# underGraduate Homework In Mathematics

## SetTheory 2


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General fire extinguisher

# 1 Question

PROBLEM I Let $(U, \leq), (V, \prec)$ be two well-orderings. Consider $f := \{(x, y) : x \in U \wedge y \in V \wedge (U_x, \leq) \cong (V_y, \prec)\}$, prove $f$ is isomorphism from some initial segment of $U$ to some initial segment of $V$.

SOLUTION . Let $f := \{(x, y) : x \in U \wedge y \in V \wedge (W_x, \leq) \cong (W_y, \prec)\}$

1. $f : \operatorname{dom} f \to V$ is a function: $\forall x \in \operatorname{dom} f$, if $\exists y_1, y_2 \in V$, s.t. $(x, y_1) \in f, (x, y_2) \in f$, w.l.o.g., $y_1 \prec_y y_2$, s.t. $(W_x, \leq_x) \cong (W_{y_1}, \prec_{y_1})$, $(W_x, \leq) \cong (W_{y_2}, \prec_{y_2})$. However, $W_{y_1} \subseteq W_{y_2}$, contradiction!

2. $f$ is isomorphic: $\forall x_1, x_2 \in U :\ x_1 \leq x_2$, $\exists y_1, y_2 \in V$, s.t. $g_1 : (W_{x_1}, \leq_{x_1}) \to (W_{y_1}, g_2 :\prec_{y_1})$, $(W_{x_2}, \leq_{x_2}) \to (W_{y_2}, \prec_{y_2})$, where $g_1, g_2$ are isomorphic. Since $W_{x_1} \subset W_{x_2}$, so $W_{y_1} \subset W_{y_2}$, so $y_1 \prec y_2$. Therefore, $f$ is isomorphic. Thus, It is trivial that $f$ is injective, moreover, $f$ is bijective.

3. $\operatorname{dom} f, \operatorname{ran} f$ are both initial segment of $U, V$ respectly:

   *Lemma* 1. $g : (x, \leq_x) \to (y, \prec_y)$ is isomorphic, then $\forall W_a \subset x, a < x$, s.t. $g[W_a] = W_{g(a)}$

   证明. $\forall u \in W_{g(a)}$ i.e $u \leq_x a$, then $g(u) \prec_y g(a)$, so $g(u) \in W_{g(a)}$. $\forall v \in W_{g(a)} \subset W_y$, $\exists u \in W_x$, s.t. $g(u) = v$. Since $v \prec_y g(a)$, then $u \leq_x a$, so $v \in g[W_a]$. Therefore, $g[W_a] = W_{g(a)}$. □

   (a) $\forall x \in \operatorname{dom} f$, $\forall a \leq_x x$, then $\exists | y \in V\ \exists g : (W_x, \leq_x) \to (W_y, \prec_y)$, where $g$ is isomorphic. By Lemma 1, $g|_{W_a} : (W_x, \leq_x) \cong (W_y, \prec_y)$ is isomorphic.

   (b) $\forall y \in \operatorname{ran} f$, $\forall b \prec_y y$, then $\exists | x \in U\ \exists h : (W_y, \prec_y) \to (W_x, \leq_x)$, where $h$ is isomorphic. Just as before, $h|_{W_b} : (W_y, \prec_y) \cong (W_x, \leq_x)$.

4. $\operatorname{dom} f, \operatorname{ran} f$ can't be both proper initial segment of $U, V :$ Otherwise, $\operatorname{dom} f \subsetneq U, \operatorname{ran} f \subsetneq V$, $u := \min U \operatorname{dom} f, v := \min V \operatorname{ran} f$, so $\tilde{f} : \operatorname{dom}(f \cup \{(u, v)\}) \to \operatorname{ran}(f \cup \{(u, v)\})$ s.t. $x \in \operatorname{dom} f$, $\tilde{f}(x) = f(x), \tilde{f}(u) = v$. Obviously, $\tilde{f}$ is isomorphic. So $u \in \operatorname{dom} f$, contradiction!

