underGraduate Homework In Mathematics

MarkovProcess 1

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PROBLEM I Assume $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration. For $t \geq 0$ we let $\mathscr{F}_{t+} := \bigcap_{s>t} \mathscr{F}_s$. Prove that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ and $(\mathscr{F}_{t+} : t \geq 0)$ is a filtration.

1. Since $(\mathscr{F}_t : t \geq 0, t \in \mathbb{R})$ is a filtration, $\forall s > t$, $\mathscr{F}_s \supset \mathscr{F}_t$, then by the definition of $\mathscr{F}_{t+}$, $\forall s > t$, $\forall x \in \mathscr{F}_t$, $x \in \mathscr{F}_s$, so $x \in \mathscr{F}_{t+}$. Therefore, $\mathscr{F}_t \subset \mathscr{F}_{t+}$.
2. Since $\forall s > t$, $\mathscr{F}_s$ is a σ-algebra, so it is obvious that $\mathscr{F}_{t+}$ is a σ-algebra. $\forall r > t$, $\forall s > r$, $\mathscr{F}_s \supset \mathscr{F}_r \supset \mathscr{F}_t$, then $\bigcap_{s>r} \mathscr{F}_s \supset \bigcap_{s>t} \mathscr{F}_s$, that is $\mathscr{F}_{r+} \supset \mathscr{F}_{t+}$.

PROBLEM II Assume $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that $\forall s, t \geq 0, \varepsilon > 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

Lemma 1. $\{\rho(X_s, X_t) < \varepsilon\} = \bigcup_{q \in D}\{\rho(X_s, q) + \rho(X_t, q) < \varepsilon\}$, where $D := E \cap Q^d$, $E = R^d$.

证明. 1. Since $\rho(X_s, X_t) < \rho(X_s, q) + \rho(X_t, q) < \varepsilon$, then $\{\rho(X_s, X_t) < \varepsilon\} \subset \bigcup_{q \in D}\{\rho(X_s, q) + \rho(X_t, q) < \varepsilon\}$,

2. Only need to prove that if $\rho(X_s, X_t) < \varepsilon$, then $\exists q \in D, \rho(X_s, q) + \rho(X_t, q) < \varepsilon$. Since D is dense in E, $\exists q \in D$ s.t. $\rho(X_s, q) \leq \frac{\varepsilon - \rho(X_s, X_t)}{4}$, so $\rho(X_t, q) + \rho(X_s, q) \leq \rho(X_t, X_s) + 2\rho(X_s, q) \leq \rho(X_t, X_s) + \frac{\varepsilon - \rho(X_s, X_t)}{2} < \varepsilon$.

□

Lemma 2. $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, $(E, \mathscr{E})$ is a measurable space, (E, ρ) is a distance space. $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(E, \mathscr{E})$. If (E, ρ) is separable, then $\mathscr{B}(E)^2 = \mathscr{B}(E^2)$. Moreover, if $\mathscr{E} = \mathscr{B}(E)$, then $\forall \varepsilon > 0, s, t \geq 0, \{\rho(X_s, X_t) \geq \varepsilon\} \in \mathscr{F}$.

