Introduction to Linear Optimization

Arnar Bjarni Arnarson

December 11, 2024

School of Computer Science
Reykjavík University

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- What did this lead to?

Linear Optimization

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- We have a multivariate linear function $f(x_1, x_2, \dots)$.
- We wish to find $\max f(x_1, x_2, \dots)$ given some constraints.
- The constraints are also linear, such as $a_1 x_1 \leq b_1$.
- We also require $x_i \geq 0$. Can shift any problem to satisfy this condition.
- Wish to find the values of x_i as well.

An Example - Nutrition

- Nutritionist is helping someone with a daily diet.
- Kilocalorie intake should be between $1\,500$ and $1\,800$ calories.
- Protein intake should be between 65 and 100 grams.
- Fat intake should be between 25 and 35 grams.
- Carbohydrate intake should be between 150 and 250 grams.
- Budget is low, want to minimize cost!
- Chicken ($1\,000$ kcal, 50 protein, 20 fat, 0 carbs) costs 2.
- Bread (500 kcal, 5 protein, 9 fat, 140 carbs) costs 1.
- Fruit (200 kcal, 2 protein, 10 fat, 50 carbs) costs 3.

An Example - Formulation

- Let x_1 be number of chicken.
- Let x_2 be number of bread.
- Let x_3 be number of fruit.
- We want to minimize $2x_1 + x_2 + 3x_3$ subject to:
    - $1\,500 \leq 1\,000x_1 + 500x_2 + 200x_3 \leq 1\,800$
    - $65 \leq 50x_1 + 9x_2 + 2x_3 \leq 100$
    - $25 \leq 20x_1 + 3x_2 + 10x_3 \leq 35$
    - $150 \leq 0x_1 + 140x_2 + 50x_3 \leq 250$
    - $0 \leq x_1, x_2, x_3$
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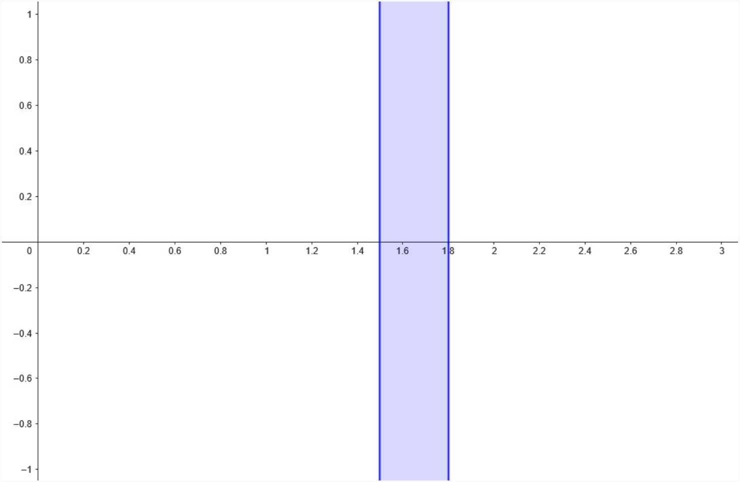
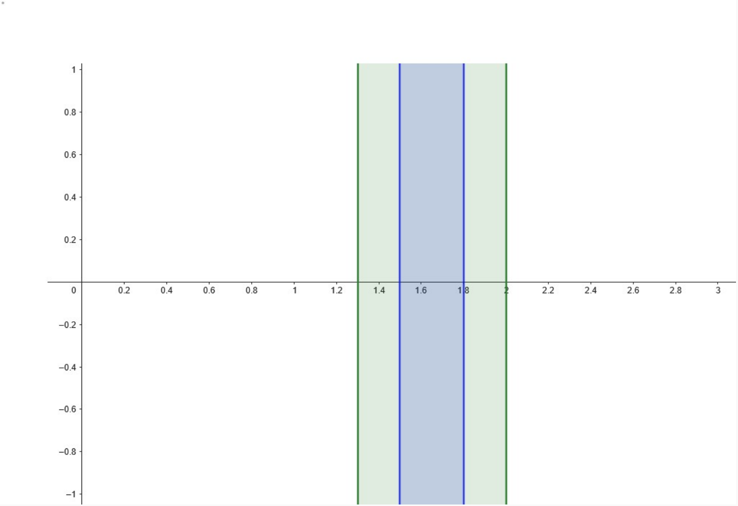
- Linear Optimization solver finds the answer!
- But how?

