PROBLEM I Assume $(N_t : t \geq 0)$ is a Possion process with parameter α. Let $P(t) := \mathbb{P}(2 \nmid N_t), Q(t) := \mathbb{P}(2 \mid N_t)$. Prove that $P(t) = \mathrm{e}^{-\alpha t}\sinh(\alpha t), Q(t) = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$.

SOLUTION. Since $(N_t : t \geq 0)$, then $\mathbb{P}(N_t = k) = \frac{(\alpha t)^k}{k!}\mathrm{e}^{-\alpha t}$. Then $\mathbb{P}(2 \nmid N_t) = \mathbb{P}(\bigcup_{k\in\mathbb{N}}\{N_t = 2k+1\}) = \sum_{k\in\mathbb{N}}\mathbb{P}(N_t = 2k+1) = \sum_{k=0}^{\infty}\frac{(\alpha t)^{2k+1}}{(2k+1)!}\mathrm{e}^{-\alpha t} = \mathrm{e}^{-\alpha t}\sinh(\alpha t)$. Then $\mathbb{P}(2 \mid N_t) = \mathbb{P}(\bigcup_{k\in\mathbb{N}}\{N_t = 2k\}) = \sum_{k\in\mathbb{N}}\mathbb{P}(N_t = 2k) = \sum_{k=0}^{\infty}\frac{(\alpha t)^{2k}}{(2k)!}\mathrm{e}^{-\alpha t} = \mathrm{e}^{-\alpha t}\cosh(\alpha t)$. $\square$

PROBLEM II Assume $(N_t : t \geq 0)$ is a Possion process with parameter α. Prove that $\lim_{t\to\infty}\frac{N_t}{t} = \alpha, a.s.$.

SOLUTION. First of all, we prove $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1$. For some $s, t \in \mathbb{Q}, 0 \leq s \leq t$, we have $\mathbb{P}(N_s > N_t) = 0$ since $N_t - N_s \sim Possion(\alpha(t-s))$. So we get $\mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 0$. Now we will prove $\exists s, t \in \mathbb{R}, 0 \leq s \leq t, N_s > N_t \iff \exists a, b \in \mathbb{Q}, 0 \leq a \leq b, N_a > N_b$. Since $\exists a, b \in \mathbb{Q} \subset \mathbb{R}, 0 \leq a \leq b, N_a > N_b$, then we only need to prove $\exists s, t \in \mathbb{R}, 0 \leq s \leq t, N_s > N.t \implies \exists a, b \in \mathbb{Q}, 0 \leq a \leq b, N_a > N_b$. Let $a_n = \frac{\lceil ns\rceil}{n}, b_n = \frac{\lceil nt\rceil}{n}$. Then $\lim a_n = s, \lim b_n = t$. Easily $a_n \geq s, b_n \geq t$. So since N_t is continuous we get $\lim N_{a_n} = N_s, \lim N_{b_n} = N_t$. Since $N_s > N_t$, we get $\exists n, N_{a_n} > N_{b_n}$. Let $a = a_n, b = b_n$ will work. So $\mathbb{P}(\forall 0 \leq s \leq t, N_s \leq N_t) = 1 - \mathbb{P}(\exists 0 \leq s \leq t, N_s > N_t) = 1 - \mathbb{P}(\exists s, t \in \mathbb{Q}, 0 \leq s \leq t, N_s > N_t) = 1 - 0 = 1$.

Let $X_0 = N_0, X_k = N_k - N_{k-1}, k = 1, \cdots, n, \cdots$, then $N_0 = 0$, a.s., $X_k \sim P(\alpha), k \in \mathbb{N}$. Then $\frac{\sum_{i=0}^{n} X_i}{n} = \frac{N_n}{n} \to \alpha, n \to \infty$, as LLN. As we have proved, N_t is increasing almost sure, then $\frac{N_{\lfloor t\rfloor}}{\lfloor t\rfloor}\frac{\lfloor t\rfloor}{t} \leq \frac{N_t}{t} \leq \frac{N_{\lceil t\rceil}}{\lceil t\rceil}\frac{\lceil t\rceil}{t}$. So $\frac{N_t}{t} \to \alpha$, a.s. . $\square$

PROBLEM III Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Prove that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$.

SOLUTION. Let $X_0 = N_0, X_k = N_k - N_{k-1}, k = 1, \cdots, n, \cdots$, then $N_0 = 0$, a.s., $X_k \sim P(\alpha), k \in \mathbb{N}$. So $\mathbb{E}(X_k) = \alpha, \mathbb{V}(X_k) = \alpha$, then by CLT, $\frac{N_n - \alpha n}{\sqrt{\alpha n}} = \frac{\sum_{k=0}^{n} X_k - \alpha n}{\sqrt{n\alpha}} \xrightarrow{d} N(0, 1)$. Noting $\frac{N_t - \alpha t}{\sqrt{\alpha t}} = \frac{N_{\lfloor t\rfloor} - \alpha\lfloor t\lfloor}{\sqrt{\alpha\lfloor t\rfloor}}\frac{\sqrt{\lfloor t\rfloor}}{\sqrt{t}} + \frac{N_t - N_{\lfloor t\rfloor} - \alpha(t - \lfloor t\rfloor)}{\sqrt{\alpha t}}$. So $t \to \infty$, $\lfloor t\rfloor, \to \infty$, and $\lfloor t\rfloor \sim t$. Since $N_t - N_{\lfloor t\rfloor} \overset{d}{=} N_{t - \lfloor t\rfloor}$, and $t - \lfloor t\rfloor \leq 1$, we easily get $\mathbb{P}(N_t - N_{\lfloor t\rfloor} = n) = \frac{((t - \lfloor t\rfloor)\alpha)^n}{n!}\mathrm{e}^{-(t - \lfloor t\rfloor)\alpha} \to 0, t \to \infty$. Then $\frac{N_t - N_{\lfloor t\rfloor}}{\sqrt{\alpha t}} \xrightarrow{d} 0$. Easily $\frac{\alpha(t - \lfloor t\rfloor)}{\alpha t} \to 0$, so finally we get that $\frac{N_t - \alpha t}{\sqrt{\alpha t}} \xrightarrow{d} N(0, 1)$ $\square$

