underGraduate Homework In Mathematics

FunctionalAnalysis 5

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General fire extinguisher

PROBLEM I $(X,||\cdot||)$ is a B space, $\{x_n\}\subset X$, $g\in L(0,\infty)$ is decreasing and $g\geq 0$, $\forall n\in\mathbb{N}_+, ||x_n||\leq g(n)$. Proof: $\sum_{n=1}^{\infty}||x_n||$ converges.

SOLUTION. Since $g\in L(0,\infty)$ is decreasing and $g\geq 0$, so $\int_0^{\infty}g(t)\,\mathrm{d}t<\infty$, and $g(t)\geq\sum_{k=0}^{\infty}g(k+1)\mathbb{1}_{[k,k+1)}(t)$. Besides, $\forall n\in\mathbb{N}_+, ||x_n||\leq g(n)$. So $\sum_{n=1}^{\infty}||x_n||\leq\sum_{k=1}^{\infty}g(k)=\sum_{k=0}^{\infty}\int_0^{\infty}g(k+1)\mathbb{1}_{[k,k+1)}(t)\,\mathrm{d}t=\lim_{m\to\infty}\sum_{k=0}^{m}\int_0^{\infty}g(k+1)\mathbb{1}_{[k,k+1)}(t)\,\mathrm{d}t=\lim_{m\to\infty}\int_0^{\infty}\sum_{k=0}^{m}g(k+1)\mathbb{1}_{[k,k+1)}(t)\,\mathrm{d}t\overset{LCDT}{=}\int_0^{\infty}\sum_{k=0}^{\infty}g(k+1)\mathbb{1}_{[k,k+1)}(t)\,\mathrm{d}t\leq\int_0^{\infty}g(t)\,\mathrm{d}t<\infty$ □

PROBLEM II $(X_k,||\cdot||_k), k\in\mathbb{N}_+$ is a sequence of $B(B^*)$ spaces on field $\mathbb{K}$, let $X=\{\{x_k\}_{k=1}^{\infty}: x_k\in X_k, k\in\mathbb{N}_+, \sum_{k=1}^{\infty}||x_k||_k^p<\infty, p\geq 1\}$, $\forall x=\{x_k\}_{k=1}^{\infty}, y=\{y_k\}_{k=1}^{\infty}\in X$, $\forall k_1,k_2\in\mathbb{K}$, let $k_1x+k_2y=\{k_1x_k+k_2y_k\}_{k=1}^{\infty}$, $||x||=(\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}}(p\geq 1)$ Proof: $(X,||\cdot||)$ is a $B(B^*)$ space.

SOLUTION.

1. X is a linear space.
   (a) The operation of add and number multiplication in X is closed: $\forall x=\{x_k\}_{k=1}^{\infty}, y=\{y_k\}_{k=1}^{\infty}\in X$, $k_1x+k_2y=\{k_1x_k+k_2y_k\}_{k=1}^{\infty}$. $\forall p\geq 1, \forall 1\leq k<\infty$, since $(X_k,||\cdot||_k)$ is B space, $\forall k_1,k_2\in\mathbb{K}$, let $||k_1x_k+k_2y_k||_k^p\leq(|k_1|||x_k||_k+|k_2|||y_k||_k)^p$
   $\leq(2\max\{|k_1|,|k_2|\}\max\{||x_k||_k,||y_k||_k\})^p\leq(2\max\{|k_1|,|k_2|\})^p(||x_k||_k^p+||y_k||_k^p)$. Therefore,
$$\begin{aligned}&\sum_{k=1}^{\infty}||k_1x_k+k_2y_k||_k^p\\ \leq&\sum_{k=1}^{\infty}(2\max\{|k_1|,|k_2|\})^p(||x_k||_k^p+||y_k||_k^p)\\ =&(2\max\{|k_1|,|k_2|\})^p\sum_{k=1}^{\infty}(||x_k||_k^p+||y_k||_k^p)\\ =&(2\max\{|k_1|,|k_2|\})^p(\sum_{k=1}^{\infty}||x_k||_k^p+\sum_{k=1}^{\infty}||y_k||_k^p)<\infty\end{aligned}\tag{1}$$
   so $k_1x+k_2y\in X$.
   (b) Let $\theta=x_k, x_k=\theta_k$, which is the 0 element in X_k. So, it is trivial that X is a linear space on $\mathbb{K}$.
2. $(X,||\cdot||)$ is B^* space:
   (a) $\forall x\in X$, since $\forall k\in\mathbb{N}_+(x_k,||\cdot||_k)$ is B^* space, so $\forall k\in\mathbb{N}_+, ||x_k||_k\geq 0, ||x_k||_k=0\iff x_k=\theta_k$. So $||x||=(\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}}\geq 0$. So $||x||=(\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}}=0\iff$

$\sum_{k=1}^{\infty}||x_k||_k^p = 0 \iff ||x_k||_k^p = 0, \forall k \in \mathbb{N}_+ \iff ||x_k||_k = 0, \forall k \in \mathbb{N}_+ \iff x_k = 0, \forall k \in \mathbb{N}_+ \iff x = \theta$

(b) $\forall x \in X, \forall a \in \mathbb{K}, p \geq 1, ||ax|| = (\sum_{k=1}^{\infty}||ax_k||_k^p)^{\frac{1}{p}} = (\sum_{k=1}^{\infty}|a|^p||x_k||_k^p)^{\frac{1}{p}} = |a|(\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}} = |a|||x||$.

