OtherItem 5

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SOLUTION. 1. 令 $\xi_i = \mathbb{1}_{\text{第 } i \text{ 次正面朝上}}$。那么 $X_n = \sum_{i=1}^n \xi_i$, 显然 X_n 是马氏链。设 $Y_n \equiv X_n \pmod 2$, 那么 Y_n 的状态空间是 $E = \{0, 1\}$, 且 Y_n 是马氏链。设 Y_n 的状态转移矩阵为 P, 那么 $P = (p_{ij})_{i,j\in E}$, 则 $p_{00} = p_{11} = \mathbb{P}\{\xi_i = 0\} = \frac{2}{3}$, $p_{10} = p_{01} = \mathbb{P}\{\xi_i = 1\} = \frac{1}{3}$。显然 P 是非周期，不可约的，且是有限维的，故 P 具有唯一平稳分布。设为 μ，μ 满足 $\mu P = P$，其中 $\mu = (\mu_0, \mu_1)$。不难算出 $\mu = (\frac{1}{2}, \frac{1}{2})$。故 $\lim_{n\to\infty} \mathbb{P}(Y_n = 0) = \lim_{n\to\infty} \mathbb{P}(X_n\text{是偶数}) = \frac{1}{2}$。

2. 令 $\xi_n^i = \mathbb{1}_{\text{第 } n \text{ 次抽出第 } i \text{ 种五福}}, i = 1, \cdots, 5$, 那么 $X_n^{(i)} = \sum_{k=1}^n \xi_k^{(i)}$。那么，$X_{2(n+1)}^{(i)} - X_{2n}^{(i)} = \xi_{2n+1}^{(i)} + \xi_{2n+2}^{(i)} \perp \{\xi_k^{(i)} : k \le 2n\}$, 故 $X_{2n}^{(i)}$ 是马氏链。令 $Y_n^{(i)} \equiv X_{2n}^{(i)} \pmod 2, i = 1, \cdots, 5$, 那么 $\sum_{i=1}^5 Y_n^{(i)} \equiv \sum_{i=1}^5 X_{2n}^{(i)} \equiv 0 \pmod 2$。令 $Y_n = (Y_n^{(1)}, \cdots, Y_n^{(5)})$, 则 Y_n 是状态空间为 $E = \mathbb{Z}_2^5\}$ 的马氏链。设 P 为 $(Y_n : n \ge 1)$ 转移矩阵，$P = (p_{ij})_{i,j\in E}$。

下计算 P: 设 $i, j \in E$, $i = (i_1, \cdots, i_5), j = (j_1, \cdots, j_5)$。

(a) 若 $\sum_{k=1}^5 i_k \not\equiv 0 \pmod 2$, 那么 $\forall j \in E$, $p_{ij} = 0$。

(b) 若 $\sum_{k=1}^5 i_k \equiv 0 \pmod 2$, 那么 $p_{ii} = \mathbb{P}(\exists i \in \{1, \cdots, 5\}, \xi_1^{(i)} = 1 = \xi_2^{(i)}, \xi_1^{(j)} = \xi_2^{(j)} = 0, \forall j \in \{1, \cdots, 5\}) = \sum_{i=1}^5 p_i^2$。若 $j \ne i$, 那么 $\sum_{k=1}^5 i_k - j_k \equiv 0 \pmod 2$, 则 $|\{k : i_k \ne j_k\}| \in \{2, 4\}$。

   i. $|\{k : i_k \ne j_k\}| = 2$, 那么 $p_{ij} = \mathbb{P}(\exists u \ne v \in \{1, \cdots, 5\}, \xi_1^{(u)} = 1, \xi_2^{(v)} = 1, \forall s \in \{1, \cdots, 5\} \setminus \{u\}, \xi_1^{(s)} = 0, \forall s \in \{1, \cdots, 5\} \setminus \{v\}, \xi_2^{(s)} = 0)$。即 $p_{ij} = \sum_{1\le u,v\le 5} p_u p_v$。

