Foundations of Reinforcement Learning

From Markov Decision Processes to Optimal Value Functions

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October 18, 2021

Today's Agenda

1. Course Information
2. What is Reinforcement Learning
3. Mathematical Framework
4. Value Functions

Course Information

- Coordinator: Matthia Sabatelli
- Lecturers: Matthia Sabatelli (m.sabatelli@rug.nl) and Nicole Orzan (n.orzan@rug.nl)
- Classroom: TBD
- Theoretical Lectures: Monday morning from 9:00-11:00
- Computer Labs: Monday afternoon from 15:00-17:00

Course Information

- Lecture 1: Foundations of Reinforcement Learning (Matthia)
- Lecture 2: Exploration and Bandit Problems (Nicole)
- Lecture 3: Dynamic Programming (Nicole)
- Lecture 4: Model-Free Reinforcement Learning (Matthia)
- Lecture 5: Function Approximators (Matthia)
- Lecture 6: Beyond Model-Free Reinforcement Learning (Matthia)
- Lecture 7: *What does it mean to do research in RL?* (Matthia & Nicole)

Course Information

All course material will be made available

- Nestor
- Github: `https://github.com/paintception/reinforcement-learning-practical`

Course Information

Textbook: Reinforcement Learning: An Introduction by Sutton & Barto

Course Information

Final course assessement:

- There is **no exam**
- Students should handle in three deliverables:
  1. Assignment 1: 25% of the grade (coding)
  2. Assignment 2: 25% of the grade (mathematics)
  3. Report: 50% of the grade (final project)
- Students can work alone or in groups of a maximum of **2** people

Reinforcement Learning

Machine Learning is typically divided into three branches:

1. Supervised Learning: learning from labeled data
2. Unsupervised Learning: learning from unlabeled data
3. Reinforcement Learning: learning from experience

Reinforcement Learning

A reminder of supervised learning:

- input space: $\mathcal{X}$
- output space: $\mathcal{Y}$
- probability distribution $p(x, y)$

The goal

We want to build a function $f : \mathcal{X} \to \mathcal{Y}$ that minimizes the expectation of a given loss ℓ

$$\mathbb{E}_{(x,y)\sim p(x,y)}\{\ell(y, f(x))\}$$

through leaning samples $LS = \{(x_i, y_i)|i = 1, ..., N\}$ of input-ouput pairs drawn from $p(x, y)$.

Reinforcement Learning

Supervised learning is therefore:

