

Linear Algebra Lecture Notes

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Chapter 1

Vectors and Matrices

1.1 Vectors and Linear Combination

Linear Combination Vectors v and w are both 2D vectors. The linear combination of v and w are the vectors $cv + dw$ for any scalars c and d :

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The linear combinations $c \begin{bmatrix} 2 \\ 4 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix}$ form xy plane.

v and w are **linearly independent**. There is exactly one solution b_1, b_2 .

The 2 by 2 matrix $A = \begin{bmatrix} v & w \end{bmatrix}$ is **invertible**.

Column Way, Row Way, Matrix Way

Column way, Linear combination:

$$c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Row way, Two equations for c and d :

$$v_1c + w_1d = b_1, \quad v_2c + w_2d = b_2$$

Matrix way, 2 by 2 matrix:

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vectors in 3D We need *three* independent vectors to span 3D space R^3 .

Identity Matrix I : denoted by I_n for an $n \times n$ identity matrix, where n is the number of rows or columns.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying any matrix by I leaves the matrix unchanged. I is the matrix

1.2 Length and Angles from Dot Products

Dot Product The dot product of two vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is $v \cdot w = v_1w_1 + v_2w_2 = w \cdot v$.

Unit Vector A unit vector is a vector with length 1. The unit vector in the direction of v is $\frac{v}{\|v\|}$.

Perpendicular Vectors Two vectors v and w are perpendicular if $v \cdot w = 0$.

$$\|v + w\|^2 = (v + w) \cdot (v + w) = v \cdot v + 2v \cdot w + w \cdot w = \|v\|^2 + \|w\|^2$$

$$\|v - w\|^2 = (v - w) \cdot (v - w) = v \cdot v - 2v \cdot w + w \cdot w = \|v\|^2 + \|w\|^2$$

Angle between Vectors The angle between two vectors v and w is $\theta = \cos^{-1} \left(\frac{v \cdot w}{\|v\| \|w\|} \right)$.

Example 1. The unit vectors $v = (\cos \alpha, \sin \alpha)$ and $w = (\cos \beta, \sin \beta)$ have $v \cdot w = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. In trigonometry, this is the formula for $\cos(\alpha - \beta)$ or $\cos(\beta - \alpha)$.

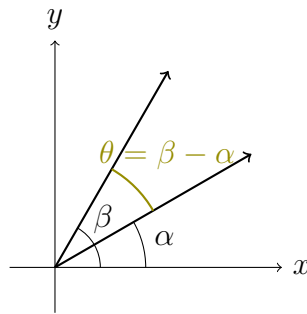


Figure 1.1: Visualization of $\cos(\beta - \alpha) = \cos(\theta)$ in the unit circle.

For unit vectors, $\cos \theta = v \cdot w$. When v and w are not unit vectors, divide by their length to get $u = v/\|v\|$ and $U = u/\|u\|$ and turn them into unit vectors.

Cosine Formula

If v and w are nonzero vectors then $\frac{v \cdot w}{\|v\| \|w\|} = \cos \theta$

Since $|\cos \theta| \leq 1$, this cosine formula gives two great inequalities:

Cauchy-Schwarz Inequality

For any vectors v and w , $|v \cdot w| \leq \|v\| \|w\|$.

Triangle Inequality

For any vectors v and w , $\|v + w\| \leq \|v\| + \|w\|$.

The triangle inequality comes directly from the Schwarz inequality:

$$\|v + w\|^2 = v^2 + 2v \cdot w + w^2 \leq v^2 + 2\|v\| \|w\| + w^2 = (\|v\| + \|w\|)^2$$

Take the square root of both sides to get $\|v + w\| \leq \|v\| + \|w\|$.

1.3 Row and Column Spaces

Column Space Think of the columns of A as vectors a_1, a_2, \dots, a_n . The column space of A is the set of all possible combinations $Ax = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$.

Row Space The row space of A is the set of all possible combinations of the rows of A . The row space is the same as the column space of A^T .

