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PROBLEM I

1. Assume $\{Y_1(n) : n \geq 0\}, \{Y_2(n) : n \geq 0\}$ are two independent migrating branch process with offspring distribution $(p(i) : i \in \mathbb{N})$ and the migrating probability respectively are $(\gamma_1(i) : i \in \mathbb{Z}_+), (\gamma_2(i) : i \in \mathbb{N})$。Prove: $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2(i) : i \in \mathbb{N}$。

2. Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$ and the migrating distribution $\gamma(i) : i \in \mathbb{N}$。$P_n^\gamma = (p_n^\gamma(i,j); i,j \in \mathbb{N})$ is the n-th transition matrix. Prove: $\forall i, n \geq 1$
$$\sum_{j=0}^{\infty} p_n^\gamma(i,j) z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$$
where h is the generating function of $(\gamma(j) : j \in \mathbb{N})$。$g$ is the generating function of $(p(j) : j \in \mathbb{N})$。

3. h, g are defined as above. Assume $m := g'(1-) < \infty, \mu := h'(1-) < \infty$。Prove: $\forall i, n \geq 1$,
$$\mathbb{P}(Y_n \mid Y_0 = i) = im^n + \mu \sum_{k=1}^{n} m^{k-1}$$

1. Let $Z_n := Y_1(n) + Y_2(n)$, $\forall \{i_0, \cdots, i_n, i_{n+1}\} \in \mathbb{N}$, $G_n = \{Z_k = i_k, 0 \leq k \leq n\}$。

$$
\begin{aligned}
&\mathbb{P}(Z_{n+1} = i_{n+1}, G_n)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_{n+1}=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, Y_2(k) = i_k - t_k, 0 \leq k \leq n+1)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_{n+1}}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n+1)\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n+1)\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n)p_1^{\gamma_1}(t_n, t_{n+1})\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n)p_1^{\gamma_2}(i_n - t_n, i_{n+1} - t_{n+1})\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_{n+1}} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n)p^{*t_n} * \gamma_1(t_{n+1})\\
\times&\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n)p^{*i_n - t_n} * \gamma_2(i_{n+1} - t_{n+1})\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_n} \mathbb{P}(Y_1(k) = t_k, 0 \leq k \leq n)\mathbb{P}(Y_2(k) = i_k - t_k, 0 \leq k \leq n)p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))\\
=&\sum_{t_0=0}^{i_0} \cdots \sum_{t_n=0}^{i_n} \mathbb{P}(Y_1(k) = t_k, Y_2(k) = i_k - t_k, 0 \leq k \leq n)p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))\\
=&\mathbb{P}(G_n)p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))
\end{aligned}
$$

Therefore, $\mathbb{P}(Z_{n+1=i_{n+1}} \mid G_n) = p^{*i_n} * (\gamma_1 * \gamma_2(i_{n+1}))$。That is $\{Y_1(n) + Y_2(n) : n \geq 0\}$ is migrating branching process with offspring distribution $p(i) : i \in \mathbb{N}$ and migrating probability $\gamma_1 * \gamma_2(i) : i \in \mathbb{N}$。

2. Let $G_n(i, z) = \sum_{j=0}^{\infty} p_n^{\gamma}(i, j)z^j$。And $G_n(i, z) = \sum_{j=0}^{\infty} p_0^{\gamma}(i, j)z^j = z^i = g_0(z)^i$。Assume $n = k$, $\sum_{j=0}^{\infty} p_n^{\gamma}(i, j)z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$。Next, we will prove $n = k + 1$, $\sum_{j=0}^{\infty} p_n^{\gamma}(i, j)z^j = g_n(z)^i \prod_{k=1}^{n} h(g_{k-1}(z)), |z| \leq 1$。Then $p_{n+1}^{\gamma}(i, j) = \sum_{l=0}^{\infty} p^{*i} * \gamma(l)p_n^{\gamma}(l, j)$。

Then

$$\begin{aligned}
\sum_{j=0}^{\infty} p_{n+1}^{\gamma}(i,j)z^j &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} p^{*i} * \gamma(l)p_n^{\gamma}(l,j)z^j \\
&= \sum_{l=0}^{\infty} p^{*i} * \gamma(l)g_n(z)^l \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= \sum_{s+t=0}^{\infty} p^{*i}(s)\gamma(t)g_n(z)^{(s+t)} \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= \sum_{s+t=0}^{\infty} p^{*i}(s)g_n(z)^s\gamma(t)g_n(z)^t \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= g_{n+1}(z)^i h(g_n(z)) \prod_{k=1}^{n} h(g_{k-1}(z)) \\
&= g_{n+1}(z)^i \prod_{k=1}^{n+1} h(g_{k-1}(z))
\end{aligned}$$

3. Since $G_n(i,z)' = ig_n(z)^{i-1}(g_n(z)')\prod_{k=1}^{n} h(g_{k-1}(z)) + g_n(z)^i \prod_{k=1}^{n} h'(g_{k-1}(z))(g_{k-1}(z))'$, then $\mathbb{P}(Y_n|Y_0=i) = G_n(i,1-)' = im^n + \mu\prod_{k=1}^{n} m^{k-1}$。

