underGraduate Homework In Mathematics

SetTheory 3

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2023 年 11 月 2 日

General fire extinguisher

PROBLEM I Prove the following statements.

1. If $x \cap y = \varnothing$ and $x \cup y \preccurlyeq y$, then $\omega \times x \preccurlyeq y$.
2. If $x \cap y = \varnothing$ and $\omega \times x \preccurlyeq y$, then $x \cup y \approx y$.

SOLUTION. 1. Let $f : x \cup y \to y$ is injective, $f_1 := f, f_{n+1} := f_n \circ f$. $g : \omega \times x \to y, g(n,t) \mapsto f_{n+1}(t)$. Next we will prove g is injective. Since f is injective, then f_n is injective obviously $\forall n \in \mathbb{N}_+$. For $(n,u),(m,v) \in \omega \times x$:

   (a) If $n = m$, then $f_n(u) \neq f_n(v)$.

   (b) If $m \neq n$, W.L.O.G. let $n < m, m = n + k$. So $f_m[x] = f_{n+k}[x] = f_n[f_k[x]] \subset f_n[y]$. Since f_n is injective, we get $f_n[x] \cap f_n[y] = \varnothing$. While $g(n,u) \in f_n[x], g(m,v) \in f_n[y]$, so $g(n,u) \neq g(m,v)$, so g is injective.

2. Let $f : \omega \times x \to y$ is injective, $x_n := \{(n,t) : t \in x\}$. Then $\omega \times x = \cup_{n \in \omega} x_n$. Consider $g : x \cup y \to y$. If $t \in x$, then $g(t) := f(0,t)$. If $t \in f[x_n]$, then $g(t) = f(n+1,t)$. If $t \notin x \cup (\cup_{n=1}^{\infty} f[x_n])$, then $g(t) = t$. Next we will prove g is a bijection.

   (a) g is injection: For $u, v \in x \cup y, u \neq v$,

   - If $u, v \in x$, since f is injective, then $g(u) = f(0,u) \neq f(0,v) = g(v)$.
   - If $u \in x, v \in f[x_n]$,for some n, then $g(u) = f(0,u) \in f[x_0]$. $g(v) = f(n+1,v) \in f[x_{n+1}]$. Since f is injective, $f[x_0] \cap f[x_{n+1}] = \varnothing$, so $g(u) \neq g(v)$.
   - If $u \in x, v \notin x \cup (\cup_{n=1}^{\infty} f[x_n])$, then we know $g(v) = v \notin f[x_0] \ni g(u)$.
   - If $u \in f[x_m], v \in f[x_n]$, then

     i. If $m = n$, then $g(u) = f(m+1,u) \neq f(n+1,v) = g(v)$.

     ii. If $m \neq n$, then $g(u) \in f[x_{m+1}], g(v) \in f[x_{n+1}]$. Since f is injective, $f[x_{m+1}] \cap f[x_{n+1}] = \varnothing$. So $g(u) \neq g(v)$.

     iii. If $u \in x_n, v \notin x \cup (\cup_{n=1}^{\infty} f[x_n])$, then $g(u) \in f[x_{n+1}]$ and $g(v) = v \notin f[x_{n+1}]$.

     iv. If $u, v \notin x \cup (\cup_{n=1}^{\infty} f[x_n])$, then $g(u) = u \neq v = g(v)$.

   (b) g is surjective.

   - If $\exists n$ s.t. $u \in f[x_n]$, then:

     i. When $n = 0$, then$\exists t \in x$ s.t. $y = f(0,t)$. Then $g(t) = u$.

     ii. When $n \geq 1$, let $n = m + 1$. Then $\exists t \in x$ s.t. $y = f(m+1,t)$. So $g(t) = u$.

   - If $u \notin f[x_n], \forall n$, then $g(u) = u$.

$\square$

PROBLEM II

1. A subset of a finite set is finite.
2. The union of a finite set of finite sets is finite.
3. The power set of a finite set is finite.

