Linear Algebra Lecture Notes

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Chapter 1

Vectors and Matrices

1.1 Vectors and Linear Combination

Linear Combination Vectors v and w are both 2D vectors. The linear combination of v and w are the vectors $cv + dw$ for any scalars c and d.:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The linear combinations $c \begin{bmatrix} 2 \\ 4 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix}$ form xy plane.

v and w are **linearly independent**. There is exactly one solution b_1, b_2.

The 2 by 2 matrix $A = \begin{bmatrix} v & w \end{bmatrix}$ is **invertible**.

Column Way, Row Way, Matrix Way
Column way, Linear combination:

$$c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Row way, Two equations for c and d:

$$v_1 c + w_1 d = b_1, \quad v_2 c + w_2 d = b_2$$

Matrix way, 2 by 2 matrix:

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vectors in 3D We need *three* independent vectors to span 3D space R^3.

Identity Matrix I: denoted by I_n for an nxn identity matrix, where n is the number of rows or columns.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying any matrix by I leaves the matrix unchanged. I is the matrix

1.2 Length and Angles from Dot Products

Dot Product The dot product of two vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is $v \cdot w = v_1 w_1 + v_2 w_2 = w \cdot v$.

Unit Vector A unit vector is a vector with length 1. The unit vector in the direction of v is $\frac{v}{\|v\|}$.

Perpendicular Vectors Two vectors v and w are perpendicular if $v \cdot w = 0$.

$$\|v+w\|^2 = (v+w)\cdot(v+w) = v\cdot v + 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$
$$\|v-w\|^2 = (v-w)\cdot(v-w) = v\cdot v - 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$

Angle between Vectors The angle between two vectors v and w is $\theta = \cos^{-1}\left(\frac{v\cdot w}{\|v\|\|w\|}\right)$.

Example 1. The unit vectors $v = (\cos\alpha, \sin\alpha)$ and $w = (\cos\beta, \sin\beta)$ have $v \cdot w = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. In trigonometry, this is the formula for $\cos(\alpha-\beta)$ or $\cos(\beta-\alpha)$.

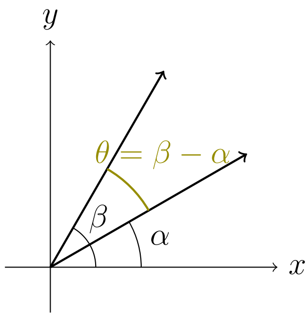

Figure 1.1: Visualization of $\cos(\beta-\alpha) = \cos(\theta)$ in the unit circle.

For unit vectors, $\cos\theta = v \cdot w$. When v and w are not unit vectors, divide by their length to get $u = v/\|v\|$ and $U = u/\|u\|$ and turn them into unit vectors.

Cosine Formula
If v and w are nonzero vectors then $\dfrac{v\cdot w}{\|v\|\|w\|} = \cos\theta$

Since $|\cos\theta| \le 1$, this cosine formula gives two great inequalities:

Cauchy-Schwarz Inequality
For any vectors v and w, $|v\cdot w| \le \|v\|\|w\|$.
Triangle Inequality
For any vectors v and w, $\|v+w\| \le \|v\| + \|w\|$.

The triangle inequality comes directly from the Schwarz inequality:

$$\|v+w\|^2 = v^2 + 2v\cdot w + w^2 \le v^2 + 2\|v\|\|w\| + w^2 = (\|v\| + \|w\|)^2$$

Take the square root of both sides to get $\|v+w\| \le \|v\| + \|w\|$.

1.3 Row and Column Spaces

Column Space Think of the columns of A as vectors a_1, a_2, ..., a_n. The column space of A is the set of all possible combinations $Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n$.

Row Space The row space of A is the set of all possible combinations of the rows of A. The row space is the same as the column space of A^T.

1.4 Matrix Multiplication AB

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

We multiply A with each column of B, b_1 and b_2, to produce columns of AB. We can multiply Ab_1 the *row way* or the *column way.*

Row way, Dot products

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} row1 \cdot b_1 \\ row2 \cdot b_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Column way, Combine columns

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Four Ways to Multiply $AB = C$

Suppose we have A as a 3 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, B as a 2 by 4 matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, then C is a 3 by 4 matrix.

