PROBLEM I Assume $(N_t : t \geq 0)$ is a Possion process with parament $\alpha \geq 0$ and initial value 0. $\{\xi_n : n \in \mathbb{N}\}$ are i.i.d. r.v. with distrubution μ and independent with $(N_t : t \geq 0)$. Let $X_t = \sum_{n=0}^{N_t} \xi_n, t \geq 0$. $\forall s \geq 0$,

1. $(N_{s+t} - N_s : t \geq 0)$ is a Possion process with parament α.
2. $\{\xi_{N_s+n} : n \in \mathbb{N}^+\}$ are i.i.d. with distrubution μ and are independent with $(N_{s+t} - N_s : t \geq 0)$.
3. $(X_t : t \geq 0)$ satisfies $\forall 0 = t_0 < t_1 < \cdots < t_n$, $X_{t_1}, X_{t_k} - X_{t_{k-1}}, k = 2, \cdots, n$ are independent.

SOLUTION . 1. Suppose $(N_t : t \geq 0)$ satisfies $N_t - N_s \sim Possion(\alpha(t-s)), \forall t \geq s \geq 0$, then $(N_{t+r} - N_r) - (N_{s+r} - N_r) = N_{t+r} - N_{s+r} \sim Possion(a(t-s))$. Besides, $\forall 0 = t_0 < t_1 < t_2 < \cdots < t_n$, $D_t := N_{t+r} - N_r$, then $D_{t_k} - D_{t_{k-1}} = N_{t_k+r} - N_{t_{k-1}+r}, \forall k = 1, \cdots, n$. So $D_{t_k} - D_{t_{k-1}} = N_{s_k} - N_{s_{k-1}}, \forall k = 1, \cdots, n$, where $s_k = t_k + r$, are independent to each other. Besides, $N_{0+r} - N_r = 0$, obviously $D_{t_k} - D_{t_{k-1}}$ is independent to $N_{0+r} - N_0$. Last, since the orbit of $(N_t : t \geq 0)$ is continuous, then $N_{t+r} - N_r$ is continuous for any $t \geq 0$. Thus, by the definition of Possion process, we get $(N_{t+r} - N_r : t \geq 0)$ is Possion process.

2. Assume $(\Omega, \mathscr{F}), (\mathbb{N}, \mathscr{P}(\mathbb{N})), (E, \mathscr{E})$ are sigma algebra. $\xi_n : \Omega \to E$, $N_t : \Omega \to \mathbb{N}$. First of all, we prove that $\forall n \in \mathbb{N}^+$, ξ_{N_s+n} has contribution μ: $\forall A \in \mathscr{E}$, $\mathbb{P}(\{\xi_{N_s+n} \in A\}) = \mathbb{P}(\bigcup_{k=0}^{\infty}\{\xi_{k+n} \in A, N_s = k\}) = \sum_{k=0}^{\infty} \mathbb{P}(\xi_{k+n} \in A, N_s = k) = \sum_{k=0}^{\infty} \mathbb{P}(\xi_{k+n} \in A)\mathbb{P}(N_s = k) = \sum_{k=0}^{\infty} \mathbb{P}(\xi_1 \in A)\mathbb{P}(N_s = k) = \mathbb{P}(\xi_1 \in A)$.

   Secondly, we prove that $\{\xi_{N_s+n} : n \in \mathbb{N}^+\}$ are independent: $\forall J \subset \mathbb{N}^+, \mathrm{card}(J) < \infty, \{A_i \in \mathscr{E} : i \in J\}$, then $\mathbb{P}(\bigcap_{i \in J}\{\xi_{N_s+i} \in A_i\}) = \mathbb{P}(\bigcup_{k=0}^{\infty}(\bigcap_{i\in J}\{\xi_{k+i} \in A_i, N_s = k\})) = \sum_{k=0}^{\infty} \mathbb{P}(\bigcap_{i \in J}\{\xi_{k+i} \in A_i\} \cap \{N_s = k\})) = \sum_{k=0}^{\infty} \mathbb{P}(\bigcap_{i\in J}\{\xi_{k+i} : i \in A_i\})\mathbb{P}(N_s = k) = \sum_{k=0}^{\infty} \mathbb{P}(\bigcap_{i \in J}\{\xi_{1+i} \in A_i\})\mathbb{P}(N_s = k) = \mathbb{P}(\bigcap_{i\in J}\{\xi_{1+i} \in A_i\}) = \prod_{i \in J} \mathbb{P}(\xi_{1+i} \in A_i) = \prod_{i\in J} \mathbb{P}(\xi_{N_s+i} \in A_i)$.

