The Multi-Armed Bandits Problem

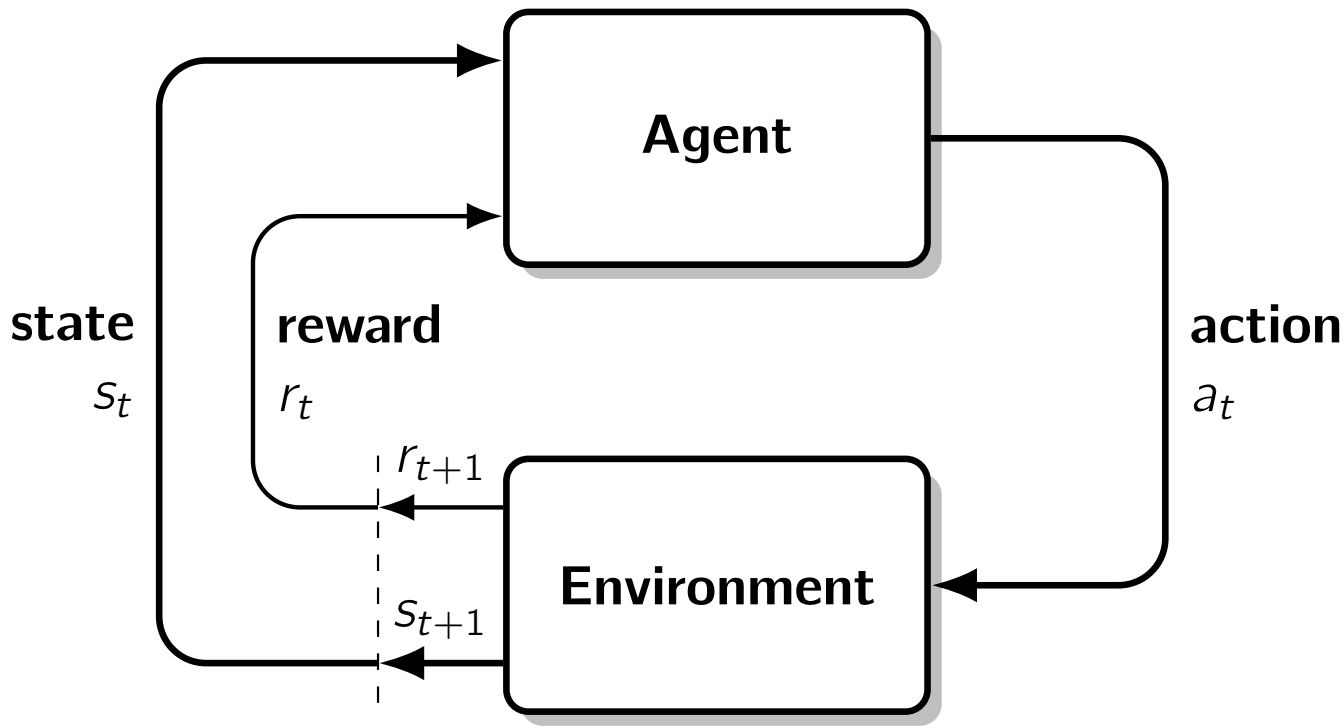
Balancing Exploration and Exploitation

Nicole Orzan

November 3, 2021

Recap

In Reinforcement Learning an agent learns how to best take actions in order to maximize some goal.



Recap

Key Concepts

- The problem of learning through interaction is framed as a Markov Decision Process $\mathcal{M}\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, r \rangle$
- The agent aims to learn a policy $\pi : \mathcal{S} \to \mathcal{A}$
- The goal of the agent is to maximize the (discounted) cumulative reward $G_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$.
- We can find the optimal policy by making use of the value functions $V^\pi(s)$ and $Q^\pi(s, a)$

Today's Agenda

1. Multi-Armed Bandits
2. Exploration-Exploitation Trade-off
3. Action-Value Methods
4. Action Preferences

Simplified Setting

The environment has a single state.

