

under Graduate Homework In Mathematics

Functional Analysis 11

王胤雅

201911010205

201911010205@mail.bnu.edu.cn

2023 年 11 月 22 日



General fire extinguisher

PROBLEM I \mathcal{X} is complexed Hilbert space, $T \in \mathcal{L}(\mathcal{X})$. If $\exists a_0 > 0$ s.t. $\langle Tx, x \rangle \geq a_0 \langle x, x \rangle$, we call T is positive definite. Prove: positive definite operator T must exist inversed operator T^{-1} and $\|T^{-1}\| \leq \frac{1}{a_0}$.

SOLUTION. 1. T is injection:

$$\begin{aligned} Tx = Ty &\iff T(x - y) = 0 \\ &\iff 0 = \langle T(x - y), x - y \rangle \leq a_0 \langle x - y, x - y \rangle \geq 0 \end{aligned} \quad (1)$$

Thus, $\|x - y\| = 0$, $x = y$.

2. T is surjection:

- (a) First of all, we prove $T\mathcal{X}$ is closed: Let $W : \mathcal{X} \rightarrow T\mathcal{X}$, $x \mapsto Tx$, we easily get W is bijection, $T\mathcal{X} \subset \mathcal{X}$ is subspace of \mathcal{X} . So $T\mathcal{X}$ is B^* space, $W^{-1} : T\mathcal{X} \rightarrow \mathcal{X}$, $y \mapsto x$ where $Tx = y$. So W^{-1} is well-defined and W^{-1} is linear operator. $\|x\| \|W^{-1}x\| = \|TW^{-1}x\| \|W^{-1}x\| \geq \langle TW^{-1}x, W^{-1}x \rangle \geq a_0 \|W^{-1}x\|^2$, so $\|W^{-1}x\| \leq \frac{1}{a_0} \|x\|$ and $W^{-1} \in \mathcal{L}(T\mathcal{X}, \mathcal{X})$. By theorem 2.3.13, there exists $\widetilde{W^{-1}} : \overline{T\mathcal{X}} \rightarrow \mathcal{X}$, where $\widetilde{W^{-1}}$ is extended of W^{-1} on $\overline{T\mathcal{X}}$ and $\|\widetilde{W^{-1}}\| = \|W^{-1}\|$. If $x \in \overline{T\mathcal{X}} \setminus T\mathcal{X}$, then $\exists \{x_n\}_{n=1}^\infty \subset T\mathcal{X}$ such that $\lim_{n \rightarrow \infty} x_n = x$. Then, $\lim_{n \rightarrow \infty} W^{-1}x_n = \lim_{n \rightarrow \infty} \widetilde{W^{-1}}x_n = \widetilde{W^{-1}}x$. So $\lim_{n \rightarrow \infty} T(W^{-1}x_n) = \lim_{n \rightarrow \infty} x_n = T(\lim_{n \rightarrow \infty} W^{-1}x_n) = T(\widetilde{W^{-1}}x) = x$. Thus, $x \in T\mathcal{X}$. Contradiction! Therefore, $T\mathcal{X} = \overline{T\mathcal{X}}$.
- (b) $\forall y \in \mathcal{X}$, $\exists |y_1, y_2 \in \mathcal{X}$ such that $y = y_1 + y_2$ where $y_1 \perp T\mathcal{X}$, $y_2 \in T\mathcal{X}$. So $0 = \langle Ty_1, y_1 \rangle \geq a_0 \langle y_1, y_1 \rangle$, i.e. $y_1 = 0$. So $y = y_2 \in T\mathcal{X}$.

