# Homework 2: First Draft

1) Coding Question, answers in file.

a) Plots uploaded to canvas.

b) Answer Included in code file.

c) - The plot shape tells us there is a strong positive correlation between the predicted and observed data.

d) 3D Plot uploaded to canvas.

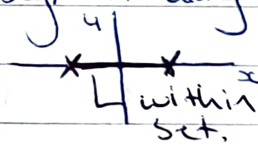
2) Determine if the sets are a convex subset of the space $\mathbb{R}^n$.

a) $\mathbb{N} \subset \mathbb{R}$, [convex]

$\mathbb{N}$ is the set of natural numbers i.e. 1, 2, 3, ....
For any 2 points $x, y \in \mathbb{N}$, is a real number and remains in $\mathbb{N}$. Therefore $\mathbb{N}$ is a convex subset of $\mathbb{R}$.

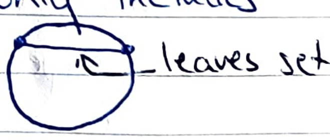
b) $X = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$. [convex]

Here we have $x$ in the set $\mathbb{R}$, with $y$ being a constant at 0. Any two points will have a line segment along the $x$ axis, which will be contained within the set, therefore the set is convex.



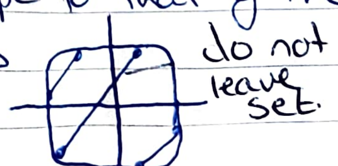

c) $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2$. [Not convex]

This represents the unit circle in $\mathbb{R}^2$. It only includes points ON the circle. Joining two points on the set will result in a line not contained within the set. Therefore the set is not convex.




d) $X = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 \leq 1\} \subset \mathbb{R}^2$, [convex]

When graphed, we have a similar shape to that of the unit circle. As the set includes points within the boundary, any line connecting the points will remain in the set. Therefore the set is convex.



e) $X = \{(x,y,z) \in \mathbb{R}^3 \mid xyz \le 1\} \subset \mathbb{R}^3$. [convex]

For 2 points in $X$, $(x_1,y_1,z_1)$ and $(x_2,y_2,z_2)$, where $x,y,z \le 1$:

The line segment connecting these points can be written as:

$$(x,y,z) = \lambda(x_2,y_2,z_2) + (1-\lambda)(x_1,y_1,z_1)$$

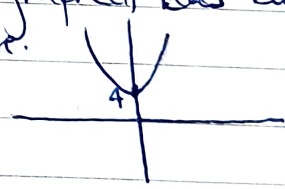
$$\begin{aligned} xyz &= \lambda(x_2,y_2,z_2) + (1-\lambda)(x_1,y_1,z_1) \\ &= \lambda(x_2) + (1-\lambda)(x_1) \cdot \lambda(y_2) + (1-\lambda)(y_1) \cdot \lambda(z_2) \\ &\quad + (1-\lambda)(z_1) \end{aligned}$$

Since both points are in $X$, $x,y,z \le 1$ for both of them. $xyz$ shows a sum of products where every term is a product of $\lambda$ & a component of $(x_1,y_1,z_1)$ and $(x_2,y_2,z_2)$ ~~of which~~. As $0 \le \lambda \le 1$, each term in $xyz$ is $\le 1$ ∴ $xyz$ is $\le 1$. The line segment is entirely within $X$ ∴ the set is convex.

3) Determine if the following functions are convex.

a) $f(x) = x^2-4$. [convex]

This is a 2-D function that when graphed, is bowl shaped, with upward facing curvature.

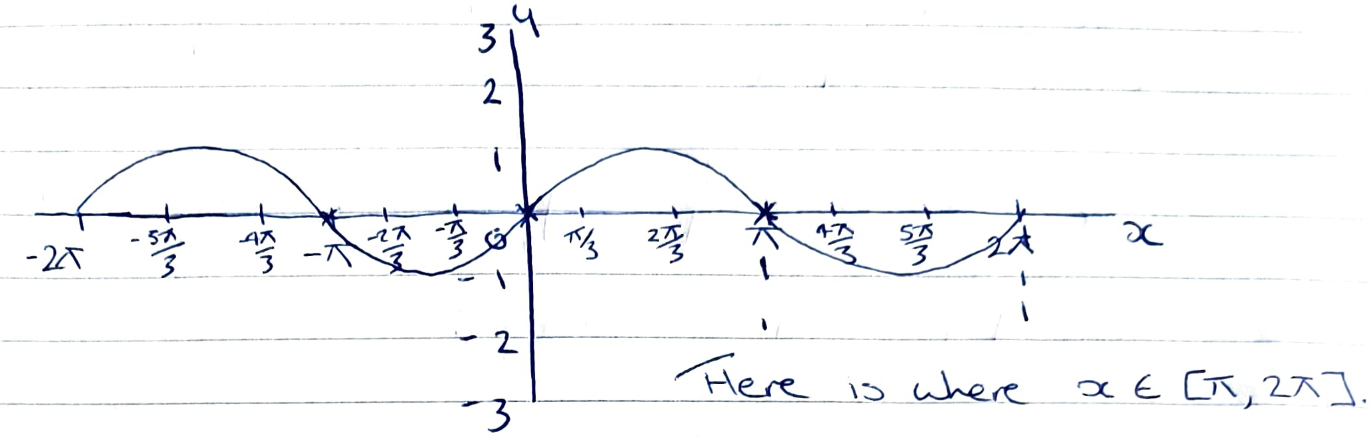



We can also see that the second derivative:

