FUNCTIONAL 4

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ProblemI. (X, d) is a distance space, $A \subset X$ is a self-sequence compact set. $\forall f \in C(A) := \{f \in \mathbb{R}^A : f \text{is continuous}\}$, $f(A) := \{f(x) : x \in A\}$. Proof: $f(A)$ is bounded and $\max f(A) = \sup f(A), \min f(A) = \inf f(A)$.

Solution. Since f is continuous and A is self-sequence-compact, then $f(A)$ is compact in $\mathbb{R}$, that means $f(A)$ is bounded and closed in $\mathbb{R}$. Let $m = \sup f(A)$, $b_n \in f(A), n \in \mathbb{N}_+$, s.t. $b_n \to m$. Let N_1 s.t. $\forall n \geq N_1$, $|b_n - m| < 1$, $a_1 \in f^{-1}(b_{N_1})$.Suppose a_{k-1} have be defined, let define a_k. $\exists N_k \geq \max\{N_1, \cdots, N_{k-1}\}$, s.t.$\forall n \geq N_k$, $|b_n - m| < \frac{1}{k}$, let $a_k \in f^{-1}(b_{N_k})$. Since A is self-sequence-compact, $\exists \{a_{n_i} : i \in \mathbb{N}_+\} \subset \{a_n : n \in \mathbb{N}_+\} \exists a \in A$ s.t. $a_{n_i} \to a, i \to \infty$. Therefore, $\forall \varepsilon > 0$, let $k > \frac{1}{\varepsilon}$, $\forall i > N_{n_k}$, $|f(a_{n_i}) - m| \leq \frac{1}{n_i} \leq \frac{1}{k} < \varepsilon$. So by the continuousness of f, $\lim_{i \to \infty} f(a_{n_i}) = f(\lim_{i \to \infty} a_{n_i}) = f(a) = m$, so $\max f(A) = \sup f(A) = f(a)$. It is the same for $\inf f(A) = \min f(A)$. □

ProblemII. (X, d) is a distance space, $M \subset X$ is a self-sequence compact set. $\forall f \in C(M) := \{f \in \mathbb{R}^M : f \text{ is continuous}\}$. Proof: f is continuous uniformly.

Solution. $\forall x \in M, \varepsilon > 0$, let $A := \{\delta > 0 : B(x, \delta) \subset \{y \in M : |f(x) - f(y)| < \frac{\varepsilon}{2}\}\}$. Since f is continuous, then $A \neq \emptyset$, besides, M is compact, then M is bounded, so A is bounded. Let $\delta_x := \frac{\sup A}{2}, \mathcal{U} := \{B(x, \delta_x) : x \in M\}$, by the compactness of M,

$\exists\{u_{x_1}\cdots u_{x_n}\}\subset\mathcal{U}$ s.t. $M\subset\cup_{i=1}^n u_i$. Let $\delta := \min_{1\le i\le n}\delta_{x_i}$. $\forall x\in M$, $\forall y:\mathrm{d}(x,y)<\delta$, $\exists i$ s.t. $x\in u_{x_i}$. W.L.O.G. $i=1$, $\mathrm{d}(y,x_1)\le \mathrm{d}(y,x)+\mathrm{d}(x,x_1)\le\delta+\delta_{x_1}\le 2\delta_{x_1}$, then $|f(y)-f(x)|\le|f(y)-f(x_1)|+|f(x_1)-f(x)|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. □

ProblemIII. $M\subset C[a,b]$, M is bounded, proof: $S=\{\int_a^x f(t)\mathrm{d}t|f\in M\}$ is a sequence compact.

Solution. Since $\forall f\in C[a,b]$, $\int_a^x f(t)\mathrm{d}t\in C[a,b]$, then $S\subset C[a,b]$. We only need to proof S is uniformly bounded and equicontinuity. Since M is bounded, then $\exists A$, $\forall f\in M$, $\max_{t\in[a,b]}|f(t)|\le A$.

- S is uniformly bounded: $\max_{t\in[a,b]}|\int_a^t f(x)\mathrm{d}x|\le\max_{t\in[a,b]}\int_a^t|f(x)|\mathrm{d}x\le\int_a^b|f(x)|\mathrm{d}x\le A(b-a)$.
- $\forall\varepsilon>0$, $0<\delta<\frac{\varepsilon}{2A}$, $\forall a\le x,y\le b:|x-y|<\delta$, $\forall f\in M$, $|\int_{[a,x]}f(t)\mathrm{d}t-\int_{[a,y]}f(t)\mathrm{d}t|\le\left(\int_{[a,x]}-\int_{[a,y]}\right)|f(t)|\mathrm{d}t\le A\delta<A\frac{\varepsilon}{2A}<\varepsilon$.

□

ProblemIV. $\varphi:\mathbb{R}^n\to\mathbb{R}$, $\forall x=(x_1,\cdots,x_n)\in\mathbb{R}^n,\varphi(x)=(\sum_{k=1}^n|x_k|^{1/2})^2$. Is $(\mathbb{R}^n,\varphi)$ a B^* space?

Solution. 1. $n=1$, then $\varphi(x)=|x|$. Obviously, φ is a norm on $\mathbb{R}$.