1. Dot product way $\Rightarrow$ 12 numbers

$$(\text{Row } i \text{ of } A) \cdot (\text{Column } j \text{ of } B) = C_{ij}$$

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 5 & 1 \cdot 2 + 2 \cdot 6 & 1 \cdot 3 + 2 \cdot 7 & 1 \cdot 4 + 2 \cdot 8 \\ 3 \cdot 1 + 4 \cdot 5 & 3 \cdot 2 + 4 \cdot 6 & 3 \cdot 3 + 4 \cdot 7 & 3 \cdot 4 + 4 \cdot 8 \\ 5 \cdot 1 + 6 \cdot 5 & 5 \cdot 2 + 6 \cdot 6 & 5 \cdot 3 + 6 \cdot 7 & 5 \cdot 4 + 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{bmatrix}$$

2. Column way $\Rightarrow$ 4 columns
For column j in C, we multiply A by B_{col_j}:

$$C_{\text{col}_j} = A \cdot \text{Column } j \text{ of } B$$

To calculate C_{col_1}:

$$C_{\text{col1}} = A \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \\ 35 \end{bmatrix}$$

The same goes for columns 2, 3, and 4 of C.
3. Row way $\Rightarrow$ 3 rows
For row i in C, we multiply A_{row_i} by B:

$$C_{\text{row}_i} = \text{Row } i \text{ of } A \cdot B$$

To calculate C_{row_1}:

$$C_{\text{row}_1} = \begin{bmatrix}1 & 2\end{bmatrix} \cdot B = \begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\end{bmatrix} = \begin{bmatrix}11 & 14 & 17 & 20\end{bmatrix}$$

The same goes for rows 2 and 3 of C.
4. Matrix way $\Rightarrow$ sum of 2 Rank 1 matrices

$$\begin{aligned}
C &= A_{\text{col}_1} \cdot B_{\text{row}_1} + A_{\text{col}_2} \cdot B_{\text{row}_2}\\
&= \begin{bmatrix}1\\3\\5\end{bmatrix} \cdot \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} + \begin{bmatrix}2\\4\\6\end{bmatrix} \cdot \begin{bmatrix}5 & 6 & 7 & 8\end{bmatrix}\\
&= \begin{bmatrix}1 & 2 & 3 & 4\\ 3 & 6 & 9 & 12\\ 5 & 10 & 15 & 20\end{bmatrix} + \begin{bmatrix}10 & 12 & 14 & 16\\ 20 & 24 & 28 & 32\\ 30 & 36 & 42 & 48\end{bmatrix}\\
&= \begin{bmatrix}11 & 14 & 17 & 20\\ 23 & 30 & 37 & 44\\ 35 & 46 & 57 & 68\end{bmatrix}
\end{aligned}$$

Associative & Distributive Law Both associative and distributive laws apply to matrix multiplication.

Associative $(AB)C = A(BC)$
Distributive $A(B + C) = AB + AC$

1.5 The Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or linearly independent columns in that matrix.
Rank-One Matrix: A matrix where all rows (or columns) are linearly dependent on one another. Example:

$$A = \begin{bmatrix}1 & 2\\ 2 & 4\\ 3 & 6\end{bmatrix}$$

Rank-Two Matrix: A matrix where there are two linearly independent rows (or columns). Example:

$$A = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$$

The two columns of A are linearly independent. The three rows of A are not all linearly independent, specifically, $r_3 = -r_1 + 2r_2$. So the maximum number of linearly independent rows or columns is 2.

Row rank equals column rank for any matrix.

1.6 $A = CR$

C represents a matrix of linearly independent columns of A. R is a reduced matrix that expresses how the columns of A can be reconstructed using the columns of C.

C contains the First r Independent Columns of A Suppose we go from left to right, looking for independent columns of A:

- If column 1 of A is not all zero, put it into the matrix C
- If column 2 of A is not a multiple of column 1, put it into C
- If column 3 of A is not a combination of columns 1 and 2, put it into C. *Continue*

C will have r columns, where r is the rank of A and C. The r columns of C are linearly independent.

Part of R is the *Identity Matrix* This corresponds to the columns of A that went straight into C. The remaining columns of R contain the coefficients that express the dependent columns of A as a combination of the columns of C. **When a column of A goes into C, a column of I goes into R.**

Example 1.

$$A = \begin{bmatrix} a_1 & a_2 & 3a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} = CR$$

Find R with Elimination The elimination process is called *row reduction.*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = CR$$

First perform $row_1 - row_2 \times 2$ and $row_3 - row_1 \times 3$ to get B. Then perform $row_3 - row_2$ to get U.

$$A \to B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & -2 & -6 \end{bmatrix} \to U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \text{upper triangular matrix } U$$

Eliminate upwards to produce more zeros in U. First divide row 2 by -2, then make the entry above the pivot element (row_1, col_2) equal to zero with $row_1 - 3 \times row_2$.

$$U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R_0$$

Removing the zero row of R_0 gives R.

Chapter 2

Solving Linear Equations

2.1 Elimination and Back Substitution