Group Representation

王胤雅

SID:201911010205

201911010205@mail.bnu.edu.cn

2023 年 10 月 13 日

Problem I. *Group G has an action on set $\Omega = \{x_1, x_2, \cdots, x_n\}$, let (φ, V) be the $n-$ dimensional K permutation representation of G, where K is the field of vector space V, and*

$$V = \left\{\sum_{i=1}^{n} a_i x_i \mid a_i \in K, i = 1, 2, \cdots, n\right\}.$$

Let $V_1 = \langle \sum_{i=1}^n x_i \rangle, V_2 = \{\sum_{i=1}^n a_i x_i \mid \sum_{i=1}^n a_i = 0, a_i \in K\}$. Proof: (1) V_1 and V_2 are invariant subspaces of G ; (2) If $\mathrm{char}K \nmid n$, then $\varphi = \varphi_{V_1} \oplus \varphi_{V_2}$.

Solution. 1. $\forall \sigma \in S_n$, $\forall v_i \in V_i, i = 1, 2$, then $v_1 = k \sum_{i=1}^n x_i, k \in K, v_2 = \sum_{i=1}^n a_i x_i, \sum_{i=1}^n a_i = 0$, so $\sigma(v_1) = k \sum_{i=1}^n x_{\sigma(i)} = k \sum_{i=1}^n x_i$, $\sigma(v_2) = \sum_{i=1}^n a_i x_{\sigma(i)} = \sum_{i=1}^n a_{\sigma^{-1}(i)} x_i$, $\sum_{i=1}^n a_{\sigma^{-1}(i)} = 0$, so $\sigma(v_2) \in V_2$ Since $\varphi(G) \cong H \leq S_n$, then $\forall g \in G$, $\varphi(g)(v_1) = v_1 \in V_1, \varphi(g)(v_2) \in V_2$.

2. We only need to proof $V = V_1 \oplus V_2$. $\forall v \in V_1 \cap V_2$, $v = k \sum_{i=1}^n x_i = \sum_{i=1}^n a_i x_i$ where $k \in K, \sum_{i=1}^n a_i = 0$, then $\sum_{i=1}^n (k - a_i) x_i = 0$, so $k - a_i = 0, \forall 1 \leq i \leq n$, which means $\sum_{i=1}^n a_i = nk = 0$. Since $\mathrm{char}K \nmid n$, then $k = 0$. Thus, $v = 0$. $\forall u \in V$, let $k = \frac{1}{n} \sum_{i=1}^n b_i$, $u = \sum_{i=1}^n b_i x_i$, so $u = k \sum_{i=1}^n b_i x_i + \sum_{i=1}^n (b_i - k) x_i$. By noting that $\sum_{i=1}^n (b_i - k) = \sum_{i=1}^n b_i - n \times k = 0$, $V = V_1 + V_2$.

□

Problem II. *Using exercise 1, calculate a $2-$dimensional complex representation of S_3 and its matrix of the representation.*

Solution. Let $\Omega = \{x_1, x_2\}, V := \{a_1 x_1 + a_2 x_2 : a_i \in K, i = 1, 2\}$, $\varphi : S_3 \to \mathrm{GL}(V)$. Since $\forall a_1 x_1 + a_2 x_2 = 0, a_1 + a_2 = 0$, then $a_1 = -a_2$, $a_1 x_1 + a_2 x_2 = a_1 x_1 - a_1 x_2$, so $V_2 = \langle x_1 - x_2 \rangle$. $\forall \sigma \in S_3$, $\varphi(\sigma) = \mathrm{id}$, when σ is an even permutation; $\varphi(\sigma) = (12)$, when σ is an odd permutation. So $\varphi(\sigma)|_{V_1} = \mathrm{id}, \varphi(\sigma)|_{V_2} = \mathrm{id}$, when σ is even; $\varphi(\sigma)|_{V_1} = \mathrm{id}, \varphi(\sigma)|_{V_2} : V_2 \to V_2, \forall v \in V_2, \varphi(\sigma)|_{V_2}(v) = -v$, when σ is odd. Therefore, the matrix of $\varphi(\sigma)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

when σ is odd; the matrix of $\varphi(\sigma)$ is I_2 when σ is even.

□

Problem III. $M_{\mathrm{n}}(K) := \{(a_{i,j})_{n \times n} : a_{ij} \in K, \forall 1 \leq i, j \leq n\}$. *Let*

$$\varphi : \mathrm{GL}_n(K) \to \mathrm{GL}(M_n(K))$$
$$A \to \varphi(A),$$

$$\varphi(A)X := AXA^{-1}; \quad \forall X \in M_n(K).$$

(1) Illustrate φ is the n^2-dimensional K representation of group $\mathrm{GL}_n(K)$; (2) $M_n^0(K) := \{A \in M_n(K) : \mathrm{tr}A = 0\}$. Illustrate $M_n^0(K)$ and $\langle I \rangle$ are invariant subspaces of φ; (3) Prove: If $\mathrm{char}K \nmid n$, then $\varphi = \varphi_{\langle I \rangle} \oplus \varphi_{M_n^0(K)}$

Solution. 1. (a) $\varphi(A)$ is an invertible linear transformation:

- $\varphi(A)$ is linear: $\forall X, Y \in M_n(K), a, b \in K$, $\varphi(A)(aX + bY) = A(aX + bY)A^{-1} = aAXA^{-1} + bAYA^{-1} = a\varphi(A)(X) + b\varphi(A)(Y)$
- $\varphi(A)$ is invertible: $\varphi(A) \in \mathrm{GL}(V)$, $\exists A^{-1} \in \mathrm{GL}_n(K)$ s.t. $\mathrm{id} = \varphi(A^{-1}) \circ \varphi(A)(X) = A^{-1}(AXA^{-1})(A^{-1})^{-1} = X$.