PROBLEM II Assume $b \in (0,1), p \in (0,1), 0 < b+p \le 1$。Let $\mu(0) = \frac{1-b-p}{1-p}, \mu(j) = bp^{j-1}, j \ge 1$。Prove:

1. $(\mu(j) : j \in \mathbb{N})$ is probability distribution and
$$g(z) := \sum_{j=0}^{\infty} \mu(j)z^j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}.$$

2. Let $b = (1-p)^2$。Prove:
   (a) $g'(1) = 1$ and
   $$g(z) = p + \frac{(1-p)^2 z}{1-pz} = \frac{p-(2p-1)z}{1-pz}.$$
   (b) $\forall n \ge 1$, then
   $$g_n(z) = \frac{np-((n+1)p-1)z}{1+(n-1)p-npz}.$$

SOLUTION. 1. Obviously, $\forall j \ge 1$, $\mu(j) = bp^{j-1} > 0$。$\mu(0) = 1 - \frac{b}{1-p} \ge 0$。And $\sum_{j=0}^{\infty} \frac{1-b-p}{1-p} + bp^{j-1} = 1$, then $(\mu(j) : j \in \mathbb{N})$ is probability distribution. $g(z) = \sum_{j=0}^{\infty}\mu(j)z^j = \frac{1-b-p}{1-p} + \sum_{j=1}^{\infty}\frac{b}{p}(pz)j = \frac{1-b-p}{1-p} + \frac{bz}{1-pz}$。

2. Let $b = (1-p)^2$, we can easily get that $g(z) = p + \frac{(1-p)^2 z}{1-pz}$。Since $g_{n+1}(z) = g(g_n(z)) = \frac{(1-p)^2 g_n(z)}{1-pg_n(z)} + p$, then $g_{n+1}(z) - 1 = \frac{(g_n(z)-1)(1-p)}{1-pz}$。Then $\frac{1}{g_{n+1}(z)} = \frac{1}{g_n(z)+1} + \frac{p}{p-1}$。So $\frac{1}{g_n(z)-1} = \frac{pn}{p-1} + \frac{1}{z-1} = \frac{1-p+pn(1-z)}{(1-p)(1-z)}$。Therefore, $g_n(z) = \frac{np-((n+1)p-1)z}{1+(n-1)p-npz}$。

$\square$

PROBLEM III Let $\{X(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$。And g is the generating function. Let $m_2 := g'(1) + g''(1) < \infty, m = g'(1) < \infty$。$\forall k \geq 1$，$X_n^{(k)} = k^{-1}X_n$。Prove: $\forall \varepsilon > 0, i, n \geq 1$，$\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) \to 0, k \to \infty$。

SOLUTION. $(Y(n,j) : n, j \in \mathbb{N})$ are branch process with offspring distribution $(p(j) : j \in \mathbb{N})$ and initial point $Y(0,j) = i, j \in \mathbb{N}$。Then $\sum_{j=1}^k Y(n,j) \overset{d}{=} X_n \mid X_0 = ki$。Since $\frac{\sum_{j=1}^k Y(n,j)}{k} \overset{\text{a.s.}}{=} \mathbb{E}(Y(n,1)) = im^n, k \to \infty$。Then $\mathbb{P}(|X_n^{(k)} - im^n| \geq \varepsilon \mid X_0^{(k)} = i) = \mathbb{P}(|\frac{X_n}{k} - im^n| \geq \varepsilon \mid X_0 = ki) = \mathbb{P}(|\frac{\sum_{j=1}^k Y(n,j)}{k} - im^n| \geq \varepsilon) = 0, k \to \infty$。 □

PROBLEM IV Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$。And g is the generating function, where $m := g'(1) \in (1, \infty), m_2 := g'(1) + g''(1) < \infty$。Let $\sigma^2 := m_2 - m^2 = \mathbb{D}(Y(1)), \lim_{n\to\infty} \frac{Y_n}{m^n} = W$。Prove:

$$\lim_{n\to\infty} \mathbb{E}_1[(m^{-n}Y_n - W)^2] = 0, \mathbb{D}_1(W) = \sigma^2 m^{(-1)}(m-1)^{-1}$$

SOLUTION. For convenience we write $\mathbb{E}, \mathbb{D}$ instead of $\mathbb{E}_1, \mathbb{D}_1$。Easy to get that $\mathbb{E}(m^{-2n}Y_n^2) = \frac{\sigma^2(1-m^{-n})}{m^2-m} + 1$。So by Fatou theorem we get that $\mathbb{E}(W^2) \leq \lim_{n\to\infty} \mathbb{E}(m^{-2n}Y_n^2) = \frac{\sigma^2}{m^2-m} + 1 < \infty$。And by Doob Stochastic Processes p317 theorem 3.4 we get that $\mathbb{E}(\max_{n\in\mathbb{N}} m^{-2n}Y_n^2) < \infty$。Thus, $m^{-2n}Y_n^2$ are integrable uniformly, and so do $(m^{-n}Y_n - W)^2$。So by LCDT we can get $\lim_{n\to\infty} \mathbb{E}((m^{-n}Y_n - W)^2) = 0$。Noting that

$$\mathbb{E}(m^{-2n}Y_n^2 - W^2) = \mathbb{E}((m^{-n}Y_n + W)(m^{-n}Y_n - W)) \leq \sqrt{\mathbb{E}((m^{-n}Y_n + W)^2)\mathbb{E}((m^{-n}Y_n - W)^2)} \to 0$$

