PROBLEM I Assume $(N(t) : t \geq 0)$ is a renewing process with renewing internal $\{\xi_n : n \geq 1\}$, $S_n = \sum_{k=1}^{n} \xi_k$, $N(t) := \sup\{n : S_n \leq t\}$, calculate $g(t) := \mathbb{E}(N(t)^2)$.

SOLUTION. Let $T_1 = \xi_1$, then $g(t) = \mathbb{E}(\mathbb{E}(N(t)^2 \mid T_1)) = \int_0^t \mathbb{E}(N(t)^2 \mid T_1 = x)dF(x)$. By the independence,

$$\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ \mathbb{E}((1 + N(t-x))^2) & , x \leq t \end{cases}$$

That is $\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ 1 + 2m(t-x) + g(t-x) & , x \leq t \end{cases}$. Therefore,

$$g(t) = F(t) + 2\int_0^t m(t-x)dF(x) + \int_0^t g(t-x)dF(x)$$

So $g(t) = 2m(t) - F(t) + \int_0^t g(t-x)dF(x)$. Thus, $g(t) = 2m(t) - F(t) + (2m - F) * m(t)$, so $g(t) = m(t) + 2m * m(t)$. $\square$

PROBLEM II Assume renewing internal time obey $U(0,1)$. $0 < t < 1$, calculate the distrubution of $S_{N(t)}$ and $\mathbb{E}(S_{N(t)})$.

SOLUTION. By calculating, $m(t) = \mathrm{e}^t - 1, 0 < t < 1$. $\forall 0 \leq s \leq t < 1$,

$$\mathbb{P}(S_{N(t)} \leq s) = 1 - t + \int_0^s (1 - t + x)\mathrm{e}^x dx = 1 - (t-s)\mathrm{e}^s$$

Therefore,

$$\mathbb{E}(S_{N(t)}) = \int_0^t s(1 - t + s)\mathrm{e}^s ds = \mathrm{e}^t - t - 1$$

$\square$

PROBLEM III Assume renewing internal time obey random variable X with distrubution function F. Let $\gamma_t = S_{N(t)+1} - t$ be the rest lifetime at time t. Prove:

$$\mathbb{P}(\gamma_t > z) = 1 - F(t+z) + \int_0^t (1 - F(t+z-x))dm(x)$$

SOLUTION. Let $A_z(t) = \mathbb{P}(\gamma_t > z)$, then

$$\mathbb{P}(\gamma_t > z \mid \xi_1 = x) = \begin{cases} 1 & , x > t + z \\ 0 & , t < x \leq t + z \\ A_z(t-x) & , 0 < x \leq t \end{cases}$$

Then,

$$A_z(t) = \int_0^\infty \mathbb{P}(\gamma_t > z \mid \xi_1 = x)dF(x) = 1 - F(t+z) + \int_0^t A_z(t-z)dF(x)$$

Thus,

$$A_z(t) = 1 - F(t+z) + \int_0^\infty (1 - F(t+z-x))dm(x)$$

□

PROBLEM IV One kind of devices are replaced as they are worn out. Let the lifetime of the devices by sequences $\{\xi_n : n \geq 1\}$, $S_n = \sum_{k=1}^n \xi_k$, $N(t) = \sup\{n : S_n \leq t\}$. $L(t) = S_{N(t)+1} - S_{N(t)}$. Prove: $\mathbb{P}(L(t) > x) \geq \mathbb{P}(\xi_1 > x)$.

SOLUTION . When $t \leq x$, easy to get that $\mathbb{P}(L(t) > x) = \mathbb{P}(\xi_1 > x)$. Now we assume $t > x$.

$$\begin{aligned}
\mathbb{P}(L(t) > x) =& \sum_{k=0}^\infty \mathbb{P}\left(\xi_{k+1} > x, N(t) = k\right) \\
=& \sum_{k=0}^\infty \mathbb{P}\left(\xi_{k+1} > x, S_k \leq t, \xi_{k+1} > t - S_k\right) \\
=& \sum_{k=1}^\infty \mathbb{P}\left(\xi_{k+1} > x, t - x < S_k \leq t\right) \\
&\quad + \sum_{k=0}^\infty \mathbb{P}\left(\xi_{k+1} > t - S_k, S_k \leq t - x\right) \\
=& \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}[(N(t) - N(t-x))] + \mathbb{P}(N(t) = N(t-x)) \\
=& \mathbb{P}\left(\xi_1 > x\right) + \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}[(N(t) - N(t-x)) - 1] \\
&+ \mathbb{P}(N(t) = N(t-x)) \\
=& \mathbb{P}\left(\xi_1 > x\right) + \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}\left[(N(t) - N(t-x) - 1) 1_{\{N(t) > N(t-x)\}}\right] \\
&- \mathbb{P}\left(\xi_1 > x\right) \mathbb{E}\left(1_{\{N(t) = N(t-x)\}}\right) + \mathbb{P}(N(t) = N(t-x)) \\
\geq& \mathbb{P}\left(\xi_1 > x\right).
\end{aligned} \tag{1}$$

□

PROBLEM V Toss a coin until we get two successively head, call it a renew. We toss the coin k times, call the number of renews $N(k)$. Find the distribution and expectation of interval time T

SOLUTION . Let $p_n := \mathbb{P}(T = n)$. Then $p_1 = 0, p_2 = \frac{1}{4}$. Easy to find that $p_{n+2} = \frac{1}{2}p_{n+1} + \frac{1}{4}p_n$. The characteristic equation of this sequence is $x^2 = \frac{1}{2}x + \frac{1}{4}$. The roots are $x_1 = \frac{1+\sqrt{5}}{4}, x_2 = \frac{1-\sqrt{5}}{4}$. So $p_n = Ax_1^n + Bx_2^n$. By p_1, p_2, easy to get that $p_n = \frac{1}{2\sqrt{5}}\left(x_1^{n-1} - x_2^{n-1}\right)$. So easily $\mathbb{E}(T) = \sum_{n=1}^\infty np_n = 6$. □