PROBLEM I

1. Assume $\{Y_1(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with offspring distribution $(p(i) : i \in \mathbb{N})$ and the migrating probability respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{N})$. Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2(i) : i \in \mathbb{N}$.

2. Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and the migrating distribution $\gamma(i) : i \in \mathbb{N}$. $P_n^\gamma = (p_n^\gamma(i,j); i,j \in \mathbb{N})$ is the $n$-th transition matrix. Prove: $\forall i, n \geq 1$
$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$
where $h$ is the generating function of $(\gamma(j) : j \in \mathbb{N})$. $g$ is the generating function of $(p(j) : j \in \mathbb{N})$.

3. $h, g$ are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$. Prove: $\forall i, n \geq 1$,
$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

1. Let $Z_n := Y_1(n) + Y_2(n)$, $\forall \{i_0, \cdots, i_n, i_{n+1}\} \in \mathbb{N}$, $G_n = \{Z_k = i_k, 0 \leq k \leq n\}$.

$$\begin{aligned}
&\mathbb{P}(Z_{n+1} = i_{n+1}, G_n)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_{n+1}=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, Y_2(k) = i_k - t_k, 0 \leq k \leq n+1)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_{n+1}}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n+1)\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n+1)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n) p_1^{\gamma_1}(t_n, t_{n+1}) \mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n) p_1^{\gamma_2}(i_n - t_n, i_{n+1} - t_{n+1})\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n) p^{*t_n} * \gamma_1(t_{n+1})\\
&\times \mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n) p^{*i_n - t_n} * \gamma_2(i_{n+1} - t_{n+1})\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_n} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n)\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n) p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_n} \mathbb{P}(Y_1(k) = t_k, Y_2(k) = i_k - t_k, 0 \leq k \leq n) p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))\\
=&\mathbb{P}(G_n) p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))
\end{aligned}$$

Therefore, $\mathbb{P}(Z_{n+1=i_{n+1}} \mid G_n) = p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))$. That is $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2(i) : i \in \mathbb{N}$.

2. Let $G_n(i,z) = \sum_{j=0}^{\infty} p_n^{\gamma}(i,j)z^j$. And $G_n(i,z) = \sum_{j=0}^{\infty} p_0^{\gamma}(i,j)z^j = z^i = g_0(z)^i$. Assume $n = k, \sum_{j=0}^{\infty} p_n^{\gamma}(i,j)z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$. Next, we will prove $n = k+1, \sum_{j=0}^{\infty} p_n^{\gamma}(i,j)z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$. Then $p_{n+1}^{\gamma}(i,j) = \sum_{l=0}^{\infty} p^{*i} * \gamma(l) p_n^{\gamma}(l,j)$. Then

$$\begin{aligned}
\sum_{j=0}^{\infty} p_{n+1}^{\gamma}(i,j)z^j &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} p^{*i} * \gamma(l) p_n^{\gamma}(l,j)z^j \\
&= \sum_{l=0}^{\infty} p^{*i} * \gamma(l) g_n(z)^l \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= \sum_{s+t=0}^{\infty} p^{*i}(s)\gamma(t) g_n(z)^{(s+t)} \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= \sum_{s+t=0}^{\infty} p^{*i}(s) g_n(z)^s \gamma(t) g_n(z)^t \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= g_{n+1}(z)^i h(g_n(z)) \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= g_{n+1}(z)^i \prod_{k=1}^{n+1} h(g_{k-1}(z))
\end{aligned}$$

3. Since $G_n(i,z)' = i g_n(z)^{i-1}(g_n(z)') \prod_{k=1}^{n} h(g_{k-1}(z)) + g_n(z)^i \prod_{k=1}^{n} h'(g_{k-1}(z))(g_{k-1}(z))'$, then $\mathbb{P}(Y_n|Y_0 = i) = G_n(i, 1-)' = im^n + \mu \prod_{k=1}^{n} m^{k-1}$.

PROBLEM II Assume $b \in (0,1), p \in (0,1), 0 < b + p \leq 1$. Let $\mu(0) = \frac{1-b-p}{1-p}, \mu(j) = bp^{j-1}, j \geq 1$. Prove:

1. $(\mu(j) : j \in \mathbb{N})$ is probability distribution and

$$g(z) := \sum_{j=0}^{\infty} \mu(j) z^j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}.$$

2. Let $b = (1-p)^2$. Prove:

   (a) $g'(1) = 1$ and

   $$g(z) = p + \frac{(1-p)^2 z}{1-pz} = \frac{p-(2p-1)z}{1-pz}.$$

   (b) $\forall n \geq 1$, then

   $$g_n(z) = \frac{np - ((n+1)p-1)z}{1+(n-1)p - npz}.$$

SOLUTION. 1. Obviously, $\forall j \geq 1$, $\mu(j) = bp^{j-1} > 0$. $\mu(0) = 1 - \frac{b}{1-p} \geq 0$. And $\sum_{j=0}^{\infty} \frac{1-b-p}{1-p} + bp^{j-1} = 1$, then $(\mu(j) : j \in \mathbb{N})$ is probability distribution. $g(z) = \sum_{j=0}^{\infty} \mu(j) z^j = \frac{1-b-p}{1-p} + \sum_{j=1}^{\infty} \frac{b}{p}(pz)j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}$.