   Last, we will prove that $\{\xi_{N_s+n} : n \in \mathbb{N}^+\}$ are independent with $(N_{t+s} - N_s : t \geq 0)$. $\forall \{A_n \in \mathscr{E} : n \in \mathbb{N}^+\}, k \in \mathbb{N}$, then $\mathbb{P}(\bigcap_{n\in\mathbb{N}^+}\{\xi_{N_s+n} \in A_n\} \cap \{N_{t+s} - N_s = k\}) = \mathbb{P}(\bigcup_{i\in\mathbb{N}}(\bigcap_{n\in\mathbb{N}^+}\{\xi_{i+n} \in A_n\} \cap \{N_{t+s} = k+i, N_s = k\})) = \sum_{i\in\mathbb{N}} \mathbb{P}(\bigcap_{n\in\mathbb{N}^+}\{\xi_{i+n} \in A_n\} \cap \{N_{t+s} = k+i, N_s = i\}) = \sum_{i\in\mathbb{N}} \mathbb{P}(\bigcap_{n\in\mathbb{N}^+}\{\xi_{i+n} \in A_n\})\mathbb{P}(N_{t+s} = k+i, N_s = i) = \sum_{i\in\mathbb{N}} \prod_{n\in\mathbb{N}^+} \mathbb{P}(\xi_{1+n} \in A_n)\mathbb{P}(N_{t+s} = k+i, N_s = i) = \prod_{n\in\mathbb{N}^+} \mathbb{P}(\xi_{1+n} \in A_n)\mathbb{P}(N_{t+s} - N_s = k) = \prod_{n\in\mathbb{N}^+} \mathbb{P}(\xi_{N_s+n} \in A_n)\mathbb{P}(N_{t+s} - N_s = k) = \mathbb{P}(\bigcap_{n\in\mathbb{N}^+}\{\xi_{N_s+n} \in A_n\})\mathbb{P}(N_{t+s} - N_s = k)$.

3. $\forall 0 = t_0 < t_1 < \cdots < t_n$, then $X_{t_1} = \sum_{i=1}^{N_{t_1}} \xi_i, X_{t_k} - X_{t_{k-1}} = \sum_{i=1}^{N_{t_k} - N_{t_{k-1}}} \xi_{N_{t_{k-1}}+i}\xi_i, k =$

$2, \cdots, n$, then $\forall \{A_k \in \mathscr{E} : k = 1, \cdots, n\}$,

$$
\begin{aligned}
&\mathbb{P}(\bigcap_{k=1}^{n} \sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k) \\
=&\mathbb{P}(\bigcup_{0\le u_1 \le \cdots \le u_n} \{\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, N_{t_k} = u_k, k = 1, \cdots, n\}) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n | N_{t_k} = u_k, k = 1, \cdots, n\})\mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k, k = 1, \cdots, n\})\mathbb{P}(N_{t_k} = u_k, k = 1, \cdots, n) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=u_{k-1}+1}^{u_k} \xi_i \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} = u_j) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k) \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = u_j - u_{j-1}) \\
=&\sum_{0\le u_1 \le \cdots \le u_n} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k)\mathbb{P}(N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\sum_{u_1-u_0\in\mathbb{N}} \cdots \sum_{u_n-u_{n-1}\in\mathbb{N}} \prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \sum_{u_k-u_{k-1}\in\mathbb{N}} \mathbb{P}(\sum_{i=1}^{u_k-u_{k-1}} \xi_{u_{k-1}+i} \in A_k, N_{t_k} - N_{t_{k-1}} = u_k - u_{k-1}) \\
=&\prod_{k=1}^{n} \mathbb{P}(\sum_{i=1}^{N_{t_k}-N_{t_{k-1}}} \xi_{i+N_{t_{k-1}}} \in A_k)
\end{aligned}
\tag{1}
$$