- The setting is non-associative: actions can not change the state of the environment
- The actions can only impact the next reward

We want to learn a policy:

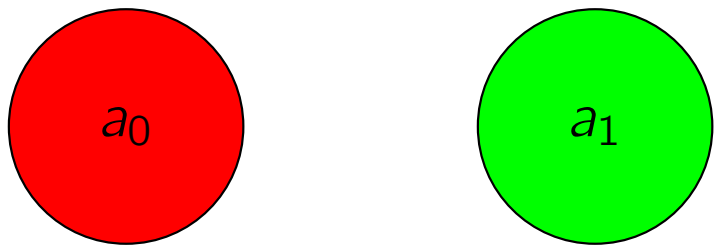
$$\pi(a) = \Pr\{a_t = a\}, \text{ for all } a \in \mathcal{A}.$$

Multi-Armed Bandits

Our agent is repeatedly faced with the choice among k different actions. After each choice, gets a reward sampled from the stationary distribution that depends on the chosen action.

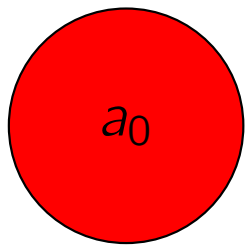
- Set of actions $\mathcal{A}$
- Fixed distribution of rewards for each action, $r_a \mid a \in \mathcal{A}$
- The goal is to maximize the expected cumulative reward
- We need to learn a policy $\pi(a) \mid a \in \mathcal{A}$

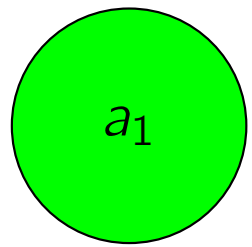
Example



$t{=}0$ $r_0{=}1$

Example

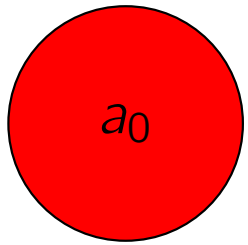


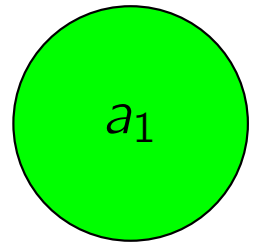


$t{=}0$ $r_0{=}1$

$t{=}1$ $r_1{=}0$

Example





t=0 r_0=1

t=1 r_1=0

t=2 r_2=2

How do we choose whether to stick to the current action, or keep exploring?

Exploration-Exploitation Trade-off

To maximize its reward in this environment, our agent must learn what the best action to take is.

- Exploration allows to gain knowledge of its actions, useful for the **long-term**
- Exploitation allows to use the current knowledge of the actions to get an **instantaneous benefit**

Action-Value Methods

The action value for action a is the reward we expect to receive by taking that action:

$$Q^*(a) \doteq \mathbb{E}[R_t | a_t = a]$$

The goal is to find the optimal value, by maximizing the expected cumulative reward:

$$a_* = \arg\max_a \mathbb{E}[R_t | a_t = a]$$

Regret of an action a:

$$\Delta_a = a_* - Q(a)$$

Action-Value Methods

In Reinforcement Learning we do not know which actions have the best value: we need to learn it.

Action Value Estimate

We denote the estimated value of action a at time step t as $Q_t(a)$.

We can use action value estimates to learn a policy.

Sample-average Method

The simpler estimate is given by the average of the sampled rewards:

$$
\begin{aligned}
Q_t(a) &\doteq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} \\
&= \frac{\sum_{i=0}^{T} R_i \cdot \mathbb{I}(A_i = a)}{\sum_{i=0}^{T} \mathbb{I}(A_i = a)}
\end{aligned}
$$

$\mathbb{I}$ is the indicator function: $\mathbb{I}(\mathit{True}) = 1$, and $\mathbb{I}(\mathit{False}) = 0$.

