underGraduate Homework In Mathematics

SetTheory 5

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 12 月 21 日

General fire extinguisher

PROBLEM I Prove: $F \subset \mathcal{N}$ is closed set $\iff F = [T]$ for some $T \subset^{<\omega} \omega$.

SOLUTION.
- $\implies$: Let $T := T_F$, by the definition of T_F and $[T]$, we get $F \subset [T]$. For $f \in [T]$, $f \restriction n \in T$, so $\forall n \in \mathbb{N}, f \restriction n = g \restriction n$, $\exists g \in F$. So $d(f,F) \le d(f,g) = \frac{1}{2^n} \to 0, n \to \infty$. Since F is closed, then $f \in F$.
- $\impliedby$: For any $[T] \in^{<\omega} \omega$, only need to prove $[T]$ is closed. Assume $f \in \overline{[T]}$, then $\forall n \in \mathbb{N}, \exists g \in [T], f \restriction n = g \restriction n$. Since $g \in [T]$, then $g \restriction n \in T$. So $f \in [T]$. So $[T]$ is closed.

$\square$

PROBLEM II Assume f is isolated point in closed set $F \subset \mathcal{N}$, then $\exists n \in \mathbb{N}, \forall g \in F, g \neq f \to g \restriction n \neq f \restriction n$.

SOLUTION. Since f is isolated, we get $\exists n \in \mathbb{N}, \forall g \in F \setminus \{f\}, d(f,g) > \frac{1}{2^n}$. Then $f \restriction n \neq g \restriction n$. $\square$

PROBLEM III A closed set $F \subset \mathcal{N}$ is perfect $\iff T_F$ is a perfect tree.

SOLUTION.
- $\implies$: For $t \in T_F$, $\exists f \in F, n \in \mathbb{N}, t = f \restriction n$. Since F is perfect, then F is not isolated, by PROBLEM II $\forall n, \exists g \in F, g \neq f$ such that $d(f,g) < \frac{1}{2^{n+1}}$. Then $t = f \restriction n \sqsubset g$. Since $f \neq g$, Then, $\exists m \in \mathbb{N}, m > n$ such that $f \restriction m \neq g \restriction m$. So $t \sqsubset f \restriction m, t \sqsubset g \restriction m$, and $f \restriction m, g \restriction m$ are incomparable. So T_F is perfect.
- $\impliedby$: For $f \in F$, only need to prove f is not isolated. Since T_F is perfect, then $\forall t := f \restriction n \in T_F$, where $f \in F, n \in \mathbb{N}$. $\exists s_1, s_2 \in T_F$ such that $t \sqsubset s_1, s_2$ and s_1, s_2 are incomparable. Then $s_1, s_2 \sqsubset f$ is impossible. Without loss of generality assume $s_1 \not\sqsubset f$. so $s_1 = g \restriction m$ for some $g \in F, m \in \mathbb{N}$. So $d(f,g) \le \frac{1}{2^{n+1}}$. So f is not isolated.

$\square$

PROBLEM IV For $\alpha < \omega_1$, we let $\Sigma_0 = \{O \subset \mathbb{R} : O \text{ is open }\}$, and $\Pi_0 = = \{F \subset \mathbb{R} : F \text{ is closed }\}$. And $\Sigma_{\alpha+1} = \{\bigcup_{n\in\mathbb{N}} A(n) : A \in^{\mathbb{N}} \Pi_\alpha\}$. $\Pi_{\alpha+1} = \{\mathbb{R} \setminus A : A \in \Sigma_\alpha\}$. $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta, \Pi_\alpha = \bigcup_{\beta<\alpha} \Pi_\beta$ for limit ordinal α. Prove that $\mathcal{B}(\mathbb{R}) = \bigcup_{\alpha<\omega_1} \Sigma_\alpha$.