□

PROBLEM II X is a possion random measure on $(E, \mathscr{E})$ with intensity μ, where μ is σ finite measure. Prove $\forall f \in (E, \mathscr{E}),\ f \ge 0$,

$$\mathbb{E}e^{-X(f)} = \exp\left\{-\int_E (1 - e^{-f(x)})\mu(dx)\right\}.$$

SOLUTION. 1. First, we consider μ is finite and $f = \sum_{i=1}^{n} \theta_i \mathbb{1}_{B_i}(x)$: Since X is a possion random measure, then $\exists \eta \xi_1, \cdots, \xi_n, \cdots$, where $\eta \sim Possion(\mu(E)), \{\xi_n : n \in \mathbb{N}^+\}$ are i.i.d., $\xi_1 \sim \overline{\mu} :=$

$\mu(E)^{-1}\mu$.

$$\begin{aligned}
\mathbb{E}e^{-X(f)} &= \sum_{m=0}^{\infty} \mathbb{E}(\exp\left\{-\sum_{j=1}^{m}\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(\xi_j)\right\})\mathbb{P}(\eta = m) \\
&= \sum_{m=0}^{\infty} e^{-\mu(E)}\frac{\mu(E)^m}{m!}(\mathbb{E}(\exp\left\{-\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(\xi_1)\right\}))^m \\
&= \sum_{m=0}^{\infty} e^{-\mu(E)}\frac{1}{m!}(\int_E \exp\left\{-\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(x)\right\}\mu(dx))^m \\
&= \exp\left\{-\mu(E) + \int_E \exp\{-\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(x)\}\mu(dx)\right\} \qquad (2) \\
&= \exp\left\{\int_E \exp\{-\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(x)\} - 1\mu(dx)\right\} \\
&= \exp\left\{\int_E -(1-\exp\{-\sum_{i=1}^{n}\theta_i \mathbb{1}_{B_i}(x)\})\mu(dx)\right\} \\
&= \exp\left\{\int_E -(1-\exp\{-f(x)\})\mu(dx)\right\}
\end{aligned}$$

2. Secondly, we consider μ is finite and $0 \le f \in (E, \mathbb{E})$: Then $\exists f_j \ge 0, j \in \mathbb{N}$ is simple measurable function, such that $f_j \to f, j \to \infty, \forall \omega \in \Omega$. So by LCDT, we get $\mathbb{E}e^{-X(f)} = \exp\left\{\int_E -(1-\exp\{-f(x)\})\mu(dx)\right\}$.
3. Lastly, we consider μ is σ finite and $0 \le f \in (E, \mathbb{E})$: Then $E = \bigcup_{i=1}^{\infty} E_i$, $\forall i, \mu(E_i) < \infty$. Let $X_i = X\mathbb{1}_{E_i}(x)$, so $\mathbb{E}e^{-X(f)} = \mathbb{E}e^{\sum_{i=1}^{\infty} X_i(f)} = \mathbb{E}\exp\{\sum_{i=2}^{\infty} X_i(f)\}\exp(\int_{E_1} -(1 - \exp(-f(x)))\mu(dx)) = \exp\left\{\sum_{i=1}^{\infty}\int_{E_i} -(1-\exp\{-f(x)\})\mu(dx)\right\}$.

□

PROBLEM III μ is a finite measure, X is a possion random measure with intensity μ on $(E, \mathscr{E})$. $\phi : (E, \mathscr{E}) \to (F, \mathscr{F})$ is measurable. Prove : $X \circ \phi^{-1}$ is a possion random measurable witn intensity $\mu \circ \phi^{-1}$ on $(F, \mathscr{F})$.

SOLUTION . Obviously, $\mu \circ \phi^{-1}$ is measurable on $(\mathscr{F})$ and finite.

1. First, we prove $X \circ \phi^{-1}(B) \sim Possion(\mu \circ \phi^{-1}(B)), \forall B \in \mathscr{F}$: Since $\forall B \in \mathscr{F}, \phi^{-1}(B) \in \mathscr{E}$, then $X \circ \phi^{-1}(B) = X(\phi^{-1}(B)) \sim Possion(\mu(\phi^{-1}(B))) = Possion(\mu \circ \phi^{-1}(B))$.
2. Secondly, $\forall B_i \in \mathscr{F}, i \in \mathbb{N}, B_i \cap B_j = \emptyset, i \ne j$, Then $\phi^{-1}(B_i) \cap \phi^{-1}(B_j) = \varnothing$, then $X(\phi^{-1}(B_i)), i \in \mathbb{N}$ are independent. Besides, $X \circ \phi^{-1}(\bigcup_{i\in\mathbb{N}} B_i) = X(\phi^{-1}(\bigcup_{i\in\mathbb{N}} B_i)) = X(\bigcup_{i\in\mathbb{N}} \phi^{-1}(B_i)) = \sum_{i\in\mathbb{N}} X(\phi^{-1}(B_i)) = \sum_{i\in\mathbb{N}} X \circ \phi^{-1}(B_i)$.

