underGraduate Homework In Mathematics

FunctionalAnalysis 9

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 12 月 20 日

General fire extinguisher

PROBLEM I $(C[0,1], \|\cdot\|_1)$, let $f : C[0,1] \to \mathbb{R}$, $x \mapsto \int_0^1 sx(s)\,\mathrm{d}s$. Prove f is continous linear functional on $C[0,1]$, calculate $\|f\|$.

SOLUTION . 1. f is continous linear functional on $C[0,1]$: $\forall a, b \in \mathbb{R}, \forall x, y \in C[0,1], f(ax+by) = \int_0^1 s(ax(s)+by(s))\,\mathrm{d}s = a\int_0^1 sx(s)\,\mathrm{d}s + b\int_0^1 sy(s)\,\mathrm{d}s = af(x)+bf(y)$, $|f(x)-f(y)| = |\int_0^1 sx(s)\,\mathrm{d}s - \int_0^1 sy(s)\,\mathrm{d}s| = |\int_0^1 s(x(s)-y(s))\,\mathrm{d}s| \le \int_0^1 |x(s)-y(s)|\,\mathrm{d}s \le \int_0^1 \|x-y\|\,\mathrm{d}s = \|x-y\|$. So f is continous linear functional.

2. $\|f\| = \sup_{\|x\|=1} |\int_0^1 sx(s)|\,\mathrm{d}s \le \sup_{\|x\|=1} \int_0^1 |x(s)|\,\mathrm{d}s = 1$. Let $x_n = (n+1)s^n$ and $\|x\| = 1$, then $\|f(x)\| = \int_0^1 (n+1)s^{n+1}\,\mathrm{d}s = \frac{n+1}{n+2} \to 1, n \to \infty$. So, $\|f\| = 1$.

□

PROBLEM II $T : (\mathbb{R}^n, l^1) \to (\mathbb{R}^n, l^1)$ is linear operation. Calculate $\|T\|$.

SOLUTION . Let A be the matrix of linear operation T. $\forall x \in \mathbb{R}^n, let x = (x_1, \cdots, x_n),, A = (a_{ij})_{n\times n}$. $\forall \|x\| = 1$, i.e. $\sum_{i=1}^n |x_i| = 1$:

$$\begin{aligned}\|Ax\| &= \sum_{i=1}^n |\sum_{j=1}^n a_{ij}x_j| \\ &\le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}||x_j| \\ &\le \sum_{j=1}^n |x_j| \sum_{i=1}^n |a_{ij}| \\ &\le \sup_{1\le j\le n} \sum_{i=1}^n |a_{ij}|\end{aligned} \tag{1}$$

While $\forall 1 \le j \le n$, $x_k = \mathbb{1}_{k=j}, \sum_{i=1}^n |a_{ij}| = \sum_{i=1}^n |a_{ij}x_j| = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}x_j| = \|Ax\|$, so $\sup_{1\le j\le n} \sum_{i=1}^n |a_{ij}| \le \|Ax\|$.

□

PROBLEM III $f : C[a,b] \to \mathbb{R}$, $x \mapsto x(a) - x(b)$. Prove f is bounded linear functional, calculate $\|f\|$.

SOLUTION . 1. f is bounded linear functional: $\forall x \in C[0,1], \|x\| = 1, |x(a)-x(b)| \le 2\max_{0\le t\le 1}|x(t)| = 2$ $\forall x, y \in C[0,1], k, s \in \mathbb{R}, f(kx+sy) = kx(a)+sy(a)-kx(b)-sy(b) = k(x(a)-x(b))+s(y(a)-y(b)) = kf(x)+sf(y)$ $x = \frac{2}{b-a}(t-a) - 1 \in C[a,b]$, and $|f(x)| = |x(a)-x(b)| = 2, \|x\| = 1$. So $\|f\| = 2$

□

PROBLEM IV $f : \mathcal{X} \to \mathbb{R}$,$x \mapsto \int_0^1 \sqrt{t}x(t^2)\,\mathrm{d}t$.Calculate $\|f\|$

1. $\mathcal{X} = C[0,1]$.
2. $\mathcal{X} = L^2[0,1]$

SOLUTION . $\int_0^1 \sqrt{t}x(t^2)\,\mathrm{d}t = \int_0^1 \frac{1}{2u^{\frac{1}{4}}}x(u)\,\mathrm{d}u$

1. $\forall x \in \mathcal{X}, \|x\| = 1, |f(x)| \leq |\int_0^1 \frac{1}{2u^{\frac{1}{4}}} x(u)\,\mathrm{d}u| \leq \int_0^1 |\frac{1}{2u^{\frac{1}{4}}} x(u)|\,\mathrm{d}u \leq \int_0^1 \frac{1}{2u^{\frac{1}{4}}}\,\mathrm{d}u = \frac{2}{3}$. $x = 1 \in C[0,1]$ and $|f(x)| = \frac{2}{3}$. So $\|f\| = \frac{2}{3}$.
2. $\forall x \in \mathcal{X}, \|x\| = 1, |f(x)| = \frac{1}{2}|\int_0^1 \frac{1}{u^{\frac{1}{4}}} x(u)\,\mathrm{d}u| \leq \frac{1}{2}\int_0^1 |\frac{1}{u^{\frac{1}{4}}} x(u)|\,\mathrm{d}u \leq \frac{1}{2}(\int_0^1 (\frac{1}{u^{\frac{1}{4}}})^2\,\mathrm{d}u)^{\frac{1}{2}} (\int_0^1 x(u)^2\,\mathrm{d}u)^{\frac{1}{2}} = \frac{1}{2}(\int_0^1 \frac{1}{u^{\frac{1}{2}}}\,\mathrm{d}u)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}$ Let $x = a\frac{1}{u^{\frac{1}{4}}}$,then $\int_0^1 a^2 \frac{1}{u^{\frac{1}{2}}}\,\mathrm{d}u = 1$, so $a = \pm\frac{\sqrt{2}}{2}$. So $\|f\| = \frac{\sqrt{2}}{2}$.

□

PROBLEM V $\Phi : C[0,1] \to \mathbb{R}$, $\Phi(f) \mapsto \int_0^1 \phi(t) f(t)\,\mathrm{d}t$, where $\phi \in C[0,1]$ Calculate $\|\Phi\|$

SOLUTION . 1. Φ is well-defined: Obviously.

2. Φ is linear: Obviously.
3. $|\int_0^1 \phi(t) f(t)\,\mathrm{d}t| \leq \int_0^1 |\phi(t) f(t)|\,\mathrm{d}t \leq \int_0^1 |f(t)|\,\mathrm{d}t\,\|\phi\|$. So $\|\Phi\| \leq \int_0^1 |f(t)|\,\mathrm{d}t$. Let $g(t) = \mathrm{Sgn}(f(t))$, so g is measurable. And $\int_0^1 g(t) f(t)\,\mathrm{d}t = \int_0^1 |f(t)|\,\mathrm{d}t$. By Lusin theorem, $\forall \varepsilon > 0$, $\exists h \in C[0,1]$, such that $A = \{x \in [0,1] : h(x) \neq g(x), |h(x)| \leq 1\}, m(A) < \varepsilon$ and $h(1) = 1$. So $|\int_0^1 g(t) f(t) - h(t) f(t)\,\mathrm{d}t| \leq \int_0^1 |g(t) - h(t)||f(t)|\,\mathrm{d}t \leq 2\varepsilon \int_0^1 |f(t)|\,\mathrm{d}t \to 0, \varepsilon \to 0$. Therefore, $\|\Phi\| = \int_0^1 |f(t)|\,\mathrm{d}t$.

□