PROBLEM I Assume $(X_n : n \geq 0)$ is an irreducible Markov chain on E. Prove that $(X_n : n \geq 0)$ is recurrent (or transient) $\iff$ $\forall i \in E$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1(\text{or } 0).$$

SOLUTION . Since $(X_n : n \geq 0)$ is irreducible, then $\forall i \in E, \exists n_i$ s.t.$0 < \mathbb{P}(X_{n_i} = i)) \leq 1$. And $\forall \omega \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}$, then $\exists k_i \geq n_i X_{k_i}(\omega) = i$. Besides, $(X_n : n \geq 1)$ is a Markov chain, let $T_i(\omega) := \min\{n \geq 1 : X_0(\omega) = X_n(\omega) = i\}$, then $\mathbb{P}_i(T_i < \infty) = \mathbb{P}(\min\{n \geq 1 : X_{n_i+n}(\omega) = i\} < \infty) \geq \mathbb{P}(\{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\})$.

$\forall \omega \in \{T_i < \infty\}$, then $\exists m_i$ s.t. $\forall 1 \leq j < m_i, X_j \neq i, X_{m_i} = i$, Then $\forall n \geq 1$, let $t_n = [\frac{n}{m_i}] + 1$, $Y_m = X_k, m \equiv k \mod m_i, 0 \leq k \leq m_i - 1$, so $Y_{t_n m_i}(\omega) = i$. Thus, $\mathbb{P}_i(T_i < \infty) \leq \mathbb{P}(\bigcap_{n=1}^{\infty} Y_{t_n m_i} = i)$. Besides, $\{\bigcup_{k=n}^{\infty}\{X_k = i\} \supset \{X_{t_n m_i} = i\} \supset \{Y_{t_n m_i} = i\}$. Then $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\} \supset \bigcap_{n=1}^{\infty}\{Y_{t_n m_i} = i\}$, thus, $\mathbb{P}_i(T_i < \infty) \leq \mathbb{P}(\bigcap_{n=1}^{\infty} Y_{t_n m_i} = i) \leq \mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} X_k = i)$.

Therefore, $\mathbb{P}_i(T_i < \infty) = \mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} X_k = i)$. Then, $(X_n : n \geq 0)$ is recurrent (or transient) $\iff$ $\mathbb{P}_i(T_i < \infty) = 1(\text{or}0)$ $\iff$ $\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} X_k = i) = 1(\text{or } 0)$. $\square$

PROBLEM II Prove: $(X_n : n \geq 0)$ is Markov chain on E, where E is finite. Then $\exists x \in E$, x is recurrent.

SOLUTION . If $\forall y \in E$, y is transient, then $\square$

PROBLEM III Assume $(X_n : n \geq 0)$ is Markov chain on $\mathbb{Z}$. Prove it is transient $\iff$ $\forall \mu_0$ is primitive distribution, $\lim_{n\to\infty} |X_n| \overset{\text{a.s.}}{=} \infty$.

SOLUTION . Only need to prove that $\forall k \in \mathbb{N}, \liminf_{n\to\infty} |X_n| > k, a.s.$. Consider the event $\liminf_{n\to\infty} |X_n| \leq k$, it means $\forall n \in \mathbb{N}, \exists t \geq n, X_t \in [-k, k]$. So we only need to prove $\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t \in [-k, k]\}\right) = 0$. It is sufficient to prove that $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. Since (X_n) is transient, it has been proved that $\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. So $\mathbb{P}(\bigcup_{u\in[-k,k]}\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0 \leq \sum_{u\in[-k,k]}\mathbb{P}(\bigcap_{n=1}^{\infty}\bigcup_{t=n}^{\infty}\{X_t = u\}) = 0$. $\square$

PROBLEM IV Assume $\{a_i : i \geq 1\} \subset (0, 1)$. Consider $E := \mathbb{N}$, P is a transition matrix on E, where $p_{ij} = a_i \mathbb{1}_{\{j=0\}} + (1 - a_i)\mathbb{1}_{\{j=i+1\}}$. Prove:

1. P is irreducible.
2. P is recurrent $\iff$ $\sum_i a_i = \infty$.
3. P is ergodici $\iff$ $\sum_{k=1}^{\infty}\prod_{i=1}^{k-1}(1 - a_i) < \infty$.

SOLUTION .