4. The image of a finite set (under a mapping) is finite.

SOLUTION. 1. (a) When $n = 0$, $A \approx 0 \to A = \varnothing$. So $B \subset A$, then $B = \varnothing \approx 0$.

   (b) If n s.t. $\forall A \approx n, \forall B \subset A, \exists m \in \omega, B \approx m$ for $n \in \omega$.
   Now we prove $n+1$. Let $A \approx n+1$, $f : A \to n+1$ is bijection. If $B = A$, then $B \approx n+1$. Else, $\exists x \in A \setminus B$.
   Let $g : A \to n+1$, where $g(t) = f(t)$, if $t \neq x$ and $g(t) \neq n$; $g(t) = n+1$, if $t = x$; $g(t) = f(x)$, if $f(t) = n$. So g is bijection. And since $x \notin B$ we get $B \subset g^{-1}[n] \approx n$, so by induction we get $\exists m \in \omega, B \approx m$.

2. (a) A and B are finite and $A \cap B = \emptyset$:
   i. For $B = \varnothing$, $A \cup B = A$ is finite.
   ii. For $B \approx 1$, assume $A \approx n$, and $B \approx \{n\}$, so $A \cup B \approx n \cup \{n\} = n+1$ is finite.
   iii. For certain n s.t. $\forall B \approx n$, $A \cup B$ is finite. Then to prove it's right for $n+1$. Let $f : B \to n+1$ is bijection, then $f^{-1}[n] \approx n$, so by induction assumption $A \cup f^{-1}[n]$ is finite. Since $B = f^{-1}[n] \cup \{f^{-1}(n)\}$, so $A \cup B = A \cup f^{-1}[n] \cup \{f^{-1}(n)\}$. Since $\{f^{-1}(n)\} \approx 1$, so by induction assumption the union is finite.

   (b) $\forall A, B$ are two finite sets, so $A \cup B = A \cup (B \setminus A)$. By II.1, $B \setminus A$ is finite, so $A \cup B$ is finite.
   Now we use MI to prove $\forall n, A_i, i \leq n$ is Finite, then $\cup_{i=1}^{n}$ is Finite.
   i. When $n = 0, 1, 2$ it's obvious.
   ii. For certain $n \geq 2$ we have $A_i, i \leq n$ is Finite, then $\cup_{i=1}^{n} A_i$ is Finite. Then we prove $n+1$. Since $\cup_{i=1}^{n} A_i$ is Finite, and so do A_{n+1}, then $\cup_{i=1}^{n+1} A_i$.

3. (a) For $x \approx 0$, so $\mathscr{P}(x) = \{\varnothing\} \approx 1$.

   (b) For certain n s.t. $\forall x \approx n$, $\mathscr{P}(x)$ is Finite, then it goes to $x \approx n+1$: Assume $f : x \to n+1$ is bijection. Let $y = f^{-1}[n]$ and $t = f^{-1}(n)$. Then $y \approx n$. Let $\theta : \mathscr{P}(x) \setminus \mathscr{P}(y) \to \mathscr{P}(y), \theta(a) := a \setminus \{t\}$. Obviously θ is bijective, so $\mathscr{P}(x) \setminus \mathscr{P}(y) \approx \mathscr{P}(y)$ is finite. By II.2, $\mathscr{P}(x) = \mathscr{P}(y) \cup (\mathscr{P}(x) \setminus \mathscr{P}(y))$ is finite.

4. (a) For $A \approx 0$ it's obvious.

   (b) For $A \approx n$ it's right.It goes for $A \approx n+1$. Let $f : A \to n+1$ is a bijection, and $g : A \to \mathbb{S}et$ is a map on A. Let $B := f^{-1}[n] \subset A, t = f^{-1}(n) \in A$. Then $B \approx n$, so $g[B]$ is finite. Since $A = B \cup \{t\}$, then $g[A] = g[B] \cup g[\{t\}] = g[B] \cup \{g(t)\}$. And $\{g(t)\} \approx 1$ is finite, by II.2, $g[A]$ is finite.

