Dynamic Programming

Finding the Optimal Policy given a model of the environment

Nicole Orzan

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Recap

The mathematical framework of Reinforcement Learning:

- A set of possible states $\mathcal{S}$ where $s_t \in \mathcal{S}$ is the current state
- A set of possible actions $\mathcal{A}$ where $a_t \in \mathcal{A}$ is the current action
- A transition function $\mathcal{P} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$
- A reward function $\mathfrak{R} : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ which returns r_t

Markov Decision Processes (MDPs)

These components allows us to define a Markov Decision Process $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, r \rangle$.

Recap

Key Concepts

- The agent aims to learn a policy $\pi : \mathcal{S} \to \mathcal{A}$
- The goal of the agent is to maximize the (discounted) cumulative reward $G_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$.
- We can find the optimal policy by making use of the value functions $V^\pi(s)$ and $Q^\pi(s, a)$
- We saw how to find an optimal policy in the specific case where one single state is available

Today we will consider the full Reinforcement Learning problem with its sequential nature.

Today's Agenda

1. Value Functions
2. Dynamic Programming
3. Control
4. Policy Evaluation
5. Iterative Dynamic Programming Algorithms

Value Functions

The state-value function $V^\pi(s)$:

$$\begin{aligned} V^\pi(s) &= \mathbb{E}\Big[G_t \;\Big|\; s_t = s, \pi \Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \;\Big|\; s_t = s, \pi \Big] \end{aligned}$$

The state-action value function $Q^\pi(s, a)$:

$$\begin{aligned} Q^\pi(s, a) &= \mathbb{E}\Big[G_t \;\Big|\; s_t = s,\ a_t = a, \pi \Big] \\ &= \mathbb{E}\Big[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \;\Big|\; s_t = s,\ a_t = a, \pi \Big] \end{aligned}$$

Bellman Equations

The state-value function $V^\pi(s)$:

$$V^\pi(s) = \mathbb{E}\left[\sum_k^\infty \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi \right]$$
$$= \sum_a \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^\pi(s_{t+1})\right]$$

The state-action value function $Q^\pi(s,a)$:

$$Q^\pi(s,a) = \mathbb{E}\left[\sum_k^\infty \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi \right]$$
$$= \sum_{s_{t+1}} p(s_{t+1}|s,a)\left[r + \gamma V^\pi(s_{t+1})\right]$$

Optimal Policy

The value functions allow us to define a partial ordering over policies.

$$\pi \geq \pi' \text{ iff } V^{\pi}(s) \geq V^{\pi'}(s) \text{ for all } s \in \mathcal{S}$$

We can find the optimal policies by maximizing the state-value and state-action value functions results in the optimal value functions.

$$V^*(s) = \max_{\pi} V^{\pi}(s) \qquad\qquad Q^*(s, a) = \max_{\pi} Q^{\pi}(s, a)$$

There is always at least one policy better than any other one: π^*.

Bellmann Optimality Equations

$$
\begin{aligned}
V^*(s) &= \mathbb{E}\left[\sum_k^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi \right] \\
&= \max_a \sum_{s_{t+1}} p(s_{t+1}|s, a)\big[r + \gamma V^*(s_{t+1})\big]
\end{aligned}
$$

$$
\begin{aligned}
Q^*(s, a) &= \mathbb{E}\left[\sum_k^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, a_t = a, \pi \right] \\
&= \sum_{s_{t+1}} p(s_{t+1}|s, a)\big[r + \gamma \max_{a_{t+1}} Q^*(s_{t+1}, a_{t+1})\big]
\end{aligned}
$$

Dynamic Programming

With the term Dynamic Programming we refer to a set of algorithms that, given a model of the environment $p(s', r|s, a)$, allow us to find the optimal policies.

The key idea of Dynamic Programming is the use of the Bellman equations to organize the search for good policies.

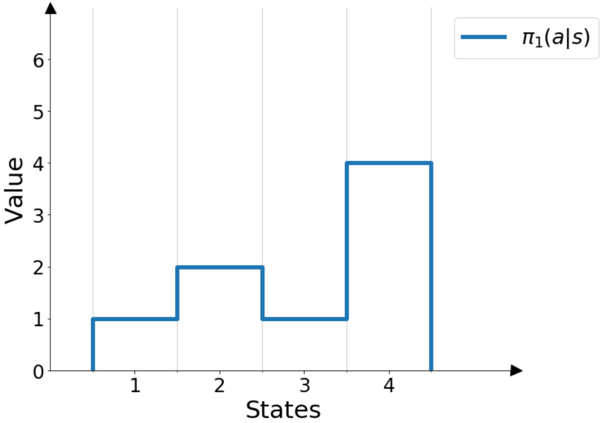
Dynamic Programming

The goal of Reinforcement Learning is to find the optimal policy $\pi(a|s)$ for each $s \in S$ for a given problem.
This task is called Control.

