# Dynamic Programming

## Finding the Optimal Policy given a model of the environment

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# Recap

The mathematical framework of Reinforcement Learning:

- A set of possible states $\mathcal{S}$ where $s_t \in \mathcal{S}$ is the current state
- A set of possible actions $\mathcal{A}$ where $a_t \in \mathcal{A}$ is the current action
- A transition function $\mathcal{P} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$
- A reward function $\Re : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ which returns $r_t$

## Markov Decision Processes (MDPs)

These components allows us to define a Markov Decision Process $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, r \rangle$.

# Recap

## Key Concepts

- The agent aims to learn a policy $\pi : \mathcal{S} \to \mathcal{A}$
- The goal of the agent is to maximize the (discounted) cumulative reward $G_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$.
- We can find the optimal policy by making use of the value functions $V^\pi(s)$ and $Q^\pi(s, a)$
- We saw how to find an optimal policy in the specific case where one single state is available

Today we will consider the full Reinforcement Learning problem with its sequential nature.

# Today's Agenda

1. Value Functions
2. Dynamic Programming
3. Control
4. Policy Evaluation
5. Iterative Dynamic Programming Algorithms

# Value Functions

The state-value function $V^\pi(s)$:

$$V^\pi(s) = \mathbb{E}\Big[ G_t \,\Big|\, s_t = s, \pi \Big]$$
$$= \mathbb{E}\Big[ \sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\Big|\, s_t = s, \pi \Big]$$

The state-action value function $Q^\pi(s, a)$:

$$Q^\pi(s, a) = \mathbb{E}\Big[ G_t \,\Big|\, s_t = s,\ a_t = a, \pi \Big]$$
$$= \mathbb{E}\Big[ \sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\Big|\, s_t = s,\ a_t = a, \pi \Big]$$

# Change in Notation

**Attention!**

So far we denoted the reward function as $\Re(s, a, s_{t+1})$.
For the sake of simplicity, hereafter we will denote it simply as $r$.

# Bellman Equations

The state-value function $V^\pi(s)$:

$$V^\pi(s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi\right]$$
$$= \sum_a \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s, a)\left[r + \gamma V^\pi(s_{t+1})\right]$$

The state-action value function $Q^\pi(s, a)$:

$$Q^\pi(s, a) = \mathbb{E}\left[\sum_{k}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi\right]$$
$$= \sum_{s_{t+1}} p(s_{t+1}|s, a)\left[r + \gamma V^\pi(s_{t+1})\right]$$
$$= \sum_{s_{t+1}} p(s_{t+1}|s, a)\left[r + \gamma \sum_{a_{t+1}} \pi(a_{t+1}|s_{t+1}) Q^\pi(s_{t+1}, a_{t+1})\right]$$

# Optimal Policy

The value functions allow us to define a partial ordering over policies.

$$\pi \geq \pi' \text{ iff } V^{\pi}(s) \geq V^{\pi'}(s) \text{ for all } s \in \mathcal{S}$$

We can find the optimal policies by maximizing the state-value and state-action value functions results in the optimal value functions.

$$V^*(s) = \max_{\pi} V^{\pi}(s) \qquad Q^*(s, a) = \max_{\pi} Q^{\pi}(s, a)$$

There is always at least one policy better than any other one: $\pi^*$.

# Bellmann Optimality Equations

$$V^*(s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi\right]$$
$$= \max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^*(s_{t+1})\right]$$

$$Q^*(s,a) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, a_t = a, \pi\right]$$
$$= \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma \max_{a_{t+1}} Q^*(s_{t+1}, a_{t+1})\right]$$

# Dynamic Programming

With the term Dynamic Programming we refer to a set of algorithms that, given a model of the environment $p(s_{t+1}|s, a)$, allow us to find the optimal policies.

## Key Idea

The key idea of Dynamic Programming is the use of the Bellman equations to organize the search for good policies.

