The Multi-Armed Bandits Problem

Balancing Exploration and Exploitation

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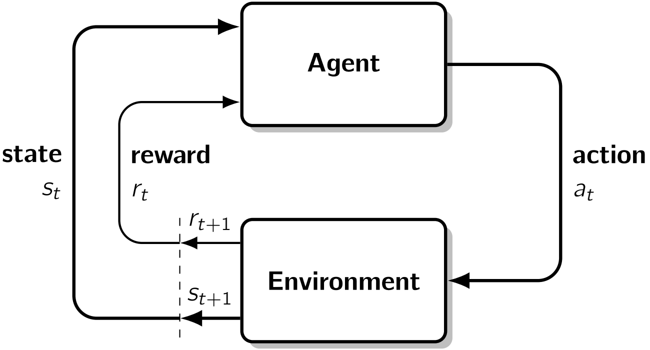
Recap

In Reinforcement Learning an agent learns how to best take actions in order to maximize some goal.



Recap

Key Concepts

- The problem of learning through interaction is framed as a Markov Decision Process $\mathcal{M}\langle\mathcal{S}, \mathcal{A}, \mathcal{P}, r\rangle$
- The agent aims to learn a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- The goal of the agent is to maximize the (discounted) cumulative reward $G_t = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$.
- We can find the optimal policy by making use of the value functions $V^\pi(s)$ and $Q^\pi(s, a)$

Today's Agenda

1. Multi-Armed Bandits
2. Exploration-Exploitation Trade-off
3. Action-Value Methods
4. Action Preferences

Simplified Setting

The environment has a single state.

- The setting is non-associative: actions can not change the state of the environment
- The actions can only impact the next reward

We want to learn a policy:

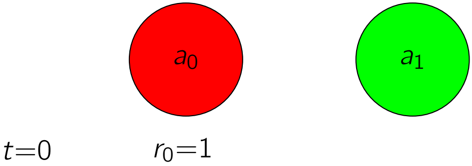
$$\pi(a) = \Pr\{a_t = a\}, \text{ for all } a \in \mathcal{A}.$$

Multi-Armed Bandits

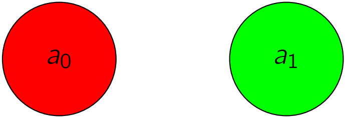
Our agent is repeatedly faced with the choice among k different actions. After each choice, gets a reward sampled from the stationary distribution that depends on the chosen action.

- Set of actions $\mathcal{A}$
- Fixed distribution of rewards for each action, $r_a \mid a \in \mathcal{A}$
- The goal is to maximize the expected cumulative reward
- We need to learn a policy $\pi(a) \mid a \in \mathcal{A}$

Example



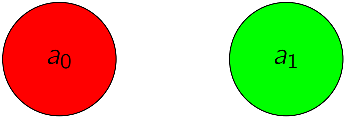
Example



t=0 r_0=1

t=1 r_1=0

Example



	a_0	a_1
$t=0$	$r_0=1$	
$t=1$		$r_1=0$
$t=2$		$r_2=2$

How do we choose whether to stick to the current action, or keep exploring?

Exploration-Exploitation Trade-off

To maximize its reward in this environment, our agent must learn what the best action to take is.

- Exploration allows to gain knowledge of its actions, useful for the **long-term**
- Exploitation allows to use the current knowledge of the actions to get an **instantaneous benefit**

Action-Value Methods

The action value for action a is the reward we expect to receive by taking that action:

$$Q^*(a) \doteq \mathbb{E}[R_t | a_t = a]$$

The goal is to find the optimal value, by maximizing the expected cumulative reward:

$$a^* = \arg\max_a \mathbb{E}[R_t | a_t = a]$$

Regret of an action a:

$$\Delta_a = Q^*(a^*) - Q^*(a)$$

Action-Value Methods

In Reinforcement Learning we do not know which actions have the best value: we need to learn it.