□

PROBLEM IV $\alpha \ge 0$ is constant, μ is probability on $\mathbb{R}$ and $\mu(\{0\}) = 0$. Let $N(ds, dz, du)$ is a possion random measure on $(0, \infty) \times \mathbb{R} \times (0, \infty)$ with intensity $ds\mu(dz)du$. Y_0 is independent with

$N(ds,dz,du)$. Let

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}} \int_0^{\alpha} zN(ds,dz,du), t > 0.$$

Prove: $(Y_t : t \geq 0)$ is a compound possion random process with rate α and jumpping distrubution μ

Solution. 1. Obvioully, since $N(ds,dz,du)$ is a possion random measure, then $\forall 0 = t_0 < t_1 < \cdots < t_n$, $Y_{t_0}, Y_{t_k} - Y_{t_{k-1}}, k = 2, \cdots, n$ are independent.

2. $\forall s,t \geq 0, \theta \in \mathbb{R}$, then

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{\mathrm{i}\theta(Y_{s+t}-Y_s)} &= \mathbb{E}\exp\{\int_s^{s+t}\int_{\mathbb{R}}\int_0^{\alpha} \mathrm{i}\theta zN(ds,du,dz)\} \\
&= \exp\left\{t\alpha\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}\theta z}-1)\mu(dz)\right\} \\
&= \exp(-t\alpha)\sum_{k=0}^{\infty}\frac{1}{k!}(t\alpha\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu(dz))^k \\
&= \mathrm{e}^{-t\alpha}\sum_{k=0}^{\infty}\frac{(\alpha k)^k}{k!}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\theta z}\mu^{*k}(dz)
\end{aligned} \tag{3}$$

□

Problem V μ is a finite measure, X is a possion random measure with intensity μ on $(E, \mathscr{E})$. Prove:

1. $\mathbb{E}[X(f)\mathrm{e}^{-X(g)}] = \mu(f\mathrm{e}^{-g})\mathbb{E}[\mathrm{e}^{-X(g)}]$
2. $\mathbb{E}[X(f)^2\mathrm{e}^{-X(g)}] = [\mu(f^2\mathrm{e}^{-g}) + \mu(f\mathrm{e}^{-g})^2]\mathbb{E}[\mathrm{e}^{-X(g)}]$

1. $\forall\theta \geq 0$, then $\mathbb{E}\mathrm{e}^{-X(\theta f+g)} = \exp\left\{-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)})\mu(dx)\right\}$. Since $\mathrm{e}^{-X(\theta f+g)} \geq 0$, then $\mathbb{E}[X(f)\mathrm{e}^{-X(\theta f+g)}] = \exp\left\{-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)})\mu(dx)\right\}\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(dx)$. Thus, when $\theta = 0$, we can get $\mathbb{E}[X(f)\mathrm{e}^{-X(g)}] = \mu(f\mathrm{e}^{-g})\mathbb{E}[\mathrm{e}^{-X(g)}]$.
2. $\forall\theta \geq 0$, then $\mathbb{E}[X(f)\mathrm{e}^{-X(\theta f+g)}] = \exp\left\{-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)})\mu(dx)\right\}\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(dx)$. Since $f \geq 0, \mathrm{e}^{-X(\theta f+g)} \geq 0$, then

$$\begin{aligned}
&\mathbb{E}[X(f)^2\mathrm{e}^{-X(\theta f+g)}] \\
=&\exp\left\{-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)})\mu(dx)\right\}(\int_E f(x)\mathrm{e}^{-\theta f(x)-g(x)}\mu(dx))^2 \\
+&\exp\left\{-\int_E(1-\mathrm{e}^{-\theta f(x)-g(x)})\mu(dx)\right\}\int_E f(x)^2\mathrm{e}^{\theta f(x)-g(x)}\mu(dx).
\end{aligned}$$

Thus, when $\theta = 0$, we can get $\mathbb{E}[X(f)^2\mathrm{e}^{-X(g)}] = [\mu(f^2\mathrm{e}^{-g}) + \mu(f\mathrm{e}^{-g})^2]\mathbb{E}[\mathrm{e}^{-X(g)}]$.