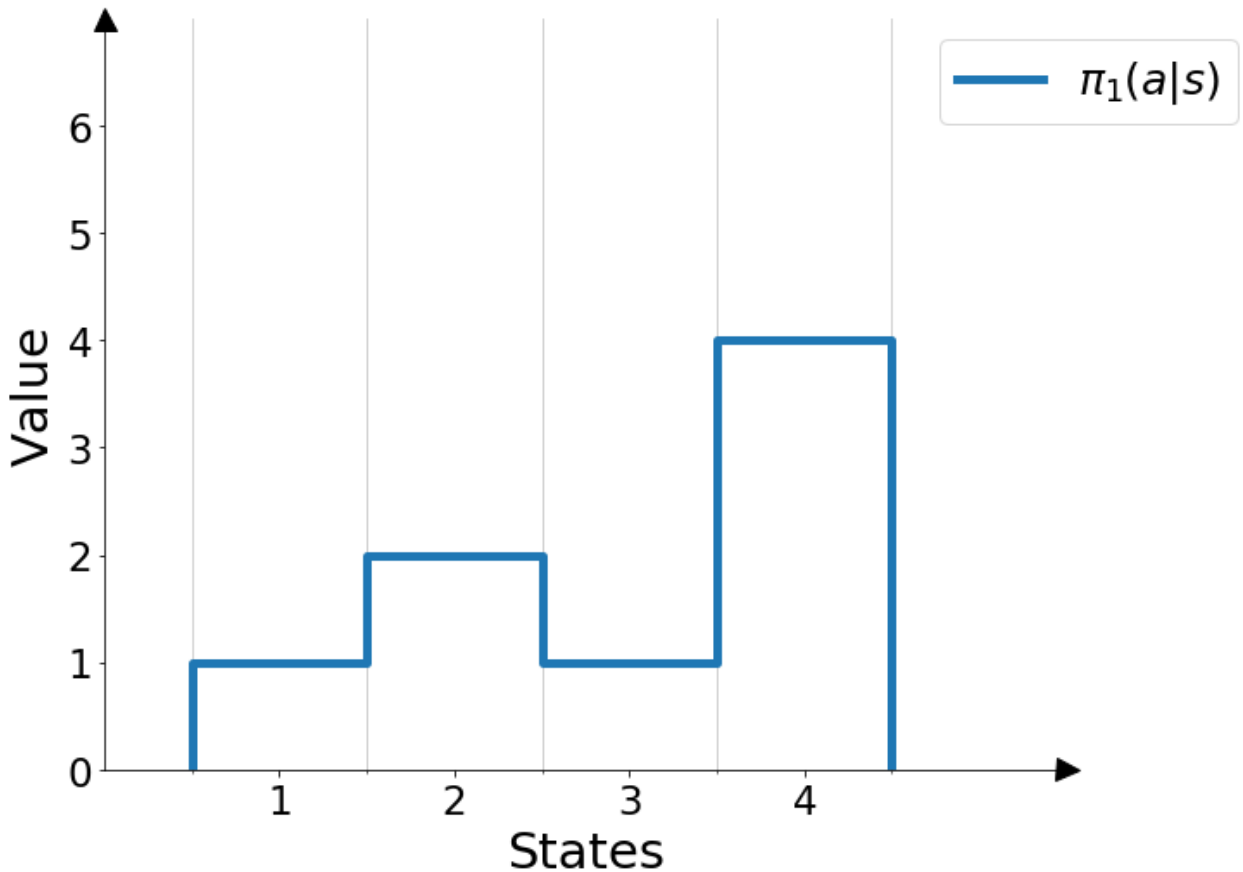
# Dynamic Programming

The goal of Reinforcement Learning is to find the optimal policy $\pi(a|s)$ for each $s \in \mathcal{S}$ for a given problem.
This task is called Control.

Often, to be able to solve the problem of Control, it is necessary to evaluate the goodness of a policy.
This task is called Policy Evaluation.

