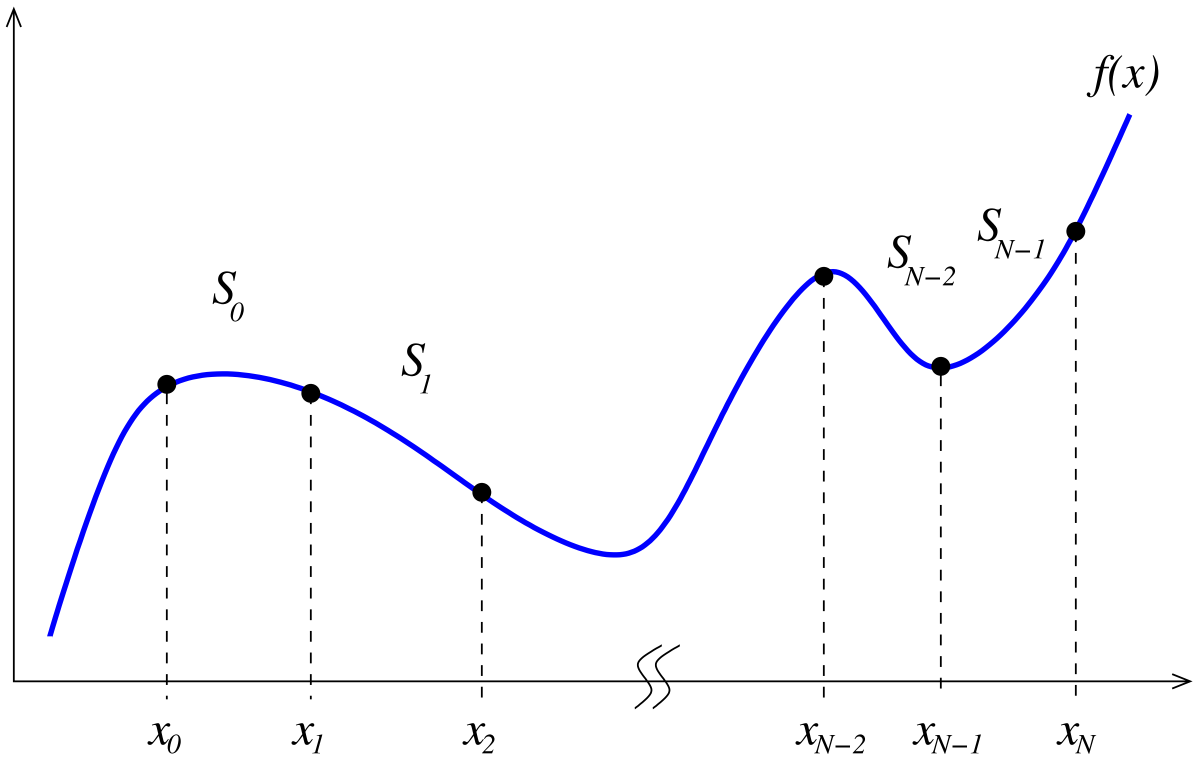
Cubic Spline Interpolation



$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \quad x \in [x_j, x_{j+1}] \;\; ; \;\; j = 0, 1, \cdots N - 1$$

Constraints:	(i)	$S_j(x_j) = f(x_j)$	$j = 0, 1, \ldots N - 1$	$\longrightarrow$	N	conditions
	(ii)	$S_j(x_{j+1}) = f(x_{j+1})$	$j = 0, 1, \ldots N - 1$	$\longrightarrow$	N	conditions
	(iii)	$S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$	$j = 0, 1, \ldots N - 2$	$\longrightarrow$	$N - 1$	conditions
	(iv)	$S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$	$j = 0, 1, \ldots N - 2$	$\longrightarrow$	$N - 1$	conditions

These constraints total $4N - 2$ conditions. We need $4N$ to get a unique solution.

The solution is to impose one condition at each endpoint, x_0 and x_N.

Common strategies:

(a)	$S'_0(x_0) = f'(x_0);\; S'_{N-1}(x_N) = f'(x_N)$	"Clamped Spline"
(b)	$S''_0(x_0) = S''_{N-1}(x_N) = 0$	"Natural Spline"
(c)	$S'''_0(x_1) = S'''_1(x_1);\; S'''_{N-1}(x_{N-1}) = S'''_{N-2}(x_{N-1})$	Other

These constraints plus *end conditions* lead to a set of relationships which can be solved for the sets of coefficients $\{a_j\}$, $\{b_j\}$, $\{c_j\}$, and $\{d_j\}$:

- From (i): $S_j(x_j) = \boxed{f(x_j) = a_j} \quad j = 0, 1, \cdots N-1$
- From (ii): $S_j(x_{j+1}) = f(x_{j+1}) = \boxed{a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1}} \quad j = 0, 1, \cdots N-1$

  where $h_j \equiv x_{j+1} - x_j$
- From (iii): $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1}) \qquad$ where $S'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$

  $\Rightarrow \boxed{b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}} \quad j = 0, 1, \cdots N-2$
- From (iv): $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1}) \qquad$ where $S''_j(x) = 2c_j + 6d_j(x - x_j)$

  $\Rightarrow \boxed{2c_j + 6d_j h_j = 2c_{j+1}} \quad j = 0, 1, \cdots N-2$

We can combine these four equations into a single equation involving only the set of $\{c_j\}$ and known quantities by various substitutions. Solve (iv) for d_j and plug into (ii), solve (ii) for b_j and plug into (iii).

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

holds for $j = 1, 2, ...N-1$ (note that we have incremented j by one), i.e., this is a system of $N-1$ equations involving $N+1$ $\{c_j\}'s$... must apply *end constraints* to determine a unique solution.

e.g., Natural Spline: $S''_0(x_0) = 0$; $S''_N(x_N) = S''_{N-1}(x_N) = 0$ by (iv)

$$S''_j = 2c_j + 6d_j(x - x_j) \quad \Rightarrow \quad \begin{aligned} S''_0(x_0) &= 2c_0 = 0 \\ S''_N(x_N) &= 2c_N = 0 \end{aligned}$$

In this case, $c_0 = c_N = 0$.

Now write as a matrix system: $\underline{\underline{A}} \cdot \underline{x} = \underline{b}$ ... (each j constitutes a row in a matrix).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & 0 & \dots & 0 \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \dots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & h_{N-2} & 2(h_{N-2} + h_{N-1}) & h_{N-1} \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-1} \\ c_N \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \frac{3}{h_2}(a_3 - a_2) - \frac{3}{h_1}(a_2 - a_1) \\ \vdots \\ \frac{3}{h_{N-1}}(a_N - a_{N-1}) - \frac{3}{h_{N-2}}(a_{N-1} - a_{N-2}) \\ 0 \end{pmatrix}$$

Tridiagonal Matrix (solve using algorithm 3.4)