PROBLEM I Prove that solution of equation

$$x^2+y^2=z^4, \gcd(x,y)=1, x>0, y>0, z>0, 2\mid x \tag{1}$$

is

$$\begin{cases} x=4ab(a^2-b^2) \\ y=|a^4+b^4-6a^2b^2| \\ z=a^2+b^2 \end{cases} \tag{2}$$

where $a>0, b>0, \gcd(a,b)=1, a\not\equiv b \mod 2$.

SOLUTION . 1. Obviously, (x,y,z) satisfying 2 satisfies 1.

2. Next, we proof the solution of 1 satisfies 2. Since $(m,n), \gcd(n,m)=1$ satisfies

$$\begin{cases} x= & 2nm \\ y= & m^2-n^2 \\ z= & m^2+n^2 \end{cases}$$

Since $y>0$ and $\gcd(x,y)=1, 2\mid x$, then $m\geq n, 2\nmid x-y$, then $2\nmid n-m$. So we just let $y=|m^2-n^2|, 2\nmid m, 2\mid n$. Since $m^2+n^2=i^2, \exists i>0$, then $(u,v)\in\mathbb{N}^2$, $\gcd(u,v)=1$, $2\nmid u-v$ satisfies

$$\begin{cases} m= & u^2-v^2 \\ n= & 2uv \end{cases}$$

So

$$\begin{cases} x= & 2uv \\ y= & |6u^2v^2-u^2-v^2| \\ z= & u^2+v^2 \end{cases}$$

□

PROBLEM II Find a method to judge whether a number can be divided by $37, 101$.

SOLUTION . 1. Since $37\equiv 1 \mod 1000$, then $\forall \sum_{k=0}^{n} a_k 10^k = \sum_{s=0}^{m}(a_{3s}+a_{3s+1}10+a_{3s+2}100)1000^s, m=[\frac{n}{3}]+1, a_{3m+i}=0$, if $3m+i>n, i=1,2$. So $\sum_{k=0}^{n} a_k 10^k \equiv \sum_{s=0}^{m}(a_{3s}+a_{3s+1}10+a_{3s+2}100)1000^s \mod 37$. Thus $37\mid \sum_{k=0}^{n} a_k 10^k \iff 37\mid \sum_{s=0}^{m} a_{3s}+a_{3s+1}10+a_{3s+2}100$.

2. Since $101\equiv -1 \mod 100$, then $\forall \sum_{k=0}^{n} a_k 10^k = \sum_{s=0}^{m}(a_{2s}+a_{s+1}10)100^s, m=[\frac{n}{2}]+1, a_{2m+1}=0$, if $2m+1>n$. So $\sum_{k=0}^{n} a_k 10^k \equiv \sum_{s=0}^{m}(a_{2s}+a_{2s+1}10)100^s \mod 101$. Thus $101 \mid \sum_{k=0}^{n} a_k 10^k \iff 101\mid \sum_{s=0}^{m}(a_{2s}+a_{2s+1}10)(-1)^s$.

□

PROBLEM III Assume $2\nmid a$, then $a^{2^n}\equiv 1 \mod 2^{n+2}$.

SOLUTION. First, we prove $a^{2^n}-1=(a^2-1)\prod_{k=1}^{n-1}(a^{2^k}+1)$. Obviously, when $n=1$, the equation is right. If n the equation is right, so $a^{2^{n+1}}-1=(a^{2^n}-1)(a^{2^n}+1)=(a^2-1)\prod_{k=1}^{n-1}(a^{2^k}+1)(a^{2^n}+1)=(a^2-1)\prod_{k=1}^{n}(a^{2^k}+1)$.

Second, since $2\nmid a$, then a is odd, then $a=2b+1$. So $8\mid a^2-1=(a-1)(a+1)=4b(b+1)$ and $a^{2^k}+1=(2b+1)^{2^k}+1=\sum_{i=1}^{2^k}(2b)^i+2=2(\sum_{i=1}^{2^k}2^{i-1}b^i+1)$, so $2\mid a^{2^k}+1$, then $8\times\prod_{k=1}^{n-1}2=2^{n+2}\mid(a^2-1)\prod_{k=1}^{n-1}(a^{2^k}+1)$. □

PROBLEM IV Let p be a prime and s,t be integers and $t\leq s$. Prove that $(u+p^{s-t}v:0\leq u\leq p^{s-t}-1, 0\leq v\leq p^t-1)$ is a Complete residue system of p^s.