□

PROBLEM II The relation "$(P, \leq) \cong (Q, \leq)$" is an equivalence relation (on the class of all partially ordered sets).

SOLUTION . Let $\mathcal{A} = \{$ All of the partially order sets $\}$.

1. $\forall (P, \leq) \in \mathcal{A}$, $\mathrm{id} : P \to P$, which is an autoisomophism on $P$. So $(P, \leq) \cong (P, \leq)$.

2. If $(P_1, \leq_1) \cong (P_2, \leq_2)$, so $\exists f : (P_1, \leq_1) \to (P_2, \leq_2)$, which is isomorphic. So $f^{-1} : (P_2, \leq_2) \to (P_1, \leq_1)$ is isomorphic, too. So $(P_2, \leq_2) \cong (P_1, \leq_1)$

3. If $(P_1, \leq_1) \cong (P_2, \leq_2), (P_2, \leq_2) \cong (P_3, \leq_3)$, so so $\exists f_1 : (P_1, \leq_1) \to (P_2, \leq_2)$, which is isomorphic, $\exists f_2 : (P_2, \leq_2) \to (P_3, \leq_3)$, which is isomorphic. So $f_2 \circ f_1 : (P_1, \leq_1) \to (P_3, \leq_3)$ is isomorphic.

□

PROBLEM III Let $\mathcal{A}$ denote the class of all well orderings. For any $a, b \in \mathcal{A}$, define $a \prec b \iff a$ is isomorphic to an initial segment of $b$. Show that $\prec$ is a well ordering on $\mathcal{A}/\cong$, where $\cong$ is the equivalence relation given in PROBLEM II.

SOLUTION .

1. $(\mathcal{B}, \leq) := (\mathcal{A}/\cong, \leq)$ is well defined: $\forall [a], [b] \in \mathcal{B}$, if $a \leq b$, then $\forall a' \in [a], b' \in [b]$, $\exists f : a \to b$, where $f$ is a order-preserving function, $\exists g_1 : a' \to a, g_2 : b \to b'$, where $g_1, g_2$ are isomorphic. So $h := g_2 \circ f \circ g_1 : a' \to b'$, where $h$ is a order-preserving function. So by Lemma 1, $h[a'] = W_{(h(a'))}$. So $a' \leq b'$.
2. $(\mathcal{B}, \leq)$ is a partially ordered set, which is obvious.
3. $(\mathcal{B}, \leq)$ is a well-ordered set. $\forall \emptyset \neq B \subset \mathcal{B}$, let $[a] \in \mathcal{B}$, $W := \{x \in a : [b] \in B \wedge [b] \leq [a], b \cong W_x\}$. So $\emptyset \neq W \subset a$, $\exists x_0 = \min W$, $x_0 \in a$. $\forall [c] \in B : [c] \leq [a]$, $\exists x \in W$, $W_{x_0} \leq c \cong W_x < a$. So $\min B = W_{x_0}$.

□

PROBLEM IV

1. If $(W, <)$ is a well ordering and $U \subset W$, then $(U, < \cap (U \times U))$ is a well ordering.
2. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings and $W_1 \cap W_2 = \varnothing$, then $W_1 \oplus W_2 = (W_1 \cup W_2, \prec)$ is a well ordering, where
$$\prec = <_1 \cup <_2 \cup \{(a, b) \mid a \in W_1 \wedge b \in W_2\}$$
3. If $(W_1, <_1)$ and $(W_2, <_2)$ are two well orderings, then $W_1 \otimes W_2 = (W_1 \times W_2, \prec)$ is a well ordering, where
$$(a_1, b_1) \prec (a_2, b_2) \leftrightarrow b_1 <_2 b_2 \vee (b_1 = b_2 \wedge a_1 <_1 a_2)$$

SOLUTION .