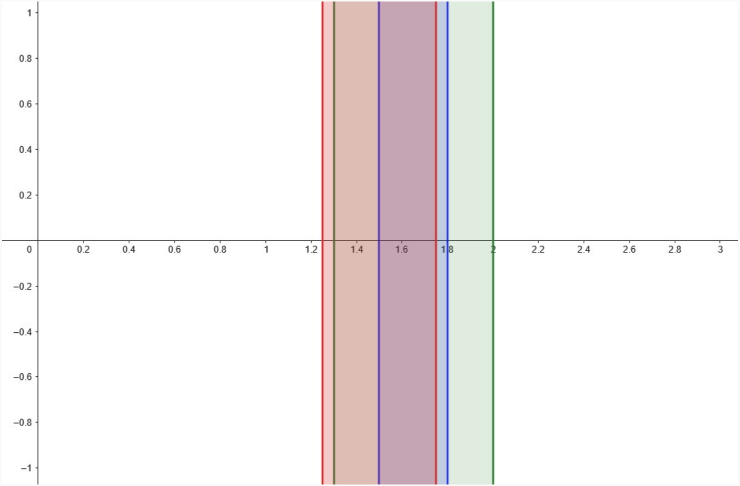
Chicken - kcal



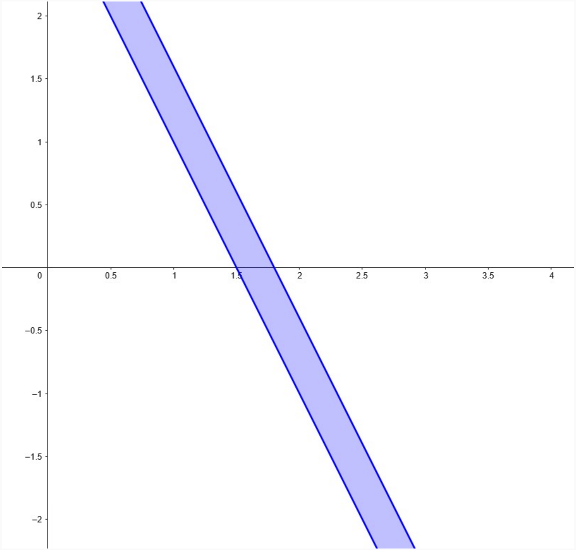
Chicken - kcal, protein



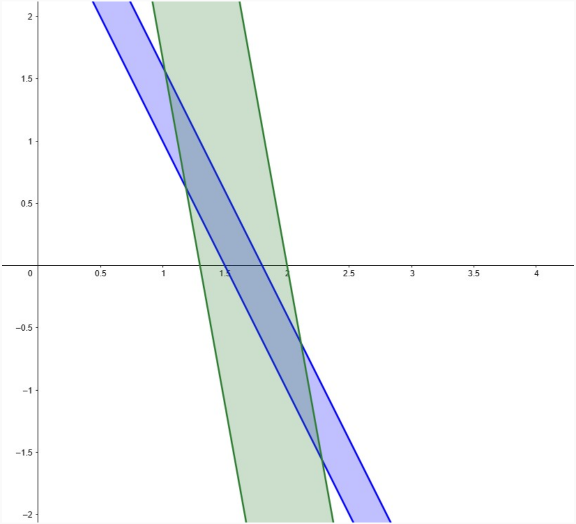
Chicken - kcal, protein, fat



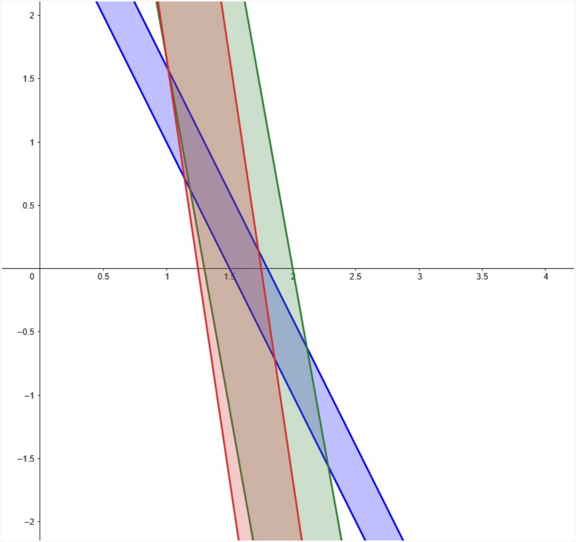
Chicken, bread - kcal



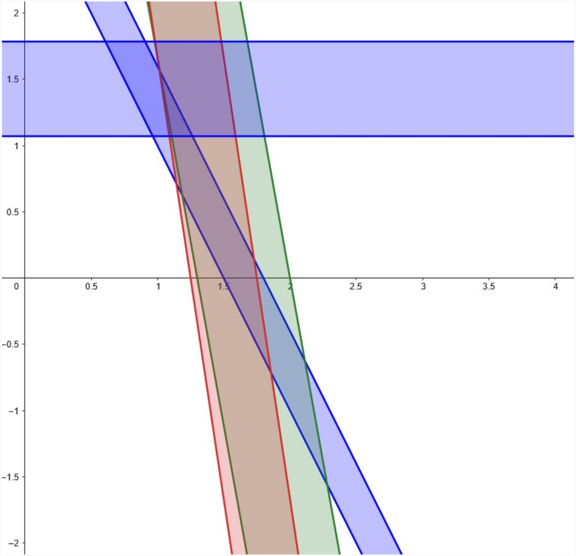
Chicken, bread - kcal, protein



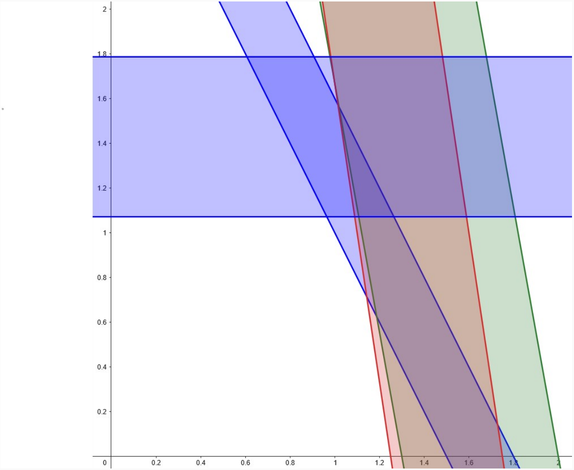
Chicken, bread - kcal, protein, fat



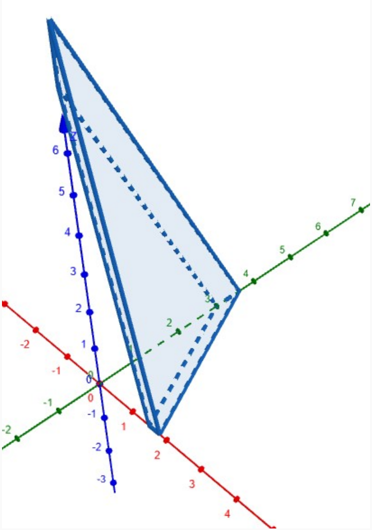
Chicken, bread - all



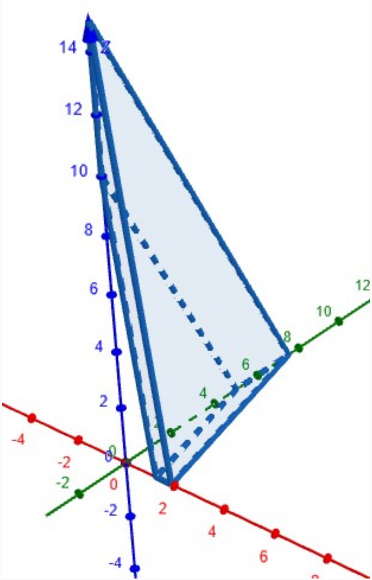
Chicken, bread - all



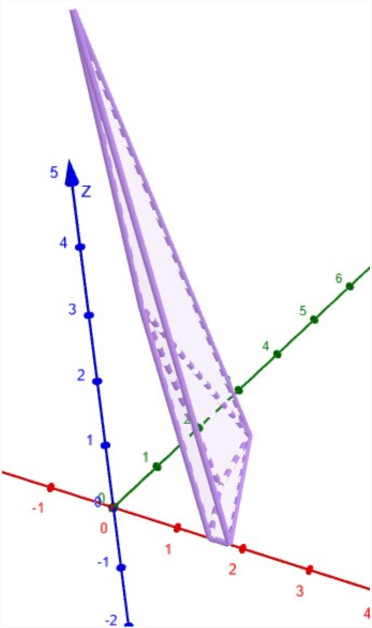
All - kcal



All - protein



All - kcal and protein



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- Half-plane intersection can give us the convex polygon $(p_1, p_2, \ldots, p_k)$.
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- Pick a segment on the polygon. It can be shown that f is convex along the segment.
- This means we can ternary search for the optimal answer on the segment.
- We have reduced our search space from an infinite set to a finite set of points.

Extending to higher dimensions

- We can walk along the edges of the polytope in higher dimensions, towards points with better objective values.
- If an extremal point does not have optimal objective value, then at least one edge lead to a strictly higher value.
- The Simplex algorithm (Dantzig) applies this method.
- Most linear optimization programs have a finite, but unreasonably large search space.
- A common constraint is to make all variables integers (or even binary values).

PuLP

- You can use the PuLP python package.
- See example code on Canvas.
- This is a new problem solving paradigm for most, if not all, of you.
- You can expect it to take a while for it to click.