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HW 2
The work in this exercise is mine alone without un-cited help. No AI was used to answer these questions.

1) The requirement for uniform random permutation means that the algorithm PermuteWithAll must produce n! permutations with equal probability. This means each permutation must have probability of 1/n!. For an input of n the probability of a each random swap would be 1/n. For the entire loop the probability of seeing a specific permutation would (1/n)^n = 1/(n^n) which does not equal 1/n!.
2)
    a. We begin with a shuffled deck of card. The number indicates the cards position in the shuffled deck not its value.

    Let card2 is chosen by the friend.

    Deck: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

    We deal the deck into 3 piles

    P1: 1 4 7 10 13 16 19
    P2: 2 5 8 11 14 17 20
    P3: 3 6 9 12 15 18 21

    Deck: 1 4 7 10 13 16 19 2 5 8 11 14 17 20 3 6 9 12 15 18 21

    Once we stacked the cards we see card2 is in position 8. We want to prove that card2 advances two positions in the pile. Doing the deal and stack operation we see.

    P1: 1 10 19 8 17 6 15
    P2: 4 13 2 11 20 9 18
    P3: 7 16 5 14 3 12 21

    Deck: 1 10 19 8 17 6 15 4 13 2 11 20 9 18 7 16 5 14 3 12 21

    Indeed card2 does move two spots. The pile containing the chosen card is moved to the middle of the deck, starting at position 8. Therefore, the top card in the chosen pile will move into position 8. When the deck is dealt, it will be dealt into the 3rd position of the second pile, two away from its starting point. Mathematically we can say that the position of card in the deck after the first turn will be the ceiling of (7+p)/3 where p is the position in the pile. 7 is added because to p because the chosen pile is put in the middle of the stack and the value is divided by 3 since the deck is dealt into three piles. Since p is 1 in our case, we get the position in the pile to be ceil(8/3) = 3.

b. Following the setup from (a) we see that card5 moves from second to third position with the pile and card8 moved from third to fourth position within the pile. This is because those two cards will be placed in deck at position 9 and 10, respectively. When dealt, as shown above, these cards will be one position away from where they started, $3^{rd}/4^{th}$ within their piles. Mathematically, the position can be represented by ceil((7+p)/3) which is 3 for p= 2 and 4 for p = 3.

c.

Deck: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

We deal the deck into 3 piles

P1: 1 4 7 10 13 16 19
P2: 2 5 8 11 14 17 20
P3: 3 6 9 12 15 18 21

Deck: 1 4 7 10 13 16 19 2 5 8 11 14 17 20 3 6 9 12 15 18 21

Lets say card11 is chosen.

P1: 1 10 19 8 17 6 15
P2: 4 13 2 11 20 9 18
P3: 7 16 5 14 3 12 21

Deck: 1 10 19 8 17 6 15 4 13 2 11 20 9 18 7 16 5 14 3 12 21

P1: 1 8 15 2 9 16 3
P2: 10 17 4 11 18 5 12
P3: 19 6 13 20 7 14 21

Deck: 1 8 15 2 9 16 3 10 17 4 11 18 5 12 19 6 13 20 7 14 21

Here we see that card11 stays in position 11. If the card is in deck position 11 that means its pile position was 4, thus its next pile position would be ceil((7+4)/3)) = 4. And since it's the chosen pile it would be gathered back into deck position 11.

d. Based on the above reasoning the following equation can be used computes the deck position (d) from the starting pile position (p)

$$p_{n+1} = ceil\left(\frac{7+p_n}{3}\right), d_n = 7 + p_n$$

Apply once for each turn:

$$p1 = x,\ d1 = d1 = 7 + x$$
$$p2 = ceil\left(\frac{d1}{3}\right) \rightarrow d2 = 7 + p2$$
$$p3 = ceil\left(\frac{d2}{3}\right) \rightarrow d3 = 7 + p3$$

Solving for d3:

$$d3 = 7 + p3 = 7 + \left(ceil\left(\frac{7+p2}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+ceil\left(\frac{7+p1}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+ceil\left(\frac{7+x}{3}\right)}{3}\right)\right)$$

Plugging in 5, 6, 7

$$x = 5 \rightarrow 7 + \left(ceil\left(\frac{7+ceil\left(\frac{7+5}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+ceil\left(\frac{12}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+4}{3}\right)\right) = 7 + 4 = 11$$
$$x = 6 \rightarrow 7 + \left(ceil\left(\frac{7+ceil\left(\frac{7+6}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+ceil\left(\frac{13}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+5}{3}\right)\right) = 7 + 4 = 11$$

$$x = 7 \rightarrow 7 + \left(ceil\left(\frac{7+ceil\left(\frac{7+7}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+ceil\left(\frac{14}{3}\right)}{3}\right)\right) = 7 + \left(ceil\left(\frac{7+5}{3}\right)\right) = 7 + 4 = 11$$

These equations show that after three turns of the trick the 5^{th}, 6^{th} and 7^{th} cards in the chosen pile are moved into the 11^{th} position of the full deck.

e. Lets use the equations from part d to find d2 for all positions in the chosen pile.

$$d2 = 7 + p2 = 7 + ceil\left(\frac{d1}{3}\right)$$

P1	**D1**	**P2**	**D2**
1	8	3	10
2	9	3	10
3	10	4	11
4	11	4	11
5	12	4	11
6	13	5	12
7	14	5	12

This shows that 3/7 or ~42% of the time the chosen card is at deck position 11 after two turns.