SOLUTION.
1. $\bigcup_{\alpha<\omega_1} \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$: Since $\cup\Sigma_0 \subset \mathcal{B}(\mathbb{R})$, if $\alpha < \omega_1$, such that $\bigcup_{\beta<\alpha} \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$. Next to prove $\bigcup_{\beta<\alpha+1} \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$, that is to prove $\bigcup \Sigma_\alpha \subset \mathcal{B}(\mathbb{R})$. Since α can be a succssor ordinal or limit ordinal, by induction assumption, in the first case, $\Pi_\alpha = \{\mathbb{R} \setminus A : A \in \Sigma_{\alpha-1}\}$, $\Sigma_{\alpha-1} \subset \mathcal{B}(\mathbb{R})$, so $\Pi_\alpha \subset \mathcal{B}(\mathbb{R})$. Therefore, $\cup\Sigma_{\alpha+1} = \bigcup_{n\in\mathbb{N}} A(n) \subset \mathbb{R}$, where $A \in \{^{\mathbb{N}}\Pi_\alpha\}$. In the second case, $\Pi_\alpha = \bigcup_{\beta<\alpha} \Pi_\beta$, $\Pi_\beta \subset \mathcal{B}(\mathbb{R})$. so $\Pi_\alpha \subset \mathcal{B}(\mathbb{R})$. Therefore, $\cup\Sigma_{\alpha+1} = \bigcup_{n\in\mathbb{N}} A(n) \subset \mathbb{R}$,
2. $\mathcal{B}(\mathbb{R}) \subset \bigcup_{\alpha<\omega_1} \Sigma_\alpha =: \mathcal{A}$: Since $\mathcal{B}(\mathbb{R})$ is $\sigma-$ algebra and $\Sigma_0 \subset \mathcal{A}$, then only need to prove $\mathcal{A}$ is a $\sigma-$ algebra. $\forall A \in \Sigma_\alpha$, then $\mathbb{R} \setminus A \in \Pi_{\alpha+1}$, then $\mathbb{R} \setminus A \in \Sigma_{\alpha+1}$, then $A \in \Pi_{\alpha+1}$, therefore, $A \in \Sigma_{\alpha+2}$. Obviously $\mathbb{R} \in \mathcal{A}$. For $A \in \mathcal{A}$, assume $A \in \Sigma_\alpha$. Then $\mathbb{R} \setminus A \in \Pi_{\alpha+1} \subset \Sigma_{\alpha+2} \subset \mathcal{A}$. Assume $A \in^{\mathbb{N}} \mathcal{A}$, let $f \in^{\mathbb{N}} \omega_1, f(n) = \min\{\alpha \in \omega_1 : A(n) \in \Sigma_\alpha\}$. Consider $\sup \operatorname{ran} f =: \gamma$. Since $\forall \alpha \in \operatorname{ran} f, \alpha$ is countable. And $\operatorname{ran} f$ is countable, so $\sup \operatorname{ran} f$ is countable, thus $\sup \operatorname{ran} f < \omega_1$. Then $\operatorname{ran} \mathcal{A} \subset \Pi_{\gamma+2}$. So we get $\bigcup_{n\in\mathbb{N}} A(n) \subset \Sigma_{\gamma+2} \subset \mathcal{A}$. So we get $\mathcal{A}$ is σ-field. So $\mathcal{B}(\mathbb{R}) \subset \mathcal{A}$, thus $\mathcal{A} = \mathcal{B}(\mathbb{R})$.

$\square$

PROBLEM V Show that $\mathcal{M} := \{A \subset \mathbb{R} : A \text{ is measurable}\}$ is a σ-field.

Lemma 1. For $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$, $|\mathcal{A}| = alpha_0$, then $\mu^*(\bigcup_{A\in\mathcal{A}} A) \le \sum_{A\in\mathcal{A}} \mu^*(A)$.