2. Let $b=(1-p)^2$, we can easily get that $g(z)=p+\frac{(1-p)^2z}{1-pz}$. Since $g_{n+1}(z)=g(g_n(z))=\frac{(1-p)^2g_n(z)}{1-pg_n(z)}+p$, then $g_{n+1}(z)-1=\frac{(g_n(z)-1)(1-p)}{1-pz}$. Then $\frac{1}{g_{n+1}(z)}=\frac{1}{g_n(z)+1}+\frac{p}{p-1}$. So $\frac{1}{g_n(z)-1}=\frac{pn}{p-1}+\frac{1}{z-1}=\frac{1-p+pn(1-z)}{(1-p)(1-z)}$. Therefore, $g_n(z)=\frac{np-((n+1)p-1)z}{1+(n-1)p-npz}$.

□

PROBLEM III Let $\{X(n):n\in\mathbb{N}\}$ be branch process with offspring distribution $p(j):j\in\mathbb{N}$. And $g$ is the generating function. Let $m_2:=g'(1)+g''(1)<\infty, m=g'(1)<\infty$. $\forall k\geq 1$, $X_n^{(k)}=k^{-1}X_n$. Prove: $\forall\varepsilon>0, i,n\geq 1$, $\mathbb{P}(|X_n^{(k)}-im^n|\geq\varepsilon\mid X_0^{(k)}=i)\to 0, k\to\infty$.

SOLUTION. $(Y(n,j):n,j\in\mathbb{N})$ are branch process with offspring distribution $(p(j):j\in\mathbb{N})$ and initial point $Y(0,j)=i, j\in\mathbb{N}$. Then $\sum_{j=1}^k Y(n,j)\overset{d}{=}X_n\mid X_0=ki$. Since $\frac{\sum_{j=1}^k Y(n,j)}{k}\overset{\text{a.s.}}{=}\mathbb{E}(Y(n,1))=im^n, k\to\infty$. Then $\mathbb{P}(|X_n^{(k)}-im^n|\geq\varepsilon\mid X_0^{(k)}=i)=\mathbb{P}(|\frac{X_n}{k}-im^n|\geq\varepsilon\mid X_0=ki)=\mathbb{P}(|\frac{\sum_{j=1}^k Y(n,j)}{k}-im^n|\geq\varepsilon)=0, k\to\infty$.

□

PROBLEM IV Let $\{Y(n):n\in\mathbb{N}\}$ be branch process with offspring distribution $p(j):j\in\mathbb{N}$. And $g$ is the generating function, where $m:=g'(1)\in(1,\infty), m_2:=g'(1)+g''(1)<\infty$. Let $\sigma^2:=m_2-m^2=\mathbb{D}(Y(1)), \lim_{n\to\infty}\frac{Y_n}{m^n}=W$. Prove:

$$\lim_{n\to\infty}\mathbb{E}_1[(m^{-n}Y_n-W)^2]=0, \mathbb{D}_1(W)=\sigma^2m^{(-1)}(m-1)^{-1}$$

SOLUTION. For convenience we write $\mathbb{E},\mathbb{D}$ instead of $\mathbb{E}_1,\mathbb{D}_1$. Easy to get that $\mathbb{E}(m^{-2n}Y_n^2)=\frac{\sigma^2(1-m^{-n})}{m^2-m}+1$. So by Fatou theorem we get that $\mathbb{E}(W^2)\leq\lim_{n\to\infty}\mathbb{E}(m^{-2n}Y_n^2)=\frac{\sigma^2}{m^2-m}+1<\infty$. And by Doob Stochastic Processes p317 theorem 3.4 we get that $\mathbb{E}(\max_{n\in\mathbb{N}}m^{-2n}Y_n^2)<\infty$. Thus, $m^{-2n}Y_n^2$ are integrable uniformly, and so do $(m^{-n}Y_n-W)^2$. So by LCDT we can get $\lim_{n\to\infty}\mathbb{E}((m^{-n}Y_n-W)^2)=0$. Noting that

$$\mathbb{E}(m^{-2n}Y_n^2-W^2)=\mathbb{E}((m^{-n}Y_n+W)(m^{-n}Y_n-W))\leq\sqrt{\mathbb{E}((m^{-n}Y_n+W)^2)\mathbb{E}((m^{-n}Y_n-W)^2)}\to 0$$

