NumberTheory 2

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PROBLEM I Assume $n \in \mathbb{N}^+$ and $2^n + 1$ is prime. Prove that $\exists k \in \mathbb{N}, n = 2^k$.

SOLUTION. If $n \neq 2^k$, then $\exists p > 0$ is prime and such that $p|n$. Then p is odd. Let $k := \frac{n}{p}$, then

$$
\begin{aligned}
&2^n + 1 \\
=&2^{kp} + 1 \\
=&2^{kp} + 1^{kp} \\
=&(2^k + 1^k - 1^k)^p + 1^{kp} \\
=&\sum_{i=0}^{p} (2^k + 1^k)^i (-1)^{p-i} + 1^{kp} \\
=&\sum_{i=1}^{p} (2^k + 1^k)^i (-1)^{p-i}
\end{aligned}
\tag{1}
$$

which is contradict with that $2^n + 1$ is prime. $\square$

PROBLEM II Find the standrad decomposition of 30!.

SOLUTION . Since $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$ are prime which is below 30, so

$$\begin{aligned}
\sum_{k=1}^{\infty}[\frac{30}{2^k}] &= 15+7+3+1=26 \\
\sum_{k=1}^{\infty}[\frac{30}{3^k}] &= 10+3+1=14 \\
\sum_{k=1}^{\infty}[\frac{30}{5^k}] &= 6+1=7 \\
\sum_{k=1}^{\infty}[\frac{30}{7^k}] &= 4 \\
\sum_{k=1}^{\infty}[\frac{30}{11^k}] &= 2 \\
\sum_{k=1}^{\infty}[\frac{30}{13^k}] &= 2 \\
\sum_{k=1}^{\infty}[\frac{30}{17^k}] &= 1 \\
\sum_{k=1}^{\infty}[\frac{30}{19^k}] &= 1 \\
\sum_{k=1}^{\infty}[\frac{30}{23^k}] &= 1 \\
\sum_{k=1}^{\infty}[\frac{30}{29^k}] &= 1
\end{aligned} \tag{2}$$

So $30! = 2^{26} \times 3^{14} \times 5^7 \times 7^4 \times 11^2 \times 13^2 \times 17 \times 19 \times 23 \times 29$.

□

PROBLEM III Assume $n \in \mathbb{N}^+$ and $\alpha \in \mathbb{R}$, prove that:

1. $\left[\frac{[n\alpha]}{n}\right] = [\alpha]$.
2. $\sum_{k=0}^{n-1}[\alpha + \frac{k}{n}] = [n\alpha]$.

SOLUTION . 1. Only need to prove $[\alpha] \leq \frac{[n\alpha]}{n}, |\alpha - \frac{[n\alpha]}{n}| < 1$. Since $n[\alpha] \leq n\alpha$, then $n[\alpha] \leq [n\alpha]$, then $[\alpha] \leq \frac{[n\alpha]}{n}$. Besides, $0 \leq \alpha - \frac{[n\alpha]}{n} = \frac{n\alpha - [n\alpha]}{n}$, then it is equal to prove $\frac{n\alpha-[n\alpha]}{n} < 1$, so this is equal to prove $n\alpha - [n\alpha] < n$, which is equal to $n(\alpha - 1) < [n\alpha]$. It is obvious that $n(\alpha - 1) < n[\alpha] \leq [n\alpha]$.

2. Let $\frac{i}{n} \leq \{\alpha\} < \frac{i+1}{n}, 0 \leq i < n-1$, so $\alpha + \frac{k}{n} = [\alpha] + \{\alpha\} + \frac{k}{n}$. Then $\forall n-1 \geq k \geq n-i$,

$1 \le \{\alpha\} + \frac{k}{n} < 2$, then

$$\begin{aligned} &\sum_{k=0}^{n-1}[\alpha + \frac{k}{n}] \\ =& \sum_{k=0}^{n-i-1}[\alpha] + \sum_{k=n-i}^{n-1}([\alpha]+1) \\ =& n[\alpha] + i \\ =& n[\alpha] + [n\{\alpha\}] \\ =& [n[\alpha]] + [n\{\alpha\}] \\ =& [n([\alpha] + \{\alpha\})] \\ =& [n\alpha] \end{aligned} \tag{3}$$

□

PROBLEM IV Assume $r > 0, r \in \mathbb{R}$. Let T be the number of integer point in $x^2 + y^2 \le r^2$. Prove that $T = 1 + 4[r] + 8\sum_{0<x\le\frac{r}{\sqrt{2}}}[\sqrt{r^2 - x^2}] - 4\left[\frac{r}{\sqrt{2}}\right]^2$.

