PROBLEM I Prove that if $(X_n : n \geq 0)$ is a simple random walk, then so is $(-X_n : n \geq 0)$.

SOLUTION . Since $(X_n : n \geq 0)$ is a simple random walk, then $\exists(\xi_i : i \geq 0)$ are i.i.d. r.v. X_0 is a r.v. which is independent with ξ_1 such that $X_n = X_0 + \sum_{i=0}^{n} \xi_i$, $\mathbb{P}(|\xi_1| = 1) = 1$. So let $Y_n = -X_n$, $Y_0 = -X_0$ is r.v., $\varepsilon_i = -\xi_i, i \geq 0$,then $(\varepsilon_i, i \geq 0)$ are i.i.d. and is independent with Y_0, and $\mathbb{P}(|\varepsilon_1| = 1) = 1$. So $(Y_n : n \geq 0)$ is a simple ramdom walk. $\square$

PROBLEM II Let $(X_n : n \geq 0)$ be a d-dimentional random walk, and $\mathbb{P}(|\xi_1| \geq 1) > 0$, prove that $\mathbb{P}(\sup_n |X_n| = \infty) = 1$.

SOLUTION . Since $\mathbb{P}(|\xi_1| \geq 1) > 0$, then $\exists t \in \mathbb{R}^d$, such that $\mathbb{P}(\xi_1 = t) > 0$. Besides,

$$\begin{aligned}
&\mathbb{P}(\sup_n |X_n| = \infty) \\
=&\mathbb{P}(\bigcap_{k\in\mathbb{N}} \{\sup_n |X_n| \geq k\}) \\
=& \lim_{k\to\infty} \mathbb{P}(\sup_n |X_n| \geq k) \\
=& \inf_{k\in\mathbb{N}} \mathbb{P}(\sup_n |X_n| \geq k)
\end{aligned} \tag{1}$$

Then, to prove $\mathbb{P}(\sup_n |X_n| = \infty) = 1$ is equal to prove $\forall k \in \mathbb{N}, \mathbb{P}(\sup_n |X_n| \geq k) = 1$. Let $v > 4\frac{k}{|t|}$ and let $A_u = \{\omega \in \Omega : \xi_{uv+1} = t, \cdots, \xi_{uv+v} = t\}$, so $\forall \omega \in A_u, |X_{uv+v} - X_{uv}| = |\sum_{m=uv+1}^{uv+v} \xi_m| = |vt| = v|t| \geq 4k$. Then $2\max\{|X_{uv+v}|, |X_{uv}|\} \geq |X_{uv+v}| + |X_{uv}| \geq |\sum_{m=uv+1}^{uv+v} \xi_m| \geq 4k$, so $\max\{|X_{uv+v}|, |X_{uv}|\} \geq 2k > k$. Thus $\sup_n |X_n| \geq k$. Besides, since $\xi_i, i \in \mathbb{N}^+$ is i.i.d., then $\mathbb{P}(A_u) = \mathbb{P}(\xi_1 = 1)^v$. And it is obvious that $A_u, u \in \mathbb{N}^+$ is independent, $\sum_{i=0}^{\infty} \mathbb{P}(A_i) = \infty$, by BC theorem, we can get that $\mathbb{P}(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j) = 1$. Since $\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j \subset \bigcup_{i=1}^{\infty} A_i \subset \{\sup_n |X_n| = \infty\}$, then $\mathbb{P}(\{\sup_n |X_n| = \infty\}) = 1$.

$\square$

PROBLEM III Let $(X_n : n \geq 0)$ be a symmtry simple random walk with $X_0 = 0$, for $d = 2$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{4^{2n}} \left(\frac{(2n)!}{(n!)^2} \right)^2$$

For $d = 3$, prove that

$$\mathbb{P}(S_{2n} = 0) = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} \sum_{i+j+k=n} \left(\frac{1}{3^n} \frac{n!}{i!j!k!} \right)^2$$

SOLUTION . 1. $d = 2$,

$$\begin{aligned}
&\mathbb{P}(S_{2n} = 0)\\
=&(\frac{1}{4^{2n}})(\sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k})\\
=&\frac{1}{4^{2n}}\sum_{k=0}^{n}\frac{(2n)!}{(k!)^2((n-k)!)^2}\\
=&\frac{1}{4^{2n}}\frac{(2n)!}{(n!)^2}\sum_{k=0}^{n}\frac{(n!)^2}{(k!)^2((n-k)!)^2}\\
=&\frac{1}{4^{2n}}\frac{(2n)!}{(n!)^2}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}\\
=&\frac{1}{4^{2n}}\frac{(2n)!}{(n!)^2}\binom{2n}{n}\\
=&\frac{1}{4^{2n}}(\frac{(2n)!}{(n!)^2})^2
\end{aligned} \tag{2}$$

2. $d = 3$,

$$\begin{aligned}
&\mathbb{P}(S_{2n} = 0)\\
=&\frac{1}{6^{2n}}(\sum_{k+j=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{j}\binom{2n-2k-j}{j}\binom{2n-2k-2j}{n-k-j})\\
=&\frac{1}{6^{2n}}(\sum_{j+k=0}^{n}\frac{(2n)!}{(k!)^2(j!)^2((n-k-j)!)^2})\\
=&\frac{1}{6^{2n}}\frac{(2n)!}{(n!)^2}\sum_{j+k=0}^{n}\frac{(n!)^2}{(k!)^2(j!)^2((n-k-j)!)^2}\\
=&\frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}\sum_{i+j+k=n}\left(\frac{1}{3^n}\frac{n!}{i!j!k!}\right)^2
\end{aligned} \tag{3}$$

□

PROBLEM IV Assume $(S_n : n \geq 0)$ is a symmtry simple random walk with $S_0 = i \in \mathbb{Z}$. Prove that $\forall a \in \mathbb{Z}$, let $\tau_a := \min\{n \in \mathbb{N} : S_n = a\}$, then $\mathbb{P}(\tau_a < \infty) = 1$.

SOLUTION . By the theorem 1.2.2 of textbook, it is obvious that $P(\tau_a < \infty) = \lim_{b\to\infty} P_i(\tau_a < \tau_b) = \frac{b-i}{b-a} = 1$. □