Bayesian Networks

CS161

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Motivation from Logic

- Starting from logic:

$$Toothache \Rightarrow\ Cavity$$

- Rule is wrong; some exceptions:

$$Toothache$$
$$\Rightarrow\ Cavity \vee GumProblems \vee Abscess \vee \cdots$$

- This is intractable to model
- Perhaps make the rule causal:

$$Cavity \Rightarrow Toothache$$

… but this rule is wrong as well…

Motivation from Logic

The monotonicity of logic is the problem… again

- Either you model everything exhaustively
  - Intractable to model; we are too ignorant
  - Even if you could, you will never be able to act
- Assume too much and get stuck when things don’t go as expected

Epistemological change: we no longer believe in a set of possible worlds (models), we believe in a probability for each world!

World View

- Propositional

  Global properties that are true or false
- Probabilistic
  - Belief is still a set of possible world
  - But now they have a degree of belief Pr(.)
  - Knowledge Base KB ≈ Pr
- *"Uncertainty is epistemological – pertaining to an agent's beliefs of the world – rather than ontological – how the world is."* [Poole et al.]
  - We can have different beliefs about the same world
  - What's the probability that the world ends tomorrow?

Decision-Making Motivation

- Acting with partial (noisy) sensor information
  - $\Rightarrow$ Consider ever logically possible explanation
- Example: drive to airport
  - $\Rightarrow$ No plan is guaranteed to achieve goal
  - $\Rightarrow$ Yet the agent must act
- Decisions depend on
  - Relative importance of goals (*utility*)
  - The likelihood of achieving them (*probability*)
  - $\Rightarrow$ Maximum expected utility

Propositions are only Boolean?

- Categorical variables
  - Weather=sunny, Weather=rainy, Weather=snowy
  - 3 Boolean variables that are mutually exclusive
    - Sometimes called "indicator variables"
    - Can all be encoded in sentences…
- Continuous variables
  - Temperature=73.514, Temperature=78.785, …
  - Infinitely many Boolean variables (and worlds).
    - In logic, see SAT Modulo Theories (SMT)
    - Special accommodations for continuous variables in statistics; we will mostly stick to the discrete world.

Sentences or "Events"

- Knowledge is a probability for every world: $\Pr(\omega)$
- What is the probability of a sentence α?
  (also called an "event" α in probability)
- Need to <u>axiomatize</u> probability [Kolmogorov]:
  1. Probabilities are non-negative: $0 \leq \Pr(\alpha)$
  2. The probability of a true event is 1: $\Pr(true) = 1$
  3. If α and β are mutually exclusive, then
     $\Pr(\alpha \vee \beta) = \Pr(\alpha) + \Pr(\beta)$.

Sentences or “Events”

- Knowledge is a probability for every world: $\Pr(\omega)$
- What is the probability of a sentence α?
  (also called an “event” α in probability)
- A sentence α is equivalent to the disjunction of its models: $\alpha \equiv \omega_1 \lor \omega_8 \lor \omega_{11} \lor \omega_{17} \lor \cdots$

$$\boxed{\Pr(\alpha) = \sum_{\omega \models \alpha} \Pr(\omega) = \sum_{\omega \in Mods(\alpha)} \Pr(\omega)}$$

Properties of Probability

- Complement events
  - $\Pr(\alpha) + \Pr(\neg\alpha) = 1$
  - Why? 
- Inclusion-exclusion
  - $\Pr(\alpha \vee \beta) = \Pr(\alpha) + \Pr(\beta) - \Pr(\alpha \wedge \beta)$
  - Why? 

$$
\begin{aligned}
\Pr(\text{Earthquake}) &= \Pr(\omega_1) + \Pr(\omega_2) + \Pr(\omega_3) + \Pr(\omega_4) = .1 \\
\Pr(\text{Burglary}) &= \Pr(\omega_1) + \Pr(\omega_2) + \Pr(\omega_5) + \Pr(\omega_6) = .2 \\
\Pr(\text{Earthquake} \wedge \text{Burglary}) &= \Pr(\omega_1) + \Pr(\omega_2) = .02 \\
\Pr(\text{Earthquake} \vee \text{Burglary}) &= .1 + .2 - .02 = .28
\end{aligned}
$$

[Darwiche '08]

Conditional Probability

- What if I observe new information in the form of a sentence β?
- Belief changes from $\Pr(\alpha)$ to $\Pr(\alpha|\beta)$
- Can also be axiomatized…
- But briefly

$$\boxed{\Pr(\alpha|\beta) = \frac{\Pr(\alpha \wedge \beta)}{\Pr(\beta)}}$$

$$\begin{aligned} \Pr(\text{Burglary}) &= .2 \\ \Pr(\text{Burglary}|\text{Earthquake}) &= .2 \end{aligned}$$

$$\begin{aligned} \Pr(\text{Alarm}) &= .2442 \\ \Pr(\text{Alarm}|\text{Earthquake}) &\approx .75 \uparrow \end{aligned}$$

[Darwiche '08]

Product Rule

Basic Properties of Probability

Betting Semantics

Inconsistent Beliefs

Agent 1		Agent 2		Outcomes and payoffs to Agent 1			
Proposition	Belief	Bet	Stakes	a, b	$a, \neg b$	$\neg a, b$	$\neg a, \neg b$
a	0.4	a	4 to 6	–6	–6	4	4
b	0.3	b	3 to 7	–7	3	–7	3
$a \vee b$	0.8	$\neg(a \vee b)$	2 to 8	2	2	2	–8
				–11	–1	–1	–1

Computing Probabilities: Example

	$toothache$		$\neg toothache$	
	$catch$	$\neg catch$	$catch$	$\neg catch$
$cavity$	0.108	0.012	0.072	0.008
$\neg cavity$	0.016	0.064	0.144	0.576

Monotonicity of Belief?

