underGraduate Homework In Mathematics

FunctionalAnalysis 12

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2023 年 11 月 30 日

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PROBLEM I $\mathcal{X}$ is a linear space on $\mathbb{C}$. p is a seminorm on $\mathcal{X}$. $p(x_0) \neq 0, x_0 \in \mathcal{X}$. Prove: $\exists f$ is a linear funcional on $\mathcal{X}$ such that

1. $f(x_0) = 1$
2. $|f(x_0)| \leq \frac{p(x)}{p(x_0)}, \forall x \in \mathcal{X}$.

SOLUTION. Consider $p^* : \mathcal{X} \to \mathbb{C}$, $x \mapsto \frac{p(x)}{p(x_0)}$. Obviously p^* is a seminorm on $\mathcal{X}$. Let $\mathcal{X}_0 := \operatorname{Span}\{x_0\} \subset \mathcal{X}$ is a subspace of $\mathcal{X}$. $f : \mathcal{X}_0 \to \mathbb{C}$, $\alpha x_0 \mapsto \alpha f(x_0)$, where $f(x_0) = 1$. So f is a linear funcional on $\mathcal{X}_0$. And $\forall x \in \mathcal{X}_0, x = \alpha x_0, |f(x)| = |\alpha||f(x_0)| \leq |\alpha| = \frac{p(\alpha x_0)}{p(x_0)} = p^*(x)$. Thus, by Hahn-Banach theorem, $\exists \tilde{f} : \mathcal{X} \to \mathbb{R}$ is a linear funcional on $\mathcal{X}$ such that

1. $\tilde{f}(x) = f(x), \forall x \in \mathcal{X}_0$.
2. $|\tilde{f}(x)| \leq p^*(x) = \frac{p(x)}{p(x_0)}, \forall x \in \mathcal{X}$.

So $\tilde{f}(x_0) = f(x_0)$. □

PROBLEM II $\mathcal{X}$ is a B^* space, $\{x_n\}_{n=1}^\infty \subset \mathcal{X}$ such that $\forall f \in \mathcal{X}^*, \{f(x_n)\}_{n=1}^\infty$ is bounded. Prove that $\{x_n\}_{n=1}^\infty$ is bounded.

SOLUTION. Since there is an embedding map from $\mathcal{X} \to \mathcal{X}^{**}$, which keeps norm. Regard $\{x_n\}_{n=1}^\infty$ as subset of $\mathcal{X}^{**}$. And $\mathcal{X}^{**} = \mathcal{L}(\mathcal{X}^*, \mathbb{K})$, $\mathcal{X}^* = \mathcal{L}(X, \mathbb{K})$. $\mathbb{K}$ is complete, so $\mathcal{X}^*$ is a B space. Besides, $\forall f \in \mathcal{X}^*, \sup_{n \in \mathbb{N}_+} |x_n(f)| = \sup_{n \in \mathbb{N}_+} |f(x_n)| < \infty$. By Banach-Steinhaus theorem, $\sup_{n \in \mathbb{N}_+} \|x_n\| < \infty$ □

PROBLEM III $\mathcal{X}$ is a B^* space, $\mathcal{X}_0$ is a closed subspace of $\mathcal{X}$. Prove that $\forall x \in \mathcal{X}, \inf_{y \in \mathcal{X}_0} \|x - y\| = \sup\{|f(x)| : f \in \mathcal{X} \ \|f\| = 1, f|_{\mathcal{X}_0} = 1\}$.

Lemma 1. $\mathcal{X}$ is a B^* space, let $H_f^\lambda := \{x \in \mathcal{Z} : f(x) = \lambda\}$ where is a linear functional on $\mathcal{X}$. If $\|f\| = 1$, then $|f(x)| = d(x, H_f^0), \forall x \in \mathcal{X}$, where $\mathrm{d}(x, H_f^0) := \inf_{z \in H_f^0} \|x - z\|$.

证明. Since $\|f\| = 1$, then $\exists z \notin H_f^0$. And $x \in H_f^0$, $f(x) = 0$, let $z = x$, then $\|x - z\| = 0$. Next consider $x \notin H_f^0$:

1. $|f(x)| \leq \mathrm{d}(x, H_f^0)$: Since $\forall \varepsilon > 0$, $\exists y \in H_f^0$ such that $\|x - y\| \leq \mathrm{d}(x, H_f^0) + \varepsilon$. And $|f(x)| = |f(x - y)| \leq \|f\| \, \|x - y\| \leq \mathrm{d}(x, H_f^0) + \varepsilon \to \mathrm{d}(x, H_f^0), \varepsilon \to 0$.
2. $|f(x)| \geq \mathrm{d}(x, H_f^0)$: $\forall 1 > \varepsilon > 0$, $\exists y \in \mathcal{X}$, $\|y\| = 1$, such that $|f(y)| + \varepsilon \geq 1$, obviously, $f(y) \neq 0$, let $t = \frac{f(x)}{f(y)}$, so $f(ty) = \frac{f(x)}{f(y)} f(y) = f(x)$, then $f(ty - x) = f(ty) - f(x) = 0$, so $ty - x \in H_f^0$. Then $ty \in x + H_f^0$ and $|f(x)| + |t|\varepsilon \geq |f(ty)| + |t|\varepsilon \geq |t| = \|ty\|$, so $|f(x)| \geq \mathrm{d}(x, H_f^0)$.