证明. Since $|\mathcal{A}| = \alpha_0$, let $\mathcal{A} = \{A_1, A_2, \cdots, A_n, \cdots\}$. $\forall n \in \mathbb{N}$, $\varepsilon > 0, \exists O_n \in \mathcal{O}, A_n \subset O_n$ and $\mu^*(A_n) \le |O_n| + \frac{\varepsilon}{2^{n+1}}$. Let $U := \bigcup_{n\in\mathbb{N}} O_n$, then $\bigcup_{n\in\mathbb{N}} A_n \subset U$. So $\mu^*(\bigcup_{n\in\mathbb{N}} A_n \le |U| \le \sum_{n\in\mathbb{N}} |O_n| \le \sum_{n\in\mathbb{N}} \mu^*(A_n) + \varepsilon$. Since ε is arbitry, then $\mu^*(\bigcup_{n\in\mathbb{N}} A_n = \sum_{n\in\mathbb{N}} \mu^*(A_n)$. □

Lemma 2. If $G \in G_\delta$, then $\forall\varepsilon > 0, \exists O \in \mathcal{O}, G \subset O \wedge \mu^*(O \setminus G) \le \varepsilon$.

证明. 1. G is bonded: Assume $G \subset [-M, M], M > 0$, and $G = \bigcap_{n\in\mathbb{N}} O_n$, where $O_n \in \mathcal{O}$. Since $G = \bigcap_{n\in\mathbb{N}} \bigcap_{k=0}^{m} O_m$, then without loss of generality, we can assume $O_n \supset O_{n+1}, n \in \mathbb{N}$. Besides, since $G = \bigcap_{n\in\mathbb{N}}(O_n \cap (-M-1, M+1))$. So, we can assume $O_n \subset (-M-1, M+1)$. So $|O_n|$ is declining and bounded. Thus, $\lim_{n\to\infty} |O_n| = a$. Therefore, if $m_k, 0 \le k < n$ have define, let we define m_n, $\forall\varepsilon > 0, \exists N, \forall l, m \ge N$, $|O_l| - |O_m| < \frac{\varepsilon}{2^{n-1}}$. Let $m_n = N$, then $\{O_{m_n}\}_{n=0}^{\infty} \subset \{O_n\}_{n=0}^{\infty}$ is a sub sequence, and $\lim_{n\to\infty} |O_{m_n}| = a$, $G = \bigcap_{n\in\mathbb{N}} O_{m_n}$, $|O_{m_n}| - |O_{m_{n+1}}| < \frac{\varepsilon}{2^{n-1}}$. Thus, we can assume $\{O_n\}_{n=0}^{\infty}$ such that $\forall n, |O_n| - |O_{n+1}| < \frac{\varepsilon}{2^n}$ By Lemma 1, so

2. G is not bounded: Let $G_n = G \cap B(0, n)$, then $G = \bigcup_{n\in\mathbb{N}} G_n$. So $\forall\varepsilon > 0$, $\exists O_n \supset G_n$ such that $\mu^*(O_n \setminus G_n) \le \frac{\varepsilon}{2^n}$. Then $O = \bigcup_{n\in\mathbb{N}} O_n \in \mathcal{O}$, $O \setminus G \subset \bigcup_{n\in\mathbb{N}} O_n \setminus G_n$, so by Lemma 1, $\mu^*(O \setminus G) \le \sum_{n\in\mathbb{N}} \frac{\varepsilon}{2^n} < \varepsilon$.

□

SOLUTION. 1. Easily, $\mathbb{R}$ is open and closed, then $\mathbb{R}$ is F_σ and G_δ, then $\mathbb{R} \in \mathcal{A}$.

2. If $A \in \mathcal{M}$, let $B := \mathbb{R} \setminus A$. Then $\exists F \in F_\sigma, G \in G_\delta$ such that $F \subset A \subset G$ and $\mu^*(G \setminus F) = 0$. Then $G^c \subset B \subset F^c$. Obviously, $G^c \in F_\sigma$, $F^c \in G_\delta$. And $\mu^*(F^c \setminus G^c) = \mu^*(G \setminus F) = 0$. So $B \in \mathcal{M}$.

