Laboratory 2

Evelyn Wilson

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1 Introduction

2 Deriving Newton's Method with Cubic Convergence

Given the equation $g(x) = x - \phi(x)f(x) - \psi(x)f^2(x)$,// we are meant to find $\psi(x)$ and $\phi(x)$ such that g(x) converges cubically. By setting f(p) = 0 and understanding that cubic convergence entails g'(p) = 0 and g"(p) = 0, we can solve for $\phi(x)$, $\psi(x)$, λ, and show using Taylor Series that $\alpha = 3$.

2.1 Solving for $\phi(x)$

2.1.1 First derivative of the function

To solve for $\phi(x)$ and $\psi(x)$, we must take the derivative of g(x) and substitute p in.

$$g(x) = x - \phi(x)f(x) - \psi(x)f^2(x)$$

$$g'(x) = \frac{d}{dx}g(x) = 1 - [\phi'(x)f(x) + \phi(x)f'(x)] - [\psi'(x)f^2(x) + 2\psi(x)f'(x)f(x)]$$

Substituting p for x, we get:

$$g'(p) = 1 - [\phi'(p)f(p) + \phi(p)f'(p)] - [\psi'(p)f^2(p) + 2\psi(p)f'(p)f(p)]$$

Knowing that $f(p) = 0$, we can take out most terms and solve for $\phi(p)$. We can set $g'(p) = 0$ because we are looking for cubic convergence, which means $g'(p)$ and $g''(p)$ are both zero.

$$g'(p) = 1 - [\cancel{\phi'(p)f(p)} + \phi(p)f'(p)] - \cancel{[\psi'(p)f^2(p) + 2\psi(p)f'(p)f(p)]}$$

$$g'(p) = 1 - \phi(p)f'(p) = 0$$

$$\phi(x) = 1/f'(x)$$

2.2 Solving for $\psi(x)$

2.2.1 Second derivative of the function

Next, we will set the second derivative to 0 as well. First we have to solve for it. Let the chain rule commence:

$$g''(x) = \frac{d}{dx}g'(x) = -[\cancel{\phi''(x)f(x)} + 2\phi'(x)f'(x) + \phi(x)f''(x)] - [\cancel{\psi''(x)f^2(x) + 4\psi'(x)f'(x)f(x)} + 2\psi(x)f'(x)^2 + \cancel{2\psi(x)f''(x)f(x)}]$$

$$g''(p) = -2\phi'(p)f'(p) - \phi(p)f''(p) - 2\psi(p)f'(p)^2 = 0$$

To find $\psi(x)$ it is clear we must first do the first derivative of $\phi(x)$ and plug it in to find $\psi(x)$:

$$\phi'(x) = \frac{f''(x)}{f'(x)^2}$$

$$g''(p) = -2(\frac{f''(p)}{f'(p)^2})f'(p) - \frac{f''(p)}{f'(p)} - 2\psi(p)f'(p)^2 = 0$$

$$g''(p) = -2(\frac{f''(p)}{f'(p)}) - \frac{f''(p)}{f'(p)} - 2\psi(p)f'(p)^2 = 0$$

$$g''(p) = -3\frac{f''(p)}{f'(p)} - 2\psi(p)f'(p)^2 = 0$$

$$\psi(x) = \frac{-3}{2}\frac{f''(x)}{f'(x)^3}$$

2.3 Confirming Cubic Convergence with Taylor Series

To derive a cubically convergent function, we will eventually need to show a convergence of $\alpha = 3$ and solve for lambda. To do this, we will expand the Taylor series around the fixed poitn p, where p is a root of f(p):

$$g(p_n) = g(p) + g'(p)(p_n - p) + \frac{g''(p)}{2!}(p_n - p)^2 + \frac{g'''(p)}{3!}(p_n - p)^3 + ...$$

Because of cubic convergence, the first and second derivatives of g(p) are both zero, so we know the second and third terms go to zero.

$$g(p_n) = g(p) + \cancel{g'(p)(p_n - p) + \frac{g''(p)}{2!}(p_n - p)^2} + \frac{g'''(p)}{3!}(p_n - p)^3 + ...$$

Recall that p is a fixed point, where $g(p) = p$:

$$g(p_n) = p + \frac{g'''(p)}{3!}(p_n - p)^3 + ...$$

Writing in terms of error, we can get rid of the p term at the beginning. Since $\epsilon_{n+1} = |g(p_{n+1}) - g(p)|$, $p_{n+1} = g(p_n)$, and $g(p) = p$ we get:

$$\epsilon_{n+1} = |\frac{g'''(p)}{3!}(p_n - p)^3 + ...|$$

We find that $\lambda = \frac{g'''(p)}{3!}$ and since the cubic term is the first time, we find that $\alpha = 3$

