Iterative 3

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PROBLEM I In the Householder implementation of the Arnoldi algorithm, show the following points of detail:

(a) Q_{j+1} is unitary and its inverse is Q_{j+1}^T.

(b) $Q_{j+1}^T = P_1 P_2 \cdots P_{j+1}$.

(c) $Q_{j+1}^T e_i = v_i$ for $i < j$.

(d) $Q_{j+1} A V_m = V_{m+1}[e_1, e_2, \ldots, e_{j+1}] \bar{H}_m$, where e_i is the i-th column of the $n \times n$ identity matrix.

(e) The vectors $v_1, v_2, \ldots, v_j$ are orthonormal.

(f) The vectors $v_1, \ldots, v_j$ are equal to the Arnoldi vectors produced by the Gram-Schmidt version, except possibly for a scaling factor.

PROBLEM II To derive the basic version of GMRES, we use the standard formula

$$\tilde{x} = x_0 + V \left(W^T A V \right)^{-1} W^T r_0, \tag{1}$$

where $V = V_m$ and $W = A V_m$.

PROBLEM III Let a matrix A have the form

$$A = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}.$$

Assume that (full) GMRES is used to solve a linear system with the coefficient matrix A. What is the maximum number of steps that GMRES would require to converge? PROBLEM IV Consider a matrix of the form

$$A = I + \alpha B \tag{2}$$

where B is skew-symmetric (real), i.e., such that $B^T = -B$.

1. Show that $\dfrac{(Ax, x)}{(x, x)} = 1$ for all nonzero x.

2. Consider the Arnoldi process for A. Show that the resulting Hessenberg matrix will have the following tridiagonal form

$$H_m = \begin{pmatrix} 1 & -\eta_2 & & & \\ \eta_2 & 1 & -\eta_3 & & \\ & \eta_3 & 1 & \ddots & \\ & & \ddots & \ddots & -\eta_m \\ & & & \eta_m & 1 \end{pmatrix}.$$

3. Using the result of the previous question, explain why the CG algorithm applied as is to a linear system with the matrix A, which is nonsymmetric, will still yield residual vectors that are orthogonal to each other.