

Some problem about Ising model

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1 The detailed setup of the problem

Ising model is used to successfully explain the phase transition between paramagnetism and ferromagnetism in Statistical Physics. The phenomenon that a magnet disappears its magnetism after being heated over a critical temperature and obtains its magnetism through cooling down the same critical temperature, is called the phase transition between paramagnetism and ferromagnetism.

This model assumes that magnet is made by small needles regularly arranged. Each needle only has two different directions (upwards and downwards). Generally we consider a magnet as a square matrix with period boundary, call it A . The direction of each needle in magnet is regarded as an element in A , and ± 1 represent two direction. Thus, we can use a matrix $A \in M_n(\mathbb{R})$ to represent the state of a magnet. The macroeconomic appearance of magnet (that is whether to appear magnetism) is depended on directions of all needles. We use Hamiltonian to measure a microstate of magnet, i.e. A :

$$H(A) = - \sum_{\sigma_i \in A} \sum_{i \sim j} J \sigma_i \sigma_j - \sum_{\sigma_i \in A} H \sigma_i \quad (1)$$

, where J represents the spin-spin interaction, H represents the external field, σ_i is individual spins on each of lattice sites. Since each needle will interact with surrounding needles (these needles on its left, right, up and down). The first sum represents the interactions between each small needle;

latter one means the external field trying to align in one direction.

How does temperature influence magnetism? It turns out that each needle can convert its direction randomly influenced by temperature T . Assume that each needle convert in different time. Whether to accept the convert of a needle is decided by the change of Hamiltonian before and after the convert, ΔH , and temperature at the same time, let's call the change as ΔH .

If $\Delta H > 0$, then accept the convert. If $\Delta H < 0$, then we accept the convert with probability of $e^{\frac{-\Delta H}{k_B T}}$ where k_B is a constant. After the convert of the needle, a convert will happen afterwards. After enough long time, the process may reach a dynamic balance. Thus, the magnet will perform may different property.

Our aim is to find out how does the temperature influence the property of magnet and determine the critical temperature. In this article, we focus on these two problem:

1. Take $J = 1, k_B = 1, H = 0$, N is the size of A . Fixing T , choosing an initial microstate and initial needle randomly, let a convert, and judge whether to accept it, and then repeat the process as before. We will get a chain of microstates. Calculate internal energy u in the process:

$$u = \frac{U}{N^2}, U = \mathbb{E}(H) \quad (2)$$

and specific heat c :

$$c = \frac{C}{N^2}, C = \frac{k_B}{T^2} \mathbb{D}(H) \quad (3)$$

where \mathbb{D} is the variance. And plot $u - T, c - T$ graph.

2. Take $J = 1, k_B = 1, H \neq 0$. Fixing T, H , choosing an initial microstate and initial needle randomly, let a convert, and judge whether to accept it, and then repeat the process as before. We will get a chain of microstates. Calculate magnetization m in the process:

$$m = \frac{M}{N^2}, M = \mathbb{E}\left(\sum_{\sigma_i \in A} \sigma_i\right). \quad (4)$$

And plot $m - (T, H)$ graph.

2 The procedure you take to do the computation

1. First of all, set the initial parameters: Set up the grid to imitate the magnet. We write a function to generate the magnet. And we set the process time.
2. Secondly, we randomly choose a point to change its direction of magnet, and calculate the changed energy by the getDeltaEnergy function. And then, we write a use the WhetherAccept function to judge whether to accept the direction change of this point.
3. Repeat the process above for times that we have set in the first step.
4. Last, we plot the object graphs and grams.

3 Analysis of the numerical results

1. In the first problem, we suppose that $J = 1, k_B = 1$, and $H = 0$. And we set the size of grid 64×64 and the program processing time 1000000. We let T ranges from $0.5 \sim 4$, with internal 0.1. Just like the picture Item 1 below, we find out the both specific temperature C - which is the variance of Hamiltonian energy - and inner energy U vary with temperature in a particular pattern. Here are the rules. As the temperature increases below the critical temperature, which is $T_c = \frac{2|J|}{k_B \ln(1+\sqrt{2})}$, the specific heat increases. And in Item 1, after temperature goes over T_c , as temperature increases, the specific heat decrease, which means the specific heat has a phase transition when it reaches critical temperature. As for the inner energy, before the critical temperature, the increasing speed of inner energy slows down as temperature increases, and the increasing speed of inner energy speeds up as temperature increases, which shows the inner energy has a phase transition when reaching critical temperature. We can infer that when T is low, the Hamiltonian will be small, when T is high, the Hamiltonian will be big, and near T_c , there is a phase.
2. And in problem 2, we set the size of grid as 64×64 and the programming process time as 1000000, T ranges from $0.5 \sim 4$ with internal 0.2, H ranges from $0 \sim 1$ with internal 0.1. We observe from Item 2 that the magnetization M change near the critical temperature, i. e. T_c . When

