PROBLEM I A radio is powered by one battery, the lifetime of the battery obey the distribution of exponential distribution with parameter $\lambda = \frac{1}{30}$. In long term, in which frequence should we change the battery?

SOLUTION. Easy to get that $\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mathbb{E}(\xi_1)} = \frac{1}{30}$. So we change battery every 30 hours in average. $\square$

PROBLEM II Consider a primitive renewing process with average renewing internal time μ. Assume every renewing time is recorded by probability p, and each record and each renew are independence. Let $N_r(t)$ be the times of renewing by recorded until time t. $\{N_r(t) : t \geq 0\}$ is a renewing process or not? And calculate $\lim_{t\to\infty} \frac{N_r(t)}{t}$.

SOLUTION. Assume $X_n : n \in \mathbb{N}$ are i.i.d r.v and $X_0 \sim Geo(p)$, and $(X_n : n \in \mathbb{N}) \perp (N(t) : t \geq 0)$. Let $Y_n := \sum_{k=1}^{n} X_n$, and $Y_0 = 0$. Let $\xi_r(n) := \sum_{k=Y_{n-1}+1}^{Y_n} \xi_k$. Then $\xi_r(n) : n \in \mathbb{N}^+$ is update time of N_r. Since $(X_n : n \in \mathbb{N}) \perp (N(t) : t \geq 0)$, we get that $(\xi_r(n) : n \in \mathbb{N}^+)$ are i.i.d. And $\mathbb{E}(\xi_r(1)) = \mathbb{E}(X_1)\mathbb{E}(\xi_1) = \frac{\mu}{p}$. So $\lim_{t\to\infty} \frac{N_r(t)}{t} = \frac{\mu}{p}$. $\square$

PROBLEM III Assume $(U_n : n \in \mathbb{N}^+)$ are i.i.d r.v. and $U_1 \sim U(0,1)$. Assume $X_{n,m} : n, m \in \mathbb{N}^+$ are r.v. and $X_{n,m} \mid U_n \sim B(U_n)$. And $(X_{n,m} \mid U_n : m \in \mathbb{N}^+)$ are i.i.d. Let $\xi_n := \inf\{m \in \mathbb{N}^+ : X_{n,m} = 1\}$ be the n-th update time of $N(t)$. Find $\lim_{t\to\infty} \frac{N(t)}{t}$.

SOLUTION. Easy to find that $\mathbb{E}(\xi_1) = \int_0^1 \mathbb{E}(\xi_1 \mid U_1 = x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}x}{x} = \infty$. So easy to find that $\lim_{t\to\infty} \frac{N(t)}{t} = \infty$. $\square$

PROBLEM IV Assume $(\xi_n : n \in \mathbb{N}^+)$ is i.i.d r.v. ranging in $\mathbb{N}$ is update time of $N(t)$. Let A_n be the event that at time n there is an update. Assume $a = \lim_{n\to\infty} \mathbb{P}(A_n)$ exists. Prove that $a = \frac{1}{\mathbb{E}(\xi_1)}$.

SOLUTION. Since $N(n) = \sum_{k=1}^{n} \mathbb{1}(A_k)$, we know that $\mathbb{E}(N(n)) = \sum_{k=1}^{n} \mathbb{P}(A_k)$. Noting that $\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{\mathbb{E}(\xi_1)}$, we obtain that $\lim_{n\to\infty} \mathbb{E}(\frac{N(n)}{n}) = \frac{1}{\mathbb{E}(\xi_1)}$. So $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbb{P}(A_k)}{n} = \frac{1}{\mathbb{E}(\xi_1)}$. By stolz, we can get that $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \mathbb{P}(A_k)}{n} = a$. So $a = \frac{1}{\mathbb{E}(\xi_1)}$. $\square$

PROBLEM V Assume $N_1(t), N_2(t)$ are two independent updating process with update time distribution $E(1), U(0,2)$. Find an estimate of $\mathbb{P}(N_1(100) + N_2(100) \geq 190)$.

SOLUTION. Easy to know the expectation and varience of the update time are $\mu_1 = 1, \sigma_1^2 = 1, \mu_2 = 1, \sigma_2^2 = \frac{1}{3}$. So by the central limit theorem of updating process we know that

$$\frac{N_1(100) - 100}{\sqrt{100}}, \frac{N_2(100) - 100}{\sqrt{\frac{100}{3}}} \sim N(0,1)$$

So $\frac{N_1(100)+N_2(100)-200}{\sqrt{\frac{400}{3}}} \sim N(0,1)$. So $\mathbb{P}(N_1(100) + N_2(100) \geq 190) \approx \mathbb{P}(N(0,1) \geq -\frac{\sqrt{3}}{2})$. $\square$