PROBLEM IV Assume $(X_t : t \geq 0), (Y_t : t \geq 0)$ are two independent Possion processes with parameter α, β respectively. Prove that $(X_t + Y_t : t \geq 0)$ is Possion process with parameter $\alpha + \beta$.

SOLUTION. Let $Z_t := X_t + Y_t, t \geq 0$. First we prove $Z_{t+s} - Z_s \sim Possion((\alpha + \beta)t)$. Since $X_{t+s} - X_s \sim Possion(\alpha t), Y_{t+s} - Y_s \sim Possion(\beta t)$, and $X_{t+s} - X_s \perp Y_{s+t} - Y_s$, by the additional property of Possion, we can get $Z_{t+s} - Z_s = X_{t+s} - X_s + Y_{s+t} - Y_s \sim Possion((\alpha + \beta)t)$.

Second we prove $\forall 0 = t_0 < t_1 < \cdots < t_n, Z_{t_{k+1}} - Z_{t_k} k = 1, \cdots, n-1, Z_0$ are independent. Easily $Z_{t_{k+1}} - Z_{t_k} = X_{t_{k+1}} - X_{t_k} + Y_{t_{k+1}} - Y_{t_k}$ and $X_{t_{k+1}} - X_{t_k}, X_0, Y_{t_{k+1}} - Y_{t_k}, Y_1$ are independent. Then $X_{t_{k+1}} - X_{t_k} + Y_{t_{k+1}} - Y_{t_k} \in \sigma(\{X_{t_{k+1}} - X_{t_k} : k = 1, \cdots, n\} \cup \{Y_{t_{k+1}} - Y_{t_k} : k = 1, \cdots, n\}), Z_0$ are independent.

Finally, we prove that $\mathbb{P}(\forall t \in [0, \infty), \lim_{s\to t+} Z_s = Z_t, \forall t \in (0, \infty), \lim_{s\to t-} Z_s \in \mathbb{R}) = 1$ Since $Z_t = X_t + Y_t$, and $\mathbb{P}(\forall t \in [0, \infty), \lim_{s\to t+} Y_s = Y_t, \forall t \in (0, \infty), \lim_{s\to t-} Y_s \in \mathbb{R}) = 1$,

$\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} X_s = X_t, \forall t \in (0,\infty), \lim_{s\to t-} X_s \in \mathbb{R}) = 1$, then we can easily get $\mathbb{P}(\forall t \in [0,\infty), \lim_{s\to t+} Z_s = Z_t, \forall t \in (0,\infty), \lim_{s\to t-} Z_s \in \mathbb{R}) = 1$.

All in all, $(X_t + Y_t : t \geq 0)$ is a Possion process with parameter $\alpha + \beta$. $\square$

PROBLEM V Assume $(\xi_n : n \in \mathbb{N}^+)$ is a sequence of i.i.d. random variable ranging in $\mathbb{Z}^d$. Let $X_n = X_0 + \sum_{k=1}^n \xi_k$, and $X_0 \perp (\xi_n : n \in \mathbb{N}^+)$ ranging in $\mathbb{Z}^d$, too. Assume $(N_t : t \geq 0)$ is a Possion process with parameter $\alpha > 0$. Discuss $\frac{X_{N_t}}{t}$ when $t \to \infty$.

SOLUTION . First we prove that $\lim_{t\to\infty} N_t = \infty, a.s.$. Since $\mathbb{P}(\sup_t N_t \geq n) \geq \mathbb{P}(N_t \geq n), \forall t, \forall n \in \mathbb{N}$. and $\lim_{t\to\infty} \mathbb{P}(N_t \geq n) = 1 - \lim_{t\to\infty} \sum_{i=1}^{n-1} \frac{(\alpha t)^i}{i!} \mathrm{e}^{-\alpha t} = 1$, so $\mathbb{P}(\sup_t N_t \geq n) = 1, \forall n \in \mathbb{N}$, then $\mathbb{P}(\sup_t N_t = \infty) = 1$. Since Problem II we know N_t is increasing almost sure, then we can get $\mathbb{P}(\lim_{t\to\infty} N_t = \infty) = 1$.

Since $\frac{X_{N_t}}{t} = \frac{X_{N_t}}{N_t} \frac{N_t}{t}$ and we have proved that $\frac{N_t}{t} \to \alpha, a.s.$ in Problem II, so we only need to find $\frac{X_{N_t}}{N_t}$. Since $N_t \to \infty, a.s.$, we only need to find $\frac{X_n}{n}$ when $n \to \infty$.

If $\mathbb{E}(\xi_1)$ exists, then by LLN $\frac{X_n}{n} \to \mathbb{E}(\xi_1), a.s.$. Then we easily get $\frac{X_{N_t}}{t} \to \alpha \mathbb{E}(\xi_1), a.s.$. $\square$