Action Value Estimate

We denote the estimated value of action a at time step t as $Q_t(a)$.

We can use action value estimates to learn a policy.

Sample-Average Method

The simplest estimate is given by the average of the sampled rewards:

$$Q_t(a) \doteq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=0}^{t-1} R_i \cdot \mathbb{I}(A_i = a)}{\sum_{i=0}^{t-1} \mathbb{I}(A_i = a)}$$

$\mathbb{I}$ is the indicator function: $\mathbb{I}(\mathit{True}) = 1$, and $\mathbb{I}(\mathit{False}) = 0$.

Incremental Update

The sample-average method can be computed incrementally:

$$
\begin{aligned}
Q_{t+1}(a) &= \frac{1}{t}\sum_{i=0}^{t} R_i \\
&= \frac{1}{t}\left[R_t + (t-1)\frac{1}{t-1}\sum_{i=0}^{t-1} R_i\right] \\
&= \frac{1}{t}\left[R_t + (t-1)Q_t(a)\right] \\
&= Q_t(a) + \frac{1}{t}\left[R_t - Q_t(a)\right]
\end{aligned}
$$

Incremental Update

$$Q_{t+1}(a) := Q_t(a) + \alpha_t (R_t - Q_t(a)) \quad \text{where} \quad \alpha_t = \frac{1}{t}$$

Update Rule

$$\mathit{NewEstimate} \leftarrow \mathit{OldEstimate} + \mathit{StepSize}\,[\mathit{Target} - \mathit{OldEstimate}]$$

We can also consider other possible step sizes α_t, but they need to respect the following conditions:

$$\sum_{t=0}^{\infty} \alpha_t = \infty \qquad \sum_{t=0}^{\infty} \alpha_t^2 < \infty$$

Action Selection: greedy

Action-value estimates can be used to select actions.

We can exploit the current knowledge by selecting the greedy action, which is the action with the highest estimated value:

$$A_t = \operatorname*{arg\,max}_{a \in \mathcal{A}} Q_t(a)$$

Downside: we can get stuck taking a suboptimal action forever, because we are not exploring better opportunities.

Action Selection: ϵ-greedy

- Behave greedily with probability $1 - \epsilon$
- Take a random action with probability ϵ

Where ϵ is a small, positive number $0 < \epsilon < 1$.

$$\pi_t(a) = \begin{cases} (1-\epsilon) + \epsilon/|\mathcal{A}| & \text{if } \ a = \arg\max_{a \in \mathcal{A}} Q_t(a) \\ \epsilon/|\mathcal{A}| & \text{otherwise} \end{cases}$$

Optimistic Initial Values

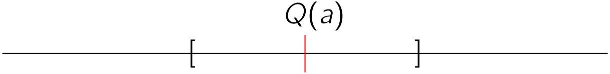
All the methods discussed up to now depend on the initial values of the action estimates, $Q_0(a) \mid a \in \mathcal{A}$.

Optimistic initial values can be used to encourage early exploration:

- For any starting action taken by the agent, the reward is smaller than the starting estimates
- The agent, disappointed, switches to other actions
- Actions are tried several times before the value estimates converge

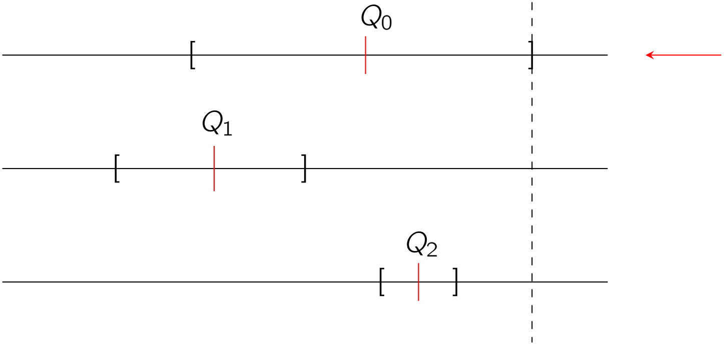
Uncertainty in Action Values Estimates

If we could explicit the uncertainty of our action-values estimates, we could select actions in a better way.

