PROBLEM I Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $C \in \mathcal{F}$ satisfy $\mathbb{P}(C) > 0$. Let $\mathbb{P}_C : \mathcal{F} \to \mathbb{R}, \mathbb{P}_C(X) = \frac{\mathbb{P}(C \cap X)}{\mathbb{P}(C)}$. Assume $A, B \in \mathcal{F}$, and $\mathbb{P}(B \cap C) > 0$, prove that $\mathbb{P}_C(A \mid B) = \mathbb{P}(A \mid B \cap C)$.

SOLUTION. Since $\mathbb{P}_C(B) = \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)} > 0$, then $\mathbb{P}_C(A \mid B)$ is well-defined. So

$$\mathbb{P}_C(A \mid B) = \frac{\mathbb{P}_C(A \cap B)}{\mathbb{P}_C(B)} = \frac{\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)}}{\frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}} = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)} = \mathbb{P}(A \mid B \cap C)$$

□

PROBLEM II Assume that $(X_n : n \geq 0)$ is 1-dimentional simple symetry random walk, prove that $(|X_n| : n \geq 0)$ is a Markov chain ranges in $\mathbb{N}$.

SOLUTION. By the definition of $(X_n : n \geq 0)$, we know that $(X_n : n \geq 0)$ is a Markov chain in $\mathbb{Z}$. Let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), \mathcal{G}_n := \sigma(|X_1|, \cdots, |X_n|)$, then $\mathcal{G}_n \subset \mathcal{F}_n$. $\forall i \in \mathbb{N}, \mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) + \mathbb{P}(X_{n+1} = -i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = i \mid X_n) + \mathbb{P}(X_{n+1} = -i \mid X_n) = \mathbb{P}(|X_{n+1}| = i \mid X_n) = \frac{1}{2}\mathbb{1}(|X_n - i| = 1)$. Noting $|X_n - i| = 1 \iff ||X_n| - i| = 1$, so $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{F}_n) = \frac{1}{2}\mathbb{1}(||X_n| - i| = 1)$ is measurable about $\sigma(|X_n|)$. Since $\sigma(|X_n|) \subset \mathcal{G}_n \subset \mathcal{F}_n$, so we finally get that $\mathbb{P}(|X_{n+1}| = i \mid \mathcal{G}_n) = \mathbb{P}(|X_{n+1}| = i \mid |X_n|)$. So $(|X_n| : n \geq 0)$ is a Markov chain on $\mathbb{N}$. □

PROBLEM III Assume $(X_n : n \geq 0)$ is a Markov chain ranges in E. Assume $\phi : E \to F$ is injection. Prove that $(\phi(X_n) : n \geq 0)$ is a Markov chain ranges in $\phi(E)$.

SOLUTION. Without loss of generality assume $F = \phi(E)$, then ϕ is bijection. Now let $\mathcal{F}_n := \sigma(X_1, \cdots, X_n), (E, \mathscr{E}), (\Omega, \mathscr{C})$ are measurable space, where $X_n : \Omega \to E$. Let $(F, \mathscr{F}), \mathscr{F} := \phi(\mathscr{E})$. Since ϕ is bijection, then $\forall A \in \mathscr{E}, \exists B \in \mathscr{C}$, s.t. $X_n(B) = A$, then $\exists A' \in \mathscr{F}$, s.t. $A = \phi^{-1}(A')$, then $X_n(B) = \phi(A)$. Thus $\sigma(X_n) \subset \sigma(\phi(X_n))$, then in the same way, we easily get that $\sigma(X_n) = \sigma(\phi(X_n))$, so $\mathcal{F}_n = \sigma(\phi(X_1), \cdots, \phi(X_n))$. Then $\mathbb{P}(\phi(X_{n+1} = i) \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} = \phi^{-1}(i) \mid X_{n+1}) = \mathbb{P}(\phi(X_{n+1}) = i \mid \phi(X_n))$. So $(\phi(X_n) : n \geq 0)$ is Markov chain. □

PROBLEM IV Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Prove that $((X_n, Y_n) : n \geq 0)$ is Markov chain on $E \times F$.

