PROBLEM I A kind of product is qualified with probability $p(0<p<1)$. We sample these product by the following way: we check all the product at first until there appears k qualified product sequently. Then we check the rest of product by probability $\alpha(0<\alpha<1)$ until there appears one unqualified product, then one circle ends. Next we restart another checking circle. Please find out the proportion of checked product after a long time.

SOLUTION. For sake of convenience, we call the k qualified products appearing sequently as k qualified sequence. Assume the proportion of checked product after a long time is β. Let N_k be the amount of product when the first k qualified sequence ends. $M_k=\mathbb{E}(N_k)$. G_k is the event that the next one is qualified after the first $k-1$ qualified sequence ends. Obviously, $\mathbb{E}(N_k-N_{k-1}\mid G_k)=1$. And $\mathbb{E}(N_k-N_{k-1}\mid G_k)=\mathbb{E}(N_k)+1$. Therefore, $\mathbb{E}(N_k-N_{k-1})=p+(1-p)(\mathbb{E}(N_k)+1)$. So $M_k-M_{k-1}=p+(1-p)(1+M_k)$. Then $pM_k=M_{k-1}+1$. Thus, $M_k=\frac{\frac{1}{p^k}-1}{1-p}$. Let $A=\{\text{The amount of product checked in one circle}\}$, $B=\{\text{The amount of product in one circle}\}$. So finding one unqualified product need average checking time $\frac{1}{1-p}$ according to geometric distribution. Then we averagely need $\frac{1}{\alpha(1-p)}$ product to find out the unqualified one. Then $\mathbb{E}(A)=M_k+\frac{1}{1-p}, \mathbb{E}(B)=M_k+\frac{1}{\alpha(1-p)}$. So

$$\beta=\frac{\mathbb{E}(A)}{\mathbb{E}(B)}=\frac{\frac{\frac{1}{p^k}-1}{1-p}+\frac{1}{1-p}}{\frac{\frac{1}{p^k}-1}{1-p}+\frac{1}{\alpha(1-p)}}=\frac{\alpha}{\alpha+p^k-\alpha p^k}$$

$\square$