } N^{\perp} \subset M^{\perp}.$