□

SOLUTION. By Lemma 1, we have that $\inf_{y \in \mathcal{X}_0} \|x - y\| \geq \sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$. Consider $\mathcal{X}_0 = \mathcal{X}$ is possible, we define: $\sup \varnothing = 0$.

1. $\mathcal{X}_0 = \mathcal{X}$: $f(x) = 0, \forall x \in \mathcal{X}$, so $\nexists f : \|f\| = 1$, so $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = \sup \varnothing$. Obviously, the conclusion is true.

2. $\mathcal{X}_0 \subsetneq \mathcal{X}$:

   (a) If $x \notin \mathcal{X}_0$, since $\mathcal{X}_0$ is closed, then $d := \inf_{y\in\mathcal{X}_0} \|x-y\| > 0$, by Hahn-Banach theorem, $\exists f \in \mathcal{X}^*$ such that $\|f\| = 1, f|_{\mathcal{X}_0} = 0, f(x) = d$. Thus, $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = |f(x)|$. So $\inf_{y\in\mathcal{X}_0 \|x-y\|} = |f(x)| = \sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$.

   (b) If $y \in \mathcal{X}_0$, take $x \notin \mathcal{X}_0$,f such that $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = |f(x)|$. So $f(y) = 0$, then $\{|f(y)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\}$ is not empty. And $\forall f \in \mathcal{X}^*$ such that $\|f\| = 1, f|_{\mathcal{X}_0} = 0$, then $f(y) = 0$. Thus, $\sup\{|f(x)| : f \in \mathcal{X}^*, \|f\| = 1, f|_{\mathcal{X}_0} = 0\} = 0 = \inf_{y\in\mathcal{X}_0} \|x-y\|$.

□

PROBLEM IV Let $\mathcal{X}$ is B^* space, $\{x_k\}_{k=1}^n \subset \mathcal{X}$ is linear independent. $\{C_k\}_{k=1}^n \subset \mathbb{K}, M \in \mathcal{X}$. Then, $\exists f \in \mathcal{X}^*$, such that $f(x_k) = C_k, 1 \le n$, and $\|f\| \le M \iff \forall a_1, a_2, \cdots, a_n \in \mathbb{K}$, $|\sum_{k=1}^n a_k C_k| \le M \|\sum_{k=1}^n a_k x_k\|$.

SOLUTION. Since $\{x_k\}_{k=1}^n$ are linear independent, then $\mathcal{X}_0 := \text{Span}\{x_1, \cdots, x_n\} \subset \mathcal{X}$ is closed subspace.

1. "$\Rightarrow$": $\forall a_1, \cdots, a_n \in \mathbb{K}$, $\sum_{k=1}^n a_k x_k \in \mathcal{X}_0$, so $f(\sum_{k=1}^n a_k x_k) = \sum_{k=1}^n a_k f(x_k) = \sum_{k=1}^n a_k C_k$. Since $\|f\| \le M$, then $|f(\sum_{k=1}^n a_k x_k)| \le M \|\sum_{k=1}^n a_k x_k\|$.

2. "$\Leftarrow$": Consider $f_0 : \mathcal{X}_0 \to \mathcal{K}$, $x \mapsto \sum_{k=1}^n a_k C_k$, $x = \sum_{k=1}^n a_k x_k$. Obviously, f_0 is a linear funcional on $\mathcal{X}_0$. So $|f_0(x)| \le M \|\sum_{k=1}^n a_k x_k\| = M \|x\|$, $\forall x \in \mathcal{X}_0$. Thus, $\|f_0\|_0 \le \infty$, by Hahn-Banach theorem, $\exists f : \mathcal{X} \to \mathbb{K}$ is linear, and $f(x) = f_0(x)$, $\forall x \in \mathcal{X}_0$, $\|f\| = \|f_0\|_0 \le M$.

□

PROBLEM V Let $\mathcal{X}$ is B^* space, $\{x_k\}_{k=1}^n \subset \mathcal{X}$ is linear independent. Then $\exists \{f_k\}_{k=1}^n \subset \mathcal{X}^*$, such that $f_i(x_j) = \delta_{i,j}, \forall 1 \le i,j \le n$.

SOLUTION. Since $\{x_k\}_{k=1}^n$ are linear independent, then $\mathcal{X}_i := \text{Span}\{\{x_k : 1 \le k \le n, k \ne i\}\} \subset \mathcal{X}$ is closed subspace. So $x_i \notin \mathcal{X}_i$, then $d_i := \rho(x_i, \mathcal{X}_i)$. So, by Hahn-Banach theorem, $\exists g_i \in \mathcal{X}^*$ such that $g_i(x_i) = d_i, g_i(x) = 0$, $\forall x \in \mathcal{X}_i$,$\|g_i\| = 1$. Let $f_i(x) = \frac{g_i(x)}{d_i}, \forall x \in \mathcal{X}$, Obviously, $f_i(x_i) = 1, f_i(x_j) = 0$, $\forall j \ne i$, $\|f_i\| = \frac{\|g_i\|}{d_i}$.

□