   ii. $|\{k : i_k \ne j_k\}| = 4$, 那么两次抽奖获得了 4 个福袋，则 $p_{ij} = 0$。

   iii. $|\{k : i_k \ne j_k\}| \in \{1, 3, 5\}$, 则 $p_{ij} = 0$。

显然，P 非周期，$\forall i, j \in E$, 则 $p_{ij}^5 \ge \prod_{k:i_k\ne j_k} \mathbb{E}(\xi_n^{(k)}) \prod_{k:i_k=j_k} (1 - \mathbb{E}(\xi_n^k)) > 0$, 故 P 是不可约的。那么 P 具有唯一的平稳分布记为 μ, 则 $\mu P = \mu$。由于 P 是对称的，则 $\mu P^T = \mu$, 即 $\sum_{k=1}^{16} \mu_k p_{ik} = 1, \sum_{k=1}^{16} p_{ik} = 1$, 由于 $\frac{1}{16} I, I = (1, \cdots, 1)_{1\times 16}$ 满足平稳分布条件，又由于唯一性知，$\mu = \frac{1}{16} I$。从而 $\lim_{n\to\infty} P(Y_n = (0, 0, 0, 0, 0)) = \lim_{n\to\infty} \mathbb{P}(X_n^{(i)}\text{是偶数}, \forall i = 1, \cdots, 5) = \frac{1}{16}$。

$\square$

SOLUTION. 1. Let $g(x) = f^{-1}(x)$. Since $f'' < 0$, then $g'' > 0$, then g' increases. Since $K(x) = f^{-1}(f(x+\varepsilon)) - x = \int_{f(x)}^{f(x)+\varepsilon} g'(t)dt$, then K increases. If Green and Red never meet in the process, suppose the Red satisfies $X(t) = 0$ and Green satisfies $Y(t) = x$. Let $x_0 = x$, $x_{n+1} = 1 - x_n - K(x_n)$. We can get to know x_{n+1} is position of the different one when the another one arrives 0. By the assumption, we can know $x_n \in (0,1)$. Next, we will prove $x_0 = x_n, \forall n \geq 1$. If not, then $\{n \geq 1 : x_0 \neq x_n\} \neq \varnothing$, w.l.o.g., we let $1 = \min\{n \geq 1 : x_0 \neq x_n\}$.

(a) If $x_1 < x_0$, then $x_2 = 1 - x_1 - K(x_1) = 1 - (1 - x_0 - K(x_0)) - K(x_1) = x_0 + K(x_0) - K(x_1)$. Since $K' > 0$, then $x_2 < x_0$. We prove $(-1)^n(x_{n+2} - x_n) < 0$ by mathematic induction. $n = 0$, we have proved, next suppose $(-1)^n(x_{n+2} - x_n) < 0$, we go to $n+1$. Then $(-1)^{n+1}(x_{n+3} + x_{n+1}) = (-1)^{n+1}(K(x_n) - K(x_{n+2}) + x_n - x_{n+2}) = (-1)^n(K(x_{n+2}) - K(x_n)) + (-1)^n(x_{n+2} - x_n)$. If $(-1)^n = 1$, then $x_{n+2} - x_n < 0$, then $K(x_{n+2}) - K(x_n) < 0$, then $(-1)^{n+1}(x_{n+3} + x_{n+1}) < 0$. If $(-1)^n = -1$, then $x_{n+2} - x_n > 0$, then $K(x_{n+2}) - K(x_n) > 0$, then $(-1)^{n+1}(x_{n+3} + x_{n+1}) < 0$.

(b) If $x_1 > x_0$, then $x_2 = 1 - x_1 - K(x_1) = 1 - (1 - x_0 - K(x_0)) - K(x_1) = x_0 + K(x_0) - K(x_1)$. Since $K' > 0$, then $x_2 > x_0$. We prove $(-1)^n(x_{n+2} - x_n) > 0$ by mathematic induction. $n = 0$, we have proved, next suppose $(-1)^n(x_{n+2} - x_n) > 0$, we go to $n+1$. Then $(-1)^{n+1}(x_{n+3} + x_{n+1}) = (-1)^{n+1}(K(x_n) - K(x_{n+2}) + x_n - x_{n+2}) = (-1)^n(K(x_{n+2}) - K(x_n)) + (-1)^n(x_{n+2} - x_n)$. If $(-1)^n = 1$, then $x_{n+2} - x_n > 0$, then $K(x_{n+2}) - K(x_n) > 0$, then $(-1)^{n+1}(x_{n+3} + x_{n+1}) > 0$. If $(-1)^n = -1$, then $x_{n+2} - x_n < 0$, then $K(x_{n+2}) - K(x_n) < 0$, then $(-1)^{n+1}(x_{n+3} + x_{n+1}) > 0$.

Therefore, x_0 satisfies $F(x) = 1 - 2x - K(x)$, $F(x_0) = 0 = x_1 - x_0 = 0$. And $F' = -1 - K' < 0$, then x_0 is the only solution satisfies of $F(x)$. By now, we get if Green and Red never meet, then x_0 s.t. $F(x_0) = 0$. So, if x_0 is not the solution of $F(x)$, then Green and Red will meet.