SOLUTION. Since there are p^s elements in $(u+p^{s-t}v:0\geq u\geq p^{s-t}-1, 0\leq v\leq p^t-1)$, we only need to prove any two of them Congruence modulo to p^s. Besides, $\forall(u_1+p^{s-t}v_1),(u_2+p^{s-t}v_2)\in\{u+p^{s-t}v:0\leq u\leq p^{s-t}-1,0\leq v\leq p^t-1\}$, $|(u_1-u_2)+p^{s-t}(v_1-v_2)|\leq p^{s-t}-1+p^{s-t}(p^t-1)=p^s-1<p^s$, so $p^s\mid(u_1+p^{s-t}v_1)-(u_2+p^{s-t}v_2)$, we can get $(u_1+p^{s-t}v_1)-(u_2+p^{s-t}v_2)=0$, that is $u_1-u_2=p^{s-t}(v_2-v_1)$, so $p^{s-t}\mid u_1-u_2$. Since $|u_1-u_2|<p^{s-t}$, then $u_1-u_2=0$. Then $v_2-v_1=0$. □

PROBLEM V Assume $m_1,\cdots,m_k$ is k integers coprime to each other. Assume $A_1,A_2,\cdots,A_k$ is Complete residue of $m_1,\cdots,m_k$ respectively. Let $m=\prod_{t=1}^k m_t$ and $M_t:=\frac{m}{m_t},t=1,\cdots,k$. Prove that $A:=\{\sum_{t=1}^k M_tx_t:x_t\in A_t,t=1,\cdots,k\}$ is a Complete residue of m.

SOLUTION. First, we prove there are m elements in A. $\forall\sum_{t=1}^k M_tx_t,\sum_{k=0}^k M_ty_t\in A$, if $\sum_{t=1}^k M_t(x_t-y_t)=0$, then $\sum_{1\leq t\leq k,t\neq i}M_t(x_t-y_t)=M_i(y_i-x_i),\forall 1\leq i\leq k$. So $m_i(\sum_{1\leq t\leq k,t\neq i}\frac{M_t}{m_i}(x_t-y_t))=M_i(y_i-x_i)$. Since $m_1,\cdots,m_k$ are coprime to each other, then $\gcd(m_i,M_i)=1$, then $m_i\mid(y_i-x_i)$. Since $y_i,x_i\in A_i$, $m_i\nmid x_i-y_i$, then $y_i-x_i=0$. Thus $\sum_{t=1}^k M_tx_t=\sum_{t=1}^k M_ty_t$.

Second, we prove if $m\mid\sum_{t=1}^k M_t(x_t-y_t)$, then $\sum_{t=1}^k M_tx_t=\sum_{t=1}^k M_ty_t$. Suppose $\sum_{t=1}^k M_t(x_t-y_t)=am$, $a\in\mathbb{N}$, then $\sum_{1\leq t\leq k,t\neq i}M_t(x_t-y_t)=am-M_i(y_i-x_i)=M_i(am_i-y_i+x_i),\forall 1\leq i\leq k$, so $m_i(\sum_{1\leq t\leq k,t\neq i}\frac{M_t}{m_i}(x_t-y_t))=M_i(am_i-y_i+x_i)$. Since $m_1,\cdots,m_k$ are coprime to each other, then $\gcd(m_i,M_i)=1$, then $m_i\mid(am_i-y_i+x_i)$, then $m_i\mid y_i-x_i$. Since $y_i,x_i\in A_i$, $m_i\nmid x_i-y_i$, then $y_i-x_i=0$. Thus $\sum_{t=1}^k M_tx_t=\sum_{t=1}^k M_ty_t$. □

PROBLEM VI Let $H=\frac{3^{n+1}-1}{2}$. Let $I=\{(x_0,\cdots,x_n):x_k\in\{-1,0,1\},k=0,\cdots,n\}$. Let $f:I\to N:=[-H,H]\cap\mathbb{Z}$, and $f(x_0,\cdots,x_n)=\sum_{k=0}^n x_k3^k$. Prove that f is bijection. Thus, we can use $n+1$ weights and a balance to weigh all integer weights between 1 and H.

SOLUTION. Since $|I|=3^{n+1}=|N|=2H+1$, so we only need to prove f is injective. If $\sum_{k=0}^n x_k3^k-\sum_{k=0}^n y_k3^k=\sum_{k=0}^n(x_k-y_k)3^k=0$, then we will prove $x_k=y_k,k=0,\cdots,n$. Obviously, $\sum_{k=1}^n(x_k-y_k)3^k=y_0-x_0$, then $3(\sum_{k=1}^n(x_k-y_k)3^{k-1})=y_0-x_0$, so $3\mid y_0-x_0$. But, $y_0,x_0\in\{-1,0,1\}$, then $|y_0-x_0|<3$. Then $y_0-x_0=0$, that is $y_0=x_0$. If $\forall k\leq m<n$, $x_k=y_k$. Then, $\sum_{k=m+2}^n(x_k-y_k)3^k=y_{m+1}-x_{m+1}$, then $3(\sum_{k=m+2}^n(x_k-y_k)3^{k-1})=y_{m+1}-x_{m+1}$, then by the same reason, $y_{m+1}=x_{m+1}$. So f is injection.