2.4 Solving for λ

2.4.1 Third derivative of the function

We need to find $g^{(3)}(p)$ to solve lambda in terms of f(x), taking out all terms multiplied by f(x):

$$g'''(x) = -3\phi''(x)f'(x) - 3\phi'(x)f''(x) - \phi(x)f'''(x) - 6\psi'(x)f'(x)^2 - 6\psi(x)f''(x)f'(x)]$$

To solve in terms of f(x) and its derivatives, we first need to find $\psi'(x)$ and $\phi''(x)$

$$\psi'(x) = \frac{9f''(x)^2 - 3f'(x)f'''(x)}{2f'(x)^4}$$

$$\phi''(x) = \frac{f'''(x)f'(x) - 2f''(x)^2}{f'(x)^3}$$

Substituting these in, we get:

$$g'''(x) = -3(\frac{f'''(x)f'(x)-2f''(x)^2}{f'(x)^3})f'(x) - 3\frac{f''(p)}{f'(p)^2}f''(x) - \frac{f'''(x)}{f'(x)} - 6\frac{9f''(x)^2-3f'(x)f'''(x)}{2f'(x)^4}f'(x)^2 - 6\frac{-3}{2}\frac{f''(x)}{f'(x)^3}f''(x)f'(x)]$$

Conveniently, we can easily make every term's denominator the same and factor it out, easily combining terms:

$$g'''(x) = -\frac{3f'''(x)f'(x) - 6f''(x)^2}{f'(x)^2} - 3\frac{f''(x)^2}{f'(x)^2} - \frac{f'''(x)f'(x)}{f'(x)^2} - 9\frac{3f''(x)^2 - f'(x)f'''(x)}{f'(x)^2} + 9\frac{f''(x)^2}{f'(x)^2}]$$

$$g'''(x) = -\frac{1}{f'(x)^2}[3f'''(x)f'(x) - 6f''(x)^2 + 3f''(x)^2 + f'''(x)f'(x) + 27f''(x)^2 + 9f'(x)f'''(x) - 9f''(x)^2]$$

Finally, we get:

$$g'''(x) = -\frac{13f'''(x)f'(x) + 15f''(x)^2}{f'(x)^2}$$

Adding 3! to the denominator we get λ in terms of f(x) and its derivatives:

$$\lambda = -\frac{13f'''(x)f'(x) + 15f''(x)^2}{3!f'(x)^2} = -\frac{13f'''(x)f'(x) + 15f''(x)^2}{6f'(x)^2}$$

3 Root Finding

3.1 Secant Method

3.1.1 Part A: $f(x) = (x + \cos x)e^{-x^2} + x\cos x$

- Solution: 1.6367225538197048
- Iterations: 3
- Absolute Error: 9.256484467812243e-15
- Estimated Order: 1.898402418187951

3.1.2 Part B: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^2$

- Solution: 1.6367226553395537
- Iterations: 23
- Absolute Error: 4.339056417214892e-14
- Estimated Order: 1.000000012060638

3.1.3 Part C: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^3$

- Solution: 1.6367418166029717
- Iterations: 18
- Absolute Error: 6.174515138860453e-14
- Estimated Order: 1.0000007184661015

3.2 Newton's Method

My computer found 1e-13 to be the smallest error it could keep up with with the cubic newton's method, so with that as my acceptable error:

3.2.1 Part A: $f(x) = (x + \cos x)e^{-x^2} + x\cos x$

the derivative of this function is:

$$f_a'(x) = e^{-x^2}(1 - \sin(x)) - 2e^{-x^2}x(x + \cos(x)) + \cos(x) - x\sin(x)$$

- Solution: 1.6367225538197003
- Iterations: 3
- Absolute Error: 1.942890293094024e-16
- Estimated Order: 0.9651762669653392

3.2.2 Part B: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^2$

Derivative:

$$f_b'(x) = 2((x + \cos(x)e^{-x^2} + x\cos(x))(e^{-x^2}(1 - \sin(x)) - 2e^{-x^2}x(x + \cos(x)) + \cos(x) - x\sin(x))$$

- Solution: 1.636722451559296
- Iterations: 16
- Absolute Error: 4.402590604644813e-14
- Estimated Order: 0.9999999678433749

3.2.3 Part C: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^3$

Derivative:

$$f_c'(x) = 3((x + \cos(x))e^{-x^2} + x\cos(x))^2(e^{-x^2}(1 - \sin(x)) - 2e^{-x^2}x(x + \cos(x)) + \cos(x) - x\sin(x)$$

- Solution: 1.6367072373739817
- Iterations: 15
- Absolute Error: 3.103911738639827e-14
- Estimated Order: 0.999998913111988

The estimated order should be around 1.6, superlinear, but it does not meet that. I couldn't figure out how to correctly calculate the convergence rates, but this could also be some other issue.

