PROBLEM I Let $S = (S_n : n \geq 0)$ be the one-dimensional symmetry simple random walk with $S_0 = c \geq 0$. Let $k \geq 1$ and τ is the time of the k-th undercrossing 0. X_b is the times of $(S_{n\wedge\tau} : n \geq 0)$ undercrossing b. Prove:

1. $(X_b : b \geq c-1)$ is branch process. And offspring distribution is presented by equation (6.3.1) on textbook.
2. $(X_{-a} : a \geq 1)$ is branch process. And offspring distribution is presented by equation (6.3.1) on textbook.
3. $(X_b : 0 \leq b \leq c-1)$ is migrating branch process. And offspring distribution is presented by equation (6.3.1) on textbook. And the migrating distribution is concentrating on 1.

PROBLEM II $c < b \in \mathbb{Z}_+$. Let $W = (W_n : n \geq 0)$ be the one-dimensional reflecting symmetry simple random walk with $W_0 = c \geq 0$ on $\mathbb{Z}^{0,b}$, whose transition matrix is $P^{0,b}$, where $a = 0, p, q > 0, p + q = 1$. Let $k \geq 1$ and τ is the time of the k-th undercrossing 0 on (W_n). $0 \leq a \leq b$, X_a is the times of $(S_{n\wedge\tau} : n \geq 0)$ undercrossing a. Prove:

1. $(X_b : c-1 \leq a \geq b-1)$ is branch process. And offspring distribution is presented by equation (6.3.1) on textbook.
2. $(X_b : 0 \leq b \leq c-1)$ is migrating branch process. And offspring distribution is presented by equation (6.3.1) on textbook. And the migrating distribution is concentrating on 1.

PROBLEM III Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ undercrossing a. $r = \frac{p}{q}$. Prove:

1. $\mathbb{P}(X_0 = i) = r^i(1-r), i \geq 0$;
2. $a \geq 0$, $\mathbb{P}(X_a = 0) = 1 - r^{a+1}$, $\mathbb{P}(X_a = i) = r^{a+1}(1-r), i \geq 1$.

PROBLEM IV Let $W = (W_n : n \geq 0)$ be the one-dimensional simple random walk with $W_0 = 0$, whose transition matrix P given by equation (4.4.3) on textbook, $0 < p < q < 1$. X_a is the times of $(W_{n\wedge\tau} : n \geq 0)$ undercrossing a. $r = \frac{p}{q}$. Prove: if $a \leq -1$, then $X_a \sim G(1-r)$, i.e. $\mathbb{P}(X_a = i) = r^{i-1}(1-r), i \geq 1$.