underGraduate Homework In Mathematics

SetTheory 6

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PROBLEM I Assume A can be well-ordered, prove that $\mathcal{P}(A)$ can be linear-orderd.

SOLUTION. Assume $(A,<)$ is a well-ordered set. For $X,Y\in\mathcal{P}(A), X\neq Y$, let $X\prec Y \iff \min X\Delta Y\in X$. Now we prove $\prec$ is linear-order.

First by defination we get $X\not\prec X,\forall X\subset A$.

Second, for $X,Y\in\mathcal{P}, X\neq Y$, we have $X\Delta Y\neq\varnothing$. Since A is well-ordered, we get $\min X\Delta Y$ exists. And $\min X\Delta Y\in X\Delta Y\in X\cup Y$. So we get $X\prec Y\vee Y\prec X$.

Finally, assume $X\prec Y, Y\prec Z$, now we prove $X\prec Z$. Let $x=\min X\Delta Y\in X, y=\min Y\Delta Z\in Y$. Easily we get $X\Delta Z=(X\Delta Y)\Delta(Y\Delta Z)$. Assume $t=\min X\Delta Z$. Only need to prove $t\in X$. If not, we get $t\in Z$. If $t\in X\Delta Y$, then $t\geq x$. Since $t\notin X\wedge x\in X$, we get $t>x$. So $x\notin X\Delta Z$, so $x\in Z$. Noting $x\notin Y$, we get $x\in Y\Delta Z$, so $x>y$. So $y\notin X\Delta Y$, so $y\in X$. Since $y\notin Z$, we get $y\in X\Delta Z$. So $t<y$. So $t<y<x<t$, contradiction! Else we get $t\in Y\Delta Z$. Since $t\in Z$ we get $t\notin Y$. Then $t>y$. So $y\notin X\Delta Z$. Since $y\notin Z$, we get $y\notin X$. So $y\in X\Delta Y$, thus $y>x$. So $t>x$, thus $x\notin X\Delta Z$. So $x\in Z$, thus $x\in Y\Delta Z$. Then $x>y$, contradiction!

So we get $\prec$ is a linear-order on $\mathcal{P}(A)$. □

PROBLEM II Assume $\{X_i : i\in I\}$ and $\{Y_i : i\in I\}$ are two disjoint families such that $X_i\approx Y_i$. Prove that $\bigcup_{i\in I}X_i\approx\bigcup_{i\in I}Y_i$

SOLUTION. Since $X_i\approx Y_i$, we get $\mathrm{bij}(X_i,Y_i)\neq\varnothing$. Let $\theta: I\to\bigcup_{i\in I}\mathrm{bij}(X_i,Y_i)$ is a choice function. i.e., $\theta(i)\in\mathrm{bij}(X_i,Y_i)$. Now consider $\tau=\bigcup\mathrm{ran}(\theta)$. We will prove τ is bijection from $X:=\bigcup_{i\in I}X_i$ to $\bigcup_{i\in I}Y_i$.

First we prove τ is a map. i.e., $\forall x\in X,\exists! y\in Y,(x,y)\in\tau$. Since $X_i\cap X_j=\varnothing,\forall i\neq j$, we get $x\in X\to\exists! i\in I, x\in X_i$. So $(x,\theta(i)(x))\in\tau$. If $(x,z)\in\tau$, we get $\exists j\in I,(x,z)\in\theta(j)$. Since $x\in\mathrm{dom}(\theta(j))=X_j$, we get $j=i$. Since $\theta(i)$ is a map, we get $z=\theta(i)(x)$.

Second we prove τ is injection. Assume $x,t\in X,\tau(x)=\tau(t)$. Now we prove $x=t$. Since $Y_i\cap Y_j=\varnothing,\forall i\neq j$, we get $\exists! i\in I,\tau(x)\in Y_i$. Since $\mathrm{ran}(\theta(j))=Y_j,\forall j\in I$, we get $(x,\tau(x))\in\theta(i)$. So $\theta(i)(x)=\tau(x)$. For the same reason we get $\theta(i)(t)=\tau(t)$. So $\theta(i)(x)=\theta(i)(t)$. Since $\theta(i)$ is bijection, we get $x=t$.

Finally we prove τ is surjection. Assume $y\in Y$, then $\exists i\in I, y\in Y_i$. So $\exists x\in X_i,\theta(i)(x)=y$. So $\tau(x)=y$. So τ is surjective. □

PROBLEM III Prove that $\prod_{0<n<\omega}n=2^{\aleph_0}$.

SOLUTION. Obviously $\prod_{0<n<\omega}n=\prod_{n<\omega}(n+1)=(\sup_{n<\omega}(n+1))^{|\omega|}=\aleph_0^{\aleph_0}=2^{\aleph_0}$. □

PROBLEM IV Prove that $\prod_{n<\omega}\aleph_n=\aleph_\omega^{\aleph_0}$.

