Laboratory 6

Due: Thursday, Nov. 6

20 pts (1) Consider a simple ecosystem of two species competing for the same food supply. The population of the two species can be modeled by the coupled pair of nonlinear first-order differential equations:

$$\frac{dN_1}{dt} = N_1(A_1 - B_1N_1 - C_1N_2)$$
$$\frac{dN_2}{dt} = N_2(A_2 - B_2N_2 - C_2N_1)$$

where t is time, $N_\alpha = N_\alpha(t)$ is the number of species α ($\alpha = 1$ or 2). In these equations $A_\alpha N_\alpha$ is the birth rate, $B_\alpha N_\alpha^2$ is the death rate due to disease, and $C_\alpha N_1 N_2$ is the death rate due to competition for the food supply. Assume that $N_1(0) = N_2(0) = 1.0 \times 10^5$, $A_1 = A_2 = 0.1$, $B_1 = B_2 = 8.0 \times 10^{-7}$, $C_1 = 1.0 \times 10^{-6}$, $C_2 = 1.0 \times 10^{-7}$, and calculate $N_1(t)$ and $N_2(t)$ for $t = 0$ to 10 years. Solve this problem using a 4^{th} order Runge-Kutta method and plot $N_1(t)$ and $N_2(t)$ versus time on the same graph. Experiment with the time-step size that you use in your calculations and by trying successively smaller values, convince yourself that the solution converges as the time-step size shrinks. For the plots you turn in with your laboratory assignment, use a time-step size that is small enough so that the computed answers are independent of the actual value used. Report the value of the time-step size that you have empirically determined is appropriate for this problem. Using this solution as the *exact* answer, now increase the time-step size by a factor of 2, 4, 8, 16 and compare the errors in $N_1(10)$ and $N_2(10)$ relative to the *exact* value you have determined. Plot the logarithm of these errors as a function of $\log_{10}(h^{-1})$. Is the convergence rate you observe in this empirical experiment in agreement with what you would expect theoretically?

15 pts (2) Analyze the stability of the equation

$$\frac{dy(t)}{dt} = -ay(t)$$

where a is a constant, for a second-order Runge-Kutta method (use the Midpoint Rule as your choice), an Adams-Bashforth two-step method, an Adams-Moulton two-step method, and a two-step Adams Predictor/Corrector method. Determine the maximum allowable step-size that will still maintain stability for each method and rank these schemes from the most to least stable method for this problem. Verify your analysis by programming the predictor/corrector scheme and solving the ODE (assume y(0) = 50.0 and use the analytic solution to produce enough values to get the method started). Use several different step sizes, one which violates the stability criterion, another which satisfies and a third which satistifies but is not very accurate. Plot your solution as a function of time in each case thereby providing graphical evidence of either stability or instability.

15 pts (3) Unlike a passive resistor, which dissipates energy at all current levels, a semiconductor operates as if it were pumping energy into a circuit at low current levels, but absorbing energy at high current levels. The interplay between energy injection and energy absorption results in a periodic oscillation in voltages and currents which can be described by the dimensionless equation

$$y'' = ay' - (y')^3 - y$$

which is known as a Van der Pol equation. In this form, y is a dimensionless voltage and y' is a dimensionless current. Solve this ODE using a 4-step Adams predictor-corrector method. Assume that $y(0) = 0.0$, $y'(0) = 0.1$, and compute the solution for dimensionless times ranging from 0 to 100. Plot voltage (i.e., y) versus current (i.e., y') as time progresses using values of $a = 0.5$, 1.5, 2.5, 3.5 and 4.5. Display all of these plots on a single graph and you should obtain outward spiraling orbits of increasing amplitude (as a increases) which approach a rectangular shape. Be careful of your step size. Note that you should use an RK-4 solver to determine what the solution should look like. Be sure to start your 4-step method with a single-step method of appropriate order.