2. $\forall n\ge 2$, Not B^*. Let $x=(1,0,0,\cdots,0),y=(0,1,0,\cdots,0)$, $\varphi(x+y)=(\sum_{i=1}^n(x_k+y_k)^{\frac{1}{2}})^2=(1^{\frac{1}{2}}+1^{\frac{1}{2}})^2=4>(1^{\frac{1}{2}})^2+(1^{\frac{1}{2}})^2=2=\varphi(x)+\varphi(y)$

□

ProblemV. $||\cdot||:\mathbb{C}^\infty\to\mathbb{R}$, $\forall x=(x_1,\cdots,x_n,\cdots)$, $||x||=\sum_{n=1}^\infty 2^{-n}\min\{1,|x_n|\}$

1. Is $||\cdot||$ a norm on $\mathbb{C}^\infty$?
2. $\mathrm{d}:C^\infty\times C^\infty\to\mathbb{R}$, $\forall x,y\in C^\infty,\mathrm{d}(x,y)=||x-y||$. Whether d is the distance on $\mathbb{C}^\infty$. If so, explain the meaning of $||x^{(n)}-x||\to 0(n\to\infty)$.

Solution. 1. Not a norm: let $x = (1, 1, \cdots, 1, \cdots) \neq 0$, then $||2x|| = \sum_{n=1}^{\infty} 2^{-n} \min\{1, 2\} = \sum_{n=1}^{\infty} 2^{-n} = ||x|| \neq 2||x||$

2. d is a distance:

   (a) $\mathrm{d}(x, y) = ||x - y|| \geq 0$, trivial. $||x - y|| = 0 \Leftrightarrow \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\} = 0 \Leftrightarrow \forall n, \min\{1, |x_n - y_n|\} = 0 \Leftrightarrow \forall n |x_n - y_n| = 0 \Leftrightarrow x = y$.

   (b) $||x - y|| = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\} = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |y_n - x_n|\} = ||y - x||$.

   (c) $\forall x, y, z \in \mathbb{C}^{\infty}$, $\forall n \in \mathbb{N}_{+}, a_n = |x_n - z_n|$, $b_n = |z_n - y_n|$ Since $\min\{1, a_n\} + \min\{1, b_n\} = 2 \vee \min\{1, a_n\} + \min\{1, b_n\} = 1 + a_n \vee \min\{1, a_n\} + \min\{1, b_n\} = 1 + b_n \vee \min\{1, a_n\} + \min\{1, b_n\} = a_n + b_n$, and $\min\{1, a_n + b_n\} \leq 1 \leq 2, 1 + a_n, 1 + b_n \wedge \min\{1, a_n + b_n\} \leq a_n + b_n$, so $\min\{1, a_n + b_n\} \leq \min\{1, a_n\} + \min\{1, b_n\}$. Therefore, $\min\{1, |x_n - y_n|\} \leq \min\{1, |x_n - z_n|\} + \min\{1, |z_n - y_n|\}$. Then, $\mathrm{d}(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min\{1, |x_n - y_n|\} \leq \sum_{n=1}^{\infty} 2^{-n} (\min\{1, |x_n - z_n|\} + \min\{1, |z_n - y_n|\}) = \sum_{n=1}^{\infty} 2^{-n} (\min\{1, |x_n - z_n|\}) + \sum_{n=1}^{\infty} 2^{-n} (\min\{1, |z_n - y_n|\}) = \mathrm{d}(x, z) + \mathrm{d}(z, y)$.

3. Let $x^{(n)} = (x_1^{(n)}, \cdots, x_k^{(n)}, \cdots), x = (x_1, \cdots, x_k, \cdots)$, then $||x^{(n)} - x|| \to 0 \Leftrightarrow \forall k, |x_k^{(n)} - x_k| \to 0, n \to \infty$.

   (a) $\Rightarrow$: Since $||x^{(n)} - x|| \to 0$, then $\forall \varepsilon > 0, \exists N, \forall n > N, \forall i \in \mathbb{N}$, $\frac{\varepsilon}{2^i} > \sum_{m=1}^{\infty} 2^{-m} \min\{1, |x_m^{(n)} - x_m|\} \geq 2^{-m} \min\{1, |x_m^{(n)} - x_m|\} \forall m \geq 1$. So let $m = i$, then $2^{-i} \min\{1, |x_i^{(n)} - x_i|\} < \frac{\varepsilon}{2^i}$ Since $1 > \lim\limits_{\varepsilon \to 0} \varepsilon$, then $2^{-i} |x_i^{(n)} - x_i| < \frac{\varepsilon}{2^i}, \forall n > N$. Therefore, $|x_i^{(n)} - x_i| \to 0, n \to \infty$.

   (b) $\Leftarrow$: $\forall \varepsilon > 0, \exists N, \forall n \geq N$, $\sum_{m=n}^{\infty} 2^{-m} < \frac{\varepsilon}{2}$, $\exists M$, $\forall 1 < i \leq N - 1$, $\forall n > M$, $|x_i^{(n)} - x_i| < \frac{\varepsilon}{2 \sum_{k=1}^{N-1} 2^{-k}}$. So $n > \max\{N, M\}$, $\sum_{m=1}^{n} 2^{-m} \min\{1, |x_m^{(n)} - x_m|\} \leq \sum_{m=1}^{N-1} 2^{-m} \min\{1, |x_m^{(n)} - x_m|\} + \sum_{m=N}^{n} 2^{-m} \min\{1, |x_m^{(n)} - x_m|\} \leq \sum_{m=1}^{N-1} 2^{-m} |x_m^{(n)} - x_m| + \frac{\varepsilon}{2} \leq \sum_{m=1}^{N-1} 2^{-m} \frac{\varepsilon}{2 \sum_{k=1}^{N-1} 2^{-k}} + \frac{\varepsilon}{2} = \varepsilon$.