3. Let $A(n) \in \mathcal{M}$, we need to prove $\bigcup_{n\in\mathbb{N}} A_n =: A \in \mathcal{M}$. By AC, $\exists F \in^{\mathbb{N}} F_\sigma, G \in^{\mathbb{N}} G_\delta$ such that $F(n) \subset A_n \subset G(n), \mu^*(G(n) - F(n)) = 0$. Let $T = \bigcup_{n\in\mathbb{N}} F(n)$. Since $F(n)$ is F_σ, we get $T \in F_\sigma$. And easily $T = \bigcup_{n\in\mathbb{N}} F(n) \subset \bigcup_{n\in\mathbb{N}} A(n) = A$. By AC and Lemma 2, $\forall n \in \mathbb{N}$, $\exists\mathcal{O} \ni O(n,m) \supset G(n), \mu(O(n,m) \setminus G(n)) \le \frac{1}{m2^n}, m \in \mathbb{N}_+$. So $\bigcap_{m\in\mathbb{N}_+} \bigcup_{n\in\mathbb{N}} O(n,m) := G \in G_\delta$, $G(n) \subset G$, $\mu(G \setminus A) \le \mu(G \setminus \bigcup_{n\in\mathbb{N}} G(n)) \le \mu(\bigcup_{n\in\mathbb{N}} O(n,m) \setminus \bigcup_{n\in\mathbb{N}} G(n)) \le \mu(\bigcup_{n\in\mathbb{N}}(O(n,m) \setminus G(n))) \le \sum_{n\in\mathbb{N}} \mu(O(n,m) \setminus G(n)) = \frac{1}{m} \to 0, m \to \infty$. So $\mu(G \setminus F) \le \mu(G \setminus A) + \mu(A \setminus F) = 0$. Therefore, A is measurable.

□

PROBLEM VI Show that $\mathcal{A} := \{A \subset \mathbb{R} : A \text{ has property of Baire}\}$ is σ-field.

SOLUTION. 1. Since $\mathbb{R}\Delta\mathbb{R} = \varnothing$ is meager, so $\mathbb{R} \in \mathcal{A}$.

2. If $A \in \mathcal{A}$, let $B := \mathbb{R} \setminus A \in \mathcal{A}$. So $\exists G \in \mathcal{O}$ such that $A\Delta G$ is meager, Let $U = \mathbb{R} \setminus \overline{G} \in \mathcal{O}$. And $B \setminus U = A \setminus \overline{G}$, so $(\overline{B \setminus U})^o = (\overline{A \setminus \overline{G}})^o \subset (\overline{A \setminus G})^o = \varnothing$, then $B \setminus U$ is meager. Since $U \setminus B = \overline{G} \setminus A = (\overline{G} \setminus G) \cup (G \setminus A)$, we only need to prove $\overline{G} \setminus G$ is meager. In fact, we can prove $\overline{G} \setminus G$ is nowhere dense. Since $\overline{G} \setminus G = \overline{G} \cap G^c$ is closed, $\forall a in \overline{G} \setminus G = \partial G \setminus G$, then

$\forall \varepsilon > 0$, $B(a,\varepsilon) \cap G \neq \varnothing$, so $\exists b \neq a, b \in B(a,\varepsilon) \cap G$. Since $(\overline{G} \setminus G)^c = G \cup \overline{G}^c$, so $b \notin \overline{G} \setminus G$. Thus, $a \notin (\overline{G} \setminus G)^o$. So $\overline{G} \setminus G$ is nowhere dense. Therfore, $B\Delta U$ is meager.

