ALGEBRAIC GEOMETRY

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PROBLEM I Let R be a Abel ring, $\mathfrak{a}$ is an ideal of R, and $\sqrt{\mathfrak{a}} := \{x \in R : \exists n \in \mathbb{N}, x^n \in \mathfrak{a}\}$. Prove that:

1. $\sqrt{\mathfrak{a}}$ is ideal.
2. $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.
3. $\sqrt{\mathfrak{a}}$ is the smallest radical ideal contain $\mathfrak{a}$.
4. If $\mathfrak{p}$ is prime ideal, then $\mathfrak{p}$ is radical.
5. $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$, where $\mathcal{P}$ is the set of all prime ideal contains $\mathfrak{a}$.

PROBLEM II An algebraically field is not finite field. PROBLEM III Let $A = K[x_1, x_2, \cdots x_n]$, and $m_p = (x_1 - a_1, \cdots x_n - a_n), p = (a_1, a_2, \cdots a_n) \in \mathbb{A}^n_K$. Then m is max ideal.

Lemma 1. If $f(x_1, x_2, \cdots x_n) \in K[x_1, x_2, \cdots x_n], f(a_1, a_2, \cdots a_n) = 0$, then $f = \sum_{k=1}^{n} (x_k - a_k) f_k(x_1, x_2, \cdots x_n)$.

证明. Use MI to n. When $n = 1$ it's obvious. If for some certain n it's right, when goes to $n+1$: Let $g(x_1, x_2, \cdots x_n) := f(x_1, x_2, \cdots x_n, a_{n+1}) \in K[x_1, x_2, \cdots x_n]$. Then $g(a_1, a_2, \cdots a_n) = 0$, so $g(x_1, x_2, \cdots x_n) = \sum_{k=1}^{n} (x_k - a_k) g_i(x_1, x_2, \cdots x_n)$. Let $h(x_{n+1}) := f(x_1, x_2, \cdots x_{n+1}) - g(x_1, x_2, \cdots x_n) \in K[x_1, x_2, \cdots x_n][x_{n+1}]$, then $h(a_{n+1}) = 0$. So $h(x_{n+1}) = (x_{n+1} - a_{n+1}) h_1(x_{n+1})$ for some $h_1(x_{n+1}) \in K[x_1, x_2, \cdots x_n][x_{n+1}]$. Then $f(x_1, x_2, \cdots x_{n+1}) = \sum_{k=1}^{n+1} (x_i - a_i) f_i(x_1, x_2, \cdots x_{n+1})$, where $f_k(x_1, x_2, \cdots x_{n+1}) = g_k(x_1, x_2, \cdots x_n), k = 1, 2, \cdots n$, and $f_{n+1}(x_1, x_2, \cdots x_{n+1}) = h_1(x_{n+1})$. □

PROBLEM IV $A \subset B \subset C$ are Abel rings. If B is f.g. $A-$module and C is f.g. $B-$module, then C is f.g. $A-$module, too.

PROBLEM V If x is integral over A then $A[x]$ is f.g. $A-$module.

PROBLEM VI Let R be an integral domain, finitely generated over a field k. If R has transcendence degree n over k, then there exist elements $x_1, \ldots, x_n \in R$, algebraically independent over k, such that R is integrally dependent on the subring $k[x_1, \ldots, x_n]$ generated by the x 's.