(c) $\forall x, y \in X$, $\forall k \in \mathbb{N}_+$, $||x_k + y_k||_k^p \leq (||x_k||_k + ||y_k||_k)^p$, so by Minkovski Inequation $(\sum_{k=1}^{\infty}||x_k + y_k||_k^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}(||x_k||_k + ||y_k||_k)^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}||y_k||_k^p)^{\frac{1}{p}}$.

3. $(X, ||\cdot||)$ is complete: $\forall\{x^{(m)} \in X : x^{(m)} = \{x_k^{(m)}\}_{k=1}^{\infty}, m \in \mathbb{N}_+\}$ s.t. $\lim_{m,n\to\infty}||x^{(m)} - x^{(n)}|| = 0$, so $\forall k \in \mathbb{N}_+, ||x_k^{(m)} - x_k^{(n)}||_k \leq (\sum_{k=1}^{\infty}||x_k^{(m)} - x_k^{(n)}||_k^p)^{\frac{1}{p}} \to 0$, as $m, n \to \infty$, by the completeness of $(X_k, ||\cdot||_k)$, $\{x_k^{(m)}\}_{m=1}^{\infty} \subset X_k$ is a Cauchy sequence in X_k. Let $x = \{x_k\}_{k=1}^{\infty}$, where $x_k = \lim_{n\to\infty} x_k^{(n)} \in X_k$, so $(\sum_{k=1}^{\infty}||x_k||_k^p)^{\frac{1}{p}} = (\sum_{k=1}^{\infty}||\lim_{m\to\infty} x_k^{(m)}||_k^p)^{\frac{1}{p}}$.

(a) $x \in X$: that is to proof $(\sum_{k=1}^{\infty}||\lim_{m\to\infty} x_k^{(m)}||_k^p)^{\frac{1}{p}} < \infty$. Since $\exists M$, $\forall i, j \geq M$, $(\sum_{k=1}^{\infty}||x_k^{(i)} - x_k^{(j)}||_k^p)^{\frac{1}{p}} \leq 1$, let $j = M$, then $\forall i > M$, $(\sum_{k=1}^{\infty}||x_k^{(i)}||_k^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}||x_k^{(i)} - x_k^{(M)}||_k^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}||x_k^{(M)}||_k^p)^{\frac{1}{p}} < (\sum_{k=1}^{\infty}||x_k^{(M)}||_k^p)^{\frac{1}{p}} + 1 < \infty$. So $\exists N, \forall i$, $(\sum_{k=1}^{\infty}||x_k^{(i)}||_k^p)^{\frac{1}{p}} < N$. So ,

$$
\begin{aligned}
N &\geq \lim_{m\to\infty}(\sum_{k=1}^{\infty}||x_k^{(m)}||_k^p)^{\frac{1}{p}} \\
&\overset{LCDT}{=}(\sum_{k=1}^{\infty}\lim_{m\to\infty}||x_k^{(m)}||_k^p)^{\frac{1}{p}} \\
&=(\sum_{k=1}^{\infty}||\lim_{m\to\infty} x_k^{(m)}||_k^p)^{\frac{1}{p}} = ||x||
\end{aligned}
\tag{2}
$$

(b) $||x^{(m)} - x|| \to 0, m \to \infty$: $\forall\varepsilon > 0$, $\exists M$, $\forall i, j \geq M$, $(\sum_{k=1}^{\infty}||x_k^{(i)} - x_k^{(j)}||_k^p)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}$. $\forall k$, $\exists M_k > M$, $\forall j_k > M_k$, $||x_k^{(j_k)} - x_k||_k \leq \frac{\varepsilon}{2^{k+1}}$, so $(\sum_{k=1}^{\infty}||x_k^{(j)} - x_k||_k^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}(||x_k^{(j_k)} - x_k||_k + ||x_k^{(j)} - x_k^{(j_k)}||_k)^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty}||x_k^{(j_k)} - x_k||_k^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty}||x_k^{(j)} - x_k^{(j_k)}||_k^p)^{\frac{1}{p}} \leq \sum_{k=1}^{\infty}\frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2} = \varepsilon$.

□