Iterative 5

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2025 年 11 月 5 日

PROBLEM I Let $\varphi_j(t)$ and $\pi_j(t)$ be the residual polynomial and the conjugate direction polynomial, respectively, for the BCG algorithm, as defined in Section 7.4.1. Let $\psi_j(t)$ be any other polynomial sequence which is defined from the recurrence

$$\psi_0(t) = 1, \qquad \psi_1(t) = (1 - \xi_0 t)\psi_0(t),$$

$$\psi_{j+1}(t) = (1 + \eta_j - \xi_j t)\psi_j(t) - \eta_j \psi_{j-1}(t).$$

1. Show that the polynomials ψ_j are consistent, i.e.,

$$\psi_j(0) = 1 \quad \text{for all } j \geq 0.$$

2. Show the following relations:

$$\psi_{j+1}\varphi_{j+1} = \psi_j\varphi_{j+1} - \eta_j(\psi_{j-1} - \psi_j)\varphi_{j+1} - \xi_j t\psi_j\varphi_{j+1},$$

$$\psi_j\varphi_{j+1} = \psi_j\varphi_j - \alpha_j t\psi_j\pi_j,$$

$$(\psi_{j-1} - \psi_j)\varphi_{j+1} = \psi_{j-1}\varphi_j - \psi_j\varphi_{j+1} - \alpha_j t\psi_{j-1}\pi_j,$$

$$\psi_{j+1}\pi_{j+1} = \psi_{j+1}\varphi_{j+1} - \beta_j\eta_j\psi_{j-1}\pi_j + \beta_j(1 + \eta_j)\psi_j\pi_j - \beta_j\xi_j t\psi_j\pi_j,$$

$$\psi_j\pi_{j+1} = \psi_j\varphi_{j+1} + \beta_j\psi_j\pi_j.$$

3. Defining

$$t_j = \psi_j(A)\varphi_{j+1}(A)r_0, \quad y_j = \big(\psi_{j-1}(A) - \psi_j(A)\big)\varphi_{j+1}(A)r_0,$$

$$p_j = \psi_j(A)\pi_j(A)r_0, \quad s_j = \psi_{j-1}(A)\pi_j(A)r_0,$$

show how the recurrence relations of the previous question translate for these vectors.

4. Find a formula that allows one to update the approximation x_{j+1} from the vectors x_{j-1}, x_j and t_j, p_j, y_j, s_j defined above.

5. Proceeding as in BICGSTAB, find formulas for generating the BCG coefficients α_j and β_j from the vectors defined in the previous question.

PROBLEM II Prove that the vectors r_j and r_i^* produced by the BCG algorithm are orthogonal to each other when $i \neq j$, while the vectors p_i and p_j^* are A-orthogonal, i.e.,

$$(Ap_j, p_i^*) = 0 \quad \text{for } i \neq j.$$

PROBLEM III Consider the linear system

$$\begin{bmatrix} A & B \\ B^T & O \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}, \tag{8.30}$$

in which $c = 0$ and B is of full rank. Define the matrix

$$P = I - B(B^T B)^{-1} B^T.$$

1. Show that P is a projector. Is it an orthogonal projector? What are the range and null spaces of P?

2. Show that the unknown x can be found by solving the linear system

$$PAPx = Pb, \tag{8.35}$$

   in which the coefficient matrix is singular but the system is consistent, i.e., there is a nontrivial solution because the right-hand side is in the range of the matrix (see Chapter 1).

3. What must be done to adapt the Conjugate Gradient Algorithm for solving the above linear system (which is symmetric, but not positive definite)? In which subspace are the iterates generated from the CG algorithm applied to (8.35)?

4. Assume that the QR factorization of the matrix B is computed. Write an algorithm based on the approach of the previous questions for solving the linear system (8.30).