SOLUTION. Let $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$ and $\mathcal{G}_n = \sigma(Y_0, \cdots, Y_n)$, Let $\mathcal{H}_n = \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Obviously, $\mathscr{H}_n = \sigma(\mathscr{F}_n, \mathscr{H}_n)$.

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathscr{H}_n) &= \mathbb{E}(\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathscr{H}_n, X_{n+1}) \mid \mathscr{H}_n) \\
&= \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} = j \mid \mathscr{H}_n, X_{n+1}) \mid \mathscr{H}_n) \\
&= \mathbb{E}(\mathbb{1}_i(X_{n+1})\mathbb{P}(Y_{n+1} \mid Y_n) \mid \mathscr{H}_n) \\
&= \mathbb{P}(Y_{n+1} \mid Y_n)\mathbb{P}(X_{n+1} = i \mid \mathscr{H}_n) \\
&= \mathbb{P}(Y_{n+1} \mid Y_n)\mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}$$

So $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathscr{H}_n) \in \sigma(X_n, Y_n) \subset \mathscr{H}_n$. And $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid X_n, Y_n) = \mathbb{E}(\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid X_n, Y_n, \mathscr{H}_n) \mid \mathscr{H}_n) = \mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid \mathscr{H}_n)$. Therefore, $\mathbb{P}(X_{n+1} = i, Y_{n+1} = j \mid X_n, Y_n) = \mathbb{P}(X_{n+1} \mid X_n)\mathbb{P}(Y_{n+1} \mid Y_n)$. So $((X_n, Y_n) : n \geq 0)$ is Markov chain. ☐

PROBLEM V Assume $(X_n : n \geq 0), (Y_n : n \geq 0)$ are two independent Markov chains on E, F respectively. Let $\mathcal{H}_n := \sigma((X_0, Y_0), \cdots, (X_n, Y_n))$. Prove that $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$.

SOLUTION. Take $\mathcal{F}_n, \mathcal{G}_n$ as above in Problem IV. Obviously $X_n \in \mathcal{F}_n \subset \mathcal{H}_n$. Easily $\mathbb{P}(X_{n+1} = i \mid \mathcal{H}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n, \mathcal{G}_n) = \mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) = \mathbb{P}(X_{n+1} \mid X_n)$. So $(X_n : n \geq 0)$ is Markov chain over $(\mathcal{H}_n : n \geq 0)$. ☐

PROBLEM VI Let μ_0 be a probability distribution on $\mathbb{N}$. For $n \geq 1$, let

$$\mu_n(0) = \mu_{n-1}^{*2}(0) + \mu_{n-1}^{*2}(1), \mu_n(j) = \mu_{n-1}^{*2}(j+1), \forall j \geq 1$$

Where $\mu^{*2} = \mu * \mu$. Let F_n be distribution function of μ_n. Let $F_{n-1}^{-1}(y) := \inf\{x \geq 0 : y \leq F_{n-1}(x)\}$ for $y \in [0,1]$. Assume $X_0 \sim \mu_0$, and $(U_n : n \geq 0)$ are i.i.d r.v. with distribution $U(0,1)$. Let $X_{n+1} := \max\{0, X_n + F_n^{-1}(U_n) - 1\}$. Then $(X_n : n \geq 0)$ is Markov chain.

SOLUTION. Let $\mathcal{F}_n := \sigma(X_0, \cdots, X_n)$. For $i > 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) = i + 1 - k) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) = i + 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 = i \mid X_n) = \mathbb{P}(X_{n+1} = i \mid X_n)
\end{aligned}$$

For $i = 0$, we have

$$\begin{aligned}
\mathbb{P}(X_{n+1} = 0 \mid \mathcal{F}_n) &= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid \mathcal{F}_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) \leq 1 - k) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{1}_k(X_n)\mathbb{P}(F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \sum_{k \in \mathbb{Z}} \mathbb{P}(X_n = k, F_n^{-1}(U_n) \leq 1 - k \mid X_n) \\
&= \mathbb{P}(X_n + F_n^{-1}(U_n) - 1 \leq 0 \mid X_n) = \mathbb{P}(X_{n+1} = 0 \mid X_n)
\end{aligned}$$

So $(X_n : n \geq 0)$ is Markov chain on $\mathscr{F}$ ☐