- Recall: monotonicity of logic
- Is it possible to observe something new and undo prior beliefs?

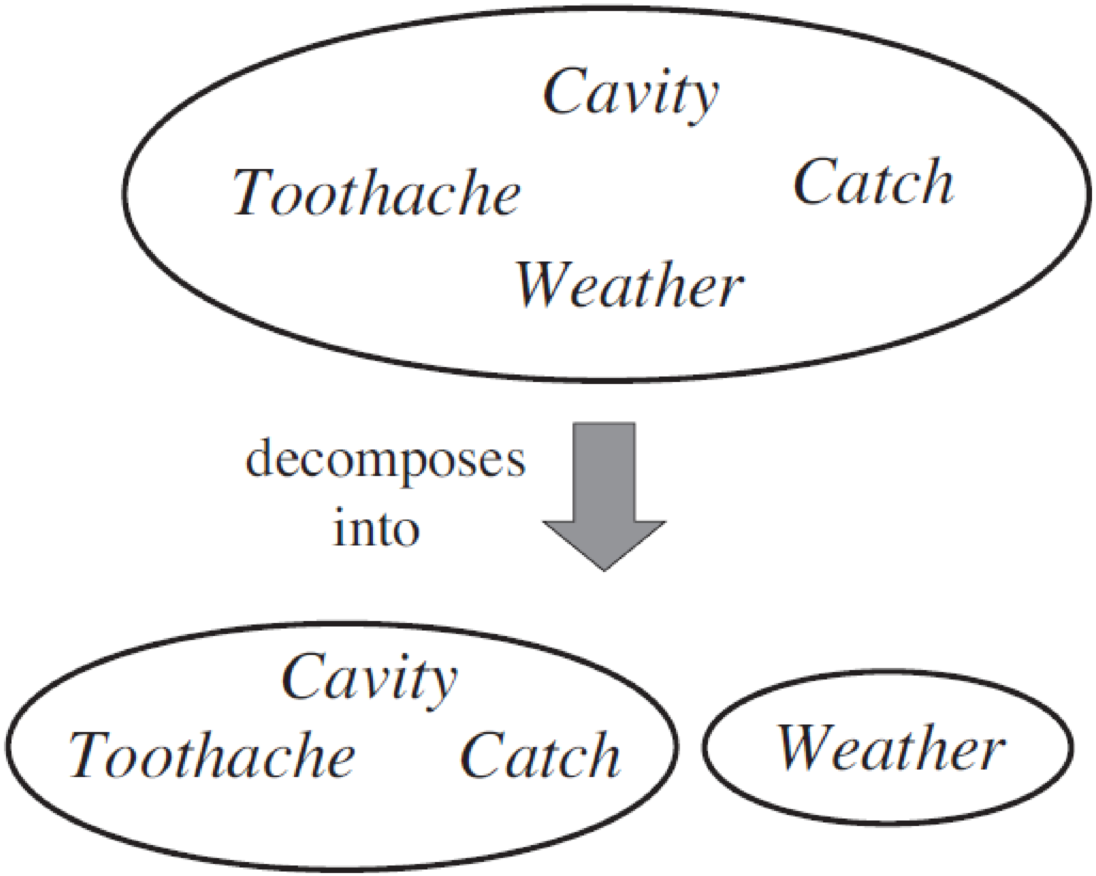
$\Pr(\text{Alarm})$	$=$	$.2442$
$\Pr(\text{Alarm}\mid\text{Earthquake})$	$\approx$	$.75 \uparrow$

world	Earthquake	Burglary	Alarm	$\Pr(.)$
ω_1	true	true	true	.0190
ω_2	true	true	false	.0010
ω_3	true	false	true	.0560
ω_4	true	false	false	.0240
ω_5	false	true	true	.1620
ω_6	false	true	false	.0180
ω_7	false	false	true	.0072
ω_8	false	false	false	.7128