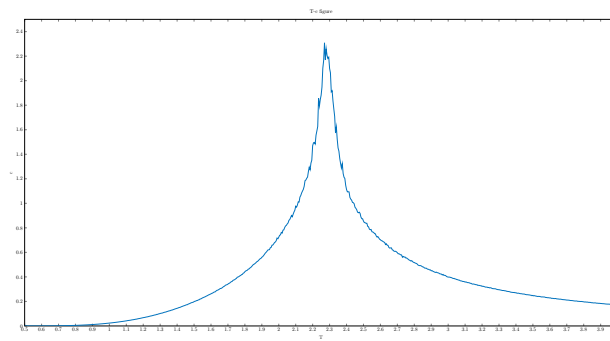


Figure 1: T-c

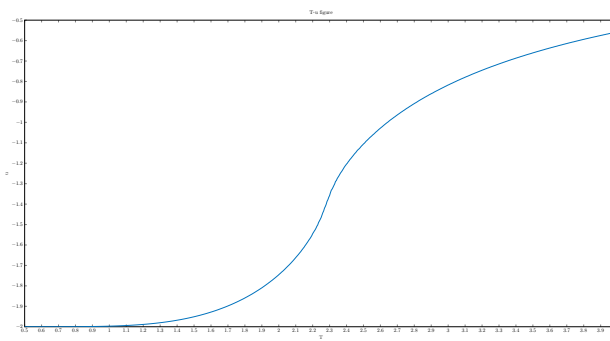


Figure 2: T-u

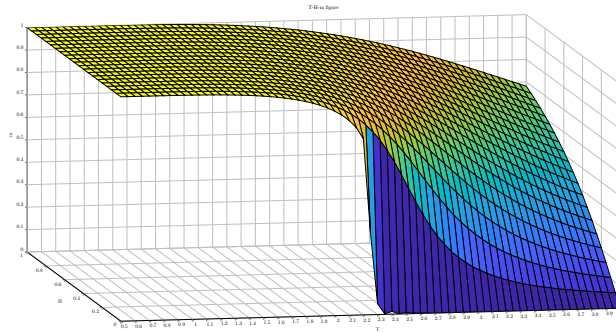


Figure 3: T-H-M

$T < T_c$, m will be so close to 1, no matter what H is. When $T > T_c$, m is very close to 0 in the condition with tiny H , and becomes bigger as H increases but will near reach 1. Besides, $T > T_c$, in the same H , as T increases, m decreases. We can infer from it that, when T is low, the magnet has magnetization, when T increases, the magnet may disappears, which means it has not permanently magnetization.

4 The issues you encounter and how you overcome

First of all, when we was running our program, our compute is not capable enough. But fortunately, we have a generous teacher, who borrowed us his compute to complete our program.

5 Possible discussion about the results and further thinking

Ising model can be applied to many other situations and fields. For example, opinion dynamics. Consider a group of people, each person can support one of two different candidates (like A and B). The candidate one support may change after exchanging ideas with sourroundings. What's more, the residents there may be influenced by advertisement of candidates. This is a typical reality situation for Ising model.

Besides, we can apply the Ising model to neuroscience. The activity of neurons in the brain can be modelled statistically. Each neuron at any time is either active or inactive. Hopfield suggested in 1982 that a dynamic Ising model would provide a first approximation to a neural network which is capable of learning. And following the general approach of a lot of scientists, we can model the process of neurons by the following model. Given a collection of neurons, we assume the energy as:

$$E = -1/2 \sum_{ij} J_{ij} S_i S_j - \sum_i h_i S_i \quad (5)$$

In this process, we can apply Ising model to simulate it.