1. $\forall \emptyset \neq A \subset U \subset W$, in $W$, $\exists a = \min A$, and $\leq$ is the same when $\leq$ in $U$. So $a$ is the minum element in $U$ of $A$.
2. $\forall \emptyset \neq A \subset W_1 \oplus W_2$, if $A \cap W_1 = \emptyset$, then $A = A \cap W_2 \neq \emptyset$, so $a = \min A \cap W_2 = \min A$. If $A \cap W_1 \neq \emptyset$, $a = \min A \cap W_1$. So, it is obvious that $a = \min A$.
3. $\forall \emptyset \neq A \subset W_1 \otimes W_2$, $b = \min \operatorname{ran} A$, $W_1 \subset W := \{a : (a, b) \in A\} \neq \emptyset$. Let $a = \min W$, $(a, b) = \min A$, obviously.

□

PROBLEM V Show that the following are equivalent:

1. $T$ is transitive;
2. $\bigcup T \subseteq T$;
3. $T \subseteq \mathscr{P}(T)$.

SOLUTION . 1. $V.1 \to V.2$:$\forall x \in \bigcup T$, $\exists y \in T$, s.t. $x \in y \in T$, since $y$ is transitive, so $y \subset T$, so $x \in T$.

2. $V.2 \to V.3$:$\forall y \in x \in T$, $y \in \bigcup T \subset T$, so $y \in T$.

3. $V.3 \to V.1$:$\forall x \in T \subset \mathscr{P}(T)$, $x \in \mathscr{P}(T)$, $x \subset T$.

□

PROBLEM VI Let $\alpha, \beta, \gamma \in \text{Ord}$ and let $\alpha < \beta$. Then

a $\alpha + \gamma \leq \beta + \gamma$.

b $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

c $\alpha^\gamma \leq \beta^\gamma$.

Given examples to show that $\leq$ cannot be replaced by $<$ in either inequality.

SOLUTION . 1. $\phi(\gamma) := \forall \alpha\beta \in \text{Ord}(\alpha+\gamma \leq \beta+\gamma)$, by Transfinite Induction, $\gamma = 0$, then $\alpha+\gamma = \alpha \leq \beta = \beta+\gamma$. If $\forall \nu \leq \gamma$, $\phi(\nu)$, when $\gamma$ is a successor ordinal, $\gamma = \nu \cup \{\nu\}$, so $\alpha+\gamma = S(\alpha+\nu) \leq S(\beta+\nu) = \beta+\gamma$. When $\gamma$ is a limit ordinal, $\alpha+\gamma = \lim_{\nu \to \gamma} \alpha+\nu \leq \lim_{\nu\to\gamma} \beta+\nu = \beta+\gamma$.
Example: $\alpha = 1, \beta = 2, \gamma = \omega$. Then $\alpha + \gamma = \omega = \beta + \gamma$

2. $\phi(\gamma) := \forall \alpha\beta \in \text{Ord}(\alpha \cdot \gamma \leq \beta \cdot \gamma)$, by Transfinite Induction, $\gamma = 0$, then $\alpha \cdot \gamma = 0 = \beta \cdot \gamma$. If $\forall \nu \leq \gamma$, $\phi(\nu)$, when $\gamma$ is a successor ordinal, $\gamma = \nu \cup \{\nu\}$, so by VI.a, $\alpha \cdot \gamma = \alpha \cdot \nu + \alpha \leq \beta \cdot \nu + \beta = \beta \cdot \gamma$. When $\gamma$ is a limit ordinal, $\alpha \cdot \gamma = \lim_{\nu\to\gamma} \alpha \cdot \nu \leq \lim_{\nu\to\gamma} \beta \cdot \nu = \beta \cdot \gamma$.
Example: $\alpha = 1, \beta = 2, \gamma = \omega$. Then $\alpha \cdot \gamma = \omega, f : \beta \cdot \gamma \to \gamma$, $f(< a, b >) = 2 * b, a = 0, f((a, b)) = 2 * b + 1, a = 1$, so $f$ is isomorphic. Then $\beta \cdot \gamma = \gamma$.