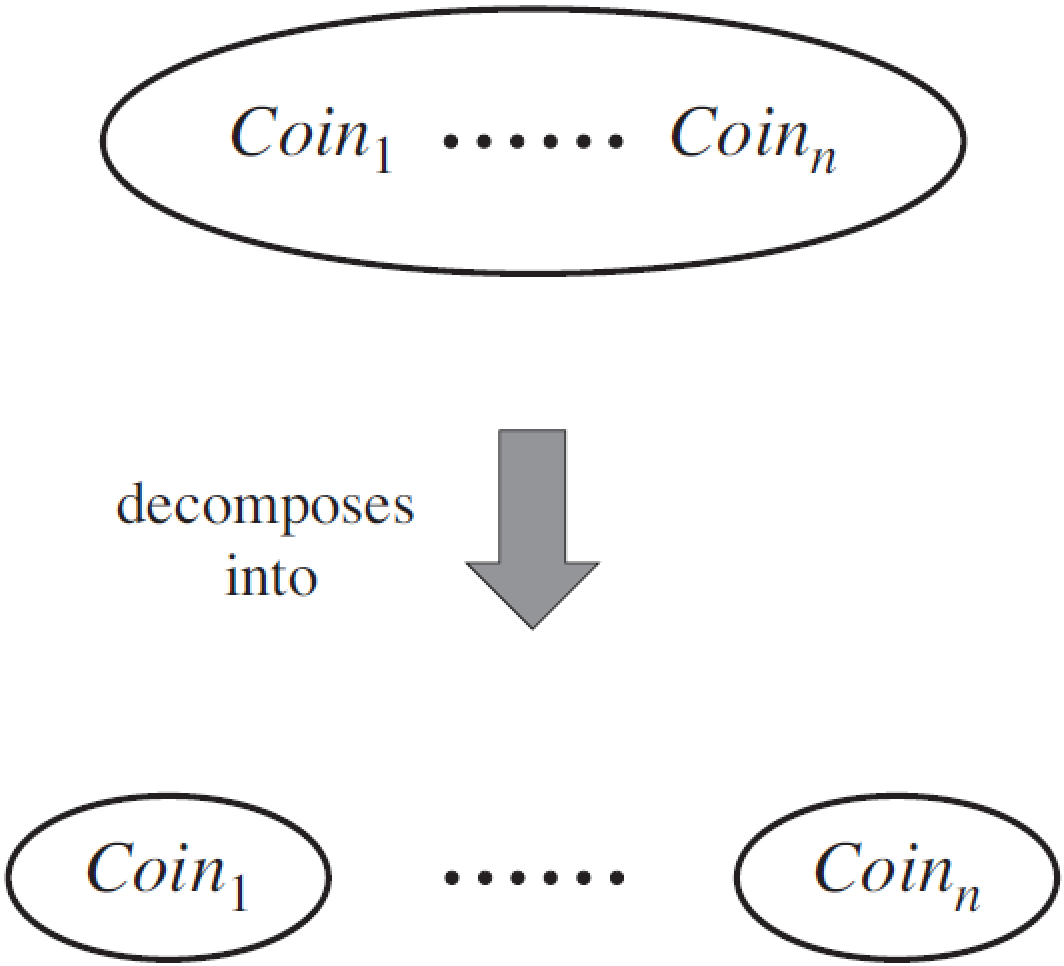
- Example:
  - Alarm and not Earthquake: .1620+.0072=0.1692
  - Not Earthquake: .9
  - Alarm given not Earthquake: .188

[Darwiche '08]

