underGraduate Homework In Mathematics

FunctionalAnalysis 9

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PROBLEM I $(C[0,1], \|\cdot\|_1)$, let $f : C[0,1] \to \mathbb{R}$, $x \mapsto \int_0^1 sx(s)\,\mathrm{d}s$. Prove f is continous linear functional on $C[0,1]$, calculate $\|f\|$.

SOLUTION. 1. f is continous linear functional on $C[0,1]$: $\forall a, b \in \mathbb{R}, \forall x, y \in C[0,1], f(ax+by) = \int_0^1 s(ax(s)+by(s))\,\mathrm{d}s = a\int_0^1 sx(s)\,\mathrm{d}s + b\int_0^1 sy(s)\,\mathrm{d}s = af(x)+bf(y)$, $|f(x)-f(y)| = |\int_0^1 sx(s)\,\mathrm{d}s - \int_0^1 sy(s)\,\mathrm{d}s| = |\int_0^1 s(x(s)-y(s))\,\mathrm{d}s| \le \int_0^1 |x(s)-y(s)|\,\mathrm{d}s \le \int_0^1 \|x-y\|\,\mathrm{d}s = \|x-y\|$. So f is continous linear functional.

2. $\|f\| = \sup_{\|x\|=1} |\int_0^1 sx(s)\,\mathrm{d}s \le \sup_{\|x\|} \int_0^1 s\,\mathrm{d}s \le \frac{1}{2}$ Let $x = 1$ and $\|x\| = 1$, then $\|f(x)\| = \int_0^1 s\,\mathrm{d}s = \frac{1}{2}$ So, $\|f\| = \frac{1}{2}$

$\square$

PROBLEM II $T : (\mathbb{R}^n, l^1) \to (\mathbb{R}^n, l^1)$ is linear operation. Calculate $\|T\|$.

SOLUTION. Let A be the matrix of linear operation T. And define $\langle x, y\rangle = \sum_{i=1}^n x_i y_i, x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$

1. A is real symmetric matrix, then let $\lambda_1, \cdots, \lambda_n, \xi_1, \cdots, \xi_n$ be the real eigenvalue and its real eigenvector, which satisfies $\lambda_1 = \max_{1\le n} |\lambda_i|, \|\xi_i\| = 1, 1 \le i \le n$.

$\square$

PROBLEM III $f : C[a,b] \to \mathbb{R}$, $x \mapsto x(a) - x(b)$. Prove f is bounded linear functional, calculate $\|f\|$.

SOLUTION. 1. f is bounded linear functional: $\forall x \in C[0,1], \|x\| = 1, |x(a)-x(b)| \le 2\max_{0\le t\le 1} |x(t)| = 2$ $\forall x, y \in C[0,1], k, s \in \mathbb{R}, f(kx+sy) = kx(a)+sy(a)-kx(b)-sy(b) = k(x(a)-x(b))+s(y(a)-y(b)) = kf(x)+sf(y)$ $x = \frac{2}{b-a}(t-a) + -1 \in C[a,b]$, and $|f(x)| = |x(a)-x(b)| = 2, \|x\| = 1$. So $\|f\| = 2$

$\square$

PROBLEM IV $f : \mathcal{X} \to \mathbb{R}$,$x \mapsto \int_0^1 \sqrt{t}x(t^2)\,\mathrm{d}t$.Calculate $\|f\|$

1. $\mathcal{X} = C[0,1]$.
2. $\mathcal{X} = L^2[0,1]$

SOLUTION. $\int_0^1 \sqrt{t}x(t^2)\,\mathrm{d}t = \int_0^1 \frac{1}{2u^{\frac{1}{4}}}x(u)\,\mathrm{d}u$

1. $\forall x \in \mathcal{X}, \|x\| = 1, |f(x)| \le |\int_0^1 \frac{1}{2u^{\frac{1}{4}}}x(u)\,\mathrm{d}u| \le \int_0^1 |\frac{1}{2u^{\frac{1}{4}}}x(u)|\,\mathrm{d}u \le \int_0^1 \frac{1}{2u^{\frac{1}{4}}}\,\mathrm{d}u = \frac{2}{3}$ $x = 1 \in C[0,1]$ and $|f(x)| = \frac{2}{3}$. So $\|f\| = \frac{2}{3}$.
2. $\forall x \in \mathcal{X}, \|x\| = 1, |f(x)| = \frac{1}{2}|\int_0^1 \frac{1}{u^{\frac{1}{4}}}x(u)\,\mathrm{d}u| \le \frac{1}{2}\int_0^1 |\frac{1}{u^{\frac{1}{4}}}x(u)|\,\mathrm{d}u \le \frac{1}{2}(\int_0^1 (\frac{1}{u^{\frac{1}{4}}})^2\,\mathrm{d}u)^{\frac{1}{2}}(\int_0^1 x(u)^2\,\mathrm{d}u)^{\frac{1}{2}} = \frac{1}{2}\int_0^1 (\frac{1}{u^{\frac{1}{2}}}\,\mathrm{d}u)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ Let $x = a\frac{1}{u^{\frac{1}{4}}}$,then $\int_0^1 a^2 \frac{1}{u^{\frac{1}{2}}}\,\mathrm{d}u = 1$, so $a = \pm\frac{\sqrt{2}}{2}$. So $\|f\| = \frac{\sqrt{2}}{2}$.

$\square$

PROBLEM V $\Phi : C[0,1] \to \mathbb{R}$, $\Phi(f) \mapsto \int_0^1 \phi(t)f(t)\,\mathrm{d}t$, where $\phi \in C[0,1]$ Calculate $\|\Phi\|$