3. $\phi(\gamma) := \forall \alpha\beta \in \text{Ord}(\alpha^\gamma \leq \beta^\gamma)$, by Transfinite Induction, $\gamma = 0$, then $\alpha^\gamma = 1 = \beta^\gamma$. If $\forall \nu \leq \gamma$, $\phi(\nu)$, when $\gamma$ is a successor ordinal, $\gamma = \nu \cup \{\nu\}$, so by VI.b, $\alpha^\gamma = \alpha^\nu \cdot \alpha \leq \beta^\nu \cdot \beta = \beta^\gamma$. When $\gamma$ is a limit ordinal, $\alpha^\gamma = \lim_{\nu\to\gamma} \alpha^\nu \leq \lim_{\nu\to\gamma} \beta^\nu = \beta^\gamma$.
Example: $\alpha = 1, \beta = 2, \gamma = 0$. Then $\alpha^\gamma = 1, \beta^\gamma = 1$.

□

PROBLEM VII Show that the following rules do not hold for all $\alpha, \beta, \gamma \in \text{Ord}$:

a If $\alpha + \gamma = \beta + \gamma$ then $\alpha = \beta$.

b If $\gamma > 0$ and $\alpha \cdot \gamma = \beta \cdot \gamma$ then $\alpha = \beta$.

c $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$.

SOLUTION . 1. Just like example in VI.a.

2. Just like example in VI.b.

3. $\beta = 1, \gamma = 1, \alpha = \omega$, then $2 \cdot \omega = \omega \neq \omega + \omega$.

□

PROBLEM VIII Find a set $A \subset \mathbb{Q}$ such that $(A, <_{\mathbb{Q}}) \cong (\alpha, \in)$, where

a $\alpha = \omega + 1$,

b $\alpha = \omega \cdot 2$,

c $\alpha = \omega \cdot \omega$,

d $\alpha = \omega^{\omega}$,

e $\alpha = \varepsilon_0$.

f $\alpha$ is any ordinal $< \omega_1$.

SOLUTION .

1. $\{-\frac{1}{n} : n \in \mathbb{N}_+\} \cup 1$
2. $\{-\frac{1}{n} : n \in \mathbb{N}_+\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}_+\}$
3. $\cup_{k \in \mathbb{N}_+} \{k - \frac{1}{n} : n \in \mathbb{N}_+\}$
4. $\{n - \sum_{l=1}^{n} \prod_{i=1}^{l} \frac{1}{2^{k_i}} : k_i \in \mathbb{N}_+\} := W_n$, it is obvious that $W_n \cong \omega^n$. While $\omega^\omega = \sum_{n \in \omega} \omega^n$ and $\bigcup_{n \in \omega} W_n \cong \sum_{n \in \omega} \omega^n$, so $\sum_{n \in \omega} \omega^n \cong \bigcup_{n \in \omega} W_n$

$\square$

PROBLEM IX An ordinal $\alpha$ is a limit ordinal iff $\alpha = \omega \cdot \beta$ for some $\beta \in \mathrm{Ord}$.

SOLUTION .

1. $\Rightarrow$: $\omega \cdot \beta$ is a limit ordinal, that is to proove $\omega \cdot \beta$ doesn't have a maximum element. If $\omega \cdot \beta$ has a maximum element $(a, b) \in \omega \cdot \beta$, but $(a+1, b) \in \omega \cdot \beta$, $(a, b) < (a+1, b)$, contradiction!

