PROBLEM I Assume $(B_t : t \geq 0)$ is Brownian motion, prove that for $r > 0$, we have $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion, too.

SOLUTION. Suppose $(B_t : t \geq 0)$ satisfies $B_t - B_s \sim N(0, a(t-r)), \forall t \geq s \geq 0$, then $(B_{t+r} - B_r) - (B_{s+r} - B_r) = B_{t+r} - B_{s+r} \sim N(0, a(t-s))$. Besides, $\forall 0 = t_0 < t_1 < t_2 < \cdots < t_n$, $D_t := B_{t+r} - B_r$, then $D_{t_k} - D_{t_{k-1}} = B_{t_k+r} - B_{t_{k-1}+r}, \forall k = 1, \cdots, n$. So $D_{t_k} - D_{t_{k-1}} = B_{s_k} - B_{s_{k-1}}, \forall k = 1, \cdots, n$, where $s_k = t_k + r$, are independent to each other. Besides, $B_{0+r} - B_r = 0$, obviously $D_{t_k} - D_{t_{k-1}}$ is independent to $B_{0+r} - B_0$. Last, since the orbit of $(B_t : t \geq 0)$ is continuous, then $B_{t+r} - B_r$ is continuous for any $t \geq 0$. Thus, by the definition of Brownian motion, we get $(B_{t+r} - B_r : t \geq 0)$ is Brownian motion. $\square$

PROBLEM II Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Prove that $\forall c > 0, (cB_{\frac{t}{c^2}} : t \geq 0)$ is standard Brownian motion start at 0, too.

SOLUTION. Since $(B_t : t \geq 0)$ is standard Brownian motion start at 0, then $\forall \omega \in \Omega$, $B_t(w)$ is continuous, so $cB_{\frac{t}{c^2}}(w)$ is continuous, and $cB_{\frac{0}{c^2}} = cB_0 = 0$. $\forall t > s \geq 0$, $cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}})$, so $\mathbb{E}(\mathrm{e}^{\mathrm{i}\theta c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}})}) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\theta cu} \frac{1}{2\pi\sqrt{t-s}} c^2 \mathrm{e}^{\frac{-u^2}{2(t-s)}} du = \mathrm{e}^{\frac{-(t-s)\theta^2}{2}}$. Besides, $\forall 0 = t_0 \leq \cdots \leq t_n$, $B_0, (B_{\frac{t_k}{c^2}} - B_{\frac{t_{k-1}}{c^2}}), k = 1, \cdots, n$ are independent. Then $B_0, (B_{\frac{t_k}{c^2}} - B_{\frac{t_{k-1}}{c^2}}), k = 1, \cdots, n$ are independent. $\square$

PROBLEM III Assume $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ are two independent standard Brownian motion, $a, b \in \mathbb{R}$ and $\sqrt{a^2 + b^2} > 0$. Prove that $(aX_t + bY_t : t \geq 0)$ is a Brownian motion with parameter $c = \sqrt{a^2 + b^2}$.

SOLUTION. Since $aX_t + bY_t - (aX_s + bY_s) = a(X_t - X_s) + b(Y_t - Y_s), \forall t > s \geq 0$, and $a(X_t - X_s) \sim N(0, a^2(t-s))$, $b(Y_t - Y_s) \sim N(0, b^2(t-s))$, $X_t \perp Y_t, t \geq 0$, then $a(X_t - X_s) + b(Y_t + Y_s) \sim N(0, a^2 + b^2(t-s))$. Since $(X_t : t \geq 0)$, $(Y_t : t \geq 0)$ are standard Brownian motion, then $\forall \omega \in \Omega$, $X_t(\omega), Y_t(\omega)$ are continuous, so $aX_t(\omega) + bY_t(\omega)$ is continuous. Besides, $\forall 0 = t_0 \leq \cdots \leq t_n$, $aX_0, a(X_{t_k} - X_{t_{k-1}}), k = 1, \cdots, n$ are independent. And $bY_0, Y_{t_k} - Y_{t_{k-1}}, k = 1, \cdots, n$ are independent, $X_t, Y_t, t \geq 0$ are independent. Then $aX_0 + bY_0, a(X_{t_k} - X_{t_{k-1}}) + b(Y_{t_k} - Y_{t_{k-1}}), k = 1, \cdots, n$ are independent with each other. So $aX_t + bY_t : t \geq 0$ is a Brownian motion with parameter $a^2 + b^2$. $\square$