And suppose we have $n+1$ weights, and their weight are different, noted A. Let $A:=\{3^k:k=1,\cdots,n\}$. $\forall n\in[1,H]\cap\mathbb{Z}$, and put n on right side of the balance. We have $n=\sum_{k=0}^n x_k3^k$, we put weights of weight in set $\{3^k:x_k=1,\cdots,n\}$ on the left side, and we put weights of weight in set $\{3^k : x_k = -1, \cdots, n\}$ on the right side, and we don't put weights of weight in set $\{3^k : x_k = 0, \cdots, n\}$ on each side. Therefore, we can weight integer n. $\square$

PROBLEM VII Assume $m_1, \cdots, m_k$ is k integers coprime to each other. Assume $A_1, A_2, \cdots, A_k$ is Complete residue of $m_1, \cdots, m_k$ respectively. Let $m = \prod_{t=1}^{k} m_t$ and $M_t := \prod_{i=1}^{t-1} m_i, t = 1, \cdots, k$. Prove that $A := \{\sum_{t=1}^{k} M_t x_t : x_t \in A_t, t = 1, \cdots, k\}$ is a Complete residue of m.

SOLUTION. First, we prove there are m elements in A. $\forall \sum_{t=1}^{k} M_t x_t, \sum_{k=0}^{k} M_t y_t \in A$, $\sum_{1\le t\le k, t\ne i} M_t(x_t - y_t) = M_i(y_i - x_i), \forall 1 \le i \le k$, so $m_1(\sum_{1\le t\le k, t\ne i} \frac{M_t}{m_1}(x_t - y_t)) = M_1(y_1 - x_1)$. Since $m_1, \cdots, m_k$ are coprime to each other, then $\gcd(m_1, M_1) = 1$, then $m_1 \mid (y_1 - x_1)$. Since $y_1, x_1 \in A_1$, $m_1 \nmid x_1 - y_1$, then $y_1 - x_1 = 0$. If $\forall i \le n-1 < k$, $y_i = x_i$, then $m_n(\sum_{n+1\le t\le k} \frac{M_t}{m_n}(x_t - y_t)) = M_n(y_n - x_n)$. Since $m_1, \cdots, m_k$ are coprime to each other, then $\gcd(m_n, M_n) = 1$, then $m_n \mid (y_n - x_n)$. Since $y_n, x_n \in A_n$, $m_n \nmid x_n - y_n$, then $y_n - x_n = 0$. Thus $\sum_{t=1}^{k} M_t x_t = \sum_{t=1}^{k} M_t y_t$.

Second, we prove if $m \mid \sum_{t=1}^{k} M_t(x_t - y_t)$, then $\sum_{t=1}^{k} M_t x_t = \sum_{t=1}^{k} M_t y_t$. Suppose $\sum_{t=1}^{k} M_t(x_t - y_t) = am$, $a \in \mathbb{N}$, then $\sum_{1\le t\le k, t\ne i} M_t(x_t - y_t) = am - M_i(y_i - x_i), \forall 1 \le i \le k$, so $m_1(\sum_{1\le t\le k, t\ne i} \frac{M_t}{m_1}(x_t - y_t)) = am - (y_1 - x_1)$. Since $m_1, \cdots, m_k$ are coprime to each other, then $\gcd(m_1, M_1) = 1$, then $m_1 \mid am - (y_1 - x_1)$, then $m_1 \mid y_1 - x_1$. Since $y_1, x_1 \in A_1$, $m_1 \nmid x_1 - y_1$, then $y_1 - x_1 = 0$. If $\forall i \le n-1 < k$, $y_i = x_i$, then $m_n(\sum_{n+1\le t\le k} \frac{M_t}{m_n}(x_t - y_t)) = M_n(a\prod_{t=n}^{k} m_t - y_n + x_n)$. Since $m_1, \cdots, m_k$ are coprime to each other, then $\gcd(m_n, M_n) = 1$, then $m_n \mid a\prod_{t=n}^{k} m_t - y_n + x_n$. Thus $m_n \mid y_n - x_n$. Since $y_n, x_n \in A_n$, $m_n \nmid x_n - y_n$, then $y_n - x_n = 0$. Thus $\sum_{t=1}^{k} M_t x_t = \sum_{t=1}^{k} M_t y_t$. $\square$