Iterative 2

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PROBLEM I

1. Prove that the iteration matrix G_ω of SSOR, as defined by

$$G_\omega = (D - \omega F)^{-1}(\omega E + (1-\omega)D)(D - \omega E)^{-1}(\omega F + (1-\omega)D).$$

   can be expressed as

$$G_\omega = I - \omega(2-\omega)(D-\omega F)^{-1}D(D-\omega E)^{-1}A.$$

2. Deduce the expression below for the preconditioning matrix associated with the SSOR iteration.

$$M_{SSOR} = \frac{1}{\omega(2-\omega)}(D-\omega E)D^{-1}(D-\omega F)$$

PROBLEM II Let A be a matrix with a positive diagonal D.

(a) Obtain an expression equivalent to that of (4.13) for G_ω, but which involves the matrices

$$S_E \equiv D^{-1/2}ED^{-1/2} \quad \text{and} \quad S_F \equiv D^{-1/2}FD^{-1/2}.$$

(b) Show that

$$D^{1/2}G_\omega D^{-1/2} = (I-\omega S_F)^{-1}(I-\omega S_E)^{-1}(\omega S_E + (1-\omega)I)(\omega S_F + (1-\omega)I).$$

(c) Now assume that, in addition to having a positive diagonal, A is symmetric. Prove that the eigenvalues of the SSOR iteration matrix G_ω are real and nonnegative.

PROBLEM III A matrix of the form

$$B = \begin{pmatrix} 0 & E & 0 \\ 0 & 0 & F \\ H & 0 & 0 \end{pmatrix}$$

is called a three-cyclic matrix.

(a) What are the eigenvalues of B? (Express them in terms of eigenvalues of a certain matrix which depends on E, F, and H.)

(b) Assume that a matrix A has the form $A = D + B$, where D is a nonsingular diagonal matrix, and B is three-cyclic. How can the eigenvalues of the Jacobi iteration matrix be related to those of the Gauss–Seidel iteration matrix? How does the asymptotic convergence rate of the Gauss–Seidel iteration compare with that of the Jacobi iteration matrix in this case?

(c) Answer the same questions as in (b) for the case when SOR replaces the Gauss–Seidel iteration.

(d) A matrix of the form

$$B = \begin{pmatrix} 0 & E_1 & 0 & \cdots & 0 \\ 0 & 0 & E_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & E_{p-1} \\ E_p & 0 & \cdots & 0 & 0 \end{pmatrix}$$

is called a p-cyclic matrix, where the blocks $E_1, \ldots, E_p$ are conformable (for instance, each E_i is an $m_i \times m_{i+1}$ block with $m_{p+1} = m_1$).

PROBLEM IV Consider the linear system $Ax = b$, where A is symmetric positive definite. Consider a projection step with $K = L = \mathrm{span}\{v\}$, where v is some nonzero vector. Let x_{new} be the new iterate after one projection step from x, and let

$$d = A^{-1}b - x, \qquad d_{\mathrm{new}} = A^{-1}b - x_{\mathrm{new}}.$$

(a) Show that

$$(Ad_{\mathrm{new}},\, d_{\mathrm{new}}) = (Ad,\, d) - \frac{(r, v)^2}{(Av, v)},$$

where $r = b - Ax$. Does this equality establish convergence of the algorithm?

(b) (Gastinel's method.) In Gastinel's method the vector v is selected so that $(v, r) = |0r|0_1$; for example one can take v with components $v_i = \mathrm{sign}(e_i^T r)$, where $r = b - Ax$ is the current residual. Show that

$$|0d_{\mathrm{new}}|0_A \le \left(1 - \frac{1}{n\kappa(A)}\right)^{1/2} |0d|0_A,$$

where $\kappa(A)$ is the spectral condition number of A. Does this prove that the algorithm converges?

(c) Compare the cost of one step of this method with that of cyclic Gauss–Seidel (see Example 5.1) and with that of an "optimal"Gauss–Seidel in which at each step $K = L = \mathrm{span}\{e_i\}$ and i is chosen to be the index of largest magnitude in the current residual vector.

PROBLEM V Consider the iteration

$$x_{k+1} = x_k + \alpha_k d_k,$$

where d_k is a nonzero vector called the search direction and α_k is a scalar. Assume a method that chooses α_k so that the residual $|0r_{k+1}|0_2$ is minimized, where $r_k = b - Ax_k$.

(a) Determine α_k that minimizes $|0r_{k+1}|0_2$.

(b) Show that the residual vector r_{k+1} obtained in this manner is orthogonal to Ad_k.

(c) Show that the residuals satisfy

$$|0r_{k+1}|0_2 \leq |0r_k|0_2 \sin\angle(r_k, Ad_k).$$

(d) Assume that at each step k we have $(r_k, Ad_k) \neq 0$. Will the method always converge?

(e) Now assume A is symmetric positive definite and choose $d_k \equiv r_k$ at each step. Prove that the method converges for any initial guess x_0.