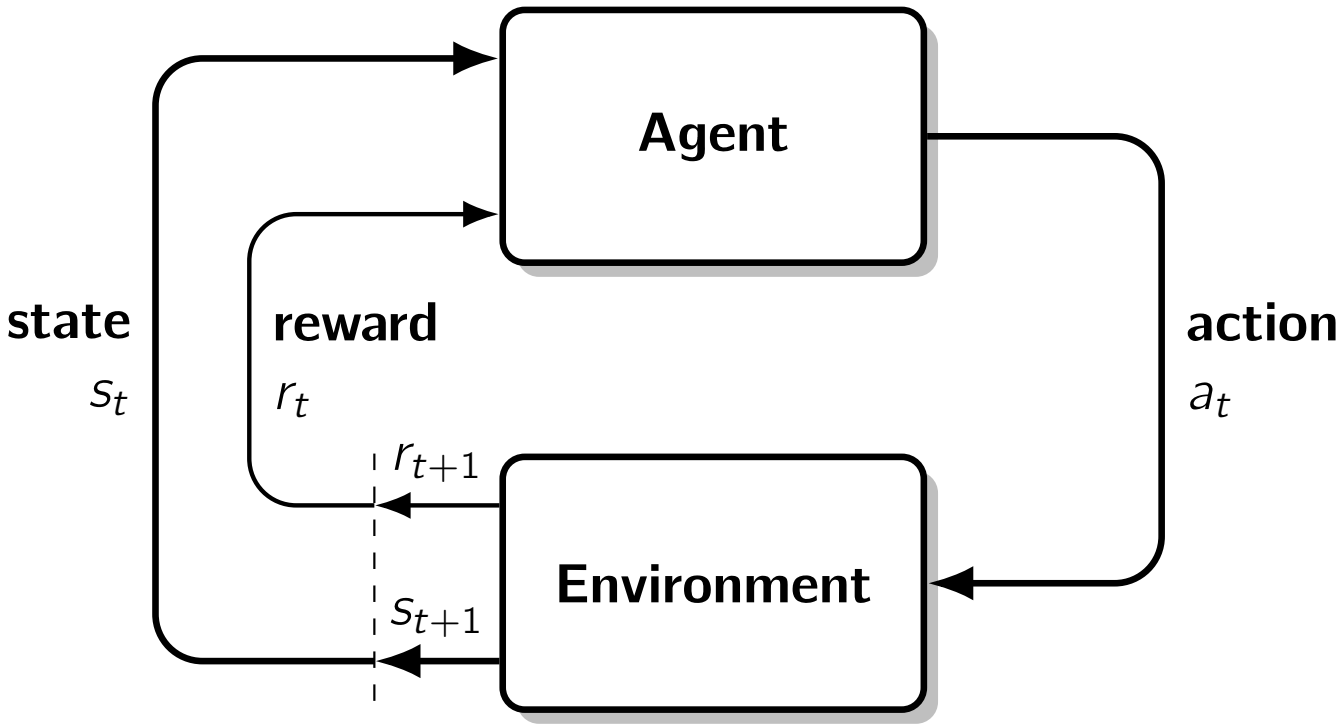
- a learning paradigm which is static
- no interaction happens between the learner and $p(x, y)$
- Assumes we have access to a knowledgable supervisor

In Reinforcement Learning however ...

Reinforcement Learning

- We would like to learn how to interact with an environment
- We do not assume any sort of supervision but only a reward signal
- The component of time plays a crucial role
- The learning process is therefore dynamic and uncertain
- Agents are goal oriented

Reinforcement Learning



Reinforcement Learning

Reinforcement Learning

The mathematical framework of Reinforcement Learning:

- a set of possible states $\mathcal{S}$ where $s_t \in \mathcal{S}$ is the current state
- a set of possible actions $\mathcal{A}$ where $a_t \in \mathcal{A}$ is the current action
- a transition function $\mathcal{P} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$
- a reward function $\Re : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ which returns r_t

Markov Decision Processes (MDPs)

These components allows us to define a Markov Decision Process $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, r \rangle$

Markov Decision Processes

What is so special about MDPs?

Markov Property

In a Markovian environment the conditional distribution of the next state of the process only depends from the current state of the process.

$$p(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, \ldots) = p(s_{t+1}|s_t, a_t).$$

Interestingly, the same property also holds for the reward that the agent will get:

$$p(r_t|s_t, a_t, \ldots, s_1, a_1) = p(r_t|s_t, a_t).$$

Markov Decision Processes

How does an agent interact with its environment?

Policy π

Through a probability distribution over $a \in \mathcal{A}(s)$ for each $s \in \mathcal{S}$:

$$\pi(a|s) = \Pr\{a_t = a | s_t = s\}, \text{ for all } s \in \mathcal{S} \text{ and } a \in \mathcal{A}.$$

which more simply can be seen as a mapping from states to actions that determines the behaviour of the agent:

$$\pi : \mathcal{S} \to \mathcal{A}$$

Episodes

Once we have policy π we can let the agent interact with the environment, which results in an episode:

$$\langle (s_t, a_t, r_t, s_{t+1}) \rangle, t = 0, \ldots, T-1$$

where T is a random variable defining the length of the episode

One $\langle s_t, a_t, r_t, s_{t+1} \rangle$ is called a trajectory τ

Goals and Returns

Why do we want an agent to interact with the environment?

- We would like it to master a certain task
- Ideally it could discover solutions that are unknown to us
- Biologically plausible form of learning

How do we formalize this mathematically?