□

PROBLEM III

1. A subset of a countable set is at most countable.
2. The union of a finite set of countable sets is countable.
3. The image of a countable set (under a mapping) is at most countable.

SOLUTION . 1. Let A is countable, so $\exists\theta$ s.t. $\theta : A \to \omega$ is bijection. Let $B \subset A$, so $B \approx \theta[B]$. So we only need to prove every subset of ω is at most countable. Let $x \subset \omega$. If x is finite, then x is at most countable. If x is infinite. Let $f(0) = \min x$ and $f(n) = \min(x \setminus f[n])$. Since x is infinite, so $f[n] \subsetneqq x$, so f is well-defined. And obviously, f is a bijection. So $x \approx \omega$ is countable.

2. That is to prove $\forall n \in \mathbb{N}_+, \{A_k\}_{k=1}^n$ is a sequence of countable sets, then $\cup_{k=1}^n A_n$ is countable.
   (a) When $n = 1$ it's obvious.
   (b) For $n = 2$, let $f : \omega \to A_1, g : \omega \to A_2$ are bijections, $h : \omega \to A_1 \cup A_2$, where $h(n) = f(\min f^{-1}[A_1 \setminus h[n]])$, if $2 \mid n$; $h(n) = g(\min g^{-1}[A_2 \setminus h[n]])$, if $2 \nmid n$. Since A_1, A_2 are infinite ,so h is well-defined.
      i. $\forall m, n \in \omega, m \neq n$, assume $m < n$, then $h(n) = f(\min f^{-1}[A_1 \setminus h[n]]) \in f[f^{-1}[A_1 \setminus h[n]]] = u \setminus h[n]$ and $h(m) \in h[n]$. So $h(m) \neq h(n)$.
      ii. For $n = 0$ it's obvious that $f[n] \subset h[2n-1]$. Assume for certain n $f[n] \subset h[2n-1]$ is right, when it is for $n+1$, we only need to prove $a := f(n) \in h[2n+1]$. If not, since $h(2n) = f(\min f^{-1}[A_1 \setminus h[2n]])$, $a \notin h[2n]$, so $a \in A_1 \setminus h[2n]$. Then $n = f^{-1}(a) \in f^{-1}[A_1 \setminus h[2n]]$. For $m < n$, $f(m) \in h[2m-1] \subset h[2n]$, so $m \notin f^{-1}[A_1 \setminus h[2n]]$, thus $n = \min f^{-1}[A_1 \setminus h[2n]]$. So $h(2n) = a$, contradiction! So, $A_1 \subset h[\omega]$, it is same to prove $A_2 \subset h[\omega]$.
   (c) Assume for certain $n \geq 2$, $\cup_{k=1}^n A_k$ is countable. It goes to $n+1$: By induction we know $\cup_{k=1}^n A_k$ is countable. And we have proved union of two countable sets is countable. So $\cup_{k=1}^{n+1} A_k = \cup_{k=1}^n A_k \cup A_{n+1}$ is countable.

3. Same as the first quesrion. We only need to prove image of ω is at most countable. For $f : \omega \to \mathbb{S}et$ is a map, let $h : \operatorname{ran}(f) \to \omega, t \mapsto \min f^{-1}[\{t\}]$. Obviously h is a injective, so $\operatorname{ran}(f)$ is at most countable.