证明. 1. Let $\mathscr{C}$ be all the open set of (E, ρ), $\mathscr{B}(E)$ be the Borel algebra of (E, ρ). Then $\mathscr{B}(E) = \sigma(\mathscr{C})$, where $\sigma(\mathscr{C})$ means the σ algebra generated from $\mathscr{C}$. Then $\mathscr{B}(E^2) = \sigma(\{A \times B : A, B \in \mathscr{C}\}) \supset \sigma(\{A \times E : A \in \mathscr{C}\}) = \sigma(\mathscr{C}) \times E = \mathscr{B}(E) \times E$. By the same way, we can get that $E \times \mathscr{B}(E) \subset \mathscr{B}(E^2)$. $\forall A, B \in \mathscr{B}(E)$, $A \times B$, then $A \times B = (A \times E) \cap (E \times B) \in \mathscr{B}(E) \times E \cap E \times \mathscr{B}(E) \subset \mathscr{B}(E^2)$. Therefore, $\mathscr{B}(E)^2 = \sigma(\{A \times B : A, B \in \mathscr{B}(E)\}) \subset \mathscr{B}(E^2)$. Since (E, ρ) is separable, then $\exists \mathscr{D} \subset \mathscr{C}$, which is a countable topology base of (E, ρ). Then $\forall A, B \in \mathscr{C}, A \times B \subset \sigma(\mathscr{D}^2)$, and $\sigma(\mathscr{D}^2) \subset \mathscr{B}(E^2)$, so $\mathscr{B}(E^2) = \sigma(D^2)$. Besides, oboviously $\sigma(\mathscr{D}^2) \subset \mathscr{B}(E)^2$. Therefore, $\mathscr{B}(E^2) \subset \mathscr{B}(E)^2$. Then $\mathscr{B}(E^2) = \mathscr{B}(E)^2$.

2. Since $\{(x, y) \in E^2 : \rho(x, y) \geq \varepsilon\} \in \mathscr{B}(E^2) = \mathscr{B}(E)^2$, then $\exists A \in \mathscr{B}(E)^2$ s.t. $\{(x, y) \in E^2 : \rho(x, y) \geq \varepsilon\} = A$. Let $\mathscr{H} := \{B \in \mathscr{B}(E)^2 : \{(X_s, X_t) \in B\} \in \mathscr{F}\}$. Next, we will prove $\mathscr{H} = \mathscr{B}(E)^2$.

   (a) $\mathscr{H}$ is a σ-algebra: obviously, $E^2 \in \mathscr{H}$. If $B \in \mathscr{H}$, then $\{(X_s, X_t) \in B\} \in \mathscr{F}$. So $\{(X_s, X_t) \in B^c\} = \{(X_s, X_t) \in B\}^c \in \mathscr{F}$. Thus, $B^c \in \mathscr{F}$. If $(B_n \in \mathscr{F} : n \in \mathbb{N}^+)$, then $\{(X_s, X_t) \in B_n\} \in \mathscr{F}$, then $\{(X_s, X_t) \in \bigcup_{n \in \mathbb{N}^+} B_n\} = \bigcup_{n \in \mathbb{N}^+}\{(X_s, X_t) \in B_n\} \in \mathscr{F}$.

   (b) $\mathscr{H} \supset \{A_1 \times A_2 : A_1, A_2 \in \mathscr{B}(E)\}$:
   Since $\{(X_s, X_t) \in A_1 \times A_2\} = \{X_s \in A_1\} \cap \{X_t \in A_2\} \in \mathscr{F}$, then $A_1 \times A_2 \in \mathscr{F}$.

   Then, $\{\rho(X_s, X_t) \geq \varepsilon\} = \{(X_s, X_t) \in A\} \in \mathscr{F}$.

□

SOLUTION. 1. First way to solve the problem:

Since $\mathscr{F}$ is a σ-algebra, then it is equal to prove that $\forall s,t \geq 0, \varepsilon > 0, \{\rho(X_s,X_t) < \varepsilon\} \in \mathscr{F}$. By Lemma 1 and D is countable, only need to prove that $\forall q \in D, \{\rho(X_s,q)+\rho(X_t,q) < \varepsilon\} \in \mathscr{F}$. And obviously, $\{\rho(X_s,q)+\rho(X_t,q) < \varepsilon\} = \bigcup_{p\in D\cap[0,\varepsilon]}\{\rho(X_s,q) < p, \rho(X_t,q) < \varepsilon - p\}$. So only need to prove that $\{\rho(X_s,q) < p, \rho(X_t,q) < \varepsilon - p\} \in \mathscr{F}$. Since $\{\rho(X_s,q) < p, \rho(X_t,q) < \varepsilon - p\}$ $= \{\rho(X_s,q) < p\} \cap \{\rho(X_t,q) < \varepsilon - p\}$, and $(X_t : t \geq 0, t \in \mathbb{R})$ is a stochastic process, then $\{\rho(X_s,q) < p\}, \{\rho(X_t,q) < \varepsilon - p\} \in \mathscr{F}$.