SOLUTION. Since

$$\begin{aligned} &T = \{(x,y) : x^2 + y^2 \le r^2\} \\ =& \{(x,y) : x = 0, y = 0\} \cup \{(x,y) : x = 0, y \ne 0, x^2 + y^2 \le r^2\} \\ &\cup \{(x,y) : x \ne 0, y = 0, x^2 + y^2 \le r^2\} \cup \{(x,y) : x \ne 0, y \ne 0, x^2 + y^2 \le r^2\} \end{aligned} \tag{4}$$

then by the symmetry $\#\{(x,y) : x = 0, y \ne 0, x^2 + y^2 \le r^2\} = \#\{(x,y) : x \ne 0, y = 0, x^2 + y^2 \le r^2\} = 2\#\{(x,y) : x = 0, y > 0, x^2 + y^2 \le r^2\} = 2[r]$. Besides, $\#\{(x,y) : x \ne 0, y \ne 0, x^2 + y^2 \le r^2\} = 8\#\{(x,y) : 0 < x \le [\frac{r}{\sqrt{2}}] < y \le r^2, x^2 + y^2 \le r^2\} + 4\#\{(x,y) : 0 < x, y \le [\frac{r}{\sqrt{2}}], x^2 + y^2 \le r^2\} = 8(\#\{(x,y) : 0 < x \le [\frac{r}{\sqrt{2}}], y < r^2, x^2 + y^2 \le r^2\} - \#\{(x,y) : 0 < x, y \le [\frac{r}{\sqrt{2}}], x^2 + y^2 \le r^2\}) + 4\#\{(x,y) : 0 < x, y \le [\frac{r}{\sqrt{2}}], x^2 + y^2 \le r^2\} = 8(\#\{(x,y) : 0 < x \le [\frac{r}{\sqrt{2}}], y < r^2, x^2 + y^2 \le r^2\} - 4\#\{(x,y) : 0 < x, y \le [\frac{r}{\sqrt{2}}], x^2 + y^2 \le r^2\} = 8\sum_{0<x\le\frac{r}{\sqrt{2}}}[\sqrt{r^2 - x^2}] - 4[\frac{r}{\sqrt{2}}]^2$. Therefore, $T = 1 + 4[r] + 8\sum_{0<x\le\frac{r}{\sqrt{2}}}[\sqrt{r^2 - x^2}] - 4\left[\frac{r}{\sqrt{2}}\right]^2$. □

PROBLEM V Find all integer solution of $306x - 360y = 630$.

SOLUTION. It is equal to find all integer solution of $17x - 20y = 35$. First of all we should find all integer solution of $17x - 20y = 1$. Obviously, we can get a special solution that is $x = -7, y = -6$. So we can get a special solution to $17x - 20y = 25$, that is $x = -245, y = -210$. Then all the integer solution of $17x - 20y = 35$ have the form that $x = -5 + 20t, y = -6 + 17t, t \in \mathbb{Z}$.

□

PROBLEM VI Assume $N, a, b \in \mathbb{N}, a, b > 0, \gcd(a,b) = 1$. Prove that the number of positive integer solutions of the equation $ax + by = N$ is $\left[\frac{N}{ab}\right]$ or $\left[\frac{N}{ab}\right] + 1$.

Lemma 1. $x, y \in \mathbb{R}$, then $[x+y] - ([x]+[y]) = 0$ or 1.

证明. Since $x, y \in \mathbb{R}$, then $[x+y] = [[x]+\{x\}+[y]+\{y\}] = [x]+[y]+[\{x\}+\{y\}]$. So $0 \le \{x\}+\{y\} < 2$, then $[x+y]-([x]+[y]) = [\{x\}+\{y\}] = 0$ or 1. $\square$

SOLUTION . Since $\gcd(a,b) = 1$,then $\exists x_0, y_0$ such that $ax_0+by_0 = 1$, then all the solution of $ax+by = N$ have the form that is $x = x_0 - bt, y = y_0 + at, t \in \mathbb{Z}$. So $\#\{t \in \mathbb{Z} : x_0 - bt > 0, y_0 + at > 0\} = [\frac{ax_0N}{ab}] - [\frac{ax_0N-N}{ab}] = [\frac{ax_0N-N}{ab} + \frac{N}{ab}] - [\frac{ax_0N-N}{ab}] = [\frac{N}{ab}]$ or $[\frac{N}{ab}]+1$. $\square$

PROBLEM VII Write $\frac{17}{60}$ as sum of three reduced fraction whose denominators are coprime to each other.

SOLUTION . Consider $\frac{17}{60} = \frac{x}{4} + \frac{y}{3} + \frac{z}{5}$, i.e., $17 = 15x + 20y + 12z$. Since $\gcd(15, 20, 12) = 1$, we know this equation has some solution. Easy to know $x = -1, y = 1, z = 1$ is a solution. So $\frac{17}{60} = -\frac{1}{4} + \frac{1}{3} + \frac{1}{5}$ satisfy the condition. $\square$