PROBLEM I Let $S = (S_n : n \geq 0)$ be the one-dimensional symmetry simple random walk with $S_0 = c \geq 0$. Let $k \geq 1$ and τ be the time of the k-th downcrossing 0. X_b is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing b. Prove:

1. $(X_b : b \geq c-1)$ is branch process. And offspring distribution has $Geo(\frac{1}{2})$.
2. $(X_{-a} : a \geq 1)$ is branch process. And offspring distribution has $Geo(\frac{1}{2})$.
3. $(X_b : 0 \leq b \leq c-1)$ is migrating branch process. And offspring distribution has $Geo(\frac{1}{2})$. And the migrating distribution is concentrating on 1.

SOLUTION . $\square$

PROBLEM II $c < b \in \mathbb{Z}_+$. Let $W = (W_n : n \geq 0)$ be the one-dimensional reflecting symmetry simple random walk with $W_0 = c \geq 0$ on $\mathbb{Z}^{0,b}$, whose transition matrix is $P^{0,b}$, where $a = 0, p, q > 0, p+q = 1$. Let $k \geq 1$ and τ is the time of the k-th downcrossing 0 on (W_n). $0 \leq a \leq b$, X_a is the times of $(S_{n\wedge\tau} : n \geq 0)$ downcrossing a. Prove:

1. $(X_b : c-1 \leq a \geq b-1)$ is branch process. And offspring distribution is presented by equation (6.3.1) on textbook.
2. $(X_b : 0 \leq b \leq c-1)$ is migrating branch process. And offspring distribution is presented by equation (6.3.1) on textbook. And the migrating distribution is concentrating on 1.

SOLUTION . $\square$

PROBLEM III Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ downcrossing a. $r = \frac{p}{q}$. Prove:

1. $\mathbb{P}(X_0 = i) = r^i(1-r), i \geq 0$;
2. $a \geq 0$, $\mathbb{P}(X_a = 0) = 1 - r^{a+1}$, $\mathbb{P}(X_a = i) = r^{a+1}(1-r), i \geq 1$.

SOLUTION . 1. Since $p < q$, then $W_n \to -\infty, n \to \infty$. Let $\tau_0 = 0, \forall k \geq 1$, $\sigma_k = \inf\{n \geq \tau_{k-1} : W_n = 1\}, \tau_k = \inf\{n \geq \sigma_k : W_n = 0\}$.

(a) If $i = 0$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_1 = \infty\}$. Then $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_1 = \infty) = r$.

(b) If $i \geq 1$, then $\{X_0 = i\} \overset{\text{a.s.}}{=} \{\sigma_i < \infty, \sigma_{i+1} = \infty\}$. Since $\{\tau_i < \infty\} \subset \{\sigma_i \infty\}, \mathbb{P}(\sigma_i < \infty, \tau_i = \infty) = 0$, then by strong markov property,

$$\begin{aligned}
\mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty) &= \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty, \tau_i < \infty) \\
&= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty) \\
&= \mathbb{P}(\sigma_{i+1} < \infty \mid \tau_i < \infty, W_{\tau_i} = 0) \\
&= \mathbb{P}(\sigma_1 < \infty) = r
\end{aligned}$$

Therefore,

$$\mathbb{P}(\sigma_{i+1} < \infty) = \mathbb{P}(\sigma_{i+1} < \infty \mid \sigma_i < \infty)\mathbb{P}(\sigma_i < \infty)$$

Then $\mathbb{P}(\sigma_i < \infty) = r^i$. Therefore, $\mathbb{P}(X_0 = i) = \mathbb{P}(\sigma_i < \infty, \sigma_{i+1} = \infty) = \mathbb{P}(\sigma_i < \infty)\mathbb{P}(\sigma_{i+1} = \infty \mid \sigma_i < \infty) = r^i(1-r)$.

2. Let $D_a = \inf(n \geq 0 : W_n = a)$, then $\mathbb{P}(D_a < \infty) = r^a$. By strong markov property, $(W_{D_a+n-a:n\geq 0})$ is a random walk starting from 0 under $\mathbb{P}(\cdot \mid D_a < \infty) = \mathbb{P}(\cdot \mid D_a < \infty, W_{D_a} = a)$. By the conclusion in **??**, $\mathbb{P}(X_a = i \mid D_a < \infty) = r^i(1-r), i \geq 0$. Then

$$\begin{aligned}
\mathbb{P}(X_a = 0) &= \mathbb{P}(D_a = \infty) + \mathbb{P}(D_a < \infty, X_a = 0) \\
&= 1 - r^a + \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = 0 \mid D_a < \infty) \\
&= 1 - r^a + r^a(1-r) = 1 - r^{a+1}
\end{aligned}$$

$\forall i \geq 1$,

$$\begin{aligned}
\mathbb{P}(X_a = i) &= \mathbb{P}(D_a < \infty, X_a = i) \\
&= \mathbb{P}(D_a < \infty)\mathbb{P}(X_a = i \mid D_a < \infty) \\
&= r^a r^i(1-r) = r^{a+i}(1-r)
\end{aligned}$$

□

PROBLEM IV Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ downcrossing a. $r = \frac{p}{q}$. Prove: if $a \leq -1$, then $X_a \sim G(1-r)$, i.e. $\mathbb{P}(X_a = i) = r^{i-1}(1-r), i \geq 1$.

SOLUTION. $D_a = \inf\{\}$ □