$\square$

PROBLEM IV $\mathbb{N} \times \mathbb{N}$ is countable.

$$[f(m,n) = 2^m(2n+1) - 1.]$$

SOLUTION . Let $f : \mathbb{N}^2 \to \mathbb{N}, (m,n) \mapsto 2^m(2n+1) - 1$

1. Let $f(a,b) = f(c,d)$, then $2^a(2b+1) = 2^c(2d+1)$. If $a \neq c$, WLOG, let $a < c$, then $2b+1 = 2^{c-a}(2d+1)$. While $2 \mid 2^{c-a}(2d+1), 2 \nmid 2b+1 = 2^{c-a}(2d+1)$, contradiction! So $a = c$, then $2b+1 = 2d+1$, so $b = d$.

2. $\forall t \in \mathbb{N}$, let $s := \sup\{k : 2^k \mid t+1\}$. Since $0 < t+1 < \omega$, then if $2^k \mid t+1$, then $2^k \leq t+1$, so $s < \omega$. Assume $t+1 = m2^s \cdot$, so $2 \nmid m$, so $m = 2n+1$. Then $t = f(m,n)$.

$\square$

PROBLEM V Prove that $\kappa^\kappa \leq 2^{\kappa\kappa\kappa}$.

SOLUTION. Let $h :{}^{\kappa} \kappa \to {}^{\kappa\times\kappa} 2$. $\forall f \in{}^{\kappa} \kappa$, let $h(f) : \kappa \times \kappa \to 2$, where $\forall u, v \in \kappa$, $h(f)(u, v) := 1$ if $u = f(v)$; $h(f)(u, v) := 0$, if $u \neq f(v)$. Assume $f, g \in{}^{\kappa} \kappa$ and $h(f) = h(g)$. Then $\forall v \in \kappa$, $h(g)(f(v), v) = h(f)(f(v), v) = 1$, so $f(v) = g(v)$. So h is injective. $\square$

PROBLEM VI If $A \preccurlyeq B$, then $A \preccurlyeq^* B$.

SOLUTION.

1. If $A = \varnothing$, then $A \preccurlyeq^* B$ is obvious.
2. If $A \neq \varnothing$, then $a \in A$. Let $f : A \to B$ is injection, $g : B \to A, g(y) : f^{-1}(y)$, if $y \in \operatorname{ran}(f)$; a, if $y \notin \operatorname{ran}(f)$. Then $\forall x \in A, h(f(x)) = x$, obviously. So h is surjective.

$\square$

PROBLEM VII If $A \preccurlyeq^* B$, then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.[2]

SOLUTION.

1. If $A = \varnothing$, then $\mathscr{P}(A) = 1$. Let $f : \mathscr{P}(A) \to \mathscr{P}(B), 0 \mapsto B$, then f is injective.
2. If $A \neq \varnothing$, then by $A \preccurlyeq^* B$, $\exists f : B \to A$ is surjective. Let $h : \mathscr{P}(A) \to \mathscr{P}(B), U \mapsto f^{-1}[U]$. Then we will prove h is injective. Let $U, V \subset A$ and $h(U) = h(V)$, i.e. $f^{-1}[U] = f^{-1}[V]$. If $U \neq V$, WLOG, let $U \setminus V \neq \varnothing$ and let $x \in U \setminus V$. Since f is surjective, so $\exists t \in B, f(t) = x$. So $t \in f^{-1}[U]$ but $t \notin f^{-1}[V]$, contradiction! So h is injective. Then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.

$\square$

PROBLEM VIII Let X be a set. If there is an injective function $f : X \to X$ such that $\operatorname{ran}(f) \subsetneq X$, then X is infinite.

SOLUTION. That is to prove $\forall n \in \omega, X \not\approx n$. By MI,

1. For $n = 0$, if $X \approx n$, then $X = 0$. So $X \subset \operatorname{ran}(f)$, contradiction!
2. Assume $n \geq 1$, $\forall m < n, X \not\approx m$ is right. If $X \approx n$, then $\exists h : X \to n$ is bijection. So $h[\operatorname{ran}(f)] \subsetneq n$, then $\exists m < n, h[\operatorname{ran}(f)] \approx m$. While f is injective, and h is bijection, so $X \approx m$. Contradiction!

$\square$