2. Second way to solve the problem:

Since $E \subset \mathbb{R}^d, \mathscr{E} = E \cap \mathscr{B}^d$, so (E,ρ) can be a separable distance space, where ρ is the distance in $\mathbb{R}^d$. By Lemma 2, we get $\{\rho(X_s,X_t) \geq \varepsilon\} \in \mathscr{F}$.

□

PROBLEM III Let $\mathscr{D}_X := \{\mu_J^X : J \in S(I)\}$ be the family of finite-dimentional distributions of a stochastic process $(X_t : t \geq 0, t \in \mathbb{R})$. $\forall (s_1,s_2) \in S(I)$ and $J = (t_1,\cdots,t_n) \in S(I)$, write $K_1 := (s_1,s_2,t_1,\cdots,t_n) \in S(I), K_2 := (s_2,s_1,t_1,\cdots,t_n) \in S(I)$. Take $A_1,A_2 \in \mathscr{E}, B \in \mathscr{E}^n$, prove that

$$\mu_{K_1}^X(A_1 \times A_2 \times B) = \mu_{K_2}^X(A_2 \times A_1 \times B)$$

and

$$\mu_{K_1}^X(E \times E \times B) = \mu_{K_2}^X(E \times E \times B) = \mu_J^X(B)$$

SOLUTION. By the definition of $\mu_J^X(H) := \mathbb{P}\{(X_{t_1},\cdots,X_{t_n}) \in H\}$, where $J = (t_1,\cdots,t_n) \in S(I)$, $H \in \mathscr{F}$. Then

$$\begin{aligned}
&\mu_{K_1}^X(A_1 \times A_2 \times B)\\
=&\mathbb{P}(\{(X_{s_1},X_{s_2},X_{t_1},\cdots,X_{t_n}) \in A_1 \times A_2 \times B\})\\
=&\mathbb{P}(\{X_{s_1} \in A_1, X_{s_2} \in A_2, (X_{t_1},\cdots,X_{t_n}) \in B\})\\
=&\mathbb{P}(\{(X_{s_2},X_{s_1},X_{t_1},\cdots,X_{t_n}) \in A_1 \times A_2 \times B\})\\
=&\mu_{K_2}^X(A_1 \times A_2 \times B)
\end{aligned} \tag{1}$$

Especially, when $A_1 = A_2 = E$, the equation is true as well. So only need to prove: $\mu_{K_1}^X(E \times E \times B) = \mu_J^X(B)$. And

$$\begin{aligned}
&\mu_{K_1}^X(E \times E \times B)\\
=&\mathbb{P}(\{X_{s_1} \in E, X_{s_2} \in E, (X_{t_1},\cdots,X_{t_n}) \in B\})\\
=&\mathbb{P}(\{X_{s_1} \in E\})\mathbb{P}(\{X_{s_2} \in E\})\mathbb{P}(\{(X_{t_1},\cdots,X_{t_n}) \in B\})\\
=&\mathbb{P}(\{(X_{t_1},\cdots,X_{t_n}) \in B\})\\
=&\mu_J^X(B)
\end{aligned} \tag{2}$$

□

PROBLEM IV Assume $(\tau_k : k \in \mathbb{N}^+)$ is an i.i.d sequence of r.v. with exponential distribution with parameter $\alpha > 0$. Let $S_n := \sum_{k=1}^n \tau_k$. For $t \geq 0, t \in \mathbb{R}$, let:

$$N_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n \leq t\}}, X_t := \sum_{n=1}^{\infty} \mathbb{1}_{\{S_n < t\}}$$

Prove that N and X are modifications of each other, but they are not indistinguishable.

