Knowledge Representation & Propositional Logic

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Knowledge Base

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- An inference engine is a set of procedures that work upon the representation and can infer new facts or answer KB queries. (e.g. resolution, forward chaining).

Logic

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Components of a logical system:

- Syntax: how to write sentences
- Semantics: how to interpret sentences
- Reasoning/Inference

Propositional Logic

It's another name for boolean logic

- Syntax:
  - Propositional symbols (**atomic sentences**): A, B, C
  - Logical connectives $\neg \wedge \vee \rightarrow \leftrightarrow$
- It is common to use standard lower-case roman letters to denote propositions

$$p, q, r, \ldots$$

Propositional Logic

Computing truth value of any sentence is done recursively

- semantic:
  - if f and g are formulas
  - $\neg f$ - True iff f is false
  - $f \vee g$ - True iff atleast one of f or g is True
  - $f \wedge g$ - True iff both f and g are True
  - $f \rightarrow g$ - False iff f is true and g is false
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Example

$$f = (\neg A \land B) \leftrightarrow C$$
$$w = \{A : 1, B : 1, C : 0\}$$

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$$w = \{A:1, B:1, C:0\}$$

Draw the truth table for f

Some Very important Definitions

- If a sentence α is true in model m, then we say model m **satisfies** α.
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Entailment:

- $\alpha \models \beta$:= if and only if every model in which α is true β is also true.
- $\alpha \models \beta \leftrightarrow M(\alpha) \subseteq M(\beta)$

Satisfiability:

- α is satisfiable if $M(\alpha) \neq \emptyset$, i.e. there is some assignment (model) that makes α true. For example, $(\alpha \wedge \neg\alpha)$ is unsatisfiable.

Validity:

- α if α is always true in *all* models. For example, $(\alpha \vee \neg\alpha)$ is valid.

Example

A knowledge base consists of $\delta = \{\alpha_1, \alpha_2, \alpha_3\}$ what is $M(\delta)$?

Some more Defn's

- Knowledgebase Δ is a set of sentences $\{\alpha_1, \alpha_2, ...\}$
- $M(\Delta)$ all possible models where all the facts hold. (That is, all these sentences are connected by conjunction.)

Example

Determine models for the following (variables R, S, C (rainy, sunny, cloudy)

$$\begin{aligned} KB = & R \vee S \vee C; \\ & R \rightarrow C \wedge \neg S; \\ & C \leftrightarrow \neg S \end{aligned}$$

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$KB = \{(R = 1, S = 0, C = 1), (R = 0, C = 1, S = 0), (R = 0, C = 0, S = 1)\}$

Models

- We say two sentences are α and β equivalent iff $M(\alpha) = M(\beta)$
- α and β are inconsistent $M(\alpha \wedge \beta) = \emptyset$
- α and β are consistent $M(\alpha \wedge \beta) \neq \emptyset$
- α and β are mutually exclusive
  - $M(\alpha) \wedge M(\beta) = \emptyset$
  - $M(\alpha \wedge \beta) = \emptyset$

Syntactic Forms

- **Conjunction Normal Form** (CNF): $(A \vee B) \wedge (B \vee \neg C \vee \neg D)$
- **Disjunction Normal Form** (DNF): $(A \wedge B) \vee (A \wedge \neg C) \vee (A \wedge \neg D)$
- Horn clause: subset of CNF where each clause has at most one positive literal
  - $A \vee B \vee \neg C$ X
  - $\neg A \vee B \vee \neg C$ ✓
  - $\neg A \vee \neg B \vee \neg C$ ✓

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    - $\neg A \vee \neg B \vee \neg C \checkmark$

Complete:

- any logic can be represented using CNF, DNF
- Horn is not complete

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sanity check: KB entails α iff it contradicts $\neg\alpha$

Inference Methods

Tables:

$$\Delta : \{A, A \vee B \rightarrow C\}$$
$$\alpha : c$$

Determine if $\Delta \models \alpha$

Inference Rules

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) \quad \text{De Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) \quad \text{De Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

Figure 7.11 Standard logical equivalences. The symbols α, β, and γ stand for arbitrary sentences of propositional logic.

Inference Rules

- Modus Ponen: $$\frac{\alpha, \alpha \rightarrow \beta}{\therefore \beta}$$
  - Example: $\Delta = \{A, B, B \vee C, B \rightarrow D\}$

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- Resolution $\dfrac{\alpha \vee \beta, \neg\beta \vee \delta}{\therefore \alpha \vee \delta}$

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Sound: is this inference rule/algorithm correct in all cases. For example $\dfrac{\alpha, \beta \to \alpha}{\therefore \beta}$ is not sound.

Complete: can it determine entailment for any $\Delta \models \alpha$.

Proof by resolution is complete.

Proof by Resolution

How do we determine whether $\Delta \models \alpha$?
Proof by refutation: $\Delta \models \alpha$ if and only if the sentence $(\Delta \wedge \neg\alpha)$ is unsatisfiable.

How do we determine whether $(\Delta \wedge \neg\alpha)$ is unsatisfiable?
Proof by Resolution (a.k.a. a resolution-based algorithm): Use the resolution inference rule. This algorithm is sound and complete. It applies to any kind of Δ and α.

Example

$$
\begin{aligned}
\Delta :\ & A \vee \neg B \rightarrow C \\
& C \rightarrow D \vee \neg E \\
& E \vee D
\end{aligned}
$$

$$\alpha : A \rightarrow D$$

Determine if $\Delta \models \alpha$

Example

$$\Delta : A \wedge B \rightarrow C, A, C \rightarrow D$$

$$\alpha : C$$

Determine if $\Delta \models \alpha$

Example

$$\Delta : P \vee Q, P \rightarrow R, Q \rightarrow R$$

$$\alpha : R$$

Determine if $\Delta \models \alpha$

Example

$$B \leftrightarrow (P \vee Q)$$

Convert the above to CNF

Example

$$\Delta : \{(P \rightarrow Q) \rightarrow Q, (P \rightarrow Q) \rightarrow R, (R \rightarrow S) \rightarrow \neg(S \rightarrow Q)\}$$

$$\alpha : R$$

Determine if (first convert to CNF) $\Delta \models \alpha$

Example

1. John is going to the store
2. That guy is going to the store
3. John, go to the store
4. Did John go to the store?

Example

Either I'll pay for the meal and you'll pay for drinks, or, if John shows up he'll pay for both

- Symbolize the above sentence into a proposition

Example

$$A = \{p \rightarrow q, q \rightarrow p, p|q, p \rightarrow \neg q\}$$

$$C = \neg p$$

Determine if the above is satisfiable

Thank You!