3)

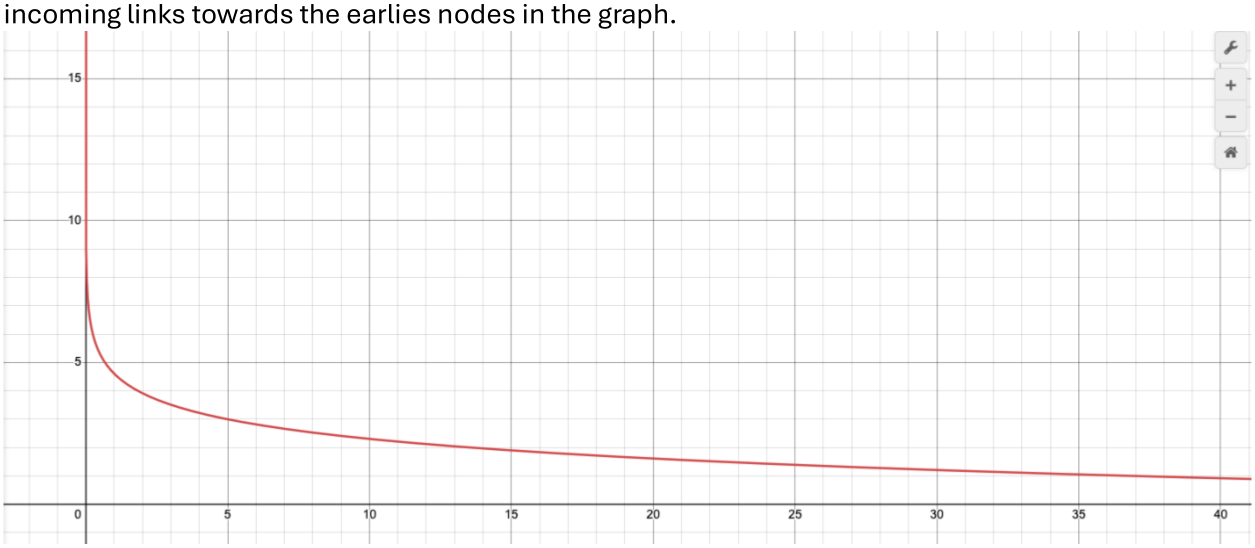
a. When inserting a new node into the graph with k-1 nodes, the probability that the node is attached to any node in the graph is 1/(k-1). Flipping to the view of a node in the graph v_j, all later nodes v_k, where k ranges from j+1 to n, added to the graph have a 1/(k-1) chance of connecting to v_j. Summing these probabilities provides the expected number of links made to v_j:

$$expected\ links\ to\ v_j = \sum_{j+1}^{n} \frac{1}{k-1}$$

Borrowing from what little of undergrad calc2 I can remember and the harmonic series expression from appendix a of the reading which states H(n) = $\sum_1^n \frac{1}{k} = \ln(n) + O(1) \leq \ln(n) + 1$ we can simplify this equation to:

$$expected_{links} = \sum_{j+1}^{n} \frac{1}{k-1} \rightarrow \frac{1}{j} + \frac{1}{j+1} \ldots + \frac{1}{n-1} = \sum_1^n \frac{1}{k-1} - \sum_1^{j+1} \frac{1}{k-1} = H(n) - H(j) = \ln(n) + O(1) - \ln(j) + O(1) = \ln(n) - \ln(j) + 2O(1) = \ln\left(\frac{n}{j}\right) + 2O(1) \rightarrow O(\ln\left(\frac{n}{j}\right))$$

Using Desmos graphing calculator we can show how the function y = ln(n/x) for n = 100 skews incoming links towards the earlies nodes in the graph.



b. The probability that a new node v_k links to a previous node v_j is 1/k-1 therefore the probability that v_k does not link to v_j is:

$$P\{v_k\ does\ not\ link\ with\ v_j\} = 1 - \frac{1}{k-1} = \frac{k-1}{k-1} - \frac{1}{k-1} = \frac{k-1-1}{k-1} = \frac{k-2}{k-1}$$

For all k between j +1 and n the probability they will not connect to v_j can be expressed below. The trick here is that in the expansion of the product successive pairs of denominator and numerator will cancel out, leaving the first numerator (j-1) and the last denominator (1/(n-1)) as the only terms not canceled out.

$$P\{v_j\ has\ no\ incoming\ links\} = \prod_{j+1}^{n} \frac{k-2}{k-1} = \frac{j-1}{j} * \frac{j}{j+1} * \frac{j+1}{j+2} \ldots \frac{n-2}{n-1} = (j-1) * \frac{1}{n-1} = \frac{j-1}{n-1}$$

The expected number of nodes with no incoming links in the whole graph $v_1 \ldots v_n$ as:

$$\sum_1^n P\{v_j\ has\ no\ incoming\ line\} = \sum_1^n \frac{j-1}{n-1} = \frac{0}{n-1} + \frac{1}{n-1} \ldots + \frac{n-1}{n-1} = \frac{1}{n-1} \sum_1^n j-1 = 0+1+2 \ldots n-1 = \frac{1}{n-1} * \frac{(n-1)(n)}{2} = \frac{n}{2}$$

The summation $\sum_1^n j-1 = \frac{n(n-1)}{2}$ comes from the formula $\sum_1^n k = \frac{n(a+b)}{2}$ where a is the first value in the series and b is the last value or 0 and n-1, respectively.