Independence



Independence

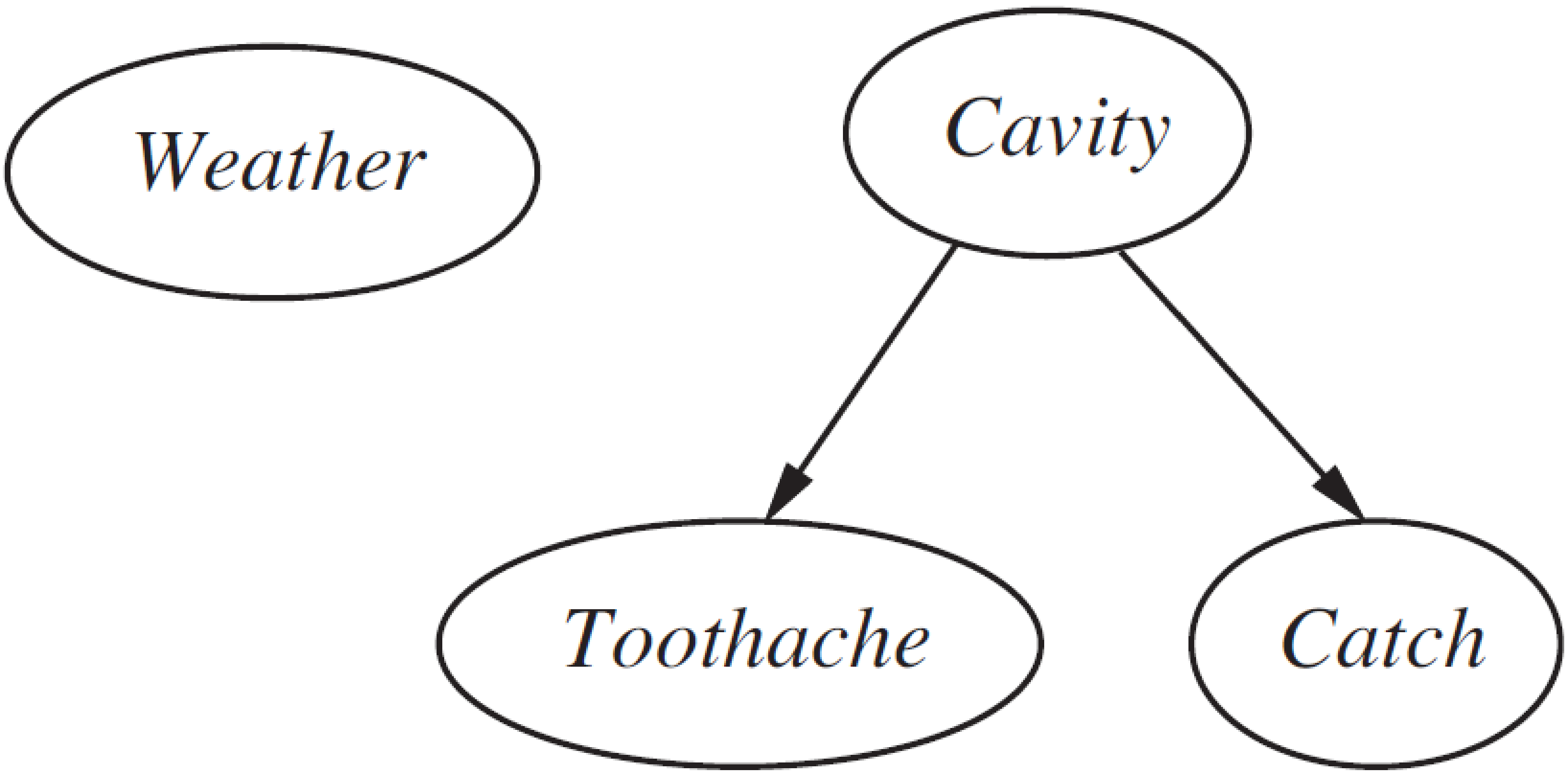


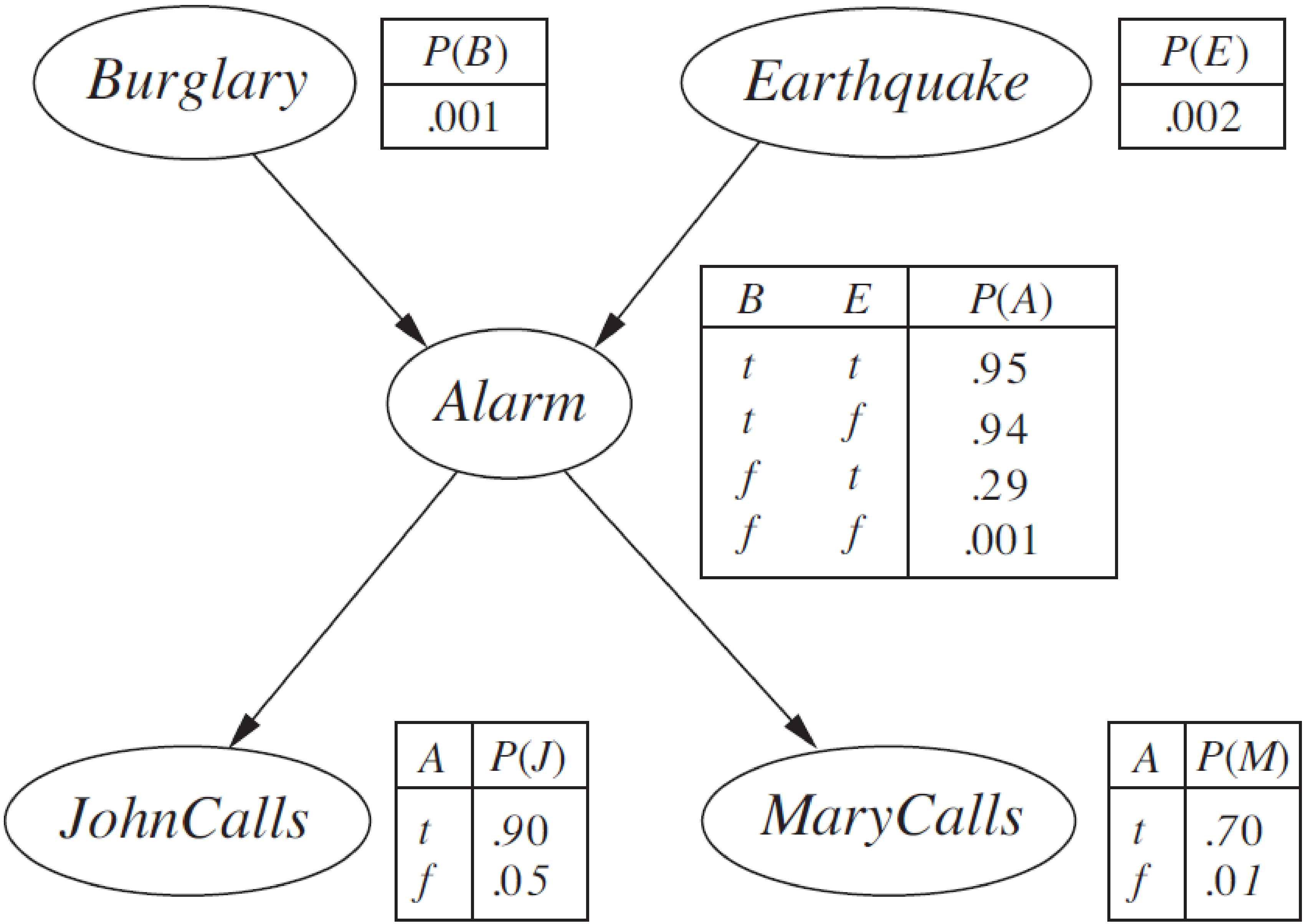
Naïve Bayes Assumption

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i \mid Cause)$$

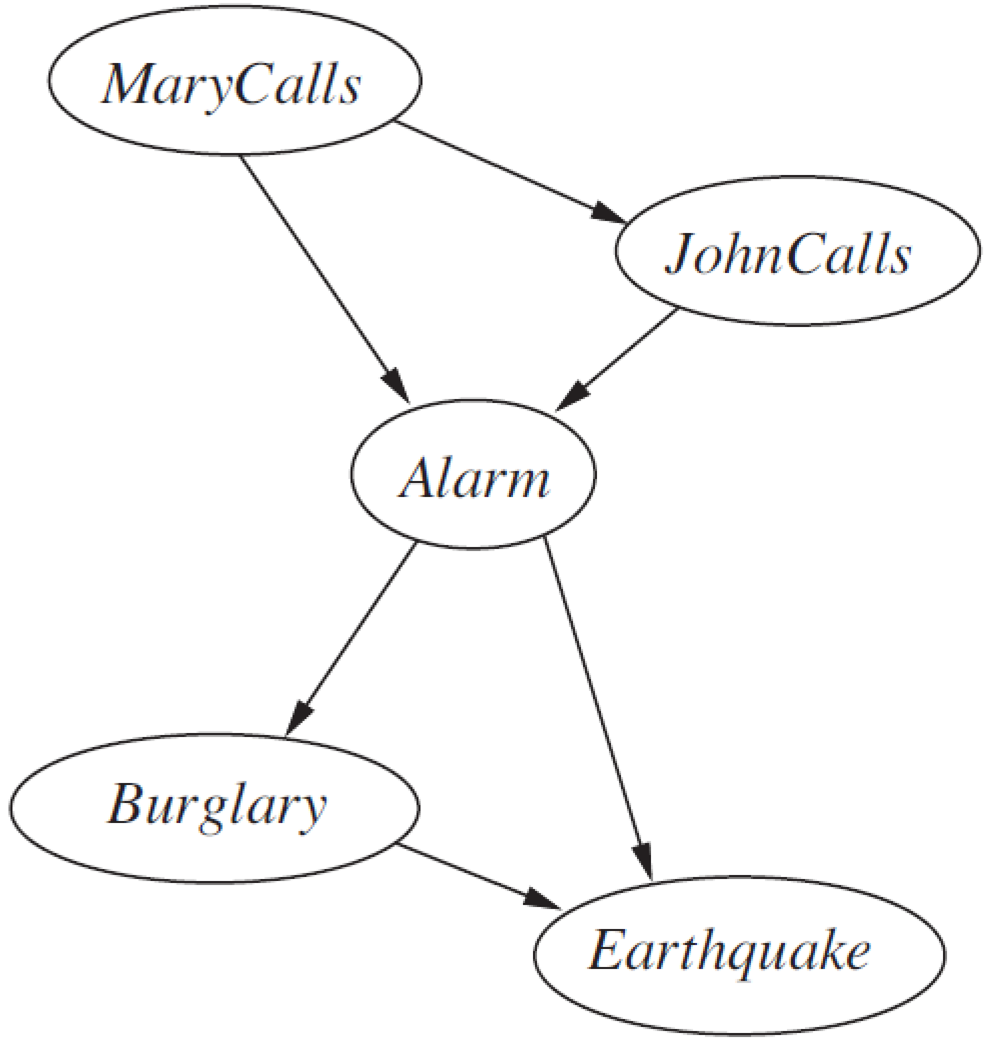
