

**PROBLEM I** Assume  $(N(t) : t \geq 0)$  is a renewing process with renewing interval  $\{\xi_n : n \geq 1\}$ ,  $S_n = \sum_{k=1}^n \xi_k$ ,  $N(t) := \sup\{n : S_n \leq t\}$ , calculate  $g(t) := \mathbb{E}(N(t)^2)$ .

**SOLUTION.** Let  $T_1 = \xi_1$ , then  $g(t) = \mathbb{E}(\mathbb{E}(N(t)^2 \mid T_1)) = \int_0^t \mathbb{E}(N(t)^2 \mid T_1 = x) dF(x)$ . By the independence,

$$\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ \mathbb{E}((1 + N(t-x))^2) & , x \leq t \end{cases}$$

That is  $\mathbb{E}(N(t)^2 \mid T_1 = x) = \begin{cases} 0 & , x > t \\ 1 + 2m(t-x) + g(t-x) & , x \leq t \end{cases}$ . Therefore,

$$g(t) = F(t) + 2 \int_0^t m(t-x) dF(x) + \int_0^t g(t-x) dF(x)$$

So  $g(t) = 2m(t) - F(t) + \int_0^t g(t-x) dF(x)$ . Thus,  $g(t) = 2m(t) - F(t) + (2m - F) * m(t)$ , so  $g(t) = m(t) + 2m * m(t)$ .  $\square$

**PROBLEM II** Assume renewing internal time obey  $U(0, 1)$ .  $0 < t < 1$ , calculate the distrubution of  $S_{N(t)}$  and  $\mathbb{E}(S_{N(t)})$ .

**SOLUTION.** By calculating,  $m(t) = e^t - 1, 0 < t < 1$ .  $\forall 0 \leq s \leq t < 1$ ,

$$\mathbb{P}(S_{N(t)} \leq s) = 1 - t + \int_0^s (1 - t + x) e^x dx = 1 - (t - s) e^s$$

Therefore,

$$\mathbb{E}(S_{N(t)}) = \int_0^t s(1 - t + s) e^s ds = e^t - t - 1$$

$\square$

**PROBLEM III** Assume renewing internal time obey random variable  $X$  with distrubution function  $F$ . Let  $\gamma_t = S_{N(t)+1} - t$  be the rest lifetime at time  $t$ . Prove:

$$\mathbb{P}(\gamma_t > z) = 1 - F(t+z) + \int_0^t (1 - F(t+z-x)) dm(x)$$

**SOLUTION.** Let  $A_z(t) = \mathbb{P}(\gamma_t > z)$ , then

$$\mathbb{P}(\gamma_t > z \mid \xi_1 = x) = \begin{cases} 1 & , x > t+z \\ 0 & , t < x \leq t+z \\ A_z(t-x) & , 0 < x \leq t \end{cases}$$

Then,

$$A_z(t) = \int_0^\infty \mathbb{P}(\gamma_t > z \mid \xi_1 = x) dF(x) = 1 - F(t+z) + \int_0^t A_z(t-z) dF(x)$$

Thus,

$$A_z(t) = 1 - F(t + z) + \int_0^\infty (1 - F(t + z - x))dm(x)$$

□

**PROBLEM IV** One kind of devices are replaced as they are worn out. Let the lifetime of the devices by sequences  $\{\xi_n : n \geq 1\}$ ,  $S_n = \sum_{k=1}^n \xi_k$ ,  $N(t) = \sup\{n : S_n \leq t\}$ .  $L(t) = S_{N(t)+1} - S_{N(t)}$ . Prove:  $\mathbb{P}(L(t) > x) \geq \mathbb{P}(\xi_1 > x)$ .

**PROBLEM V** Toss a coin until we get two successively head, call it a renew. We toss the coin  $k$  times, call the number of renews  $N(k)$ . Find the distribution and expectation of interval time  $T$