1.4 Matrix Multiplication AB

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

We multiply A with each column of B , b_1 and b_2 , to produce columns of AB . We can multiply Ab_1 the *row way* or the *column way*.

Row way, Dot products

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \text{row1} \cdot b_1 \\ \text{row2} \cdot b_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Column way, Combine columns

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Four Ways to Multiply $AB = C$

Suppose we have A as a 3 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, B as a 2 by 4 matrix $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$,

then C is a 3 by 4 matrix.

1. Dot product way \Rightarrow 12 numbers

$$(\text{Row } i \text{ of } A) \cdot (\text{Column } j \text{ of } B) = C_{ij}$$

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 5 & 1 \cdot 2 + 2 \cdot 6 & 1 \cdot 3 + 2 \cdot 7 & 1 \cdot 4 + 2 \cdot 8 \\ 3 \cdot 1 + 4 \cdot 5 & 3 \cdot 2 + 4 \cdot 6 & 3 \cdot 3 + 4 \cdot 7 & 3 \cdot 4 + 4 \cdot 8 \\ 5 \cdot 1 + 6 \cdot 5 & 5 \cdot 2 + 6 \cdot 6 & 5 \cdot 3 + 6 \cdot 7 & 5 \cdot 4 + 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{bmatrix}$$

2. Column way \Rightarrow 4 columns

For column j in C , we multiply A by B_{col_j} :

$$C_{\text{col}_j} = A \cdot \text{Column } j \text{ of } B$$

To calculate C_{col_1} :

$$C_{\text{col}_1} = A \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \\ 35 \end{bmatrix}$$

The same goes for columns 2, 3, and 4 of C .

3. Row way \Rightarrow 3 rows

For row i in C , we multiply A_{row_i} by B :

$$C_{\text{row}_i} = \text{Row } i \text{ of } A \cdot B$$

To calculate C_{row_1} :

$$C_{\text{row}_1} = [1 \ 2] \cdot B = [1 \ 2] \cdot \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = [11 \ 14 \ 17 \ 20]$$

The same goes for rows 2 and 3 of C .

4. Matrix way \Rightarrow sum of 2 Rank 1 matrices

$$\begin{aligned} C &= A_{\text{col}_1} \cdot B_{\text{row}_1} + A_{\text{col}_2} \cdot B_{\text{row}_2} \\ &= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \cdot [1 \ 2 \ 3 \ 4] + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \cdot [5 \ 6 \ 7 \ 8] \\ &= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 5 & 10 & 15 & 20 \end{bmatrix} + \begin{bmatrix} 10 & 12 & 14 & 16 \\ 20 & 24 & 28 & 32 \\ 30 & 36 & 42 & 48 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{bmatrix} \end{aligned}$$

Associative & Distributive Law Both associative and distributive laws apply to matrix multiplication.

Associative $(AB)C = A(BC)$
Distributive $A(B + C) = AB + AC$

1.5 The Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or linearly independent columns in that matrix.

Rank-One Matrix: A matrix where all rows (or columns) are linearly dependent on one another. Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Rank-Two Matrix: A matrix where there are two linearly independent rows (or columns). Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

The two columns of A are linearly independent. The three rows of A are not all linearly independent, specifically, $r_3 = -r_1 + 2r_2$. So the maximum number of linearly independent rows or columns is 2.

Row rank equals column rank for any matrix.

1.6 $A = CR$

C represents a matrix of linearly independent columns of A . R is a reduced matrix that expresses how the columns of A can be reconstructed using the columns of C .

C contains the First r Independent Columns of A Suppose we go from left to right, looking for independent columns of A :

- If column 1 of A is not all zero, put it into the matrix C
- If column 2 of A is not a multiple of column 1, put it into C
- If column 3 of A is not a combination of columns 1 and 2, put it into C . *Continue*

C will have r columns, where r is the rank of A and C . The r columns of C are linearly independent.