1. $\forall i, j \in E$, if $i > j$, then $p_{ij}(j - i) = \prod_{k=i}^{j-1}(1 - a_k) > 0$. If $i \leq j$, then $p_{ij}(j + 1) = a_i \prod_{k=0}^{j-1}(1 - a_k) > 0$. So P is recurrent.
2. Since P is irreducible, then we only need to concern the circumstance when $X_0 = 0$. Then $\{T_0 > n\} \overset{\text{a.s.}}{=} \{X_k = k, k = 0, \cdots, n\}$. Then $\mathbb{P}_0(T_0 = \infty) = \mathbb{P}_0(\bigcap_n\{T_0 > n\}) = \lim_{n\to\infty}\mathbb{P}_0(X_k = k, k = 0, \cdots, n) = \lim_{n\to\infty}\prod_{k=0}^{n-1}(1 - a_k) = \prod_{k=0}^{\infty}(1 - a_k)$. Then $\mathbb{P}_0(T_0 = \infty) = 1 \iff \prod_{k=0}^{\infty}(1 - a_k) = 1 \iff \sum_k a_k = \infty$.

3. Since $\mathbb{E}_0(T_0) = \sum_{n\in E} \mathbb{P}_0(T_0 > n) = \sum_{n=0}^{\infty} \prod_{k=0}^{n-1}(1-a_k)$, then P is ergodici $\iff \mathbb{E}_0(T_0) < \infty$ $\iff \sum_{n=0}^{\infty} \prod_{k=0}^{n-1}(1-a_k) < \infty$.

$\square$

PROBLEM V Assume P is a transition matrix on E and P is irreducible, $j \in E$. Prove: P is recurrent $\iff$ 1 is the minimum non negtive solution of

$$y_i = \sum_{k\neq j} p_{ik} y_k + p_{ij}, i \in E \setminus \{j\} \tag{1}$$

SOLUTION. "$\implies$": If P is recurrent, then the bounded solution of $y_i = \sum_{k\in E} p_{ik} y_k, i \in E\setminus\{j\}$ is constant. so the bounded solution of Equation (1) is constant. Sepcially, 1 is the minimum non negetive solution.

"$\impliedby$": If P is transient, then the bounded solution of $y_i = \sum_{k\in E} p_{ik} y_k, i \in E\setminus\{j\}$ has non constant ones. Let it be $\{z_i : i \in E\}$. W.L.O.G., we can assume $z_j = 0, z_1 \neq z_2, |z_i| \leq 1, \forall i \in E, \exists i_0 \in E, z_{i_0} < 0$. Let $y_i = 1 + z_i, i \in E$, then $\{y_i : i \in E\}$ is the bounded solution of Equation (1). But $y_{i_0} < 1, y_i \geq 0, i \in E$, which is a contradiction. $\square$

PROBLEM VI Let $\{a_k : k \geq 0\}$ satisfies $\sum_{k\geq 0} a_k = 1, a_k \geq 1, a_0 > 0$, $\mu := \sum_{k=1}^{\infty} k a_k > 1$. Define $p_{ij} = \begin{cases} a_j & , i = 0 \\ a_{j-i+1} & , i \geq 1 \wedge j \geq i-1 \\ 0 & , \text{otherwise} \end{cases}$. Prove: P is transient.

SOLUTION. First of all, we prove that P is irreducible: Since $\sum_{k=1}^{\infty} k a_k > 1$, then $\exists a_m > 0$. And $\forall i \geq 1$, $p_{i-1,i} = a_0 > 0$. Then $\forall i, j$, if $i < j$, then $p_{ij}(j-i) = a_0^{j-i} > 0$. If $i \geq j$, let $t \equiv i - j \pmod{m}, 1 \leq t \leq m$, then $p_{ij}(t+1) = a_0^t a_m > 0$.

Second, we prove that P is transient: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k, z \in [0,1]$, $F(z) = f(z) - z$. Then $F(z) \in C^{\infty}[0,1]$, $F(0) = F(1) = 0$, $F(1)' = f(1) - 1 = \mu - 1 > 0$. Then $\exists \eta \in (0,1)$, $F(\eta) < 0$. Besides, $F^{(m)}(0) = a_m > 0$, then easily we can get $F'(0) > 0$, then $\exists \varepsilon\xi \in (0,1), F(\xi) > 0$. Thereofore, $\exists \mid c \in (0,1)$, s.t. $c = f(c)$, then $\forall i \geq 1$,

$$c^i = \sum_{k=0}^{\infty} a_k c^{k+i-1} = \sum_{j=i-1}^{\infty} a_{j-i+1} c^j = \sum_{j=0}^{\infty} p_{ij} c^j$$

So P is transient. $\square$