underGraduate Homework In Mathematics

SetTheory 2

白永乐

202011150087

202011150087@mail.bnu.edu.cn

2023 年 10 月 9 日

General fire extinguisher

1 Question

PROBLEM I Let $(U, \leq), (V, \prec)$ be two well-orderings. Consider $f := \{(x, y) : x \in U \wedge y \in V \wedge (U_x, \leq) \cong (V_y, \prec)\}$, prove f is isomorphism from some initial segment of U to some initial segment of V.

PROBLEM II The relation "$(P, \leq) \cong (Q, \leq)$" is an equivalence relation (on the class of all partially ordered sets).

PROBLEM III Let $\mathcal{A}$ denote the class of all well orderings. For any $a, b \in \mathcal{A}$, define $a \prec b \iff a$ is isomorphic to an initial segment of b. Show that $\prec$ is a well ordering on $\mathcal{A}/\cong$, where $\cong$ is the equivalence relation given in PROBLEM II.

PROBLEM IV

1. If $(W, <)$ is a well ordering and $U \subset W$, then $(U, < \cap (U \times U))$ is a well ordering.
2. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings and $W_1 \cap W_2 = \varnothing$, then $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$ is a well ordering, where
$$\prec = <_1 \cup <_2 \cup \{(a, b) \mid a \in W_1 \wedge b \in W_2\}$$
3. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, then $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$ is a well ordering, where
$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \vee (b_1 = b_2 \wedge a_1 <_1 a_2)$$

PROBLEM V Show that the following are equivalent:

1. T is transitive;
2. $\bigcup T \subseteq T$;
3. $T \subseteq \mathscr{P}(T)$.

PROBLEM VI Let $\alpha, \beta, \gamma \in \mathrm{Ord}$ and let $\alpha < \beta$. Then

a $\alpha + \gamma \leq \beta + \gamma$.

b $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

c $\alpha^\gamma \leq \beta^\gamma$.

Given examples to show that $\leq$ cannot be replaced by $<$ in either inequality.

Example 1. a Let $\alpha = 0, \beta = 1, \gamma = \omega$, then $\alpha < \beta$ but $\alpha + \gamma = \omega = 1 + \omega = \beta + \gamma$.

b Let $\alpha = 1, \beta = 2, \gamma = \omega$, then $\alpha \cdot \gamma = \omega = 2 \cdot \omega = \omega$.

c Let $\alpha = 2, \beta = 3, \gamma = \omega$, then $\alpha^\gamma = \beta^\gamma$.

PROBLEM VII Show that the following rules do not hold for all $\alpha, \beta, \gamma \in \mathrm{Ord}$:

a If $\alpha + \gamma = \beta + \gamma$ then $\alpha = \beta$.

b If $\gamma > 0$ and $\alpha \cdot \gamma = \beta \cdot \gamma$ then $\alpha = \beta$.

c $(\beta+\gamma)\cdot\alpha = \beta\cdot\alpha+\gamma\cdot\alpha$.

PROBLEM VIII Find a set $A \subset \mathbb{Q}$ such that $(A, <_{\mathbb{Q}}) \cong (\alpha, \in)$, where

a $\alpha = \omega + 1$,

b $\alpha = \omega \cdot 2$,

c $\alpha = \omega \cdot \omega$,

d $\alpha = \omega^{\omega}$,

e $\alpha = \varepsilon_0$.

f α is any ordinal $< \omega_1$.

PROBLEM IX An ordinal α is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some $\beta \in \mathrm{Ord}$.

PROBLEM X Find the first three $\alpha > 0$ s.t. $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

PROBLEM XI Find the least ξ such that

a $\omega + \xi = \xi$.

b $\omega \cdot \xi = \xi, \xi \neq 0$.

c $\omega^{\xi} = \xi$.

(Hint for (1): Consider a sequence $\langle \xi_n \rangle$ s.t. $\xi_{n+1} = \omega + \xi_n$.)

Lemma 1. If $f : \mathrm{Ord} \to \mathrm{Ord}$ and $a \le b \to f(a) \le f(b)$ and $f(\sup B) = \sup f(B)$ for any B is subset of Ord, let $a_0 = 0, a_{n+1} = f(a_n)$, then $\xi = \sup\{a_n : n \in \mathbb{N}\}$ is the least ξ such that $f(\xi) = \xi$.

证明. First we prove $a_{n+1} \ge a_n$. Use MI it's obvious.

Second we prove $f(\xi) = \xi$. Obviously $f(\xi) = f(\sup\{a_n\}) = \sup\{f(a_n)\} = \sup\{a_{n+1}\} = \lim a_{n+1} = \lim a_n = \xi$.

Finally we prove ξ is the least. Assume $f(\alpha) = \alpha$, then use MI we can easily prove $\alpha \ge a_n \forall n < \omega$. So $\alpha \ge \sup\{a_n\} = \xi$. □