$f'(x) = 2x$

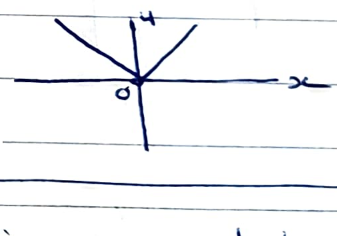
$f''(x) = 2$, is ~~constant~~ positive, further proving convexity.

b) $f(x) = \sin(x)$, $x \in [\pi, 2\pi]$. [convex]
Graphing this function:



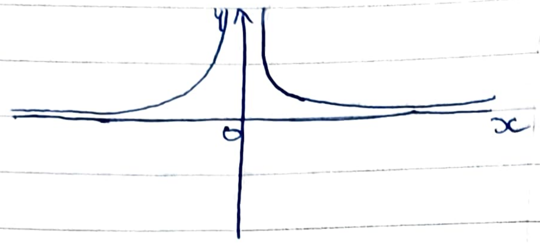

The $\sin(x)$ function in general is neither convex nor concave, as is the case with all trigonometric functions. Where $x \in [\pi, 2\pi]$, the graph is bowl shaped with an upward curve, so in this instance the function is convex.

c) $f(x) = |x|$. [convex]
This function describes the Euclidean norm (also modulus $x$). All norms are convex by definition and so this function is convex.



3d) $f(x) = 1/x^2$ [NOT CONVEX]

Graphing this function:




In this function we see that $x=0$ is undefined. Picking two points, one where $x$ is negative and one where $x$ is positive would result is a line not contained within the set. For that reason, the function is not convex.

e) $f(x,y) = x^2 + y^2$. [convex]
We can use a hessian matrix of partial derivatives to check for convexity here.
First partial derivatives:

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$$

Second partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y \partial x} = 0$$

$$\therefore \quad H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Compute the Eigenvalues:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \det(A - \lambda I) = 0$$

$$\rightarrow \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\rightarrow \det\left(\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}\right) = 0$$

$$\rightarrow (2-\lambda)(2-\lambda) = 0$$

$$\rightarrow (2-\lambda)^2 = 0$$

$$\rightarrow 2 - \lambda = 0$$

$$\rightarrow \underline{\lambda = 2}$$

The eigenvalue(s) is non-negative, therefore the ~~function is convex.~~ Hessian matrix is positive definite, meaning the function is convex.

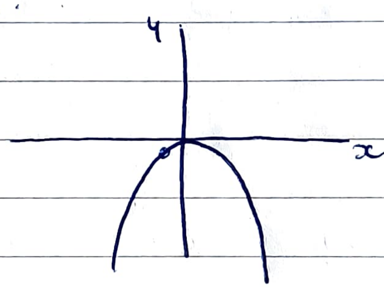
4) Suppose $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are both convex. Is their composition $g \circ f(x) = g(f(x))$ convex? Either provide a proof or counter-example.

A counterexample to the claim that the composite of two convex functions is always convex:

Let $f(x) = x^2$ and let $g(x) = -x$

$$g(f(x)) = g(x^2)$$
$$= -x^2$$

graphing this:




The product of this composition is not convex.

5) Minimise $3x_1^2 + 4x_2^2 + 2x_1x_2 - 2x_1 - 3x_2$

subject to

$3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0$

a) Writing the problem is matrix form.

Objective function matrix $= \begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix} = Q$

Objective function coefficient vector $= \begin{bmatrix} -2 \\ -3 \end{bmatrix} = c$

coefficient matrix for inequality constraints $= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = A$

RHS vector for inequality constraints $= \begin{bmatrix} 6 \\ 2 \end{bmatrix} = b$

bounds on X variable $= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq x$

standard form:

minimize $\frac{1}{2}x^T Gx + c^T x$

subject to $Ax \leq b$

$x \geq 0$

b) Answer written in code script.

c) To determine if the optimal solution is unique, we prove that the problem is convex. Convex optimisation problems have a unique solution as any optimal solution (a local min / max) is a global optimal solution - therefore is unique.

Take the hessian matrix $G$:

$$\begin{bmatrix} 6 & 2 \\ 2 & 8 \end{bmatrix}$$

To find convexity, we look for whether the matrix is positive definite, or positive semi-definite. To check for this property, we look at the eigenvalues.

$\det(A - \lambda I) = 0$ (use this for our matrix $G$).

$$\det\left[\begin{pmatrix} 6-\lambda & 2 \\ 2 & 8-\lambda \end{pmatrix}\right] = 0$$

$$(6-\lambda)(8-\lambda) - (2 \cdot 2) = 0$$

$$(48 - 6\lambda - 8\lambda + \lambda^2) - 4 = 0$$

$$(48 - 14\lambda + \lambda^2) - 4 = 0$$

$$\lambda^2 - 14\lambda + 44 = 0$$

Using the quadratic formula will solve the eigenvalues.

P.T.O $\longrightarrow$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{14 \pm \sqrt{14^2 - 4(1)(44)}}{2 \cdot 1}$$

$$\lambda = \frac{14 \pm \sqrt{196 - 176}}{2}$$

$$\lambda = \frac{14 \pm \sqrt{20}}{2}$$

$$\lambda = \frac{14 \pm 2\sqrt{5}}{2}$$

$$\lambda = 7 \pm \sqrt{5}$$

$\therefore \lambda = 7 + \sqrt{5}, \quad \lambda = 7 - \sqrt{5}$

The eigenvalues are both positive, therefore the Hessian Matrix is positive definite, and the problem is convex,
Meaning, we have a unique optimal solution.