2. Also, we consider $g = f^{-1}$, then $g(y) = \frac{e^{by}-1}{e^b-1}$. And we define $\{x_n : n \in \mathbb{N}\}$ as above. So we find the circles that Red did before they meet, we need to find $\lim_{n\to\infty} \frac{1}{|x_{n+3}-x_{n+1}|} = \lim_{n\to\infty} \frac{1}{|K(x_{n+2})-K(x_{n+1})|}$. Since $K(u) - K(v) = g(f(u)+\varepsilon) - g(f(u)) - (g(f(v)+\varepsilon) - g(f(v))) = \frac{(e^{bf(u)}-e^{bf(v)})(e^{b\varepsilon}-1)}{e^b-1}$, and $x_n \to 0, n \to \infty$, then $\lim_{n\to\infty} \frac{1}{|x_{n+3}-x_{n+1}|} = \lim_{n\to\infty} \frac{1}{|K(x_{n+2})-K(x_{n+1})|} = C\frac{1}{b\varepsilon}$, where C is a constance.

$\square$

SOLUTION . 1. Since $\mathrm{tr}(A)=0$, so we can assume $A=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. And $\det(A)\neq 0$, then $-a^2-bc\neq 0$. So $A^2=\begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix}$. So $A^2=-(\det(A))I_2$, where $I_2=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\forall w=(w_1,w_2)\in \mathbb{R}^2 \forall n\geq 1$, $A^{-n}w=(\overline{w_1},\overline{w_2})\in\mathbb{R}^2$, let $v=([\overline{w_1}],[\overline{w_2}])\in\mathbb{Z}^2$, so $|v-A^{-n}w|<2$. Consider $A^n v-w=A^n(v-A^{-n}w)$:

(a) If $n=2k, k\in\mathbb{N}$, then $|A^nv-w|=|a^2+bc|^k|v-A^{-n}w|\leq 2|a^2+bc|^k=2|\det(A)|^{\frac{n}{2}}$. Then $\frac{\inf_{v\in\mathbb{Z}^2}|A^nv-w|}{|\det(A)|^{\frac{n}{2}}}\leq 2$.

(b) If $n=2k+1$, then we get that $|A^nv-w|=|a^2+bc|^k|A(v-A^{-n}w)|\leq 2|a^2+bc|^k||A||=\frac{2||A||}{|\det(A)|^{1/2}}|\det(A)|^{\frac{n}{2}}$. Where $||A||=\sup_{x\in\mathbb{R}^2,x\neq 0}\frac{|Ax|}{|x|}$, it must be well-defined, since $\dim(Mn_2(\mathbb{R}))=2$. Then $\frac{\inf_{v\in\mathbb{Z}^2}|A^nv-w|}{|\det(A)|^{\frac{n}{2}}}\leq\frac{2||A||}{|\det(A)|^{\frac{1}{2}}}$.

Then, $C=\max\{2,\frac{2||A||}{|\det(A)|^{\frac{1}{2}}}\}$, we get what we want.

2. Let$f(x)=x^2+ax+b$, where $a,b\in\mathbb{Z}$ is the characteristic polynomial of A. Since $f(x)$ is irreducible in $\mathbb{Q}$, we get $f(x)$ has no rational root. Therefore, $f(x)$ has no shigene in $\mathbb{R}$ if it has roots. If $x_1,x_2\in\mathbb{C}$ are roots of f:

(a) If $x_1=x_2$, then $x_1+x_2=2x_1=-a$, then $x_1=-\frac{a}{2}\in\mathbb{Q}$, contradiction!

(b) $x_1\neq x_2$, then $x_1=\overline{x_2}$ and $|x_1|=|x_2|=|\det(A)|^{\frac{1}{2}}$ and $\exists P\in M_2(\mathbb{C})$ $PAP^{-1}=\Lambda$, where $\Lambda=\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$. Then $A^n=P^{-1}\Lambda^nP$. And $||\Lambda||_{\mathbb{C}}=|x_1|$, then $||A^n||_{\mathbb{C}}=|\det(A)|^{\frac{n}{2}}||P||||P^{-1}||$, then $\forall w\in\mathbb{R}^2,\exists v\in\mathbb{Z}^2,|v-A^{-n}w|<2$, then $|v-A^nw|\leq ||A^n|||v-A^{-n}w|\leq 2||p||||p^{-1}|||\det(A)|^{\frac{n}{2}}$.

□

SOLUTION . 1.

2. $\dim(W\cap U_0)=d+1$.

3.

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