, we get $\mathbb{E}(W^2)=\lim\mathbb{E}(m^{-2n}Y_n^2)=\frac{\sigma^2}{m^2-m}+1$. Also, $\mathbb{E}(|m^{-n}Y_n-W|)^2\leq\mathbb{E}((m^{-n}Y_n-W)^2)$, so $\mathbb{E}(W)=\lim\mathbb{E}(m^{-n}Y_n)=1$. So $\mathbb{D}(W)=\mathbb{E}(W^2)-\mathbb{E}(W)^2=\frac{\sigma^2}{m(m-1)}$.

□

PROBLEM V Let $\{Y(n):n\in\mathbb{N}\}$ be branch process with offspring distribution $p(j):j\in\mathbb{N}$, And $g$ is the generating function, where $m:=g'(1)\leq 1$. Prove $(p^\gamma(j):j\in\mathbb{N})$ is the steady-state vector of transition matrix $P_n^\gamma$, that is $\sum_{i=0}^\infty p^\gamma(i)p_n^\gamma(i,j)=p^\gamma(j), j\geq 0$.

SOLUTION. Since $\lim_{m\to\infty}p_m^\gamma(i,j)=p^\gamma(j),\forall i\in\mathbb{N}$ and $\forall k,j\in\mathbb{N}$, then $\sum_{l=0}^\infty p_n(k,l)p_m(l,j)=p_{n+m}(k,j)$. Then we only need to prove

$$\lim_{m\to\infty}\sum_{i=0}^\infty(p^\gamma(i)-p_m^\gamma(k,i))p_n^\gamma(i,j)=0$$

Since $\sum_{j=0}^{\infty} p_m^\gamma(i,j) = 1, \forall i \in \mathbb{N}$, then by LCDT we can get that $\sum_{j=0}^{\infty} p^\gamma(i,j) = 1, \forall i \in \mathbb{N}$. Then $\forall \varepsilon > 0, \exists N > 0$, $\sum_{j=N}^{\infty} p^\gamma(i,j) < \varepsilon$ and $\exists M > 0, \forall 0 \le i \le N-1, \forall m \ge M, |p_m^\gamma(k,i) - p^\gamma(i)| < \frac{\varepsilon}{N}$. Then,

$$
\begin{aligned}
&|\sum_{i=0}^{\infty}(p^\gamma(i) - p_m^\gamma(k,i))p_n^\gamma(i,j)| \\
&\le \sum_{i=0}^{\infty} |p^\gamma(i) - p_m^\gamma(k,i)| p_n^\gamma(i,j) \\
&\le \sum_{i=0}^{N-1} |p^\gamma(i) - p_m^\gamma(k,i)| p_n^\gamma(i,j) + \sum_{i=N}^{\infty} |p^\gamma(i) - p_m^\gamma(k,i)| p_n^\gamma(i,j) \\
&\le \varepsilon + \sum_{i=N}^{\infty} p^\gamma(i) p_n^\gamma(i,j) + \sum_{i=N}^{\infty} p_m^\gamma(k,i) p_n^\gamma(i,j) \\
&\le \varepsilon + \sum_{i=N}^{\infty} p^\gamma(i) + \sum_{i=N}^{\infty} p_m^\gamma(k,i) \\
&\le 2\varepsilon + 1 - \sum_{i=0}^{N-1} p^\gamma(i) + \sum_{i=0}^{N-1} |p_m^\gamma(k,i) - p^\gamma(i)| \\
&\le 2\varepsilon + \sum_{i=N}^{\infty} p^\gamma(i) + \varepsilon \\
&\le 4\varepsilon < \varepsilon
\end{aligned}
$$

□

PROBLEM VI Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$, and migrating distribution $\gamma(i) : i \in \mathbb{N}$. And $g$ is the generating function, where $m := g'(1) \le 1$. Discuss $\lim_{n\to\infty} \mathbb{E}(Y_n \mid Y_0 = i)$.

SOLUTION. $\lim_{n\to\infty} \mathbb{E}(Y_n \mid Y_0 = i) = \lim_{n\to\infty} im^n + \mu \sum_{k=1}^{n} m^{k-1} = \begin{cases} \infty & m = 1 \\ \frac{\mu}{1-m} & m < 1 \end{cases}$. □