FUNCTIONAL2

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2023 年 10 月 8 日

Problem I. 证明存在闭区间 $[0,1]$ 上的连续函数 $x(t)$，使得

$$x(t) = \frac{1}{2}\sin x(t) - a(t).$$

其中 $a(t)$ 是给定的 $[0,1]$ 上的连续函数

证明. $\theta : C[0,1] \to C[0,1]$, $x(t) \mapsto 1/2\sin x(t) - a(t)$, $\rho(x(t), y(t)) := \max_{0\le t\le 1}|x(t) - y(t)|$ is the distance on $C[0,1]$.

$$\begin{aligned}
&|\theta(x(t)) - \theta(y(t))| \\
=&|1/2\sin x(t) - a(t) - (1/2\sin y(t) - a(t))| \\
=&|1/2\sin x(t) - 1/2\sin y(t)| \\
=&|\sin\frac{x(t)-y(t)}{2}\cos\frac{x(t)+y(t)}{2}| \\
\le&|\frac{x(t)-y(t)}{2}|
\end{aligned} \tag{1}$$

Therefore, $\rho(\theta(x(t)), \theta(y(t))) \le \frac{1}{2}\rho(x(t), y(t))$. Thus, θ is a contraction mapping on $(C[0,1], \rho)$. By contraction mapping principle, $\exists x(t) \in C[0,1]$ statisfies $x(t) = \frac{1}{2}\sin x(t) - a(t)$.

□

Problem II.

$$x(t) - \lambda\int_0^1 \mathrm{e}^{t-s}x(s)\mathrm{d}s = y(t),$$

其中 $y(t) \in C[0,1], \lambda$ 为常数，$|\lambda| < 1$. 证明存在唯一解 $x(t) \in C[0,1]$.

证明. $x(t) - \lambda\int_0^1 \mathrm{e}^{t-s}x(s)\mathrm{d}s = y(t), \Leftrightarrow \mathrm{e}^{-t}x(t) - \lambda\int_0^1 \mathrm{e}^{-s}x(s)\mathrm{d}s = \mathrm{e}^{-t}y(t)$.
Let $z(t) = \mathrm{e}^{-t}x(t), w(t) = \mathrm{e}^{-t}y(t)$, $\theta : C[0,1] \to C[0,1]$, $z(t) \mapsto \lambda\int_0^1 z(s)\mathrm{d}s - w(t)$,

$\rho(x(t), y(t)) := \max_{0\leq t\leq 1} |x(t) - y(t)|$ is the distance on $C[0,1]$.

$$\begin{aligned}
&|\theta(u(t)) - \theta(v(t))| \\
=&|(\lambda \int_0^1 u(s)\mathrm{d}s - w(t)) - (\lambda \int_0^1 v(s)\mathrm{d}s - w(t))| \\
=&|\lambda \int_0^1 u(s)\mathrm{d}s - \lambda \int_0^1 v(s)\mathrm{d}s| \\
\leq&|\lambda| \int_0^1 |u(s) - v(s)|\mathrm{d}s \\
\leq&|\lambda|\rho(u(s), v(s))
\end{aligned} \tag{2}$$

Therefore, $\rho(\theta(x(t)), \theta(y(t))) \leq |\lambda|\rho(x(t), y(t))$, where $|\lambda| < 1$. Thus, θ is a contraction mapping on $(C[0,1], \rho)$. By contraction mapping principle, $\exists | z(t) \in C[0,1]$ statisfies $z(t) = \lambda \int_0^1 z(s)\mathrm{d}s - w(t)$. $x(t) = \mathrm{e}^t z(t)$ is what we need. $\square$