PROBLEM IV Assume $(B_t : t \geq 0)$ is standard Brownian motion start at 0. Let $X_0 = 0$ and $X_t := tB_{\frac{1}{t}}$. Given

$$\limsup_{t \to \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1$$

Prove that $(X_t : t \geq 0)$ is standard Brownian motion start at 0.

SOLUTION.

1. First, we get the distribution of $X_t - X_s \forall t > s \geq 0$. $X_t - X_s = tB_{\frac{1}{t}} - sB_{\frac{1}{s}} = s(B_{\frac{1}{t}} - B_{\frac{1}{s}}) + (t-s)B_{\frac{1}{t}}$, and $\frac{1}{s} > \frac{1}{t}$. Since $(B_t : t \geq 0)$ is standard Brownian motion starting at 0, then $N(0, \frac{1}{t}) \sim B_{\frac{1}{t}} \perp B_{\frac{1}{t}} - B_{\frac{1}{s}} \sim N(0, \frac{1}{t} - \frac{1}{s})$. So $X_t - X_s$ is normal distribution and $\mathrm{Var}(X_t - X_s) = s^2(\frac{1}{t} - \frac{1}{s}) + (t-s)^2 * \frac{1}{t} = t - s$. So $X_t - X_s \sim N(0, t-s)$.

2. We prove $X_0 = 0$ a.s. and the orbit of X_t is continuous. Since $\forall \omega \in \Omega$, $X_t(\omega)$ is continuous when $t \neq 0$, we only need to prove $X_t = 0$ a.s.. Since $(-B_t : t \geq 0)$ is standard Brownian motion too, then $\limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2t \log lligt}} = 1$. So $\limsup_{t\to 0} |tB_{\frac{1}{t}}| = \limsup_{t\to\infty} |\frac{B_t}{t}| \leq \limsup_{t\to\infty} \frac{\sqrt{2t\log\log t}}{t} = 0$.

3. We prove $\forall 0 = t_0 < t_1 < \cdots < t_n, X_0, X_{t_k} - X_{t_{k-1}}, k = 0, \cdots, n$ are independent. We only need to prove $X_{t_k} - X_{t_{k-1}}, k = 0, \cdots, n$ are independent since $X_0 = 0$. Since $X_{t_k} - X_{t_{k-1}} \sim N(0, t_k - t_{k-1}), k = 0, \cdots, n$, then we only need to prove $\mathbb{E}((X_{t_k} - X_{t_{k-1}})(X_{t_j} - X_{t_{j-1}})) = 0, \forall 0 \leq k < j \leq n$. Since $\frac{1}{t_{k-1}} > \frac{1}{t_k} \geq \frac{1}{t_{j-1}} > \frac{1}{t_j}$, so $X_{t_j} - X_{t_{j-1}} \in \sigma(\{B_s : 0 \leq s \leq t_k\})$. So we only need to prove $\mathbb{E}((X_{t_k} - X_{t_{k-1}})B_{\frac{1}{r}}) = 0, r \geq t_k$. Since $\mathbb{E}(t_k B_{\frac{1}{t_k}} B_{\frac{1}{r}}) = \mathbb{E}(t_k(B_{\frac{1}{r}} - B_{t_k})B_{t_k} + t_k B_{t_k}^2) = 1$ and $\mathbb{E}(t_{k-1} B_{\frac{1}{t_{k-1}}} B_{\frac{1}{r}}) = \mathbb{E}(t_{k-1}(B_{\frac{1}{r}} - B_{t_{k-1}})B_{t_{k-1}} + t_{k-1}B_{t_{k-1}}^2) = 1$, so $\mathbb{E}(X_{t_k} - X_{t_{k-1}})(B_{\frac{1}{r}}) = 0$.

$\square$