PROBLEM I Let $X=\{X(n):n\geq 0\}$ be Markov chain defined on probability space $(\Omega,\mathscr{F},\mathbb{P})$, with state space E and transition probability matrix $P=(p(i,j):i,j\in E)$. Let $a,b\in E$, $\tau_0=0$, $\sigma_k=\inf\{n\geq\tau_{k-1}:X(n)=b\},\tau_k=\inf\{n\geq\sigma_k:X(n)=a\}$. Prove: $\tau_n,\sigma_n,n\geq 1$ are all stopping time on $(\mathscr{F}_n:n\geq 0)$.

SOLUTION. Since $\sigma_1=\inf\{n\geq 0:X(n)=b\}$, then $\{\sigma_1=m\}=\{X(i)\neq b,0\leq i\leq m-1,X(m)=b\}\in\sigma\{X(i),0\leq i\leq m-1,X_m\}=\mathscr{F}_m$. Then σ_1 is stopping time. Next, we will prove $\sigma_k,\tau_{k-1},k\in\mathbb{N}$ are stopping time. Since σ_1,τ_0 are stopping time, which we have proved. Assume σ_k,τ_{k-1} are stopping time, we will prove σ_{k+1},τ_k are stopping time. Let $m\in\mathbb{N}^+$, $\{\tau_k=m\}=\bigcup_{i\in[0,m-1]\cap\mathbb{N}}\{\sigma_k=i,X(i+l)\neq a,1\leq l\leq m-i-1,X(m)=a\}\in\mathscr{F}_m$, since $\{\sigma_k=i\}\in\mathscr{F}_i\subset\mathscr{F}_m,\forall 0\leq i\leq m-1,\sigma(X_j)\subset\mathscr{F}_m,j\leq m$. Let $m\in\mathbb{N}^+$, $\{\sigma_{k+1}=m\}=\bigcup_{i\in[0,m-1]\cap\mathbb{N}}\{\tau_k=i,X(i+l)\neq a,1\leq l\leq m-i-1,X(m)=b\}\in\mathscr{F}_m$, since $\{\tau_k=i\}\in\mathscr{F}_i\subset\mathscr{F}_m,\forall 0\leq i\leq m-1,\sigma(X_j)\subset\mathscr{F}_m,j\leq m$. Therefore, σ_{k+1},τ_k are stopping time. $\square$

PROBLEM II Let $(X_n:n\geq 0)$ is a one-dimension simple random walk starting at 1. Let $e(n)=\{X_{n\wedge\tau_1}:n\geq 0\}$, where $\tau_1=\inf\{n\geq 0:X_n=0\}$. Find the distribution of $\sup_{n\geq 0}e(n)$.

SOLUTION. Assume $\mathbb{P}(X_{n+1}-X_n=1)=p,\mathbb{P}(X_{n+1}-X_n=-1)=q,p,q>0,p+q=1$. Let $E:=\sup_{n\geq 0}e(n)$. Let $\gamma=\inf\{n\geq 0:X_n=0 or X_n=m\}\geq 1$, where $m\in\mathbb{N}^+$. First of all, if $p=q=\frac{1}{2}$: Easy to get that $\gamma<\infty,a.s.$, so $X_{n\wedge\gamma}\overset{\text{a.s.}}{\to}X_\gamma$. And $0\leq X_{n\wedge\gamma}\leq m$, so $\mathbb{E}(X_\gamma)=\mathbb{E}(X_{n\wedge\gamma})=\mathbb{E}(X_0)$. Noting that $\{X_\gamma=0\}\overset{\text{a.s.}}{=}\{E<m\}$ and $\{X_\gamma=m\}\overset{\text{a.s.}}{=}\{E\geq m\}$, we get two equations:

$$\begin{cases}\mathbb{P}(E<m)+\mathbb{P}(E\geq m)=1\\ 0\mathbb{P}(E<m)+m\mathbb{P}(E\geq m)=1\end{cases}.$$

Solve this equation, we get $\mathbb{P}(E\geq m)=\frac{1}{m}$. So $\mathbb{P}(E=m)=\frac{1}{m(m+1)}$, and easily $\mathbb{P}(E=\infty)=0$.