This is how spam filters work!

Bayesian Networks

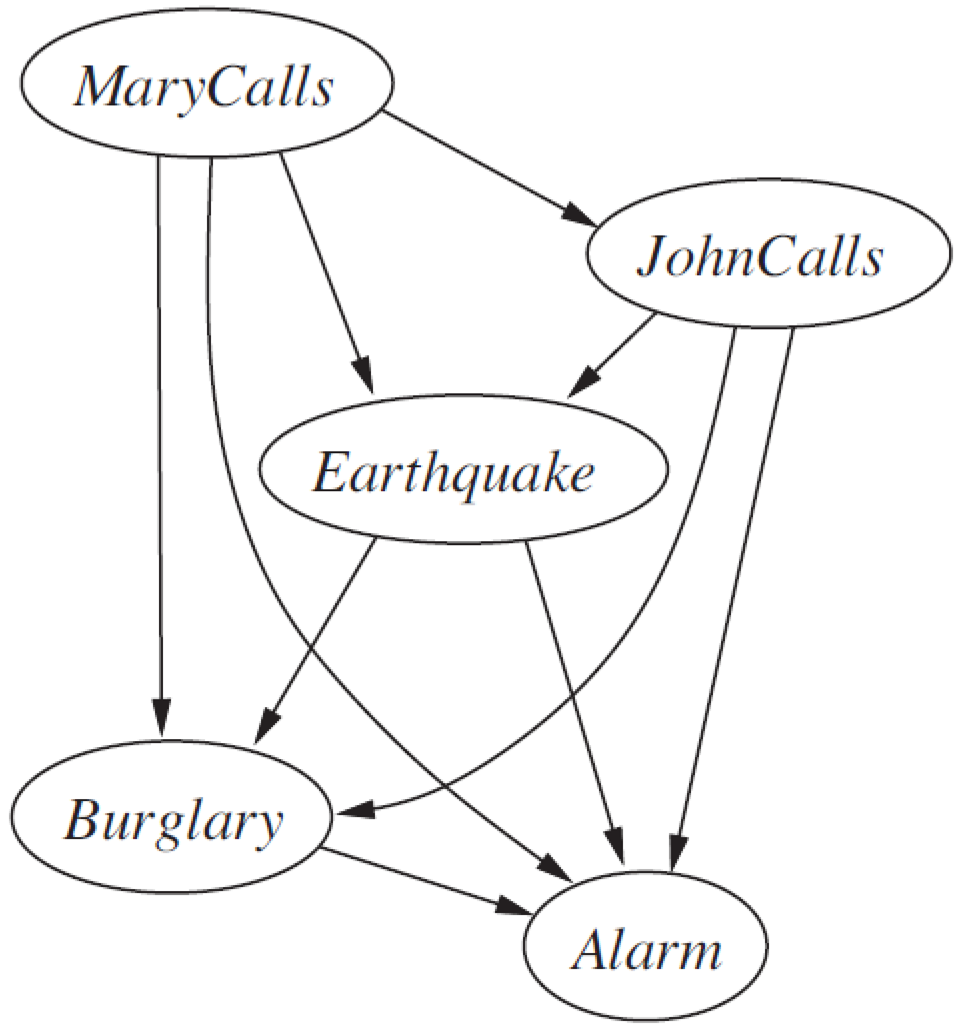



Burglary
P(B)
.001
Earthquake
P(E)
.002
Alarm
B E P(A)
t t .95
t f .94
f t .29
f f .001
JohnCalls
A P(J)
t .90
f .05
MaryCalls
A P(M)
t .70
f .01