SOLUTION. Obviously $\aleph_n>0$, so we get $\prod_{n<\omega}\aleph_n=(\sup_{n<\omega}\aleph_n)^{\omega}=\aleph_\omega^{\aleph_0}$. □

PROBLEM V Prove that $\prod_{n<\omega+\omega}\aleph_n=\aleph_{\omega+\omega}^{\aleph_0}$.

SOLUTION. Let $f:\omega\to\omega+\omega$ be a bijection. Then $\prod_{n<\omega+\omega}\aleph_n=\prod_{n<\omega}\aleph_{f(n)}$. So we get $\prod_{n<\omega+\omega}=\left(\sup_{n<\omega}\aleph_{f(n)}\right)^{\aleph_0}=\aleph_{\omega+\omega}^{\aleph_0}$. □

PROBLEM VI For every ordinal α less than ω_1, prove that $\exists X : \omega \to \mathcal{P}(\alpha)$ such that $\operatorname{ordertype}(X(n)) \leq \alpha^n$ and $\alpha = \bigcup \operatorname{ran} X$.

SOLUTION . If not, assume β is the least ordinal less than ω_1 don't meet the requirement. If $\beta = \alpha+1$, Since $\alpha < \beta$, we get $\exists X \in^\omega \mathcal{P}(\alpha)$ meet the requirement. Now we let $Y : \omega \to \mathcal{P}(\beta)$ and $Y(0) = \{\alpha\}, Y(n+1) = X(n)$. Then easily Y meet the requirement, contradiction! Else, β is limit ordinal. Since $\beta < \omega_1$ we get $\operatorname{cf}(\beta) \leq \omega$. Since β is limit ordinal we get $\operatorname{cf}(\beta) = \omega$. Consider $\theta \operatorname{cf}(\beta) \to \beta$ is unbounded, then $\beta = \bigcup \operatorname{ran} \theta$. For $n \in \operatorname{cf}(\beta)$, we have $\theta(n) < \beta$, so by AC, $\exists X : \operatorname{cf}(\beta) \times \omega \to \mathcal{P}(\beta)$ such that $\operatorname{ordertype}(X(n,m)) \leq \theta(n)^m$ and $\theta(n) = \bigcup_{m\in\omega} X(n,m)$. Now let $Y : \omega \to \mathcal{P}(\beta)$ and $Y(2^n(2m+1)-1) = X(n,m)$. Then easily $\operatorname{ordertype}(Y(k)) \leq \beta^k$ and $\beta = \bigcup \operatorname{ran} \theta = \bigcup_{n,m\in\omega} X(n,m) = \bigcup_{k\in\omega} Y(k)$. contradiction! So such β doesn't exist. $\square$

PROBLEM VII If κ is a cardinal and $\lambda < \operatorname{cf}(\kappa)$, then $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$.

SOLUTION . When $\lambda = 0$ it's obvious, now we assume $\lambda > 0$. Easily $\kappa \geq \omega$ is a cardinal, so we get $\sum_{\alpha<\kappa} |\alpha|^\lambda = \kappa \sup_{\alpha<\kappa} |\alpha|^\lambda \leq \kappa \cdot \kappa^\lambda = \kappa^\lambda$. Now consider $f \in^\lambda \kappa$, we get f is bounded. So ${}^\lambda\kappa = \bigcup^\lambda_{\alpha<\kappa} \alpha$. So we get $\kappa^\lambda \leq \sum_{\alpha<\kappa} |\alpha|^\lambda$. Finally we get $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$. $\square$

PROBLEM VIII Prove that $\aleph_\omega^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$.

SOLUTION . Since $2^{\aleph_1}, \aleph_\omega^{\aleph_0} \leq \aleph_\omega^{\aleph_1}$, we get $\aleph_\omega^{\aleph_1} \geq 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$. Since $\aleph_\omega = \sup_{n<\omega} \aleph_n$, we get $\aleph_\omega = \prod_{n<\omega} \aleph_n^{\aleph_1}$. By Hausdoff formula we get $\aleph_{n+1}^{\aleph_1} = \aleph_n^{\aleph_1} \cdot \aleph_{n+1}$. By MI we can easily get $\aleph_n^{\aleph_1} \leq \aleph_\omega \cdot \aleph_0^{\aleph_1}$. So finally we get $\aleph_\omega^{\aleph_1} \leq \prod_{n<\omega} \aleph_\omega \cdot \aleph_0^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot \aleph_0^{\aleph_1 \cdot \aleph_0}$. Easily we get $\aleph_0^{\aleph_1 \cdot \aleph_0} = \aleph_0^{\aleph_1} = 2^{\aleph_1}$, so finally we get $\aleph_\omega^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_\omega^{\aleph_0}$. $\square$