PROBLEM I Assume $(X_n : n \geq 0)$ is an irreducible Markov chain on E. Prove that $(X_n : n \geq 0)$ is recurrent (or transient) $\iff \forall i \in E$,

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{X_k = i\}\right) = 1(\text{or } 0).$$

PROBLEM II Let $(X_n : n \geq 0)$ is a one dimension simple random walk, and P is it's transition matrix. Let $a \leq b \in \mathbb{Z}$ satisfies $\mathbb{P}(a \leq X_0 \leq b) = 1$. Define $\tau = \inf\{n \geq 0 : X_n = a \text{or } b\}, Y_n = X_{n\wedge\tau}$. Prove: $(Y_n : n \geq 0)$ is Markov chain on $[a, b] \cap \mathbb{Z}$, and give its transition matrix and the classification. PROBLEM III Prove: $(X_n : n \geq 0)$ is Markov chain on E, where E is finite. Then $\exists x \in E$, x is recurrent. PROBLEM IV Assume $(X_n : n \geq 0)$ is Markov chain on $\mathbb{Z}$. Prove it is transient $\iff \forall\mu_0$ is primitive distribution, $\lim_{n\to\infty} |X_n| \overset{\text{a.s.}}{=} \infty$. PROBLEM V Assume P is a transition matrix on $\mathbb{Z}^+$, which has a first line $\{a_0, a_1, \cdots\}$, $\forall i \geq 1$, $p_{i,i-1} = 1$, and $\forall j \neq i-1$, $p_{i,j} = 0$. Discuss the irreducibility, recurrence, ergodicity and periodicity of 0. PROBLEM VI Assume P is a transition matrix on E. Prove: $\forall i \in E$, $\lim_{n\to\infty} p_{ii}(n)$ exists, and $\lim_{n\to\infty} p_{ii}(n) = \frac{1}{F'_{ii}(1)} = \frac{1}{\mathbb{P}_i(T_i)}$. PROBLEM VII Assume P is a transition matrix on E and P is irreducible, $j \in E$. Prove: P is recurrent $\iff$ 1 is the minimum non negtive solution of

$$y_i = \sum_{k\neq j} p_{ik}y_k + p_{ij}, i \in E$$

PROBLEM VIII Let $\{a_k : k \geq 0\}$ satisfies $\sum_{k\geq 0} a_k = 1, a_k \geq 1, a_0 > 0$, $\mu := \sum_{k=1}^{\infty} ka_k > 1$. Define $p_{ij} = \begin{cases} a_j & , i = 0 \\ a_{j-i+1} & , i \geq 1 \wedge j \geq i-1 \\ 0 & , \text{otherwise} \end{cases}$. Prove: P is transient.