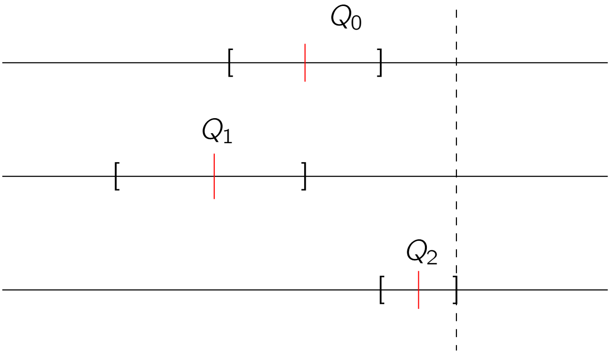


Select actions by leveraging their uncertainty: if we are uncertain about the value of an action, we optimistically assume that it is good.

Optimism in the Face of Uncertainty



Optimism in the Face of Uncertainty



Upper-Confidence Bound (UCB)

We want to estimate an upper confidence bound for every action value:

$$Q^*(a) \leq Q_t(a) + U_t(a)$$

And select greedily the action maximizing the UCB:

$$a_t \doteq \underset{a \in \mathcal{A}}{\arg\max} \left[Q_t(a) + U_t(a) \right]$$

Upper-Confidence Bound (UCB)

$$a_t \doteq \underset{a \in \mathcal{A}}{\arg\max} \left[Q_t(a) + c\sqrt{\frac{ln(t)}{N_a(t)}} \right]$$

- Each time we select a, $N_a(t)$ is increased and therefore the uncertainty is reduced
- Each time we select $a' \neq a$, t increases but $N_a(t)$ does not, therefore the uncertainty increases

The parameter $c > 0$ regulates the amount of exploration.

Action Preferences

All methods described until now rely on the estimate of action values to learn a policy. Another approach relies on learning a preference for each action, $H_t(a)$.

- The larger the preference, the more probably a certain action should be selected
- The preference has no meaning in terms of reward

Boltzmann distribution

From the action preferences, we can define a policy using the Softmax or Boltzmann distribution:

$$\pi(a) = \frac{e^{H_t(a)}}{\sum_{k=0}^{N} e^{H_t(k)}}$$

Note: You will use this distribution in deep learning when it comes to multi-class classification problems!

Action Preferences

If at time step t we choose action a', we can update the action preferences as:

$$
\begin{aligned}
H_{t+1}(a') &\doteq H_t(a') + \alpha\,(r_t - \bar{r}_t)(1 - \pi_t(a')) \\
H_{t+1}(a) &\doteq H_t(a) - \alpha\,(r_t - \bar{r}_t)\,\pi_t(a) \quad \text{if} \quad a \neq a'
\end{aligned}
$$

Where the formula above is based on the idea of stochastic gradient ascent.

Associative Search: Contextual Bandits

Until now, we considered contexts where the agent has to learn how to behave in a single situation. However, the general problem of Reinforcement Learning deals with learning how to behave in a number of different situations.

Associative Search: Contextual Bandits

Simple way to extend bandits:

- You face N different Multi-Armed bandit problems
- This is an associative task: $\pi : \mathcal{S} \rightarrow \mathcal{A}$
- Similar to the full Reinforcement Learning problem as they involve learning a policy, but like the k-armed bandit problem in that each action affects only the immediate reward.

Final Slide!

Lecture Takeaway

1. Multi-Armed bandit is a simplified setting with a single state
2. To learn a policy we need to balance Exploration and Exploitation
3. We can learn the policy by leveraging action-values estimates $Q(a)$
4. Or by leveraging action preferences $H(a)$