PROBLEM I

1. Assume $\{Y_i(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with descending distribution $(p(i) : i \in \mathbb{Z}_+)$ and the migrating probobility respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{Z}_+)$. Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with descending distribution $p(i) : i \in \mathbb{Z}_+$ and migrating probability $\gamma_1 * \gamma_2(i) : i \in \mathbb{Z}_+$.

2. Let $\{Y(n) : n \in \mathbb{Z}_+\}$ be migrating branch process with descending distribution $p(j) : j \in \mathbb{Z}_+$ and the migrating distribution $\gamma(i) : i \in \mathbb{Z}_+$. $P_n^\gamma = (p_n^\gamma(i,j); i,j \in \mathbb{Z}_+)$ is the n-th transition matrix. Prove: $\forall i, n \geq 1$
$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$
where h is the generating function of $(\gamma(j) : j \in \mathbb{Z}_+)$. g is the generating funtion of $(p(j) : j \in \mathbb{Z}_+)$.

3. h, g are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$. Prove: $\forall i, n \geq 1$,
$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

PROBLEM II Assume $b \in (0,1), p \in (0,1)$. Let $\mu(0) = \frac{1-b-p}{1-p} \mu(j) = bp^{j-1}, j \geq 1$. Prove:

1. $(\mu(j) : j \in \mathbb{Z}_+)$ is probability distribution and
$$g(z) := \sum_{j=0}^{\infty} \mu(j) z^j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}.$$

2. Let $b = (1-p)^2$. Prove:

   (a) $g'(1) = 1$ and
   $$g(z) = p + \frac{(1-p)^2 z}{1-pz} = \frac{p-(2p-1)z}{1-pz}.$$

   (b) $\forall n \geq 1$, then
   $$g_n(z) = \frac{np - ((n+1)p-1)z}{1+(n-1)p - npz}.$$

PROBLEM III Let $\{Y(n) : n \in \mathbb{Z}_+\}$ be branch process with descending distribution $p(j) : j \in \mathbb{Z}_+$. And g is the generating function. Let $m_2 := g'(1) + g''(1) < \infty$. $\forall k \geq 1$, $X_n^{(k)} = k^{-1} X_n$. Prove: $\forall \varepsilon > 0, i, n \geq 1$, $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) \to 0, k \to \infty$. PROBLEM IV Let $\{Y(n) : n \in \mathbb{Z}_+\}$ be branch process with descending distribution $p(j) : j \in \mathbb{Z}_+$. And g is the generating function, where $m := g'(1) \in (1, \infty), m_2 := g'(1) + g''(1) < \infty$. Let $\sigma^2 := m_2 - m^2 = \mathbb{D}(Y(1))$. Prove:
$$\lim_{n\to\infty} \mathbb{E}_1[(m^{-n} X_n - W)^2] = 0, \mathbb{D}_1(W) = \sigma^2 m^{(-1)} (m-1)^{-1}$$

PROBLEM V Let $\{Y(n) : n \in \mathbb{Z}_+\}$ be branch process with descending distribution $p(j) : j \in \mathbb{Z}_+$. And g is the generating function, where $m := g'(1) \leq 1$. Prove $(p^\gamma(j) : j \in \mathbb{Z}_+)$ is the steady-state vector of transition matrix P_n^γ, that is $\sum_{i=0}^{\infty} p^\gamma(i) p_n^\gamma(i,j) = p^\gamma(j), i \geq 0$. PROBLEM VI Let $\{Y(n) : n \in \mathbb{Z}_+\}$ be branch process with descending distribution $p(j) : j \in \mathbb{Z}_+$. And g is the generating function, where $m := g'(1) \leq 1$. Discuss $\lim_{n\to\infty} \mathbb{E}(Y_n \mid Y_0 = i)$.