Secondly, $p\neq q$: Let $Y_n:=(\frac{q}{p})^{X_n}$, then $\mathbb{E}(Y_{n+1})=\mathbb{E}(\mathbb{E}(Y_{n+1}\mid Y_n))=\mathbb{E}((\frac{q}{p})^{X_n+1}p+(\frac{q}{p})^{X_n-1}q)=\mathbb{E}((\frac{q}{p})^{X_n})=\mathbb{E}(Y_n)$. Obviously, γ is stopping time. Then $\mathbb{E}(Y_{n+1\wedge\gamma})=\mathbb{E}(\mathbb{E}(Y_{n+1\wedge\gamma}\mid Y_{n\wedge\gamma}))=\mathbb{E}(\mathbb{1}_{n<\gamma}\mathbb{E}(Y_{n+1}\mid Y_n)+\mathbb{1}_{n\geq\gamma}\mathbb{E}(Y_\gamma\mid Y_\gamma))=\mathbb{E}(\mathbb{1}_{n<\gamma}Y_n+\mathbb{1}_{n\geq\gamma}\mathbb{E}(Y_\gamma))=\mathbb{E}(Y_{\gamma\wedge n})$. Then $\mathbb{E}(Y_{n\wedge\gamma})=\mathbb{E}(Y_0)=\frac{q}{p}$. Besides, $\gamma<\infty,a.s.$, $Y_{n\wedge\gamma}\overset{\text{a.s.}}{\to}Y_\gamma$, $Y_n<\infty a.s.$. Then $\mathbb{E}(Y_{n\wedge\gamma})\to\mathbb{E}(Y_\gamma)=\mathbb{P}(T_0<T_m)+(\frac{q}{p})^m\mathbb{P}(T_0>T_m)$, where $T_i=\inf\{k\geq 1:X_k=i\},i\in\mathbb{N}$. Additionally, $\{T_0<T_m\}\overset{\text{a.s.}}{=}\{E<m\},\{T_0>T_m\}\overset{\text{a.s.}}{=}\{E\geq m\}$. Then $\mathbb{P}(E<m)+(\frac{q}{p})^m\mathbb{P}(E\geq m)=\frac{q}{p},\mathbb{P}(E<m)+\mathbb{P}(E\geq m)=1$. Thus, $\mathbb{P}(E\geq m)=\frac{p^m-qp^{m-1}}{p^m-q^m}$, $\mathbb{P}(E<m)=\frac{qp^{m-1}-q^m}{p^m-q^m}$. Then $\mathbb{P}(E=m)=\frac{(\frac{p}{q})^m(\frac{p}{q}-1)}{((\frac{p}{q})^m-1)((\frac{p}{q})^{m+1}-1)}$. Furthermore, easily $\mathbb{P}(E=\infty)=\lim_{m\to\infty}\mathbb{P}(E\geq m)=\begin{cases}0 & \frac{q}{p}>1\\ 1-\frac{q}{p} & \frac{q}{p}<1\end{cases}$. $\square$

PROBLEM III Prove:

1. When $0<p\leq q$, the reflecting random walk with transition matrix Q_+^a is recurrent.
2. When $0<q\leq p$, the reflecting random walk with transition matrix Q_-^a is recurrent.

SOLUTION. By symmetry, only need to prove the first question. Without loss of generality we can assume $a = 0$. We consider the equation

$$y_0 = y_1, \forall i \geq 1, y_i = qy_{i-1} + py_{i+1}$$

Only need to prove its all bounded solution are all constant. Easy to get $y_{i+2} = \frac{1}{p}y_{i+1} - \frac{q}{p}y_i$. Consider the charasteristic equation of this sequence, $x^2 - \frac{x}{p} + \frac{q}{p} = 0$. We get $x_1 = 1, x_2 = \frac{q}{p} \geq 1$. If $x_2 > 1$, then $y_n = c_1 x_1^n + c_2 x_2^n$ is bounded $\iff$ $c_2 = 0$, so $y_n = c_1 x_1^n = c_1$ is constant. Else, $x_2 = x_1 = 1$, then $y_n = (an + b)x_1^n = an + b$ is bounded $\iff$ $a = 0$, so $y_n = b$ is constant. So the Markov chain is recurrent. $\square$

PROBLEM IV Let $\phi_0(n : n \in \mathbb{N}^+)$ be simple random walk begin at $\phi_0(0) \geq a + 1$, let $\zeta_0 := \inf\{m : \phi_0(m) = a + 1\}$, let $(W_n : n \in \mathbb{N})$ be reflecting simple random walk on $\mathbb{Z}_+^a$, starting at $a + 1$, indenpendent with ϕ_0. Let $X_n := \begin{cases} \phi_0(n) & n \leq \zeta_0 \\ W_n - \zeta_0 & n \geq \zeta_0 \end{cases}$. Prove that $(X_n : n \in \mathbb{N})$ is reflecting random walk on $\mathbb{Z}_+^a$ begin at $\phi_0(0)$.

SOLUTION. Now we consider $n \in \mathbb{N}^+$ and $i_0, i_1, i_2, \cdots, i_{n+1} \in \mathbb{Z}_+^a$.

1. If $\forall k : 1 \leq k \leq n, i_k \neq a + 1$, then we have

$$\begin{aligned}
\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) &= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n+1) = i_{n+1}) \\
&= \mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(n) = i_n)\mathbb{P}(\phi_0(n+1) = i_{n+1} \mid \phi_0(n) = i_n) \\
&= \mathbb{P}(X_0 = i_0, \cdots, X_n = i_n)q_+^a(i_n, i_{n+1})
\end{aligned}$$

2. Else, we let $k := \inf\{m : 1 \leq m \leq n, i_m = a + 1\}$. Then we have

$$\begin{aligned}
&\mathbb{P}(X_0 = i_0, \cdots, X_{n+1} = i_{n+1}) \\
=&\mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k, W_0 = a + 1, W_{k+2+i} = i_{k+i}, i = 1, \cdots, n - k + 1) \\
=&\mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(W_0 = a + 1, W_{k+2+i} = i_{k+i}, i = 1, \cdots, n - k + 1) \\
=&\mathbb{P}(\phi_0(0) = i_0, \cdots, \phi_0(k) = i_k)\mathbb{P}(W_0 = a + 1, W_{k+2+i} = i_{k+i}, i = 1, \cdots, n - k)q_+^a(i_n, i_{n+1}) \\
=&\mathbb{P}(X_0 = i_0, \cdots, X_n = i_n)q_+^a(i_n, i_{n+1})
\end{aligned}$$

So we get $(X_n : n \geq 0)$ is reflecting simple random walk on $\mathbb{Z}_+^a$. $\square$