2. $\Leftarrow$: $A := \{\gamma < \alpha : \gamma \text{ is a limit ordinal }\}$, $f : \alpha \to A$, $f(x) := \inf\{y : \exists n : x = y + n\}$, if $\inf\{y : \exists n : x = y + n\}$ is a successor ordinal of $z$, then $x = y + n = z + 1 + n$, so $z \in \{y : \exists n : x = y + n\}$, contradiction! So $\inf\{y : \exists n : x = y + n\}$ is a limit ordinal, then, $f$ is well-defined. Let $\beta =$OrderType$(A)$, next to proof $\omega \cdot \beta = \alpha$, i.e. $\omega \otimes A \cong \alpha$. $g : \alpha \to \omega \otimes A$, $g(x) = (n, f(x))$, where $x = f(x) + n$, so $g$ is isomorphic. Since $\alpha$ is a limit ordinal, then $\forall (n, \gamma) \in \omega \otimes A$, $\gamma + n < \alpha$, while $f(\gamma + n) = \gamma$, so $g$ is surjection. Thus, $\omega \otimes A \cong \alpha$.

$\square$

PROBLEM X Find the first three $\alpha > 0$ s.t. $\xi + \alpha = \alpha$ for all $\xi < \alpha$.

SOLUTION . The first one is 0, since $\forall \xi < 0$ is false, so $\xi + 0 = 0$ is true. The second one is 1, since $\xi < 1$, then $\xi = 0$, so $0 + 1 = 1$. The third one is $\omega$, since $\forall \xi < \omega$, $\xi + \omega = \omega$. $\forall 1 < n < \omega$, then $1 + n \cong n + 1 \neq n$. $\square$

PROBLEM XI Find the least $\xi$ such that

a $\omega + \xi = \xi$.

b $\omega \cdot \xi = \xi, \xi \neq 0$.

c $\omega^{\xi} = \xi$.

(Hint for (1): Consider a sequence $\langle \xi_n \rangle$ s.t. $\xi_{n+1} = \omega + \xi_n$.)

SOLUTION .

*Lemma* 2. If $f$ : Ord $\to$ Ord, s.t. $\forall x < y$, $f(x) < f(y)$, $\sup f(B) = f(\sup B)$, let $a_0 = 0$, $a_{n+1} = f(a_n)$, then $\sup_{n\in\omega} a_n$ is the minimum ordinal s.t. $f(\sup_{n\in\omega} a_n) = \sup_{n\in\omega} f(a_n)$.

证明. 1. Since the increasing of $f$, so $a_{n+1} = f(a_n) > a_n$, so $\sup_{n\in\omega} f(a_n) = \sup_{n\in\omega} a_{n+1} = f(\sup_{n\in\omega} a_n)$. $\forall \alpha$, $f(\alpha) = \alpha$, $\alpha > a_n, \forall n \in \omega$, so $\alpha > \sup_{n\in\omega} a_n$.

□

1. $f$ : Ord $\to$ Ord, $f(x) = \omega + x$, $f$ is increasing. $a_0 = 0$, $a_{n+1} = f(a_n)$, $\sup_{n\in\omega} a_{n+1} = \sup_{n\in\omega} f(a_n) = \sup_{n\in\omega} \omega + a_n = \omega + \sup_{n\in\omega} a_n$. So, by Lemma 2 $\xi = \omega \cdot \omega$.
2. $f$ : Ord $\to$ Ord, $f(x) = \omega \cdot x$, $f$ is increasing. $a_0 = 0$, $a_{n+1} = f(a_n)$, $\sup_{n\in\omega} a_{n+1} = \sup_{n\in\omega} f(a_n) = \sup_{n\in\omega} \omega \cdot a_n = \omega \cdot \sup_{n\in\omega} a_n$. So, by Lemma 2 $\xi = \omega^\omega$.
3. $f$ : Ord $\to$ Ord, $f(x) = \omega^x$, $f$ is increasing. $a_0 = 0$, $a_{n+1} = f(a_n)$, $\sup_{n\in\omega} a_{n+1} = \sup_{n\in\omega} f(a_n) = \sup_{n\in\omega} \omega_n^a = \omega^{\sup_{n\in\omega} a_n}$. So, by Lemma 2 $\xi = \epsilon_0$.

□