Incremental Update

The sample-average method can be computed incrementally:

$$
\begin{aligned}
Q_{t+1}(a) &= \frac{1}{t+1}\sum_{i=0}^{t+1} R_i \\
&= \frac{1}{t}\left[R_t + (t-1)\frac{1}{t-1}\sum_{i=0}^{t-1} R_i\right] \\
&= \frac{1}{t}\Big[R_t + (t-1)Q_t(a)\Big] \\
&= Q_t(a) + \frac{1}{t}\Big[R_t - Q_t(a)\Big]
\end{aligned}
$$

Incremental Update

$$Q_{t+1}(a) = Q_t(a) + \alpha_t(R_t - Q_t(a)) \quad \text{where} \quad \alpha_t = \frac{1}{t}$$

Update Rule

$$NewEstimate \leftarrow OldEstimate + StepSize[Target - OldEstimate]$$

Later we will consider other possible step sizes α_t.

Action Selection: greedy

Action-value estimates can be used to select actions.

We can exploit the current knowledge by selecting the greedy action, which is the action with the highest estimated value:

$$A_t = \operatorname*{arg\,max}_{a \in \mathcal{A}} Q_t(a)$$

Downside: we can get stuck taking a suboptimal action forever, because we are not exploring better opportunities.

Action Selection: $\epsilon-$greedy

- Behave greedily with probability $1 - \epsilon$
- Take a random action with probability ϵ

Where ϵ is a small, positive number $0 < \epsilon < 1$.

$$\pi_t(a) = \begin{cases} (1-\epsilon) + \epsilon/|\mathcal{A}| & \text{if } a = \underset{a \in \mathcal{A}}{\arg\max}\ Q_t(a) \\ \epsilon/|\mathcal{A}| & \text{otherwise} \end{cases}$$

Optimistic Initial Values

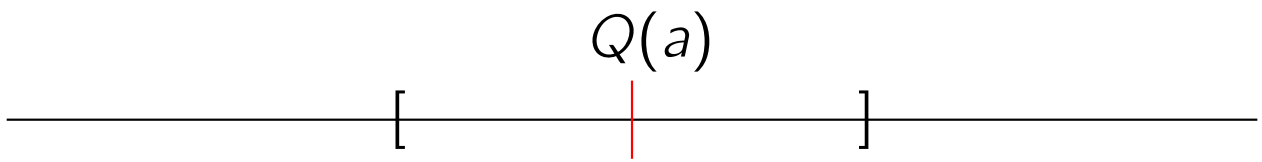
All the methods discussed up to now depend on the initial values of the action estimates, $Q_0(a) \,|a \in A$.

Optimistic initial values can be used to encourage early exploration:

- For any starting action taken by the agent, the reward is smaller than the starting estimates
- The agent, disappointed, switches to other actions
- Actions are tried several times before the value estimates converge