Part of R is the *Identity Matrix* This corresponds to the columns of A that went straight into C . The remaining columns of R contain the coefficients that express the dependent columns of A as a combination of the columns of C . **When a column of A goes into C , a column of I goes into R .**

Example 1.

$$A = \begin{bmatrix} a_1 & a_2 & 3a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} = CR$$

Find R with Elimination The elimination process is called *row reduction*.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = CR$$

First perform $row_1 - row_2 \times 2$ and $row_3 - row_1 \times 3$ to get B . Then perform $row_3 - row_2$ to get U .

$$A \rightarrow B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & -2 & -6 \end{bmatrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \text{upper triangular matrix } U$$

Eliminate upwards to produce more zeros in U . First divide row 2 by -2, then make the entry above the pivot element (row_1, col_2) equal to zero with $row_1 - 3 \times row_2$.

$$U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R_0$$

Removing the zero row of R_0 gives R .

Chapter 2

Solving Linear Equations $Ax = b$

A systematic way to solve the equation $Ax = b$:

1. Apply elimination to $Ax = b$
2. Get upper triangular U
3. Solve $Ux = c$ by back substitution

2.1 The number of solutions to $Ax = b$

1. Exactly one solution A has independent columns. A has an inverse matrix A^{-1} .
Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. No solution b is not in the column space of A . Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 15 \end{bmatrix} \Rightarrow 0 = 3$$

3. Infinitely many solutions A has dependent columns. Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 3\alpha \\ 6 - 2\alpha \end{bmatrix} \text{ for any number } \alpha$$

2.2 Elimination Matrix E_{ij}

We move column by column from left to right. Typically, the first non-zero element in each column is chosen as the pivot (one row below the row of the pivot we just used). The pivots are used to eliminate the elements below them. The elimination matrix E_{ij} eliminates the element a_{ij} in the i th row and j th column.

Example:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix}, \quad b = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$$

Step 1 The first pivot is $a_{11} = 2$. Eliminate the elements below it.

$$\text{Eliminate } a_{21} \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \quad E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 50 \end{bmatrix}$$

$$\text{Eliminate } a_{31} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix} \quad E_{31}E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 31 \end{bmatrix}$$

Step 2 The second pivot is $a_{22} = 5$. Eliminate the elements below it.

$$\text{Eliminate } a_{32} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \quad E_{32}E_{31}E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 7 \end{bmatrix}$$

Step 3 Solve $Ux = c$ by back substitution.

Now we have $U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$ and $c = \begin{bmatrix} 19 \\ 17 \\ 7 \end{bmatrix}$. We can go from the bottom up to solve for x .

An easy way to come up with elimination matrix

To go from $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix}$ to $B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix}$, we need to come up with E_{31} :

1. For the first row in E_{31} :

A_{row_1} is $[2 \ 3 \ 4]$, B_{row_1} is $[2 \ 3 \ 4]$. We take $1 \times A_{\text{row}_1}$, $0 \times A_{\text{row}_2}$, and $0 \times A_{\text{row}_3}$.
So the first row of E_{31} is $[1 \ 0 \ 0]$.

2. For the second row in E_{31} :

A_{row_2} is $[0 \ 5 \ 6]$, B_{row_2} is $[0 \ 5 \ 6]$. We take $0 \times A_{\text{row}_1}$, $1 \times A_{\text{row}_2}$, and $0 \times A_{\text{row}_3}$.
So the second row of E_{31} is $[0 \ 1 \ 0]$.

3. For the third row in E_{31} :

A_{row_3} is $[2 \ 8 \ 17]$, B_{row_3} is $[0 \ 5 \ 13]$. We take $-1 \times A_{\text{row}_1}$, $0 \times A_{\text{row}_2}$, and $1 \times A_{\text{row}_3}$.
So the third row of E_{31} is $[-1 \ 0 \ 1]$.

4. Put it together:

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

2.3 Permutation Matrix P