Probability 6

王胤雅

25114020018

yinyawang25@m.fudan.edu.cn

2025 年 11 月 18 日

SOLUTION. 令 $A := \{\xi > 0\}$，若 $\mathbb{P}(A) = 0$，由 $\xi \geq 0$，那么 $\xi \overset{\text{a.s.}}{=} 0$。从而 $\mathbb{E}(\xi^2) = \mathbb{E}(\xi)^2 = 0$.

$\square$

SOLUTION. 1. 令 $A := \{\xi - m \geq t\}$，$\mu := \mathbb{P}(A)$。若 $\mu = 0$，那么 $0 = \mathbb{P}(\xi - m \geq t) \leq \frac{\sigma^2}{\sigma^2+t^2}$。若 $\mu > 0$，那么 $x_1 = \mathbb{E}(\xi - m \mid A), x_2 = \mathbb{E}(\xi - m \mid A^c)$. 又由于 $\mathbb{E}(\xi - m) = 0$，那么 $\mathbb{E}(\xi - m) = \mathbb{E}(\xi - m \mid A)\mathbb{P}(A) + \mathbb{E}(\xi - m \mid A^c)\mathbb{P}(A) = x_1\mu + x_2(1-\mu) = 0$. 又由于 $x_1 = \mathbb{E}(\xi - m \mid A) \geq \mathbb{E}(t \mid A) = t$，故 $x_1 \geq t$。

$$\begin{aligned}\sigma^2 = \mathbb{E}(|\xi - m|^2) &= \mathbb{E}(|\xi - m|^2 \mid A)\mathbb{P}(A) + \mathbb{E}(|\xi - m|^2 \mid A^c)\mathbb{P}(A) \\ &\overset{??}{\geq} (\mathbb{E}(\xi - m \mid A))^2\mathbb{P}(A) + (\mathbb{E}(\xi - m \mid A^c))^2\mathbb{P}(A^c) \\ &= x_1^2\mu + x_2^2(1-\mu)\end{aligned}$$

- 若 $\mu = 1$，那么 $x_1\mu + x_2(1-\mu) = 0$，知 $x_1 = 0$，那么 $t \leq 0$. 又由于 $t \geq 0$，那么 $t = 0$，此时 $\mathbb{P}(\xi - m \geq t) = \mu = 1 = \frac{\sigma^2}{\sigma^2}$.
- 若 $0 < \mu < 1$，那么 $\sigma^2 \geq \frac{\mu}{1-\mu}x_1^2$。又由于 $x_1 = \mathbb{E}(\xi - m \mid A) \geq \mathbb{E}(t \mid A) = t$，故 $\sigma^2 \geq \frac{\mu}{1-\mu}t^2$. 从而 $\mu \leq \frac{\sigma^2}{t^2+\sigma^2}$，即 $\mathbb{P}(\xi - m \geq t) \leq \frac{\sigma^2}{t^2+\sigma^2}$.

2. 令 $\eta = -\xi$，那么 $\mathbb{E}(\eta) = -m, \mathrm{Var}(\eta) = \sigma^2$，那么 $\mathbb{P}(\eta + m \geq t) \leq \frac{\sigma^2}{t^2+\sigma^2}$，故 $\mathbb{P}(-\xi + m \geq t) = \mathbb{P}(\xi - m \leq -t) \leq \frac{\sigma^2}{\sigma^2+t^2}$. 故 $\mathbb{P}(|\xi - m| \geq t) \leq \mathbb{P}(\xi - m \geq t) + \mathbb{P}(\xi - m \leq -t) \leq \frac{2\sigma^2}{\sigma^2+t^2}$.

$\square$