Goal

In its easiest form we can define such goal as maximizing the sequence of r_t returned by $\Re$

$$G_t = r_t + r_{t+1} + r_{t+2}, \ldots, r_T.$$

Goals and Returns

However the definition of G_t has some limitations:

- Assumes that episodes are finite
- Many real world applications are instead continuous, therefore $T = \infty$
- As a result G_t can be infinite as well

We need a slightly more complex definition of G_t based on the concept of discounting modeled by γ

Goals and Returns

Discounted Return

γ allows us to define the notion of discounted cumulative reward

$$G_t = r_t + \gamma r_{t+1}, \gamma^2 r_{t+2} + \ldots$$
$$= \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}.$$

- γ determines the value of future rewards as $0 \leq \gamma \leq 1$
- makes G_t finite as long as $\gamma < 1$ and r_t is constant

$$G_t = \sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}$$

Value Functions

One of the most concepts of Reinforcement Learning: value

- directly connected to the notion of G_t
- we can define the value of a state s, of an action a, and even of a policy π
- allows the introduction of value functions
  - state-value function $V(s)$
  - state-action value function $Q(s, a)$

Value Functions

The state-value function $V^{\pi}(s)$

$$V^{\pi}(s) = \mathbb{E}\Big[G_t \;\Big|\; s_t = s, \pi \Big]$$
$$= \mathbb{E}\Big[\sum_{k}^{\infty} \gamma^k r_{t+k+1} \;\Big|\; s_t = s, \pi \Big]$$

- The easiest value function to learn
- Intuitively it tells us *"how good/bad"* every state s visited by policy π is
- This *"goodness"* is expressed with respect to G_t

Value Functions

The state-value function is only conditioned on the states that are being visited

The state-action value function $Q^{\pi}(s, a)$

$$Q^{\pi}(s, a) = \mathbb{E}\left[G_t \,\middle|\, s_t = s,\ a_t = a, \pi \right]$$
$$= \mathbb{E}\left[\sum_{k}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s,\ a_t = a, \pi \right]$$

- Is much more informative compared to $V^{\pi}(s)$
- Intuitively tells us *how good/bad* taking action a in state s is
- Plays a crucial role in the developement of Reinforcement Learning algorithms

Value Functions

An interesting property of these value functions is that they satisfy a recursive equality

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{k}^{\infty} \gamma^{k} r_{t+k+1} \,\middle|\, s_t = s, \pi \right]$$

$$= \sum_{a} \pi(s, a) \sum_{s+1} p(s_{t+1}|s, a) \left[\Re(s_t, a, s_{t+1}) + \gamma V^{\pi}(s_{t+1}) \right]$$

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which is key for the creation of Reinforcement Learning algorithms (Lectures 3 and 4)!

Optimal Value Functions

Solving a Reinforcement Learning problem intuitively means finding a policy π that achieves a lot of reward:

- What makes a certain policy π better than another policy $\pi^{'}$?
- How can we rank different policies?

We can answer these questions thanks to the $V(s)$ and the $Q(s, a)$ functions:

Optimal Policy π^*

There is always one policy that is better or equal than all other policies

$$\pi \geq \pi^{'} \text{ iff } V^{\pi}(s) \geq V^{\pi^{'}}(s) \text{ for all } s \in \mathcal{S}$$

The best possible policy is the optimal policy π^*

Optimal Value Functions

- How do we find the optimal policy π^*?
- By maximizing the state-value and state-action value functions results in the optimal value functions

$$V^*(s) = \max_\pi V^\pi(s) \qquad Q^*(s, a) = \max_\pi Q^\pi(s, a)$$

which, if expressed in a recursive form, result in the Bellman optimality equations (for the coming lectures!)

Final Slide!

We aim to teach an agent how to interact with its environment:

Lecture Takeaway

1. The environment is modelled as a Markov Decision Process $\mathcal{M}\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, r \rangle$
2. The interaction is governed by the agent's policy π
3. The goal of the agent is measured wrt discounted cumulative reward G_t
4. G_t allows us to quantify how good a certain state s is, and how good a certain action a, in a certain state is: $V^{\pi}(s)$ and $Q^{\pi}(s, a)$