underGraduate Homework In Mathematics

FunctionalAnalysis 6

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PROBLEM I $\mathscr{X}$ is B^* space. Prove: $\mathscr{X}$ is B space $\iff$ $\forall \{x_n\}_{n=1}^{\infty} \subset \mathscr{X}$, $\sum_{n=1}^{\infty} ||x_n|| < \infty \to \sum_{n=1}^{\infty} x_n$ exists in $\mathscr{X}$.

SOLUTION .

1. "$\Rightarrow$": Let $\{x_n\}_{n=1}^{\infty} \subset \mathscr{X}$, $\sum_{n=1}^{\infty} ||x_n|| < \infty$, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $\forall k \in \mathbb{N}_+$, $|| \sum_{i=1}^{n+k} x_i - \sum_{i=1}^{n} x_i|| \leq \sum_{i=1}^{k} ||x_{n+i}|| < \varepsilon$. So $\{\sum_{i=1}^{n} x_i\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $\mathscr{X}$ is B space, then $\exists x \in \mathscr{X}$ s.t. $\lim_{n\to\infty} \sum_{i=1}^{n} x_i = x \in \mathscr{X}$.
2. "$\Leftarrow$": Let $\{x_n\}_{n=1}^{\infty} \subset \mathscr{X}$ is a Cauchy seqence. We only need to prove that exist $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ that converges. Let $k \in \mathbb{N}_+$, assuming $N_i, n_i, i = 1, \cdots, k-1$ have defined, we'll define N_k, n_k. Since $\exists N_k \geq \max_{i=1,\cdots,k-1} N_i$, $\forall n, m \geq N_k$, $||x_m - x_n|| < \frac{1}{2^k}$, let $n_k = N_k + 1$. Obviously, $n_k > n_i, i < k$, $\forall k \in \mathbb{N}_+$, $||x_{n_{k+1}} - x_{n_k}|| < \frac{1}{2^k}$. So $\sum_{i=1}^{\infty} ||x_{n_{i+1}} - x_{n_i}|| < \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty$, so $\lim_{k\to\infty} x_{n_k} = \sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k}) + x_{n_1} \in \mathscr{X}$. Thus, $x = \lim_{n\to\infty} x_n \in \mathscr{X}$.

$\square$

PROBLEM II $[a, b] \subset \mathbb{R}$, let $\mathbb{P}_n := \{f \in \mathbb{R}^{[a,b]} : \exists g \in \mathbb{R}[x], \deg g \leq n, \forall t \in [a, b], f(t) = g(t)\}$. Prove: $\forall f \in C[a, b], \exists P_0 \in \mathbb{P}_n$ s.t.

$$\max_{a\leq x\leq b} |f(x) - P_0(x)| = \min_{P\in\mathbb{P}_n} \max_{a\leq x\leq b} |f(x) - P(x)|. \tag{1}$$

SOLUTION . Since $(C[a, b], \|\cdot\|)$ is B space, where $\|f\| = \max_{t\in[a,b]} |f(t)|, \forall f \in C[a, b]$, and $\dim_{\mathbb{R}} \mathbb{P}_0 = n + 1$, so by optimal approximation theorem $\forall f \in C[a, b]$, $\exists P_0 \in \mathbb{P}_n$ s.t.

$$\max_{a\leq x\leq b} |f(x) - P_0(x)| = \min_{P\in\mathbb{P}_n} \max_{a\leq x\leq b} |f(x) - P(x)| \tag{2}$$

. $\square$

PROBLEM III $\mathscr{X}$ is B^* space, $\mathscr{X}_0 \subset \mathscr{X}$ is a subspace. Assume $\exists c \in (0, 1)$, s.t.

$$\inf_{x\in\mathscr{X}_0} ||y - x|| \leq c||y|| \quad (\forall y \in \mathscr{X}). \tag{3}$$

Proof: $\mathscr{X}_0$ is dense in $\mathscr{X}$.

SOLUTION . Since $\forall y : \|y\| = 1$, $\rho(y, \mathscr{X}_0) := \inf_{x\in\mathscr{X}_0} \|y - x\| \leq c\,\|y\| = c$, and $\mathscr{X}_0 \subset \overline{\mathscr{X}_0}$, so $\rho(y, \overline{\mathscr{X}_0}) := \inf_{x\in\overline{\mathscr{X}_0}} \|y - x\| \leq \inf_{x\in\mathscr{X}_0} \|y - x\| = \rho(y, \mathscr{X}_0) \leq c$. If $\overline{\mathscr{X}_0} \subsetneq \mathscr{X}$, By Riesz theorem, $\forall \varepsilon > 0, \forall y \in \mathscr{X} \setminus \overline{\mathscr{X}_0} : \|y\| = 1$, $\exists x \in \mathscr{X}_0$, s.t. $\|y - x\| > 1 - \varepsilon$, let $\varepsilon = \frac{1-c}{2}$, then $\exists x \in \mathscr{X}_0$, s.t. $\|y - x\| > 1 - \varepsilon = 1 - \frac{1-c}{2} > 1 - (1 - c) = c$, contradiction! $\square$