(b) φ is a group homomorphism: $\varphi(AB) : V \to V$, $\forall X$, $\varphi(A) \circ \varphi(B)(X) = A(BXB^{-1})A^{-1} = (AB)X(AB)^{-1} = \varphi(AB)(X)$

2. $\forall X \in M_n^0(K)$, $\mathrm{tr}(\varphi(A)(X)) = \mathrm{tr}(AXA^{-1}) = \mathrm{tr}(A^{-1}AX) = \mathrm{tr}(X) = 0$, so $\varphi(A)(X) \in M_n^0(K)$, $\varphi(A)(kI) = AkIA^{-1} = kI \in \langle I \rangle$

3. $\forall X \in M_n^0(K) \cap \langle I \rangle$, then $X = kI$ and $\mathrm{tr}(X) = nk = 0$. Since $\mathrm{char}K \nmid n$, $k = 0$. So $X = 0$. $\forall X \in M_n(K)$, $k = \frac{1}{n}\mathrm{tr}(X)$, then $\mathrm{tr}(X - kI) = \mathrm{tr}(X) - nk = 0$, so $X = kI + (X - kI)$, that means $M_n(K) = M_n^0(K) + \langle I \rangle$. Therefore, $M_n(K) = M_n^0(K) \oplus \langle I \rangle$

$\square$

Problem IV. *$\mathcal{O}(n) := \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ is the set of all n-dimensional orthogonal matrix over $\mathbb{R}$. Let:*

$$\begin{aligned} \varphi : \mathcal{O}(n) &\to \mathrm{GL}\left(M_n(\mathbb{R})\right) \\ A &\mapsto \varphi(A), \end{aligned} \tag{2}$$

$$\varphi(A)X := AXA^{-1} : \quad \forall X \in M_n(\mathbb{R}) \tag{3}$$

$M_n^+(\mathbb{R}) := \{A \in M_n^0(\mathbb{R}) : A = A^T\}$, $M_n^-(\mathrm{R}) := \{A \in M_n^0(\mathbb{R}) : A^T = -A\}$. (1) Proof: $M_n^+(\mathrm{R})$ and $M_n^-(\mathrm{R})$ are invariant spaces of φ; (2) Let the subrepresentation of φ on $\langle I \rangle$,$M_n^+(\mathbb{R}), M_n^-(\mathbb{R})$ be $\varphi_0, \varphi_1, \varphi_2$. Proof: $\varphi = \varphi_0 + \varphi_1 + \varphi_2$. (3) calculate a $\frac{1}{2}n(n-1)-$ dimensional $\mathbb{R}$ representation of $\mathcal{O}(n)$.

Solution. 1. Since $\mathcal{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$, $M_n^+(\mathbb{R}), M_n^-(\mathbb{R}) \subset M_n^0(\mathbb{R})$, then $\forall A \in \mathcal{O}(n), X \in M_n^+(\mathbb{R})(or M_n^-(\mathbb{R}))$, by problem 3,$\varphi(A)(X) \in M_n^0(\mathbb{R})$. Since $AA^T = I_n$, then $A^T = A^{-1}$, so $\forall X \in M_n^+(\mathbb{R})$, $(\varphi(A)(X))^T = (AXA^{-1})^T = (A^{-1})^T X^T A^T = (A^T)^{-1} X A^T = AXA^{-1} = \varphi(A)(X)$, so $\varphi(A)(X) \in M_n^+(\mathbb{R})$. $\forall X \in M_n^-(\mathbb{R})$, $(\varphi(A)(X))^T = (AXA^{-1})^T = (A^{-1})^T X^T A^T = -(A^T)^{-1} X A^T = -AXA^{-1} = -\varphi(A)(X)$, so $\varphi(A)(X) \in M_n^-(\mathbb{R})$.

2. By problem 3(3), we get $M_n(K) = M_n^0(K) \oplus \langle I \rangle$, so we only need to proof $M_n^0(\mathbb{R}) = M_n^+(\mathbb{R}) \oplus M_n^-(\mathbb{R})$. $\forall Y \in M_n^0(\mathbb{R})$, $Z^+ = \frac{Y + Y^T}{2}$, $Z^- = \frac{Y - Y^T}{2}$, so $Z^+ \in M_n^+(\mathbb{R}), Z^- \in M_n^-(\mathbb{R})$ and $Y = Z^+ + Z^-$. Therefore $M_n^0(\mathbb{R}) = M_n^+(\mathbb{R}) + M_n^-(\mathbb{R})$. $\forall X \in M_n^+(\mathbb{R}) \cap M_n^-(\mathbb{R})$, $X^T = X = -X$, so $X = 0$.

3. Let $\psi = \varphi|_{M_n^+(\mathbb{R})}$, since $\dim_{\mathbb{R}} M_n^+(\mathbb{R}) = \frac{1}{2}n(n-1)$, so $(\psi, M_n^+(\mathbb{R}))$ is a $\frac{1}{2}n(n-1)-$ dimensional representation of $\mathcal{O}(n)$.

$\square$