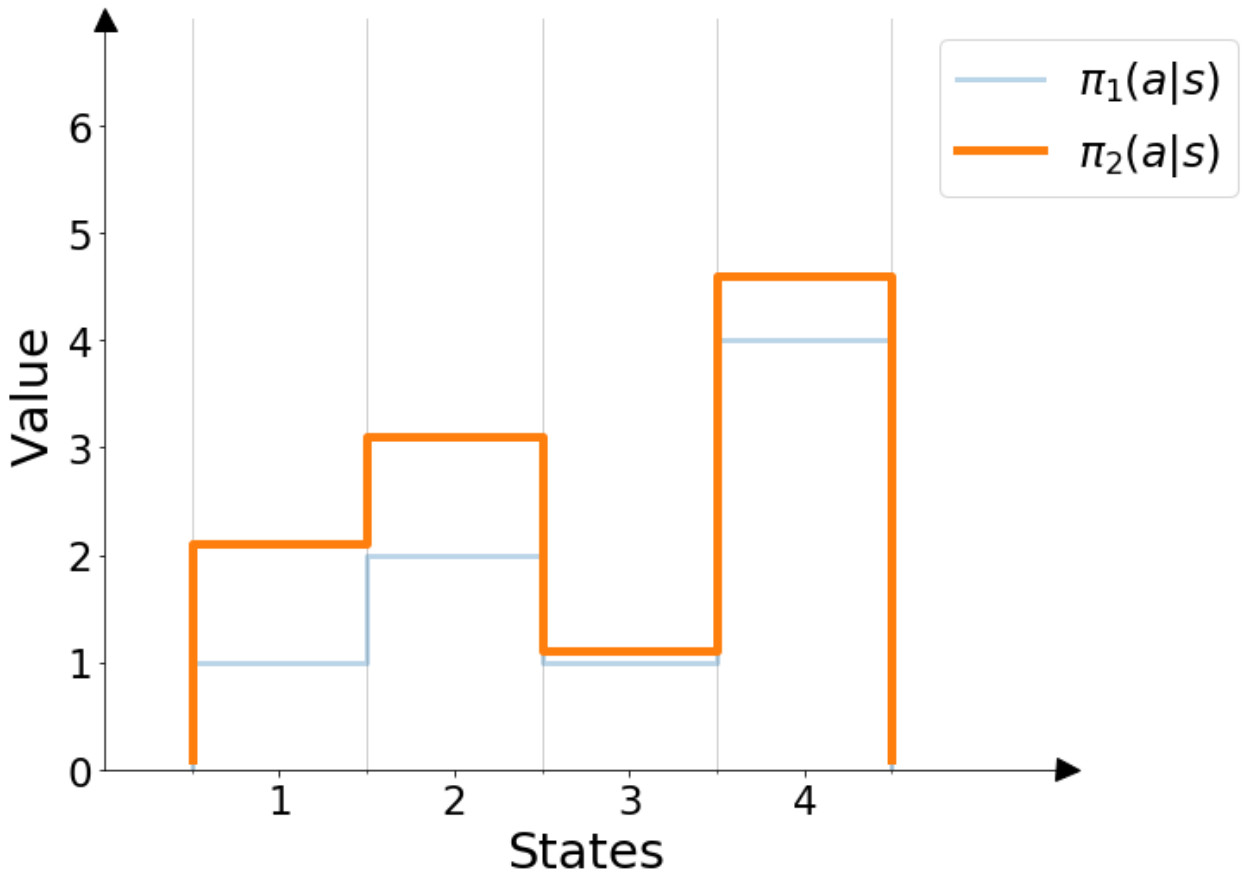
# Control

## Improving a policy



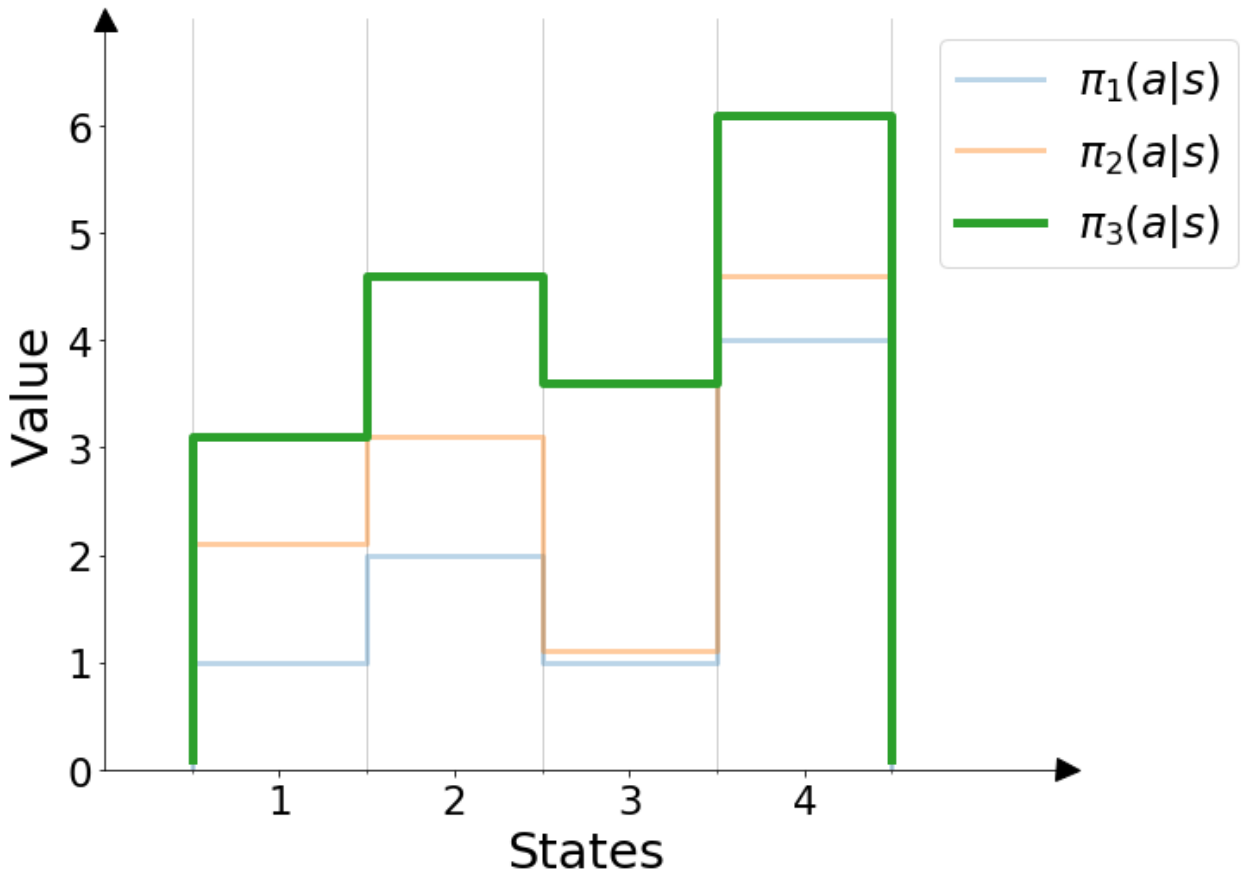
# Control

## Improving a policy



# Control

## Improving a policy



# Policy Evaluation

Evaluating a policy $\pi$ means computing its state-value function (or state-action value function).

If the environment dynamic is completely known, this problem is reduced to finding the solution to a set of $|\mathcal{S}|$ linear equations:

$$
\begin{aligned}
V^{\pi}(s) &= \mathbb{E}\left[ \sum_{k=0}^{\infty} \gamma^{k} r_{t+k+1} \,\middle|\, s_t = s, \pi \right] \\
&= \sum_{a} \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s,a) \left[ r + \gamma V^{\pi}(s_{t+1}) \right]
\end{aligned}
$$

# Iterative Policy Evaluation

Since the above computation is complex, we can compute the value function for a given policy iteratively:

- We choose $V^0$ arbitrarily
- We compute a sequence of improved approximations $V^1, V^2, V^3 \ldots$ by applying the Bellman equation to the previous estimate:

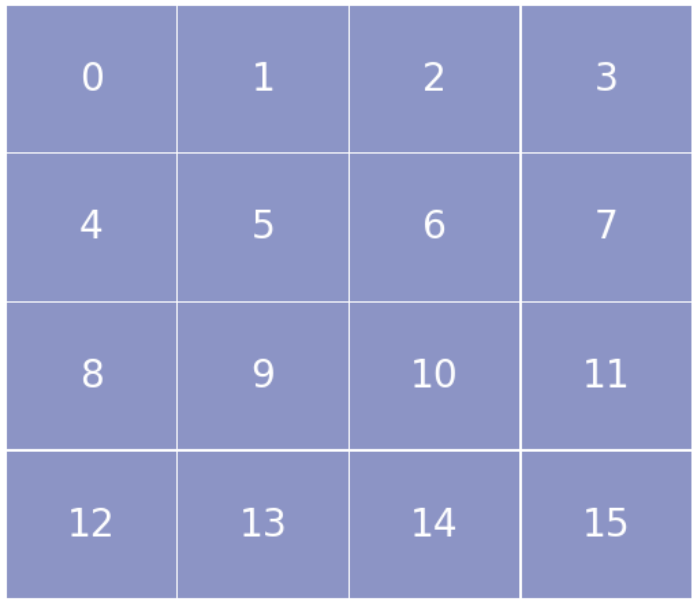
$$V^{k+1}(s) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi\right]$$

$$= \sum_a \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^k(s_{t+1})\right] \text{ for each } s \in S$$

The sequence $\{V_k\}$ is proven to converge to $V^\pi$ as $k \to \infty$.

# Gridworld

- $\mathcal{S}$: 16 states, with 2 terminal ones (0 and 15)
- $\mathcal{A}$: Four possible actions: $\uparrow$ (up), $\downarrow$ (down), $\leftarrow$ (left), $\rightarrow$ (right)
- the function $p(s_{t+1}|s, a)$ is deterministic
- $\mathfrak{R}(s, a)$: $r_t = -1$ for every step



# Iterative Policy Evaluation

Let's evaluate the uniform random policy, which has probability 0.25 of taking each of the 4 possible actions. $\gamma = 1$.

$$V^{k+1}(s) := \sum_a \pi(a|s) \sum_{s_{t+1}, r} p(s_{t+1}|s, a) \Big[ r + \gamma \, V^k(s_{t+1}) \Big]$$

$$V^1(s=1) = \pi(\uparrow | s=1)[\, -1 + V^0(s=1) \,] + \pi(\downarrow | s=1)[\, -1 + V^0(s=5) \,] +$$
$$\pi(\leftarrow | s=1)[\, -1 + V^0(s=0) \,] + \pi(\rightarrow | s=1)[\, -1 + V^0(s=2) \,]$$

$V^0$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

$V^1$

| | | | |
|---|---|---|---|
| 0 | -1 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

# Iterative Policy Evaluation

Let's evaluate the uniform random policy, which has probability 0.25 of taking each of the 4 possible actions. $\gamma = 1$.

$$V^{k+1}(s) := \sum_a \pi(a|s) \sum_{s_{t+1}, r} p(s_{t+1}|s, a) \Big[ r + \gamma \, V^k(s_{t+1}) \Big]$$

$$V^1(s = 1) = 0.25 \, [\, -1 + 0 \,] + 0.25 \, [\, -1 + 0 \,] + 0.25 \, [\, -1 + 0 \,] + 0.25 \, [\, -1 + 0 \,] = -1$$

$V^0$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

$V^1$

| | | | |
|---|---|---|---|
| 0 | -1 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

# Iterative Policy Evaluation

Let's evaluate the uniform random policy, which has probability 0.25 of taking each of the 4 possible actions. $\gamma = 1$.

$$V^{k+1}(s) := \sum_a \pi(a|s) \sum_{s_{t+1}} \sum_r p(s_{t+1}|s,a) \Big[ r + \gamma \, V^k(s_{t+1}) \Big]$$

$$V^1(s=1) = 0.25\,[\,-1+0\,] + 0.25\,[\,-1+0\,] + 0.25\,[\,-1+0\,] + 0.25\,[\,-1+0\,] = -1$$

$V^0$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

$V^1$

| | | | |
|---|---|---|---|
| 0 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | 0 |

# Iterative Policy Evaluation

$$V^{k+1}(s) := \sum_a \pi(a|s) \sum_{s_{t+1}} \sum_r p(s_{t+1}|s,a) \Big[ r + \gamma \, V^k(s_{t+1}) \Big]$$

$V^0$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

$V^N$

| | | | |
|---|---|---|---|
| 0 | -14 | -20 | -22 |
| -14 | -18 | -20 | -20 |
| -20 | -20 | -18 | -14 |
| -22 | -20 | -14 | 0 |