Conditional Independence and Order



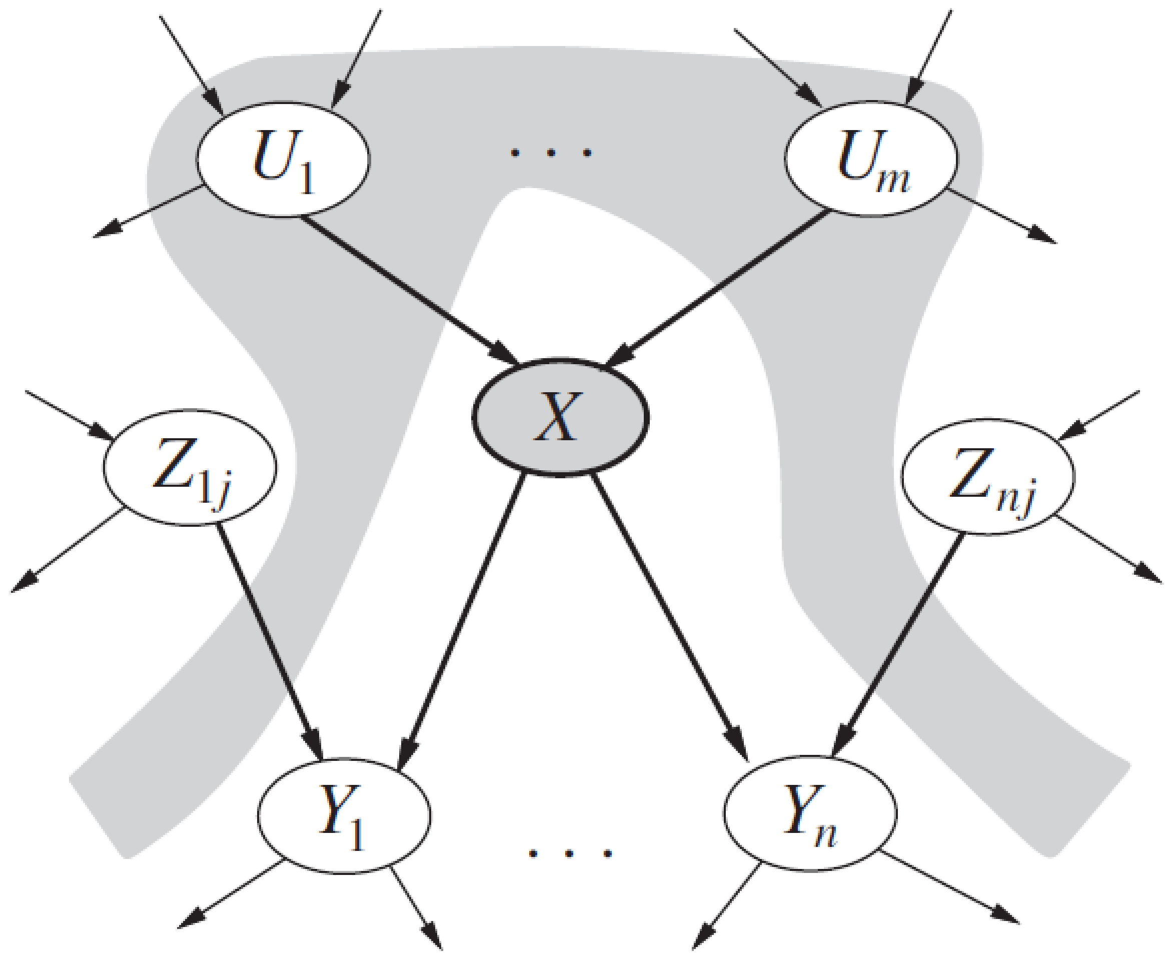
Conditional Independence and Order



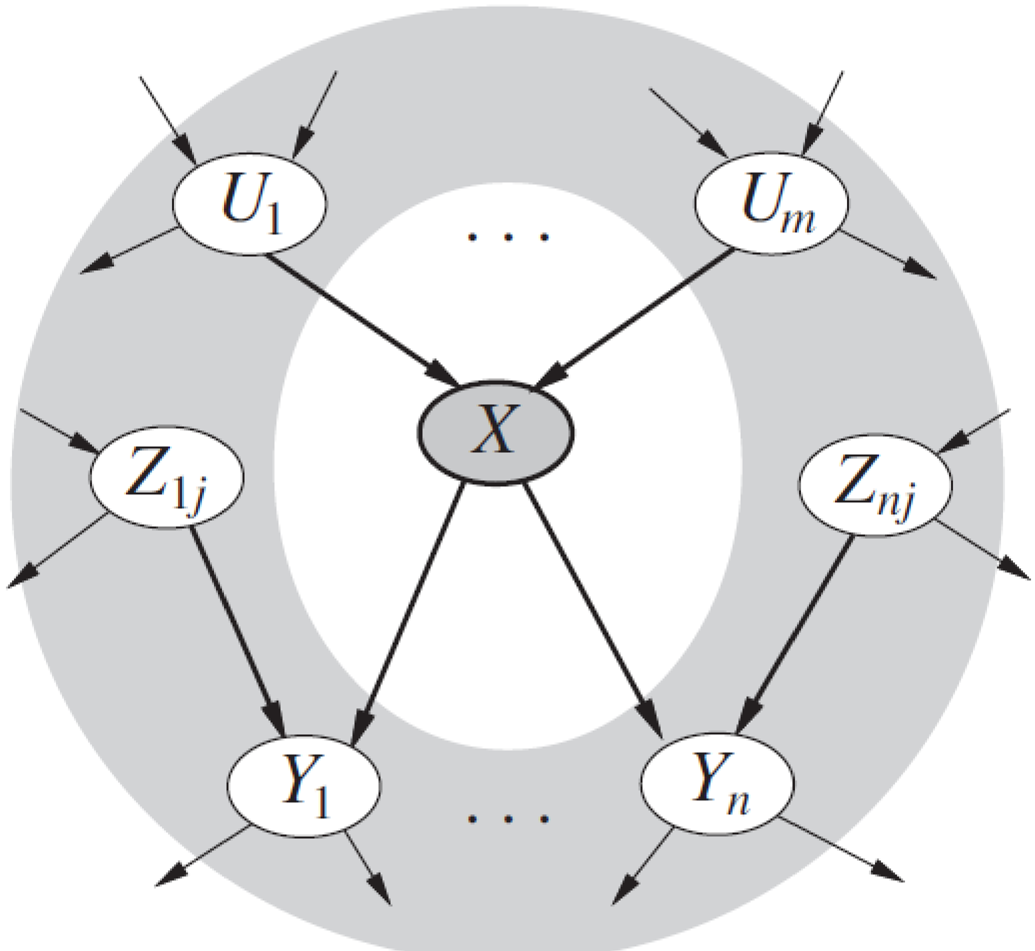
Topological Semantics

What knowledge is encoded in Bayesian network structure?

