underGraduate Homework In Mathematics

FunctionalAnalysis 11

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PROBLEM I $\mathcal{X}$ is complexed Hilbert space, $T \in \mathcal{L}(\mathcal{X})$. If $\exists a_0 > 0$ s.t. $(Tx, x) \geq a_0(x, x)$, we call T is positive definite. Prove: positive definite operator T must exist inversed operator T^{-1} and $\|T^{-1}\| \leq \frac{1}{a_0}$.

SOLUTION. 1. T is injection:

$$\begin{aligned} Tx = Ty &\iff T(x-y) = 0 \\ &\iff 0 = \langle T(x-y), x-y\rangle \leq a_0\langle x-y, x-y\rangle \geq 0 \end{aligned} \tag{1}$$

Thus, $\|x-y\| = 0$, $x = y$.

2. T is surjection:

   (a) First of all, we prove $T\mathcal{X}$ is closed: Let $W : \mathcal{X} \to T\mathcal{X}, x \mapsto Tx$, we easily get W is bijection, $T\mathcal{X} \subset \mathcal{X}$ is subspace of $\mathcal{X}$. So $T\mathcal{X}$ is B^* space, $W^{-1} : T\mathcal{X} \to \mathcal{X}, y \mapsto x$ where $Tx = y$. So W^{-1} is well-defined and W^{-1} is linear operator. $\|x\|\,\|W^{-1}x\| = \|TW^{-1}x\|\,\|W^{-1}x\| \geq \langle TW^{-1}x, W^{-1}x\rangle \geq a_0\,\|W^{-1}x\|^2$, so $\|W^{-1}x\| \leq \frac{1}{a_0}\|x\|$ and $W^{-1} \in \mathcal{L}(T\mathcal{X}, \mathcal{X})$. By theorem 2.3.13, there exists $\widetilde{W^{-1}} : \overline{T\mathcal{X}} \to \mathcal{X}$, where $\widetilde{W^{-1}}$ is extended of W^{-1} on $\overline{T\mathcal{X}}$ and $\left\|\widetilde{W^{-1}}\right\| = \|W^{-1}\|$. If $x \in \overline{T\mathcal{X}} \setminus T\mathcal{X}$, then $\exists\{x_n\}_{n=1}^{\infty} \subset T\mathcal{X}$ such that $\lim_{n\to\infty} x_n = x$. Then, $\lim_{n\to\infty} W^{-1}x_n = \lim_{n\to\infty} \widetilde{W^{-1}}x_n = \widetilde{W^{-1}}x$. So $\lim_{n\to\infty} T(W^{-1}x_n) = \lim_{n\to\infty} x_n = T(\lim_{n\to\infty} W^{-1}x_n) = T(\widetilde{W^{-1}}x) = x$. Thus, $x \in T\mathcal{X}$. Contradiction! Therefore, $T\mathcal{X} = \overline{T\mathcal{X}}$.

   (b) $\forall y \in \mathcal{X}$, $\exists | y_1, y_2 \in \mathcal{X}$ such that $y = y_1 + y_2$ where $y_1 \perp T\mathcal{X}, y_2 \in T\mathcal{X}$. So $0 = \langle Ty_1, y_1\rangle \geq a_0\langle y_1, y_1\rangle$, i.e. $y_1 = 0$. So $y = y_2 \in T\mathcal{X}$.

   Another way to prove $T\mathcal{X}$ is closed: $\forall\{x_n\}_{n=1}^{\infty} \subset T\mathcal{X}$ such that $\lim_{n\to\infty} x_n = x \in \mathcal{X}$. Without loss of generality, $\forall n, \|x_n - x\| \leq \frac{1}{2^{n+1}}$. Let $x_0 = 0$, $y_n = x_{n+1} - x_n \in T\mathcal{X}$, $z_n = Tx_n$, $n \geq 1$, so $\|y_n\| \leq \frac{1}{2^n}$, $\|z_n\|\,\|y_n\| \geq \langle y_n, z_n\rangle \geq a_0\langle z_n, z_n\rangle = a_0\,\|z_n\|^2$. So $\|z_n\| \leq \frac{\|y_n\|}{a_0}$. Thus $\sum_{n=1}^{\infty}\|z_n\| < \infty$, then $\exists z \in \mathcal{X}$ such that $\sum_{n=1}^{\infty} z_n = z \in \mathcal{X}$. Besides, $Tz = T\sum_{n=1}^{\infty} z_n = \lim_{n\to\infty}\sum_{k=1}^{n} Tz_n = \lim_{n\to\infty} x_{n+1} = x \in T\mathcal{X}$.

Thus, by inversed operator theorem, we get that T^{-1} exists and $\|T^{-1}\| = \|W^{-1}\| \leq \frac{1}{a_0}$. □

PROBLEM II Assume $\{a_k\}_{k=1}^{\infty}$ such that $\sup_{k\geq 1}|a_k| < \infty$. $T : l^1 \to l^1$, $x = \{\xi_k\}_{k=1}^{\infty} \in l^1, T(x) = \{a_k\xi_k\}_{k=1}^{\infty}$. Prove: $T^{-1} \in \mathcal{L}(l^1) \iff \inf_{k\geq 1}|a_k| > 0$.

SOLUTION. 1. "$\Leftarrow$:" Since $a := \inf_{k\geq 1}|a_k| > 0, b := \sup_{k\geq 1}|a_k| < \infty$, then $0 \neq a \leq |a_k| \leq b, \forall k \in \mathbb{N}^+$. So $x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty} \in l^1$.

   (a) T is injection: $Tx = Ty \iff a_n x_n = a_n y_n, n \in \mathbb{N}_+, \iff x_n = y_n, n \in \mathbb{N}$.
   (b) T is surjection: $z = \{z_n\}_{n=1}^{\infty}, z_n = \frac{x_n}{a_n}$, then $\sum_{n=1}^{\infty}|z_n| = \sum_{n=1}^{\infty}|\frac{x_n}{a_n}| \leq \frac{\sum_{n=1}^{\infty}|x_n|}{a} < \infty$.
   (c) T is bounded: $\|Tx\| = \sum_{k=1}^{\infty}|a_k x_k| \leq b\sum_{k=1}^{\infty}|x_k| = b\,\|x\|$.

   By inversed operator theorem, we get T^{-1} exists, and $T^{-1} \in \mathcal{L}(X)$.

2. "$\Rightarrow$:"

   (a) If $\exists a_n = 0$, without loss of generality, let $a_1 = 0$, then by Item 1a, we get T is not injection. So T^{-1} doesn't exists.

   (b) If $\forall a_n \neq 0, n \in \mathbb{N}_+, \inf_{k\geq 1} |a_k| = 0$, without loss of generality, $\lim_{n\to\infty} a_n = 0$. Consider $\{x_n\}_{n=1}^{\infty}, x_n = (1, \cdots, 1, 0, \cdots), \sum_{k=1}^{\infty} x_n(k) = n$, where $x_n(k)$ is the k- th number of x_n. Obviously, $\{x_n\}_{n=1}^{\infty} \subset l^1$. $(T^{-1}x_n)(k) = \frac{1}{a_k}\mathbb{1}_{1\leq k\leq n}(k)$. So $\|T^{-1}x_n\| = \sum_{k=1}^{n} |\frac{1}{a_k}| \to \infty, n \to \infty$. That is $\|T^{-1}\| = \infty$, which is contradict with $T^{-1} \in \mathcal{L}(\mathcal{X})$.

$\square$