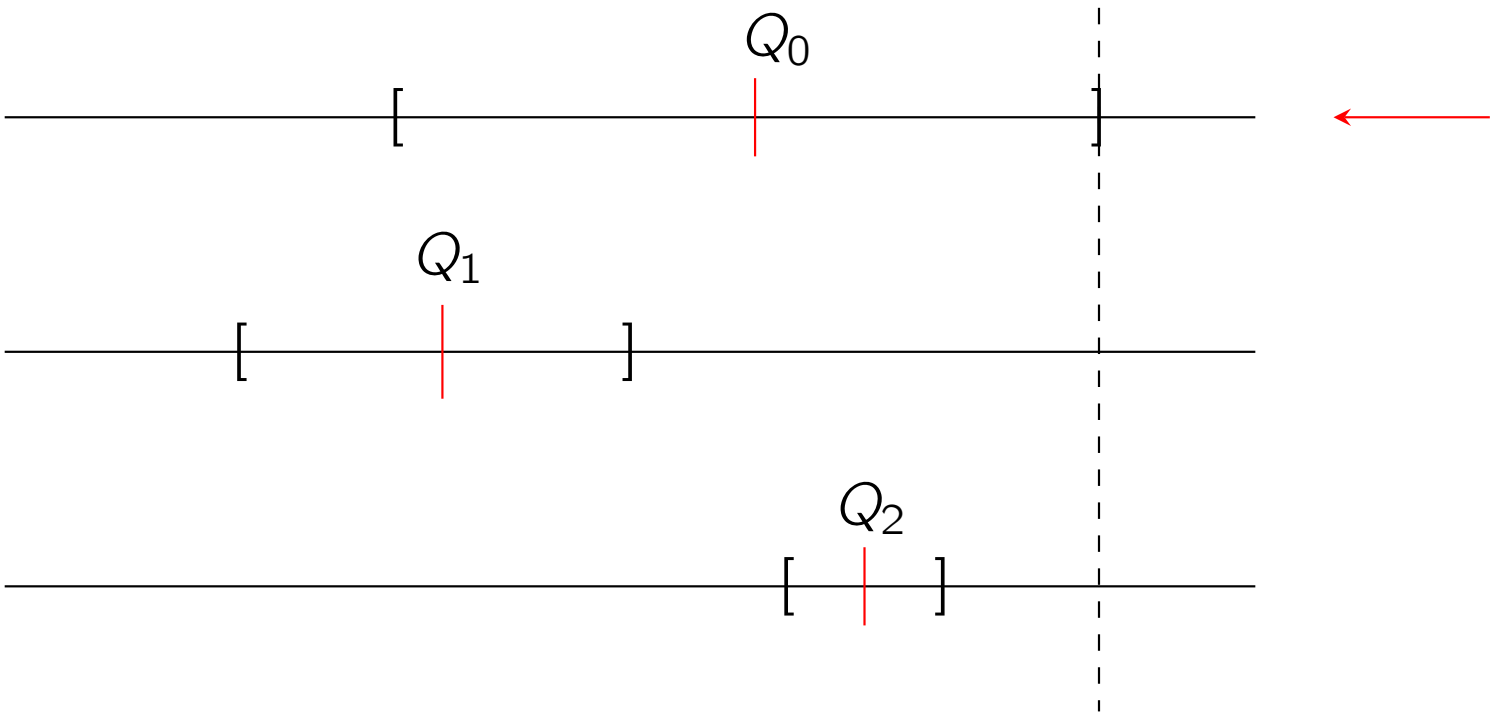
Uncertainty in Action Values Estimates

If we could explicit the uncertainty of our action-values estimates, we could select actions in a better way.