3.3 Modified Newton's Method

I assumed m = 1 because there was only one root between x = 1 and 2. I estimated g(p) = m * u(p).

3.3.1 Part A: $f(x) = (x + \cos x)e^{-x^2} + x\cos x$

- Solution: 1.6367225538197003
- Iterations: 3
- Absolute Error: 1.942890293094024e-16
- Estimated order: 0.9651762669653392

3.3.2 Part B: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^2$

- Solution: 1.636722451559296
- Iterations: 16
- Absolute Error: 4.402590604644813e-14
- Estimated order: 0.9999999678433749

3.3.3 $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^3$

- Solution: 1.6367072373739817
- Iterations: 15
- Absolute Error: 3.103911738639827e-14
- Estimated order: 0.999998913111988

The estimated order should be near 2, but it is incorrect.

3.4 Cubic Newton's Method

I had to figure out how to write this out as a function in python. My (probably incorrect) Cubic Newton's Method used second derivatives of f(x), all done within python.

3.4.1 Part A: $f(x) = (x + \cos x)e^{-x^2} + x\cos x$

- Solution: 1.6367225538197003
- Iterations: 3
- Absolute Error: 1.942890293094024e-16
- Estimated order: 1.6384519091611984

3.4.2 Part B: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^2$

- Solution: 1.636722401480589
- Iterations: 80
- Absolute Error: 9.770484877315221e-14
- Estimated order: 0.9999999621117293

3.4.3 Part C: $f(x) = [(x + \cos x)e^{-x^2} + x\cos x]^3$

- Solution: 1.6366999322818843
- Iterations: 574146
- Absolute Error: 9.99998985412954e-14
- Estimated order: 0.9999965677365105

The estimated order should be near 3, but it is incorrect.

4 Four bar linkage

$f_1(\theta_2, \theta_3) = r_2\cos\theta_2 + r_3\cos\theta_3 + r_4\cos\theta_4 - r_1 = 0$
$f_2(\theta_2, \theta_3) = r_2\sin\theta_2 + r_3\sin\theta_3 + r_4\sin\theta_4 = 0$
$r_1 = 43, r_2 = 23, r_3 = 33, r_4 = 9, \theta = 65\circ$
$\theta_2 =$
$\theta_3 =$

Measuring the diagram with a protractor, I came up with estimates of $\theta_2 = 52\circ$ and $\theta_3 = 360\circ$. First, I initialized these values (converting to radians), I input the link lengths, I defined the functions f_1 and f_2. Then, I put f_1 and f_2 into a matrix and created a Jacobian matrix taking partial derivatives of each function in relation to each of the two thetas. Then, using this equation:

$$G(x) = x - J(x)^{-1}F(x)$$

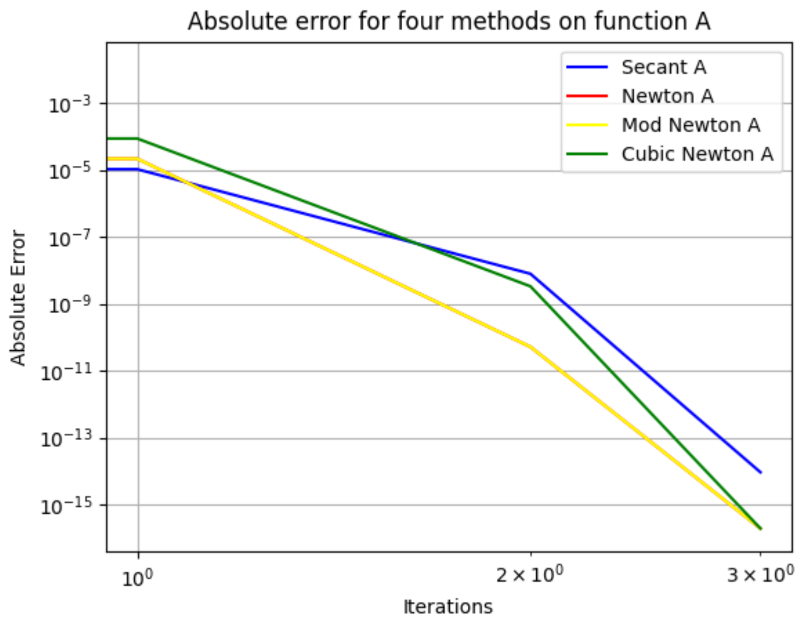

Figure 1: Comparing Methods in Part A

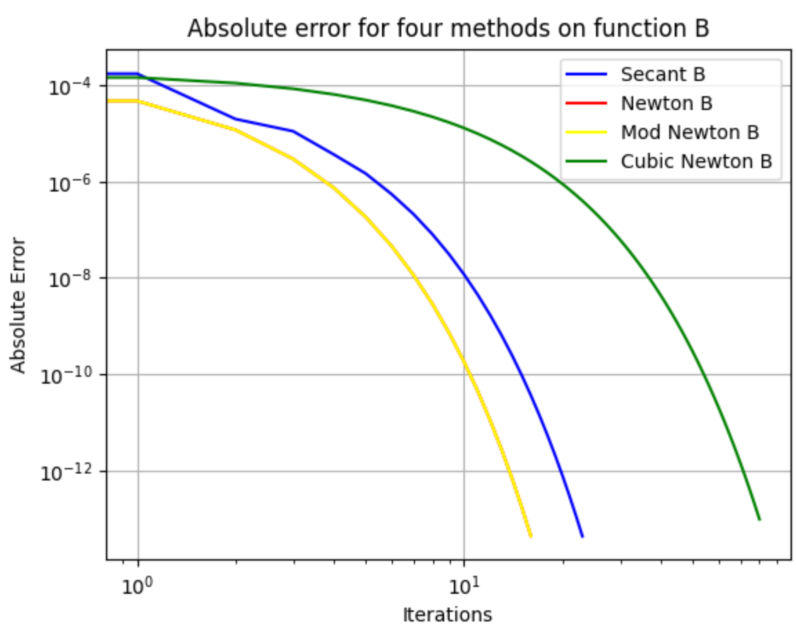

Figure 2: Comparing Methods in Part B

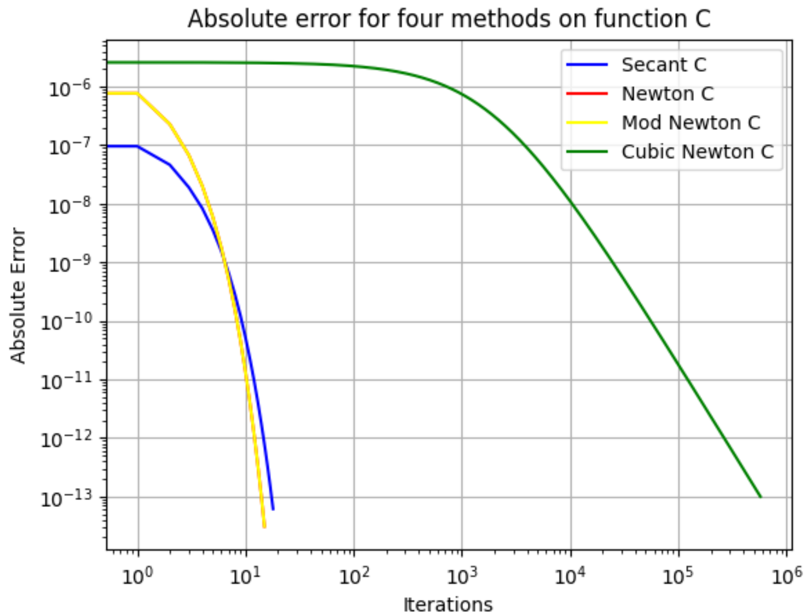

Figure 3: Comparing Methods in Part C

I solved for the $J(x)^{-1}F(x)$ term, which I called delta. I split the delta into two parts and used each part as offset for the last x value. Using this formula:

$$\theta_{2,n} = \theta_{2,n-1} + \delta_{2,n-1}$$

$$\theta_{3,n} = \theta_{3,n-1} + \delta_{3,n-1}$$

I iterated this a set number of times, which ended up being 100,000 times. Here are my results:

$$\theta_2 = 71.7888298545806, \theta_3 = 327.1605417974648$$

My calculated convergence rates for θ_2 and θ_3 were respectively 0.7981155233919234 and 0.1751304701914585.