Another way to prove $T\mathcal{X}$ is closed: $\forall \{x_n\}_{n=1}^\infty \subset T\mathcal{X}$ such that $\lim_{n \rightarrow \infty} x_n = x \in \mathcal{X}$. Without loss of generality, $\forall n, \|x_n - x\| \leq \frac{1}{2^{n+1}}$. Let $x_0 = 0$, $y_n = x_{n+1} - x_n \in T\mathcal{X}$, $z_n = Tx_n$, $n \geq 1$, so $\|y_n\| \leq \frac{1}{2^n}$, $\|z_n\| \|y_n\| \geq \langle y_n, z_n \rangle \geq a_0 \langle z_n, z_n \rangle = a_0 \|z_n\|^2$. So $\|z_n\| \leq \frac{\|y_n\|}{a_0}$. Thus $\sum_{n=1}^\infty \|z_n\| < \infty$, then $\exists z \in \mathcal{X}$ such that $\sum_{n=1}^\infty z_n = z \in \mathcal{X}$. Besides, $Tz = T \sum_{n=1}^\infty z_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n Tz_k = \lim_{n \rightarrow \infty} x_{n+1} = x \in T\mathcal{X}$.

Thus, by inversed operator theorem, we get that T^{-1} exists and $\|T^{-1}\| = \|W^{-1}\| \leq \frac{1}{a_0}$. \square

PROBLEM II Assume $\{a_k\}_{k=1}^\infty$ such that $\sup_{k \geq 1} |a_k| < \infty$. $T : l^1 \rightarrow l^1$, $x = \{\xi_k\}_{k=1}^\infty \in l^1$, $T(x) = \{a_k \xi_k\}_{k=1}^\infty$. Prove: $T^{-1} \in \mathcal{L}(l^1) \iff \inf_{k \geq 1} |a_k| > 0$.

SOLUTION. 1. “ \Leftarrow .” Since $a := \inf_{k \geq 1} |a_k| > 0$, $b := \sup_{k \geq 1} |a_k| < \infty$, then $0 \neq a \leq |a_k| \leq b, \forall k \in \mathbb{N}^+$. So $x = \{x_n\}_{n=1}^\infty, y = \{y_n\}_{n=1}^\infty \in l^1$.

- (a) T is injection: $Tx = Ty \iff a_n x_n = a_n y_n, n \in \mathbb{N}_+, \iff x_n = y_n, n \in \mathbb{N}$.
- (b) T is surjection: $z = \{z_n\}_{n=1}^\infty, z_n = \frac{x_n}{a_n}$, then $\sum_{n=1}^\infty |z_n| = \sum_{n=1}^\infty \left| \frac{x_n}{a_n} \right| \leq \frac{\sum_{n=1}^\infty |x_n|}{a} < \infty$.
- (c) T is bounded: $\|Tx\| = \sum_{k=1}^\infty |a_k x_k| \leq b \sum_{k=1}^\infty |x_k| = b \|x\|$.

By inversed operator theorem, we get T^{-1} exists, and $T^{-1} \in \mathcal{L}(X)$.

2. “ \Rightarrow :”

- (a) If $\exists a_n = 0$, without loss of generality, let $a_1 = 0$, then by Item 1a, we get T is not injection. So T^{-1} doesn't exist.
- (b) If $\forall a_n \neq 0, n \in \mathbb{N}_+, \inf_{k \geq 1} |a_k| = 0$, without loss of generality, $\lim_{n \rightarrow \infty} a_n = 0$. Consider $\{x_n\}_{n=1}^\infty, x_n = (1, \dots, 1, 0, \dots)$, $\sum_{k=1}^\infty x_n(k) = n$, where $x_n(k)$ is the k -th number of x_n . Obviously, $\{x_n\}_{n=1}^\infty \subset l^1$. $(T^{-1}x_n)(k) = \frac{1}{a_k} \mathbb{1}_{1 \leq k \leq n}(k)$. So $\|T^{-1}x_n\| = \sum_{k=1}^n |\frac{1}{a_k}| \rightarrow \infty, n \rightarrow \infty$. That is $\|T^{-1}\| = \infty$, which is contradict with $T^{-1} \in \mathcal{L}(\mathcal{X})$.

□