3. Let $A(n) \in \mathcal{A}, n \in \mathbb{N}$, we need to prove $\bigcup_{n\in\mathbb{N}} A(n) =: A \in \mathcal{A}$. Let $G(n) \in \mathcal{O}$ and $A(n)\Delta G(n)$ is meager. Consider $G := \bigcup_{n\in\mathbb{N}} G(n)$. Since $G \setminus A \subset \bigcup_{n\in\mathbb{N}} G(n) \setminus A(n)$ and $G(n) \setminus A(n)$ is meager, we get $G \setminus A$ is meager. For the same reason, we get $A \setminus G \subset \bigcup_{n\in\mathbb{N}} A(n) \setminus G(n)$ is meager.

   So $\mathcal{A}$ is σ-field. □

PROBLEM VII Assume $A \subset^\omega \omega$ has the property of Baire, prove A is nonmerger $\iff \exists O \in \mathcal{O}(^\omega\omega), O \neq \varnothing \wedge O \setminus A$ is meager.

SOLUTION. $\implies$: Since A has the property of Baire, then $\exists O \in \mathcal{O}, O\Delta A$ is meager. So $O \setminus A, A \setminus O$ are meager. Since A is nonmeager, $A \setminus O$ is meager, then $O \neq \varnothing$.

$\impliedby$: Assume $O \in \mathcal{O}, O \neq \varnothing, O \setminus A$ is meager. If A is meager, then $O \setminus A \cup A = \bigcup_{k\in\mathbb{N}} A_k$, where $\overline{A_k}^o = \varnothing$. And $\overline{O} \cap A_k \subset A_k$, so $\overline{O} \cap A_k$ is nowhere dense. Since $\overline{O} = \bigcup_{k\in\mathbb{N}} \overline{O} \cap A_k$ is nonmeager and meager at the same time. Contradiction! Therefore, A is nonmeager. □

PROBLEM VIII Let $C_A := \bigcup\{O_s : s \in^{<\omega} \omega, O_s \setminus A \text{ is meager}\}$. Prove that $C_A \setminus A$ is meager.

SOLUTION. Since $\mathbb{R}$ satisfies the second countable axiom, i.e., $\exists \mathcal{B} \subset \mathcal{O}(^\omega\omega)$ such that $\forall O \in \mathcal{O}, \forall x \in O, \exists B \in \mathcal{B}, x \in B \subset O$. And $\mathcal{B}$ is countable. i.e. $\mathcal{B}$ is countable topology basis of $\mathcal{O}(^\omega\omega)$. Consider $\mathcal{X} := \{X \in \mathcal{B} : \exists O_s, X \subset O_s \wedge O_s \setminus A \text{ is meager }\}$. Let $Y = \bigcup \mathcal{X}$, we will prove $C_A = Y$.

1. $x \in Y$, then $\exists X \in \mathcal{X}$ such that $x \in X$. So $\exists O_s$ such that $x \in X \subset O_s \wedge O_s \setminus A$ is meager. So $x \in C_A$.

2. $x \in C_A$, then $\exists O_s, x \in O_s, O_s \setminus A$ is meager. Since O_s is open, then $\exists B \in \mathcal{B}, x \in B \subset O_s$. So $B \in \mathcal{X}$. Thus $x \in Y$.

So we get $Y = C_A$. So $C_A \setminus A = Y \setminus A = \bigcup_{X\in\mathcal{X}} X \setminus A$. Since $\forall X \in \mathcal{X}$, then $X \setminus A$ is meager. Besides, $\mathcal{X} \subset \mathcal{B}$, so $\mathcal{X}$ is countable. Therfore, $C_A \setminus A = \bigcup_{X\in\mathcal{X}} X \setminus A$ is meager. □

PROBLEM IX Let $\pi :^\omega \omega \to^\omega 2, \pi(x) = s_{x(0)}{}^\frown s_{x(1)}{}^\frown \cdots$. Where $s_{x(k)} = 11\cdots10$ for even k, there is k "1" in total, and $s_{x(k)} = 00\cdots01$ for odd k, there is k "0" in total. Prove that $^\omega 2 \setminus \operatorname{ran} \pi$ is countable.