, we get $\mathbb{E}(W^2) = \lim \mathbb{E}(m^{-2n}Y_n^2) = \frac{\sigma^2}{m^2-m} + 1$。Also, $\mathbb{E}(|m^{-n}Y_n - W|)^2 \leq \mathbb{E}((m^{-n}Y_n - W)^2)$，so $\mathbb{E}(W) = \lim \mathbb{E}(m^{-n}Y_n) = 1$。So $\mathbb{D}(W) = \mathbb{E}(W^2) - \mathbb{E}(W)^2 = \frac{\sigma^2}{m(m-1)}$。

□

PROBLEM V Let $\{Y(n) : n \in \mathbb{N}\}$ be branch process with offspring distribution $p(j) : j \in \mathbb{N}$，And g is the generating function, where $m := g'(1) \leq 1$。Prove $(p^\gamma(j) : j \in \mathbb{N})$ is the steady-state vector of transition matrix P_n^γ，that is $\sum_{i=0}^\infty p^\gamma(i)p_n^\gamma(i,j) = p^\gamma(j), j \geq 0$。

SOLUTION. Since $\lim_{m\to\infty} p_m^\gamma(i,j) = p^\gamma(j), \forall i \in \mathbb{N}$ and $\forall k, j \in \mathbb{N}$，then $\sum_{l=0}^\infty p_n(k,l)p_m(l,j) = p_{n+m}(k,j)$。Then we only need to prove

$$\lim_{m\to\infty} \sum_{i=0}^\infty (p^\gamma(i) - p_m^\gamma(k,i))p_n^\gamma(i,j) = 0$$

Since $\sum_{j=0}^\infty p_m^\gamma(i,j) = 1, \forall i \in \mathbb{N}$，then by LCDT we can get that $\sum_{j=0}^\infty p^\gamma(i,j) = 1, \forall i \in \mathbb{N}$。Then $\forall \varepsilon > 0, \exists N > 0$，$\sum_{j=N}^\infty p^\gamma(i,j) < \varepsilon$ and $\exists M > 0, \forall 0 \leq i \leq N-1, \forall m \geq M, |p_m^\gamma(k,i) - p^\gamma(i)| < \frac{\varepsilon}{N}$。

Then,

$$
\begin{aligned}
&|\sum_{i=0}^{\infty}(p^{\gamma}(i) - p_m^{\gamma}(k,i))p_n^{\gamma}(i,j)| \\
&\leq \sum_{i=0}^{\infty} |p^{\gamma}(i) - p_m^{\gamma}(k,i)|p_n^{\gamma}(i,j) \\
&\leq \sum_{i=0}^{N-1} |p^{\gamma}(i) - p_m^{\gamma}(k,i)|p_n^{\gamma}(i,j) + \sum_{i=N}^{\infty} |p^{\gamma}(i) - p_m^{\gamma}(k,i)|p_n^{\gamma}(i,j) \\
&\leq \varepsilon + \sum_{i=N}^{\infty} p^{\gamma}(i)p_n^{\gamma}(i,j) + \sum_{i=N}^{\infty} p_m^{\gamma}(k,i)p_n^{\gamma}(i,j) \\
&\leq \varepsilon + \sum_{i=N}^{\infty} p^{\gamma}(i) + \sum_{i=N}^{\infty} p_m^{\gamma}(k,i) \\
&\leq 2\varepsilon + 1 - \sum_{i=0}^{N-1} p^{\gamma}(i) + \sum_{i=0}^{N-1} |p_m^{\gamma}(k,i) - p^{\gamma}(i)| \\
&\leq 2\varepsilon + \sum_{i=N}^{\infty} p^{\gamma}(i) + \varepsilon \\
&\leq 4\varepsilon < \varepsilon
\end{aligned}
$$

$\square$

PROBLEM VI Let $\{Y(n) : n \in \mathbb{N}\}$ be migrating branch process with offspring distribution $p(j) : j \in \mathbb{N}$, and migrating distribution $\gamma(i) : i \in \mathbb{N}$。And g is the generating function, where $m := g'(1) \leq 1$。Discuss $\lim_{n\to\infty} \mathbb{E}(Y_n \mid Y_0 = i)$。

SOLUTION. $\lim_{n\to\infty} \mathbb{E}(Y_n \mid Y_0 = i) = \lim_{n\to\infty} im^n + \mu \sum_{k=1}^{n} m^{k-1} = \begin{cases} \infty & m = 1 \\ \frac{\mu}{1-m} & m < 1 \end{cases}$。 $\square$