# Iterative Policy Evaluation

**Algorithm 1** Iterative Policy Evaluation

Input the policy to evaluate $\pi$

Define threshold for accuracy $\theta$

Initialize $V(s)$, for all $s \in \mathcal{S}$ arbitrarily, except V(terminal) $= 0$

$\Delta \leftarrow 0$

**while** True

  **for** each $s \in \mathcal{S}$

    $v \leftarrow V(s)$

    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s_{t+1}|s,a)[r + \gamma V(s_{t+1})]$

    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

  if $\Delta < \theta$ return V

# Policy Improvement

What happens if, instead of following policy $\pi$ in state $s$, we decide to take action $a$?

$$Q^{\pi}(s,a) = \mathbb{E}\left[\sum_{k}^{\infty} \gamma^{k} r_{t+k+1} \,\middle|\, s_t = s, a_t = a, \pi\right]$$
$$= \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^{\pi}(s_{t+1})\right]$$

If this value is greater than $V^{\pi}(s)$, we should definitely take action $a$.

# Policy Improvement

## Policy Improvement Theorem

Given any pair of deterministic policies $\pi$ and $\pi'$ such that:

$$Q^{\pi}(s, \pi'(s)) \geq Q^{\pi}(s, \pi(s))$$

Which is the same as:

$$Q^{\pi}(s, \pi'(s)) \geq V^{\pi}(s)$$

Then policy $\pi'$ is as good as or better than $\pi$:

$$V^{\pi'}(s) \geq V^{\pi}(s)$$

An easy way to find a new policy $\pi'$ which is always better than $\pi$ is:

$$\pi' = \arg\max_{a \in A} Q^{\pi}(s, a)$$

# Policy Iteration

$$\pi_0 \xrightarrow{E} V^{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} V^{\pi_1} \xrightarrow{I} \ldots \xrightarrow{I} \pi_* \xrightarrow{E} V^{\pi_*}$$

Since a finite MDP has only a finite number of policies, this process must converge to an optimal policy in a finite number of iterations.

# Policy Iteration

**Algorithm 2** Policy Iteration for estimating $\pi \sim \pi^*$

1) **Initialization**

Initialize $V(s)$ and $\pi(s)$ arbitrarily for all $s \in \mathcal{S}$

2) **Policy Evaluation Step** (Algorithm 1)

3) **Policy Improvement Step:**

policy-stable $\leftarrow$ true

**for** each $s \in \mathcal{S}$ **do**

old-action $\leftarrow \pi(s)$

$\pi(s) \leftarrow \arg\max_a \sum_{s_{t+1}} p(s_{t+1}|s, a)[r + \gamma V(s_{t+1})]$

if old-action $\neq \pi(s)$, policy-stable $\leftarrow$ false

if policy-stable = true then return $\pi \sim \pi^*$ and $V \sim V^*$

# Value Iteration

Drawback of Policy Iteration: in each iteration we need to perform policy evaluation.

The policy evaluation step can be truncated without losing convergence guarantees.

$$V^{k+1}(s) = \max_a \mathbb{E}\left[\sum_k^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, a_t = a\right]$$

$$= \max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^k(s_{t+1})\right] \text{ for each } s \in \mathcal{S}$$

# Value Iteration

**Algorithm 3** Value Iteration Algorithm

Define threshold for accuracy $\theta$
Initialize $V(s)$ for all $s \in \mathcal{S}$ arbitrarily, except V(terminal) $= 0$

$\Delta \leftarrow 0$
**while** True **do**
    **for** each $s \in \mathcal{S}$ **do**
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\Big[r + \gamma\, V(s_{t+1})\Big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy $\pi \sim \pi^*$, such that:
$\pi(s) \leftarrow \arg\max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\Big[r + \gamma\, V(s_{t+1})\Big]$

# Generalised Policy Iteration

We define generalized policy iteration as the alternative execution of policy evaluation and policy improvement, regardless of the granularity of the two processes.

This process stabilizes only when we found a policy that is greedy with respect to its own value function.

# Final Slide!

## Lecture Takeaway

1. Given $p(s_{t+1}|s, a)$ we can find the optimal policy mathematically
2. But better use Dynamic Programming algorithms
3. It works by alternating the tasks of Policy Evaluation and Control