PROBLEM I Assume $N(t)$ is updating process. X is the time interval distrabution of $N(t)$. Assume $\mathbb{D}(X) < \infty$. Let $R(t) := S_{N(t)+1} - t$. Find:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\,\mathrm{d}t$$

SOLUTION. Easily $N(t) + 1 \geq T \geq N(t)$. So $\int_0^T R(t)\,\mathrm{d}t \leq \sum_{i=1}^{N(T)+1} \int_{S_{i-1}}^{S_i} (S_i - t)\,\mathrm{d}t = \frac{1}{2}\sum_{i=1}^{N(t)+1}(S_i - S_{i-1})^2 = \frac{1}{2}\sum_{i=1}^{N(T)+1} \xi_i^2$. For the same reason, we get that $\int_0^T R(t)\,\mathrm{d}t \geq \frac{1}{2}\sum_{i=1}^{N(T)} \xi_i^2$.

Easy to know that $\lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{N(T)} \xi_i^2 = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{N(T)+1} \xi_i^2 = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}$. So finally we get that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R(t)\,\mathrm{d}t = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)^2}$$

. $\square$

PROBLEM II Assume the number of people arriving the cinema is distributed as a Possion process with parameter λ. Assume the film begin at a fixed time $t \geq 0$. Let $A(t)$ be the sum of waiting time of all people arriving in $(0, t]$, find $\mathbb{E}(A(t))$.

SOLUTION. Let V_k be the arriving time of k-th people. Let $N(t)$ be the number of people in $(0, t]$. Then $A(t) = \sum_{k=1}^{N(t)}(t - V_k)$. Let $\xi_k := V_k - V_{k-1}$. Then $\sum_{k=1}^{N(t)} V_k = \sum_{k=1}^{N(t)}(N(t) - k)\xi_k = \sum_{k=0}^{N(t)-1} k\xi_{N(t)-k}$. So $\mathbb{E}(A(t)) = t\mathbb{E}(N(t)) - \mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k})$. Easy to get that $\mathbb{E}(\sum_{k=0}^{N(t)-1} k\xi_{N(t)-k} \mid N(t) = n) = \frac{nt}{2}$. So $\mathbb{E}(A(t) \mid N(t) = n) = nt - \frac{nt}{2} = \frac{nt}{2}$. So finally we have $\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t) \mid N(t))) = \mathbb{E}(\frac{N(t)t}{2}) = \frac{\lambda t^2}{2}$. $\square$

PROBLEM III Assume a machine has life distrabuted p. When machine is broken or has been used T years, we will change a new machine. The price of new machine is C_1, and if the machine is broken, it would cause loss C_2.

1. Give the long-time average fee of this machine.
2. Let $C_1 = 10, C_2 = 0.5$, and $p(x) = \mathbb{1}_{(0,10)}(x)\frac{1}{10}$. Which T can let the fee be minimum.

SOLUTION.

1. Let ξ be the time when the machine will broken. Let $\gamma := \xi \wedge T$. Then the machine will be changed at γ. Obviously $\mathbb{E}(\gamma) = T\mathbb{P}(\xi > T) + \mathbb{E}(\xi\mathbb{1}(\xi \leq T)) = T\int_T^\infty p(x)\,\mathrm{d}x + \int_0 Txp(x)\,\mathrm{d}x$. Let η be the fee of this machine, then we have $\eta = C_1\mathbb{1}(\xi > T) + (C_1 + C_2)\mathbb{1}(\xi \leq T) = C_1 + C_2\mathbb{1}(\xi \leq T)$. So $\mathbb{E}(\eta) = C_1 + C_2\int_0^T p(x)\,\mathrm{d}x$. So the long-time average fee is

$$g(T) = \frac{C_1 + C_2\int_0 Tp(x)\,\mathrm{d}x}{T\int_T^\infty p(x)\,\mathrm{d}x + \int_0^T xp(x)\,\mathrm{d}x}$$

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2. Easy to get that $g(T) = \frac{200+T}{20T-T^2}$ when $T \in (0, 10)$. And $g'(T) = \frac{T^2+400T-4000}{(20T-T^2)^2}$. Let $g'(T) = 0$, then $T^2 + 400T - 4000 = 0$, then $T = 20\sqrt{110} - 200 \approx 9.76$. So $T = 9.76$ can make the fee get minimum.

$\square$

PROBLEM IV A kind of product is qualified with probability $p(0 < p < 1)$. We sample these product by the following way: we check all the product at first until there appears k qualified product sequently. Then we check the rest of product by probability $\alpha(0 < \alpha < 1)$ until there appears one unqualified product, then one circle ends. Next we restart another checking circle. Please find out the proportion of checked product after a long time.

SOLUTION. For sake of convenience, we call the k qualified products appearing sequently as k qualified sequence. Assume the proportion of checked product after a long time is β. Let N_k be the amount of product when the first k qualified sequence ends. $M_k = \mathbb{E}(N_k)$. G_k is the event that the next one is qualified after the first $k-1$ qualified sequence ends. Obviously, $\mathbb{E}(N_k - N_{k-1} \mid G_k) = 1$. And $\mathbb{E}(N_k - N_{k-1} \mid G_k) = \mathbb{E}(N_k) + 1$. Therefore, $\mathbb{E}(N_k - N_{k-1}) = p + (1-p)(\mathbb{E}(N_k) + 1)$. So $M_{k-}M_{k-1} = p + (1-p)(1 + M_k)$. Then $pM_k = M_{k-1} + 1$. Thus, $M_k = \frac{\frac{1}{p^k}-1}{1-p}$. Let $A = \{\text{The amount of product checked in one circle}\}$, $B = \{\text{The amount of product in one circle}\}$. So finding one unqualified product need average checking time $\frac{1}{1-p}$ according to geometric distribution. Then we averagely need $\frac{1}{\alpha(1-p)}$ product to find out the unqualified one. Then $\mathbb{E}(A) = M_k + \frac{1}{1-p}, \mathbb{E}(B) = M_k + \frac{1}{\alpha(1-p)}$. So

$$\beta = \frac{\mathbb{E}(A)}{\mathbb{E}(B)} = \frac{\frac{\frac{1}{p^k}-1}{1-p} + \frac{1}{1-p}}{\frac{\frac{1}{p^k}-1}{1-p} + \frac{1}{\alpha(1-p)}} = \frac{\alpha}{\alpha + p^k - \alpha p^k}$$

$\square$