underGraduate Homework In Mathematics

GroupRepresentation 7

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General fire extinguisher

PROBLEM I R is a ring with identity element. If every non zero element in R is inversible, we call R is division ring. Prove: if D is a division ring, then $M_n(D)$ is a monocycle.

SOLUTION. I is non zero two sides ideal of $M_n(D)$. $\forall E_{ij}, 1 \leq i, j \leq n$, let $A = (a_{ij}) \neq 0$, $a_{st} \neq 0$. So $E_{is} A E_{tj} = a_{st} E_{ij} \in I$. So $E_{ij} \in I$. So $M_n(D) \subset I \subset M_n(D)$. $\square$

PROBLEM II V is right module of division ring D, $f : D \times V \to V$ is an action of D on V. Let $\forall v \in V, d \in D$, $vd := f(d, v)$, we call V is right linear space of division ring D. All of module homomerphism from V to V is noted as $\hom_D(V, V)$. Prove: $\dim_D V = n$, then $\exists g : \hom_D(V, V) \to M_n(D)$ is ring isomorphism.

SOLUTION. $a_1, \cdots, a_n \subset V$ is a set of V. $\forall A \in \hom_D(V, V)$, $Aa_i = \sum_{k=1}^n d_{ki} a_k, d_{ki} \in D, \forall 1 \leq k, i \leq n$. Let $B = (d_{ij}) \in M_n(D)$, $f : \hom_D(V, V) \to M_n(D), A \mapsto B$. Obviously f is well-defined and a bijection.

1. f preserves addition: $A_1, A_2 \in \hom_D(V, V), (A_1 + A_2)(a_i) = A_1 a_i + A_2 a_i = \sum_{k=1}^n d_{ki}^{(1)} a_k + \sum_{k=1}^n d_{ki}^{(2)} a_k = \sum_{k=1}^n (d_{ki}^{(1)} + d_{ki}^{(2)}) a_k$. So $f(A_1 + A_2) = f(A_1) + f(A_2)$.
2. f preserves multiplication: $A, B \in \hom_D(V, V), (AB)(a_i) = A(\sum_{k=1}^n d_{ki}^{(2)} a_k) = \sum_{k=1}^n d_{ki}^{(2)} A a_k = \sum_{k=1}^n d_{ki}^{(2)} \sum_{j=1}^n d_{jk}^{(1)} a_j = \sum_{j=1}^n \sum_{k=1}^n d_{jk}^{(1)} d_{ki}^{(2)} a_j$. So $f(AB) = f(A)f(B)$.

$\square$