SOLUTION. Since $S_n = \sum_{k=1}^n \tau_k$, where $(\tau_k : k \in \mathbb{N}^+)$ is i.i.d., then by SLLN $\frac{S_n}{n} \to \mathbb{E}(\tau_1) < \infty$. So $S_n \to \infty$, then N_t, X_t are all well-defined r.v.. Since $\forall t \geq 0, t \in \mathbb{R}$, $\mathbb{P}(\{N_t \neq X_t\}) = \mathbb{P}(\{S_n = t, \exists n \in \mathbb{N}^+\})$. Since $(\tau_k, k \in \mathbb{N}^+)$ are all continuous i.i.d. r.v. , then $(S_n, n \in \mathbb{N}^+)$ are all continuous r.v.. Therefore, $\mathbb{P}(\{S_n = t, \exists n \in \mathbb{N}^+\}) = \sum_{n=1}^{\infty} \mathbb{P}(\{S_n = t\}) = 0$. So N and X are modifications. And $\forall \omega \in \Omega$, $t := \tau_1(\omega)$, $N_t = 1 \neq X_t = 0$, then $\{N_t = X_t, \forall t \geq 0, t \in \mathbb{R}\} = \emptyset$. Thus, N and X are not indistinguishable. $\square$

PROBLEM V Assume T is non-negetive r.v. with distribution function F continuous on $\mathbb{R}$. Let $X_t = \mathbb{1}_{\{T \leq t\}}$. Prove that X is stochastically continuous.

SOLUTION. $\forall \varepsilon > 0, t \geq 0, t \in \mathbb{R}$, $s \to t^+$, $\mathbb{P}(|X_s - X_t| \geq \varepsilon) = \mathbb{P}(\{X_s - X_t \geq \varepsilon\}) = \mathbb{P}(\{X_s = 1, X_t = 0\}) = \mathbb{P}(\{t < T \leq s\}) = F(s) - F(t) \to 0$. In the same way, we can get $s \to t^-$, $\mathbb{P}(|X_s - X_t| \geq \varepsilon) \to 0$. Therefore, X is stochastically continuous. $\square$

PROBLEM VI Assume $I = \mathbb{Z}^+$, then the stochastic process $X = (X_0, X_1, \cdots)$ is a r.v. from Ω to E^{∞}. Define the distribution of X, μ_X, as follows:

$$\mu_X(A) = \mathbb{P}(X \in A), A \in \mathscr{E}^{\infty}$$

Then stochastic process X, Y are equivalent $\iff \mu_X = \mu_Y$.

SOLUTION.
1. If X, Y are equivalent, so $\forall J \in S(\mathbb{N}), \forall A \in E^{|J|}, \mu_J^X(A) = \mathbb{P}(\{(X_{t_1}, \cdots, X_{t_n}) \in A, (t_1, \cdots, t_n) = J\}) = \mathbb{P}(\{(Y_{t_1}, \cdots, Y_{t_n}) \in A\})$. Since $\mathscr{E}^{\infty} = \sigma(\mathscr{C})$, where $\mathscr{C} := \{A_J \times \prod_{i \in J^c} E_i : J \in S(\mathbb{N})\}$ is a semialgebraic set. Then $\forall A_J \times \prod_{i \in J^c} E_i \in \mathscr{C}$,

$$\begin{aligned}
&\mu_X(A) \\
=&\mathbb{P}(X \in A_J \times \prod_{i \in J^c} E_i) \\
=&\mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in A_J, X_i \in E_i, i \in J^c) \\
=&\mathbb{P}((X_{t_1}, \cdots, X_{t_n}) \in A_J) \\
=&\mu_J^X(A) \\
=&\mu_J^Y(A) \\
=&\mu_Y(A)
\end{aligned} \tag{3}$$

   By the measure extension theorem, $\mu_X(A) = \mu_Y(A), \forall A \in \mathscr{E}^{\infty}$.

2. If $\mu_X(A) = \mu_Y(A), \forall A \in \mathscr{E}^{\infty}$, then by the discussion above, we get easliy X, Y are equivalent.

$\square$