FUNCTIONAL 3

王胤雅
SID:201911010205
201911010205@mail.bnu.edu.cn

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Problem I. 设 F 是只有有限项不为零的实数列全体，在 F 上引入距离

$$\rho(x,y)=\sup_{k\geq 1}|\xi_k-\eta_k|,\quad \forall x=(\xi_k)_{k\geq 1},y=(\zeta_k)_{k\geq 1}\in F.$$

求证 (F,ρ) 不完备. 并求其完备化空间.

证明.

- (F,ρ) is not complete:
  $\{x^{(n)}\in F: x_k^{(n)}=\frac{1}{k}\mathbb{1}_{k\leq n}, n\in\mathbb{N}_+, k\in\mathbb{N}_+\}, x:=\{\frac{1}{n}\}_{n=1}^{\infty}$. Since $\forall x^{(n)},x^{(m)}\in F, n\leq m$, $\rho(x^{(n)},x^{(m)})=\sup_{k\geq 1}|x_k^{(n)}-x_k^{(m)}|=\frac{1}{n+1}\to 0$, when $n,m\to\infty$, then $\{x^{(n)}\}_{n=1}^{\infty}$ is a cauchy sequence on (F,ρ). However, $\rho(x^{(n)},x)=\frac{1}{n+1}\to 0, n\to\infty$, by the uniqueness of limit, $\lim\limits_{n\to\infty}x^{(n)}=x\notin F$.
- Consider (E,ρ) where $E:=\{\{x_n\}_{n=1}^{\infty}: x_n\in\mathbb{R}\}$, which is a complete metric space. It is obvious that $(E,\rho)\supset(F,\rho)$. We need to find the closer of F in E. $\forall\varepsilon>0, x\in F$, $B(x,\varepsilon):=\{y\in E:\rho(y,x)<\varepsilon\}$, let $m:=\sup\{k: x_k\neq 0\}$,$\forall y\in B(x,\varepsilon)$, $\rho(x,y)=\sup\{\max_{1\leq k\leq m}|x_k-y_k|,\sup_{k>m}|y_k|\}<\varepsilon$, then $\sup_{k>m}|y_k|<\varepsilon$, so $\lim\limits_{k\to\infty}y_k=0$, as $\varepsilon\to 0$.
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  - $H:=\{\{x_n\}_{n=1}^{\infty}\in E:\lim\limits_{k\to\infty}x_k=0\}$ is complete. We need to proof H is closed in E. $\forall x^{(n)}\in H, n\in\mathbb{N}_+$, and $x^{(n)}\to x:=\{x_n\}_{n=1}^{\infty}$, then $\forall\varepsilon>0$, $\exists N$, $\forall n\geq N$, $\sup_{k\geq 1}|x_k^{(n)}-x_k|<\varepsilon/3$, $\exists G$, $\forall k,j\geq G$, $|x_k^{(N)}-x_j^{(N)}|<\varepsilon/3$, $\exists M$, $\forall j\geq M$, satisfies $|x_j^{(N)}|\leq\varepsilon/3$. Therefore $L:=\max\{G,M\},\forall k,j\geq L$, $|x_k|\leq|x_k-x_k^{(N)}|+|x_k^{(N)}-x_j^{(N)}|+|x_j^{(N)}|<\varepsilon$. Thus, $x\in H$.
  - Consider $\varphi: F\to H$ which is an embedding map. $\forall x\in H$, then $y_k:=\{x_k\mathbb{1}_{k\leq n}\}\in F$. And $\rho(y_k,x)=\sup_{i\geq k+1}|x_i|\to 0$ means F is dense in H.

$\square$

Problem II. 设 (X,ρ) 完备, $\{F_n\}$ 是 X 内的单调下降非空闭集序列, 即

$$F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots, F_n \neq \varnothing$$

且 F_n 的直径 $d_n = d(F_n) \to 0$. 证明: $\bigcap_{n\geq 1} F_n \neq \varnothing$. 如果没有条件 $d_n \to 0$, 结论成立吗? 如果 X 为列紧空间时, 关于直径的条件是否还需要?

证明. 1. Let $x_1 \in F_1$, $x_{k+1} \in F_{k+1} \setminus F_k$ if $F_{k+1} \setminus F_k \neq \emptyset$, otherwise $x_{k+1} = x_k$, if $F_{k+1} \setminus F_k = \emptyset$. Consider $\{x_n\}_{n=1}^{\infty}$. $\forall \varepsilon > 0$, $\exists N$, $\forall n > m \geq N$, $\rho(x_n, x_m) \leq d_m < \varepsilon$. So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (x,ρ) is complete, $x = \lim_{n\to\infty} x_n \in X$. Besides, $\forall n$, F_n is close, so $\{x_k\}_{k\geq n} \in F_n$, so $x \in F_n$. Then $x \in \cap_{n=1}^{\infty} F_n$.

2. $(X,\rho) = (\mathbb{R},\rho)$, where $\rho(x,y) := |x-y|$, $F_n := [n,\infty)$, $\cap_{n=1}^{\infty} F_n = \emptyset$. Another example, $(\mathbb{R},\rho)$, where $\rho(x,y) = 1, x \neq y, \rho(x,y) = 0, x = y$. $(\mathbb{R},\rho)$ is complete, let $F_1 := \{x_1, \cdots, x_n, \cdots\} \subset \mathbb{R} : |F_1| = \aleph_0$, $F_{k+1} = F_k \setminus \{x_k\}$, $\cap_{k=1}^{n} F_k = \emptyset$

3. Unnecessary. Since X is sequence compact and complete, X is compact. If $B = \cap_{n\geq 1} F_n = \emptyset$, then $\exists F_n$, $F_n \cap B = \emptyset$, then $\forall x \in F_n$, $\exists u_x : x \in u_x \subset X$ is open, and $\exists F_x \in \{F_n\}$ satisfies $u_x \cap F_x =$. Let $\mathcal{U} := \{u_x : x \in F_n\}$, which is an open covery of F_n. Since X is compact, F_n is close in X, then F_n is compact. So, $\exists \{u_{x_1}, \cdots, u_{x_n}\} \subset \mathcal{U}$ is a finite open covery of F_n, and $u_{x_k} \cap F_{x_k} = \emptyset$, thus $\emptyset = \cup_{k=1}^{n}(u_{x_k} \cap F_{x_k}) \supset \cup_{k=1}^{n} u_{x_k} \cap (\cap_{k=1}^{n} F_{x_k}) \supset F_n \cap (\cap_{k=1}^{n} F_{x_k}) = F_m \neq \emptyset$, where $m = \min\{n, x_1, \cdots x_n\}$, contraction!

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