E-402-STFO Problems for Module 3

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100 points give you full marks. This set of problems revolve around combinatorics. This includes state transfer methods and generating functions. A large portion of this module is turned in via PDF files which are not autograded, programmatic use of generating functions and related concepts will be explored in the next module.

We recommend reading the first chapter or two of generatingfunctionology for this module and the next one. It is available legally for free from the University of Pennsylvania webpage, click here. If you do read it we recommend skipping subchapter 1.4.

The first four problems are programming exercises to be turned in on Kattis, the others are to be turned in on Canvas.

`m3p1`, 10 points Implement a general program to solve state transfer problems, see details on Kattis.

`m3p2`, 10 points Count the number of paths from A1 to H8 on a chessboard of a given length using the state transfer matrix method.

`m3p3`, 10 points Colour lightposts in several colours using state transfer matrix, see details on Kattis.

`m3p4`, 10 points Find the number of subsets of a $n \times k$ grid of squares that have no two adjacent squares using state transfer matrix.

`m3p5`, 10 points Find the ordinary and exponential generating functions of the following sequences:

* $a_n = \alpha n + \beta$ for some constant α, β
* $a_n = \alpha^n$ for some constant α
* $a_n = n \pmod 2$, so $a_i = 1$ if i is odd and $a_i = 0$ if i is even
* $a_n = f_n$ where $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$

Note one of these problems will produce a differential equation. You do

not need to solve this differential equation yourself, you may simply use a computer solver like Mathetmatica/Wolframalpha or Sagemath.

`m3p6`, 10 points Assume $A(x)$ is the ordinary generating function for $a_0, a_1, \ldots$. Find expressions for the ordinary generating functions of the following sequences in terms of A:

* $b_n = \alpha a_n + \beta$ for some constants α, β
* $b_n = a_{n+1}$, so the sequence starts $a_1, a_2, \ldots$ with a_0 cut off
* $b_i = a_i$ if i is even and $b_i = 0$ otherwise, so the sequence starts $a_0, 0, a_2, 0, a_4, \ldots$
* $b_n = a_0 + a_1 + \cdots + a_n$

`m3p7`, 10 points Let $D(x)$ be the ordinary generating function of partitions into distinct parts, i.e. writing n as a sum of integers with no two values equal. Let $O(x)$ be the ordinary generating function of partitions into odd parts, i.e. writing n as the sum of odd integers. Calculate $D(x)$ and $O(x)$ and then show that they are equal.

`m3p8`, 10 points Write a recursive formula for the generating function of unary-binary trees, that is to say trees where each vertex has $0, 1$ or 2 children.

`m3p9`, 10 points Find the exponential generating function for derangements (permutations where no element stays in place) by writing out a generating function formula that it satisfies. Consider the generating functions for sets and permutations.

`m3p10`, 10 points Find the exponential generating function for Fubini numbers by considering the composition of other generating functions. Fubini numbers count the number of weak orders, i.e. the way to order n elements but allowing ties.