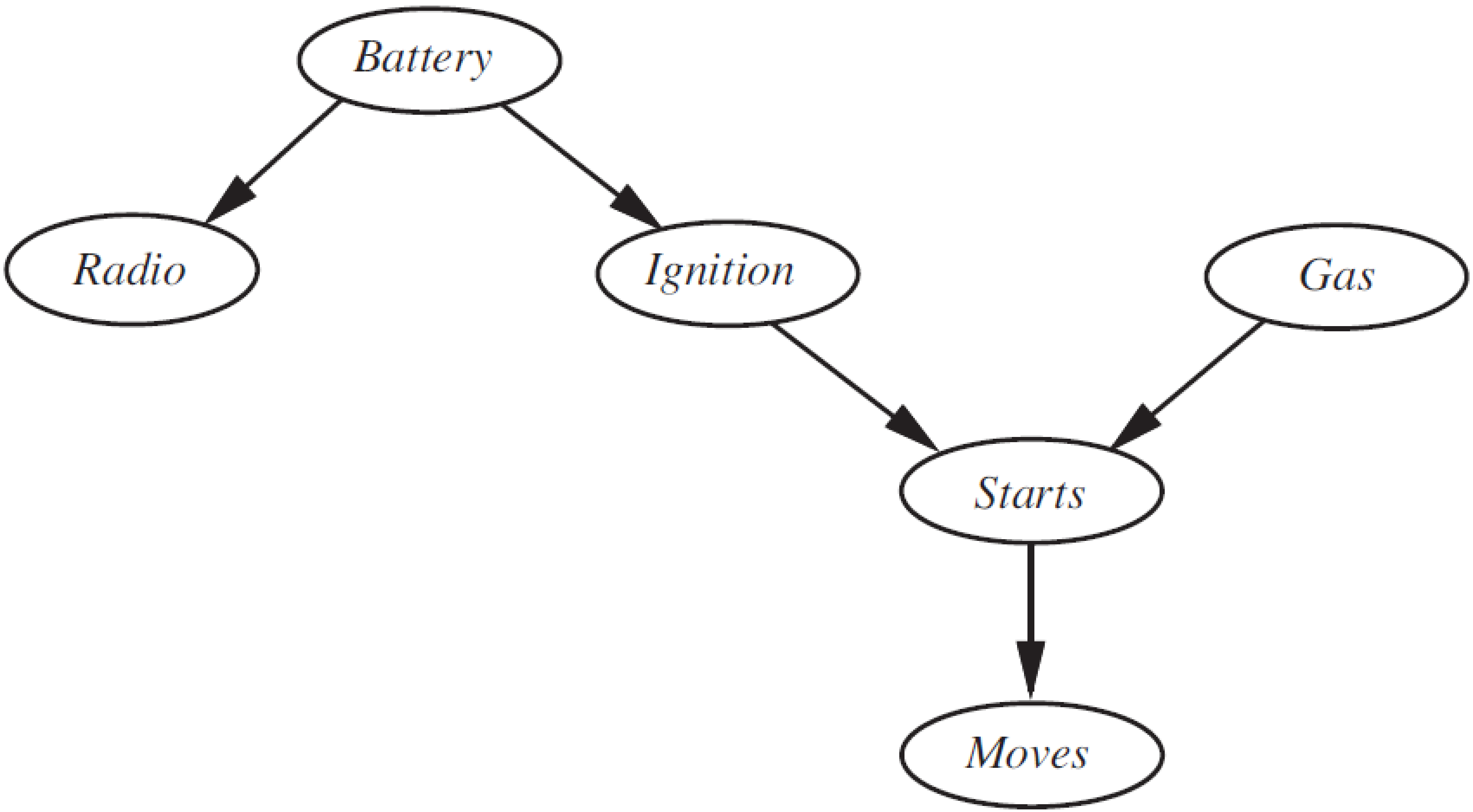
Markovian Assumptions



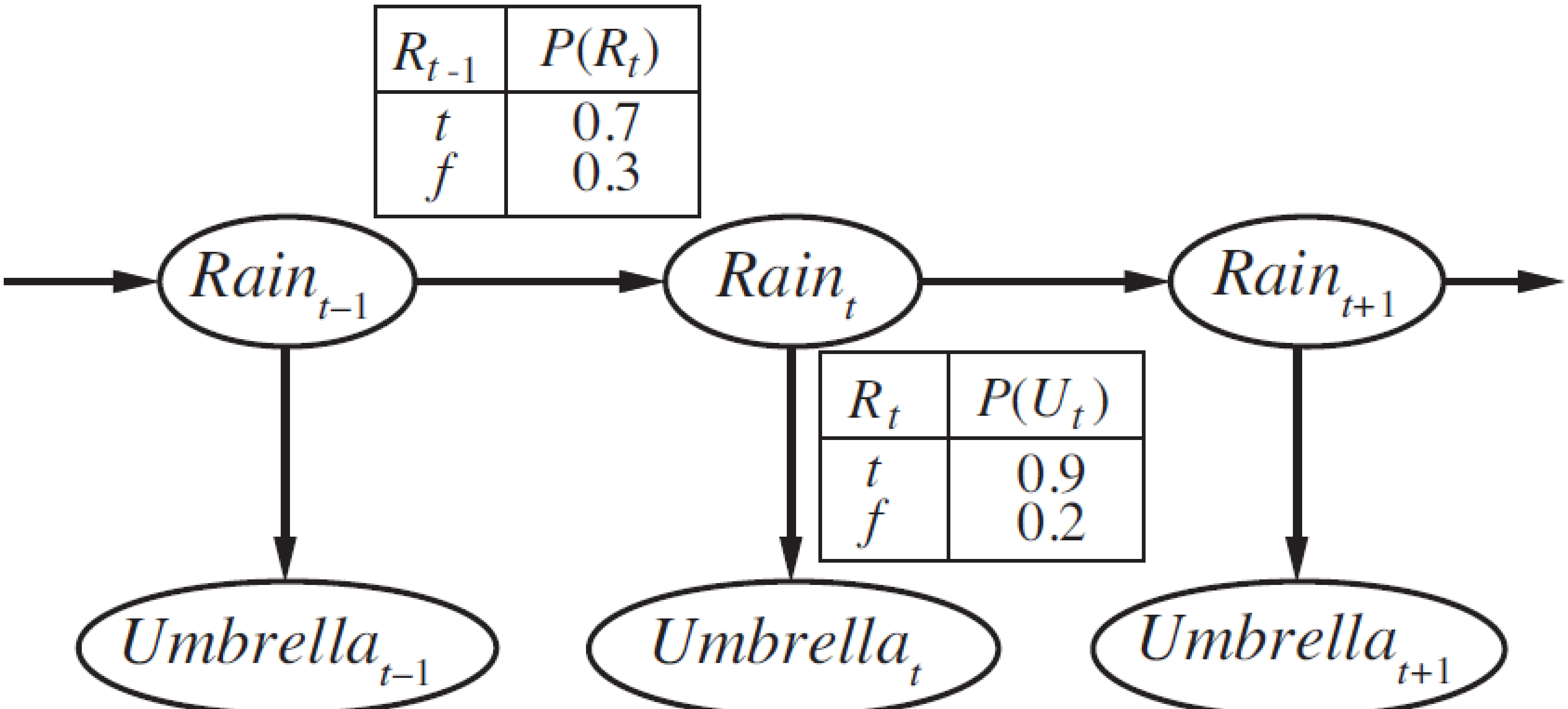
Markov Blanket



Example Network



Markov Chains and Hidden Markov Models



R_{t-1}	$P(R_t)$
t	0.7
f	0.3

R_t	$P(U_t)$
t	0.9
f	0.2

Inference by Enumeration

Factors Multiplication

A	B	$\mathbf{f}_1(A,B)$
T	T	.3
T	F	.7
F	T	.9
F	F	.1

X

B	C	$\mathbf{f}_2(B,C)$
T	T	.2
T	F	.8
F	T	.6
F	F	.4

=

A	B	C	$\mathbf{f}_3(A,B,C)$
T	T	T	$.3 \times .2 = .06$
T	T	F	$.3 \times .8 = .24$
T	F	T	$.7 \times .6 = .42$
T	F	F	$.7 \times .4 = .28$
F	T	T	$.9 \times .2 = .18$
F	T	F	$.9 \times .8 = .72$
F	F	T	$.1 \times .6 = .06$
F	F	F	$.1 \times .4 = .04$

Summing out Variable from Factor

A	B	C	$\mathbf{f}_3(A,B,C)$
T	T	T	$.3 \times .2 = .06$
T	T	F	$.3 \times .8 = .24$
T	F	T	$.7 \times .6 = .42$
T	F	F	$.7 \times .4 = .28$
F	T	T	$.9 \times .2 = .18$
F	T	F	$.9 \times .8 = .72$
F	F	T	$.1 \times .6 = .06$
F	F	F	$.1 \times .4 = .04$

$$\mathbf{f}(B,C) = \sum_a \mathbf{f}_3(A,B,C) = \mathbf{f}_3(a,B,C) + \mathbf{f}_3(\neg a,B,C)$$

$$= \begin{pmatrix} .06 & .24 \\ .42 & .28 \end{pmatrix} + \begin{pmatrix} .18 & .72 \\ .06 & .04 \end{pmatrix} = \begin{pmatrix} .24 & .96 \\ .48 & .32 \end{pmatrix}.$$

Variable Elimination