PROBLEM I Prove: If $m \in \mathbb{Z}^+, a \in \mathbb{Z}, \gcd(a,m) = 1$, A is reduced residue system of m, then

$$\sum_{i \in A} \left\{ \frac{ai}{m} \right\} = \frac{1}{2}\phi(m)$$

SOLUTION. Let $f : \mathbb{Z} \to \{1, \cdots, m-1\}, f(x) \equiv x \mod m$, then $\left\{\frac{ai}{m}\right\} = \frac{f(ai)}{m}$. Then $\sum_{i\in A} \left\{\frac{ai}{m}\right\} = \sum_{i\in A} \frac{f(ai)}{m}$. Obviously, we can get $\{f(ai) : i \in A\} = \{f(i) : i \in A\} =: B$, then $\sum_{a\in A} \left\{\frac{ai}{m}\right\} = \sum_{a\in A} \frac{f(i)}{m} = \sum_{x\in B} \frac{x}{m}$ and $\mathrm{card}(B) = \phi(m)$. And $\forall x \in B$, $(x, m) = (m - x, m) = 1$, then $\exists y \in A$, s.t. $x + y = m$, then $\sum_{x\in B} \frac{x}{m} = \frac{m\mathrm{card}(B)}{2m} = \frac{1}{2}\phi(m)$. □

PROBLEM II

1. Prove: $\sum_{i=0}^{a} \phi(p^i) = p^a$, where p is prime.
2. Prove: $\sum_{d\in\mathbb{N}:d|a} \phi(d) = a$.

SOLUTION.

1. Obviously, $\phi(p^i) = p^i - p^{i-1}, i = 1, \cdots, a$. So $\sum_{i=0}^{a} \phi(p^i) = 1 \sum_{i=1}^{a}(p^i - p^{i-1}) = p^a$.
2. Since a can be decomposed by primes, let $a = p_1^{r_1} \cdots p_s^{r_s}$, where p_i are primes, $p_i \neq p_j, i \neq j$, $r_i \in \mathbb{N}$, $i = 1, \cdots, s$. So $A := \{d \in \mathbb{N} : d \mid a\} = \{p_1^{t_1} \cdots p_s^{t_s} : 0 \le t_i \le r_i, i = 1, \cdots, s\}$, then $\phi(p_1^{t_1} \cdots p_s^{t_s}) = \phi(\prod_{i=1}^{s} p_i^{t_i}) = \prod_{i=1}^{s} \phi(p_i^{t_i})$. So $\sum_{d\in A} \phi(d) = \sum_{0\le t_i\le r_i, i=1,\cdots,s} \prod_{i=1}^{s} \phi(p_i^{t_i})$.

$$\begin{aligned}
\sum_{d\in A} \phi(d) &= \sum_{0\le t_i\le r_i, i=1,\cdots,s} \prod_{i=1}^{s} \phi(p_i^{t_i}) \\
&= \sum_{0\le t_1\le r_1} \phi(p_1^{t_1})(\sum_{0\le t_i\le r_i, i=2,\cdots,s} \prod_{i=2}^{s} \phi(p_i^{t_i})) \\
&= p_1^{r_1} \sum_{0\le t_i\le r_i, i=2,\cdots,s} \prod_{i=2}^{s} \phi(p_i^{t_i}) \\
&= \prod_{i=1}^{s} p_i^{r_i} = a
\end{aligned} \tag{1}$$

□

PROBLEM III If today is Monday, then what day is it $10^{10^{10}}$ days after today?

SOLUTION. Since $10^{10} \equiv 1 \mod 3$, $10^{10} \equiv 0 \mod 2$, by Chinese Remainder Theorem, we only need to find a integer $n \le 5$ which satisfies $n \equiv 1 \mod 3$ and $n \equiv 0 \mod 2$. So $n = 4$, then $10^{10} \equiv 4 \mod 6$. And $\gcd(10, 7) = 1$, then $10^{10^{10}} \equiv 3^4 \equiv 4 \mod 7$. So it is Friday $10^{10^{10}}$ days later. □

PROBLEM IV Find the remainder of $(12371^{56} + 34)^{28} \mod 111$.

SOLUTION. Since $111 = 3 \times 37$, then we can compute the remainder of $(12371^{56} + 34)^{28} \mod 3$, $(12371^{56} + 34)^{28} \mod 37$ at first.

1. Since $34 \equiv 1 \mod 3$, $12371 \equiv 2 \mod 3$, then $12371^{56} \equiv 2^{56} \equiv 1 \mod 3$, then $12371^{56} + 34 \equiv 2 \mod 3$. So $\gcd(12371^{56} + 34, 3) = 1$, so $(12371^{56} + 34)^{28} \equiv 2^{28} \equiv 1 \mod 3$.
2. Since $12371 \equiv 13 \mod 37$, then $\gcd(12371, 37) = 1$, and $56 \equiv 20 \mod 36$, then $12371^{56} \equiv 13^{20} \equiv 16 \mod 37$. Then $12371^{56} + 34 \equiv 13 \mod 37$, so $\gcd(12371^{56} + 34, 37) = 1$, then $(12371^{56} + 34)^{28} \equiv 13^{28} \equiv 33 \mod 37$.

So by Chinese Remainder Theorem, we only need to find a integer $n \leq 110$ which satisfies $n \equiv 1 \mod 3, n \equiv 33 \mod 37$. Assuming $n = 33 + 37k$, then $k = 0, 1, 2$, then $33 + 37k \equiv 1 \mod 3$, then $k = 1$, so $n = 70$. Thus $(12371^{56} + 34)^{28} \equiv 70 \mod 111$.

$\square$

PROBLEM V Prove: $\frac{a}{b} \in \mathbb{Q}, 0 < a < b, \gcd(a, b) = 1$ is pure recurring decimal $\iff$ $\exists t \in \mathbb{N}^+$ s.t. $10^t \equiv 1 (\mod b)$, and $\min\{t \in \mathbb{N}^+ : 10^t \equiv 1 (\mod b)\}$ is the length of cycle section.

SOLUTION. Let l be the length of cycle section of $\frac{a}{b}$. " $\Longrightarrow$ ": Assume $\frac{a}{b} = \sum_{k=1}^{\infty} 10^{-kl} x$, where $x \in \mathbb{N}, 0 < x < 10^l$, so $\frac{a}{b} = x \frac{1}{10^l} \frac{1}{1 - 10^{-l}} = \frac{x}{10^l - 1}$. Then $a(10^l - 1) = bx$. Since $\gcd(a, b) = 1$, we get $b \mid 10^l - 1$. And we get $l \in \{t \in \mathbb{N}^+ : 10^t \equiv 1 \mod b\}$.

" $\Longleftarrow$ ": Assume $10^t \equiv 1 \mod b$, where $t \in \mathbb{N}^+$. Let $10^t - 1 = bk$, where $k \in \mathbb{N}^+$. Let $x = ak$, we will prove $\frac{a}{b} = \sum_{k=1}^{\infty} 10^{-kt} x$. Easily $\sum_{k=1}^{\infty} 10^{-kt} x = \frac{x}{10^t - 1} = \frac{ak}{bk} = \frac{a}{b}$. So $\frac{a}{b}$ is pure recurring decimal and $l \mid t$.

So obviously $l = \min\{t \in \mathbb{N}^+ : 10^t \equiv 1 \mod b\}$. $\square$