Often, to be able to solve the problem of Control, it is necessary to evaluate the goodness of a policy.
This task is called Policy Evaluation.

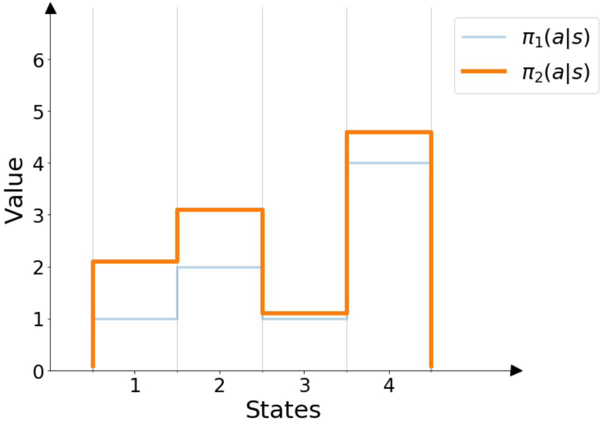
Control

Improving a policy



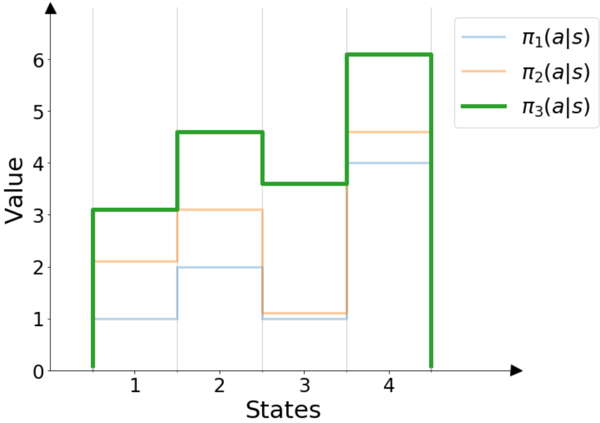
Control

Improving a policy



Control

Improving a policy



Policy Evaluation

Evaluating a policy π means computing its state-value function (or state-action value function).

If the environment dynamic is completely known, this problem is reduced to find the solution to a set of $|\mathcal{S}|$ linear equations:

$$\begin{aligned} V^{\pi}(s) &= \mathbb{E}\left[\sum_{k}^{\infty} \gamma^{k} r_{t+k+1} \,\middle|\, s_t = s, \pi\right] \\ &= \sum_{a} \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s, a)\left[r + \gamma V^{\pi}(s_{t+1})\right] \end{aligned}$$

Iterative Policy Evaluation

Since the above computation is complex, we can compute the value function for a given policy iteratively:

- We choose V^0 arbitrarily
- We compute a sequence of improved approximations $V^1, V^2, V^3...$ by applying the Bellman equation to the previous estimate:

$$V^{k+1}(s) = \mathbb{E}\left[\sum_{k}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, \pi\right]$$

$$= \sum_{a} \pi(a|s) \sum_{s_{t+1}} p(s_{t+1}|s,a)\big[r + \gamma V^k(s_{t+1})\big] \text{ for each } s \in S$$

The sequence $\{V_k\}$ is proven to converge to V_π as $k \to \infty$.

Gridworld

- $\mathcal{S}$: 16 states, with 2 terminal ones (0 nd 16)
- $\mathcal{A}$: Four possible actions: $\uparrow$ (up), $\downarrow$ (down), $\leftarrow$ (left), $\rightarrow$ (right)
- $\mathcal{R}(s, a)$: $r_t = -1$ for every step in a non-terminal state, 0 in both terminal states

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

Iterative Policy Evaluation

We want to evaluate the uniform random policy, which has probability 0.25 of taking each of the 4 possible actions.

$$V(s_{t+1}) := \sum_a \pi(a|s) \sum_{s_{t+1}} \sum_r p(s_{t+1}, r|s, a)\Big[r + \gamma\, V(s_{t+1})\Big]$$

$$0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) = -1$$

V^0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V^1

0	-1	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Iterative Policy Evaluation

We want to evaluate the uniform random policy, which has probability 0.25 of taking each of the 4 possible actions.

$$V(s_{t+1}) := \sum_{a} \pi(a|s) \sum_{s_{t+1}} \sum_{r} p(s_{t+1}, r|s, a)\Big[r + \gamma\, V(s_{t+1})\Big]$$

$$0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) + 0.25 \cdot (-1 + 0) = -1$$

V^0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

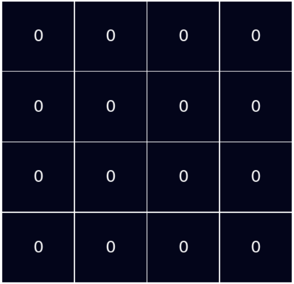
V^1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