SOLUTION. Consider $g \in^\omega 2$ and $\forall N in \mathbb{N}, \exists n, m \in \mathbb{N}, n, m \geq N$ such that $g(n) = 1, g(n) = 0$. Next, prove $\exists h \in^\omega \omega, \pi(h) = g$. Let $h(0) := \min\{n \in \omega : g(n) = 0\}$. If $h \restriction n$ is defined. Let $M(n) = \sum_{k=0}^{n-1}(h(k) + 1)$. Let $h(n) = \min\{k - 1 : g(M(n) + k) = a_n\}$, where $a_n = 0$ for even n and $a_n = 1$ for odd n. By the definition of g, h is well-defined. Now we prove $\pi(h) = g$. For $k < h(0)$, by the defintion of $h(0)$, $g(k) = 1 = \pi(h)(k)$. For $k = h(0)$, then $g(k) = 0 = \pi(h)(k)$. $\forall k : \sum_{i=0}^{n}(h(i) + 1) < k \leq \sum_{i=0}^{n+1}(h(i) + 1)$. $\pi(h)(k) = s_{h(n)}(k - M(n))$. By the definition of h, if n is even, $s_{h(n)}(k - M(n)) = 1 = g(n), k \leq M(n) + h(n), s_{h(n)}(h(n) + 1) = 0 = g(k)$, Thus, $\pi(h)(k) = g(k)$. So $\pi(h) = g$. Since $\mathcal{A} = \{g \in^\omega 2 : \exists N, n > N, g(n) = g(N)\}$ is countable, $^\omega 2 \setminus \operatorname{ran} f \subset \mathcal{A}$, then $^\omega 2$ is countable. □

PROBLEM X Assume AD, then $\mathrm{AC}_\omega({}^\omega\omega)$. Consequently, ω_1 is regular.

SOLUTION. 1. Assume $X : \omega \to \mathcal{P}({}^\omega\omega)$ and $\forall n \in \omega, X(n) \neq \varnothing$. Let $\theta :^\omega \omega \to^\omega \omega, \theta(f)(n) := f(2n+1)$. Consider $G(A)$, where $A^c := \{x \in^\omega \omega : \theta(x) \in X(x(0))\}$. So I have no w.s since $\forall n \in \omega, X(n) \neq \varnothing$. By AD we get II has a w.s., write τ. Now consider $\gamma : \omega \to^\omega \omega, \gamma(n) := \theta((n, 0, 0, \cdots) * \tau)$. Since $\theta((n, 0, \cdots) * \tau) \in X(n)$. So γ is the choose function.

2. Now we prove ω_1 is regular. Only need to prove union countable many countable ordinal is countable. Assume $f : \omega \to \omega_1$, now we only need to prove $\bigcup \operatorname{ran} f \in \omega_1$. Consider $F : \operatorname{ran} f \to \mathcal{P}(\omega \times \omega)$, $F(\alpha) := \{R \subset \omega \times \omega : (\omega, R) \cong \alpha\}$. Since ${}^\omega\omega \approx \mathcal{P}(\omega \times \omega)$, we get $\mathrm{AC}_\omega(\mathcal{P}(\omega\times\omega))$. So $\exists \theta : \operatorname{ran} f \to \omega\times\omega, \theta(\alpha) \in F(\alpha), \forall \alpha \in \operatorname{ran} f$. Consider $G : \omega \to^\omega \omega_1, G(n)$ is the isomorphic from $(\omega, \theta(\omega))$ to $f(n)$. Let $h : \omega\times\omega \to \bigcup \operatorname{ran} f, h(n,m) := G(n)(m)$. Easily h is surjective. And since we have $\mathrm{AC}_\omega(\mathcal{P}(\omega \times \omega))$, we get $\bigcup \operatorname{ran} f \approx A$ for some $A \subset \omega \times \omega$. So we get $\bigcup \operatorname{ran} f$ is countable. So ω_1 is regular.

$\square$