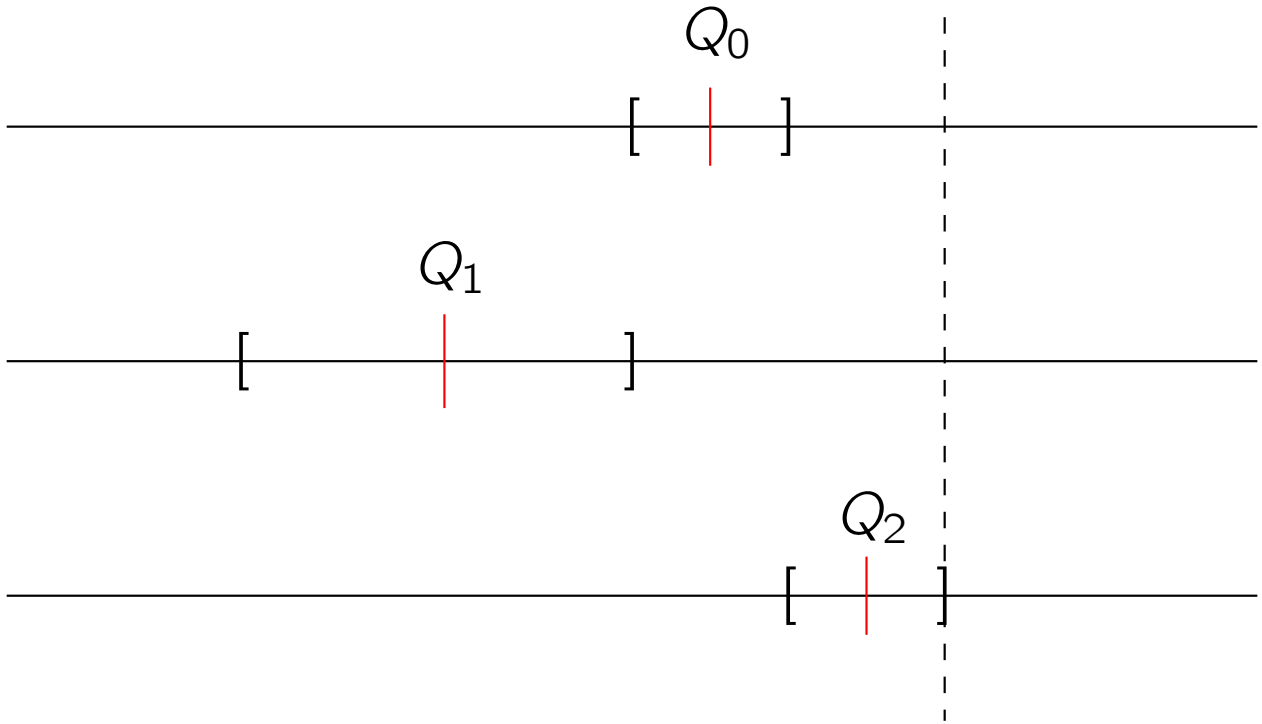


Select actions leveraging their uncertainty: if we are uncertain about the value of an action, we optimistically assume that it is good.

Optimism in the Face of Uncertainty



Optimism in the Face of Uncertainty



Upper-Confidence Bound (UCB)

We want to estimate an upper confidence bound for every action value:

$$q_t \leq Q_t(a) + U_t(a)$$

And select greedily the action maximizing the UCB:

$$a_t \doteq \underset{a \in \mathcal{A}}{\arg\max}\, Q_t(a) + U_t(a)$$

Upper-Confidence Bound (UCB)

$$a_t \doteq \underset{a \in \mathcal{A}}{\arg\max} \left[Q_t(a) + c \sqrt{\frac{ln(t)}{N_a(t)}} \right]$$

- Each time we select a, $N_a(t)$ is increased and therefore the uncertainty is reduced
- Each time we select $a' \neq a$, t increases but $N_a(t)$ does not, therefore the uncertainty increases

The parameter $c > 0$ regulates the amount of exploration.

Action Preferences

All methods described until now rely on the estimate of action values to learn a policy. Another approach relies on learning a preference for each action, $H_t(a)$.

- The larger the preference, the more probably a certain action should be selected
- The preference has no meaning in terms of reward

Boltzmann distribution

From the action preferences, we can define a policy using the Softmax or Boltzmann distribution:

$$\pi(a) = \frac{e^{H_t(a)}}{\sum_{k=0}^{N} e^{H_t(k)}}$$

Reminder: You will use this distribution in deep learning, to convert the values into a probability distribution.

Action Preferences

If at time step t we choose action a', we can update the action preferences as:

$$H_{t+1}(a') \doteq H_t(a') + \alpha\, r_t(1 - \pi_t(a'))$$
$$H_{t+1}(a) \doteq H_t(a) + \alpha\, r_t\, \pi_t(a') \;\text{ if }\; a \neq a'$$

Where the formula above is based on the idea of stochastic gradient ascent.

Associative Search: Contextual Bandits

Until now, we considered contexts where the agent has to learn how to behave in a single situation. However, the general problem of reinforcement learning deals with learning how to behave in a number of different situations.

Associative Search: Contextual Bandits

Simple way to extend bandits:

- You face N different Multi-Armed bandit problems
- This is an associative task: $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- Similar to the full reinforcement learning problem as they involve learning a policy, but like the k-armed bandit problem in that each action affects only the immediate reward.

Final Slide!

Lecture Takeaway

1. Multi-Armed bandit is a simplified setting with a single state
2. To learn a policy we need to balance Exploration and Exploitation
3. We can learn the policy by leveraging action-values estimates $Q(a)$
4. Or leveraging action preferences $H(a)$