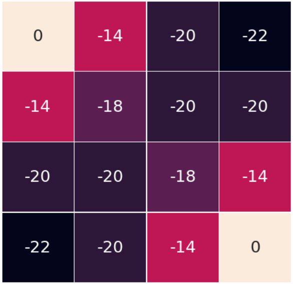
Iterative Policy Evaluation

$$V(s_{t+1}) := \sum_a \pi(a|s) \sum_{s_{t+1}} \sum_r p(s_{t+1}, r|s, a)\left[r + \gamma\, V(s_{t+1})\right]$$

V^0



V^N



Iterative Policy Evaluation

Algorithm 1 Iterative Policy Evaluation

Input the policy to evaluate π

Define threshold for accuracy θ

Initialize $V(s)$, for all $s \in \mathcal{S}$ arbitrarily, except V(terminal) $= 0$

$\Delta \leftarrow 0$
while True
  for each $s \in S$
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
  if $\Delta < \theta$ return V

Policy Improvement

How can we improve our policy once we know its value function?

$$Q_\pi(s, a) = \mathbb{E}\left[\sum_{k}^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, a_t = a, \pi\right]$$

$$= \sum_{s_{t+1}} p(s_{t+1}|s, a)\left[r + \gamma V^k(s_{t+1})\right]$$

Policy Improvement

Policy Improvement Theorem

Given any pair of deterministic policies π and π' such that:

$$Q^{\pi}(s, \pi'(s)) \geq Q^{\pi}(s, \pi(s))$$

Which is the same as:

$$Q^{\pi}(s, \pi'(s)) \geq V^{\pi}(s)$$

Then policy π' is as good as or better than π:

$$V^{\pi'}(s) \geq V^{\pi}(s)$$

The new policy π' can be found as:

$$\pi' = \arg\max_{a \in A} Q^{\pi}(s, a)$$

Policy Iteration

$$\pi_0 \xrightarrow{E} V^{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} V^{\pi_1} \xrightarrow{I} \dots \xrightarrow{I} \pi_* \xrightarrow{E} V^{\pi_*}$$

Since a finite MDP has only a finite number of policies, this process must converge to an optimal policy in a finite number of iterations.

Policy Iteration

Algorithm 2 Policy Iteration for estimating $\pi \sim \pi^*$

1) **Initialization**

Initialize $V(s) and \pi(s)$ arbitrarily for all $s \in S$

2) **Policy Evaluation Step** (Algorithm 1)

3) **Policy Improvement Step:**

policy-stable $\leftarrow$ true

for each $s \in \mathcal{S}$ **do**

old-action $\leftarrow \pi(s)$

$\pi(s) \leftarrow \arg\max_a \sum_{s_{t+1}} p(s_{t+1}|s, a)[r + \gamma V(s^{t+1})]$

if old-action $\neq \pi(s)$, policy-stable $\leftarrow$ false

if policy-stable $=$ true then return $\pi \sim \pi^*$

Value Iteration

Drawback of Policy Iteration: in each iteration we need to perform policy evaluation.

The policy evaluation step can be truncated without losing convergence guarantees.

$$
\begin{aligned}
V^{k+1}(s) &= \max_a \ \mathbb{E}\left[\sum_k^{\infty} \gamma^k r_{t+k+1} \,\middle|\, s_t = s, a_t = a \right] \\
&= \max_a \sum_{s_{t+1}} p(s_{t+1}|s, a)\big[r + \gamma V^k(s_{t+1})\big] \text{ for each } s \in S
\end{aligned}
$$

Value Iteration

Algorithm 3 Value Iteration Algorithm

Define threshold for accuracy θ
Initialize $V(s)$, for all $s \in \mathcal{S}$, arbitrarily

$\Delta \leftarrow 0$
while True **do**
  for each $s \in \mathcal{S}$ **do**
    $v \leftarrow V(s)$
    $V(s) \leftarrow \max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\Big[r + \gamma\, V(s_{t+1})\Big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy $\pi \sim \pi^*$, such that:
$\pi(s) \leftarrow \arg\max_a \sum_{s_{t+1}} p(s_{t+1}|s,a)\Big[r + \gamma\, V(s_{t+1})\Big]$

Generalised Policy Iteration

We define generalized policy iteration as the alternative execution of policy evaluation and policy improvement, regardless of the granularity of the two processes.

This process stabilizes only when we found a policy that is greedy with respect to its own value function.

Final Slide!

Lecture Takeaway

1. Given $p(s'|s, a)$ we can find the optimal policy mathematically
2. But better use Dynamic Programming algorithms
3. It works by alternating the tasks of Policy Evaluation and Control