Linear Algebra Lecture Notes

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Chapter 1

Vectors and Matrices

1.1 Vectors and Linear Combination

Linear Combination Vectors v and w are both 2D vectors. The linear combination of v and w are the vectors $cv + dw$ for any scalars c and d.:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The linear combinations $c \begin{bmatrix} 2 \\ 4 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix}$ form xy plane.

v and w are **linearly independent**. There is exactly one solution b_1, b_2.

The 2 by 2 matrix $A = \begin{bmatrix} v & w \end{bmatrix}$ is **invertible**.

Column Way, Row Way, Matrix Way
Column way, Linear combination:

$$c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Row way, Two equations for c and d:

$$v_1 c + w_1 d = b_1, \quad v_2 c + w_2 d = b_2$$

Matrix way, 2 by 2 matrix:

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vectors in 3D We need *three* independent vectors to span 3D space R^3.

Identity Matrix I: denoted by I_n for an nxn identity matrix, where n is the number of rows or columns.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying any matrix by I leaves the matrix unchanged. I is the matrix

1.2 Length and Angles from Dot Products

Dot Product The dot product of two vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is $v \cdot w = v_1 w_1 + v_2 w_2 = w \cdot v$.

Unit Vector A unit vector is a vector with length 1. The unit vector in the direction of v is $\frac{v}{\|v\|}$.

Perpendicular Vectors Two vectors v and w are perpendicular if $v \cdot w = 0$.

$$\|v+w\|^2 = (v+w)\cdot(v+w) = v\cdot v + 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$

$$\|v-w\|^2 = (v-w)\cdot(v-w) = v\cdot v - 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$

Angle between Vectors The angle between two vectors v and w is $\theta = \cos^{-1}\left(\frac{v\cdot w}{\|v\|\|w\|}\right)$.

Example 1. The unit vectors $v = (\cos\alpha, \sin\alpha)$ and $w = (\cos\beta, \sin\beta)$ have $v\cdot w = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. In trigonometry, this is the formula for $\cos(\alpha-\beta)$ or $\cos(\beta-\alpha)$.

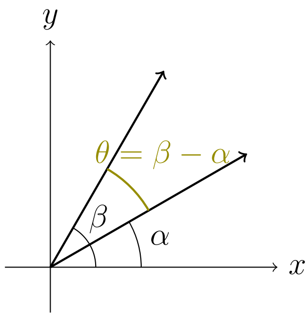

Figure 1.1: Visualization of $\cos(\beta-\alpha) = \cos(\theta)$ in the unit circle.

For unit vectors, $\cos\theta = v\cdot w$. When v and w are not unit vectors, divide by their length to get $u = v/\|v\|$ and $U = u/\|u\|$ and turn them into unit vectors.

Cosine Formula
If v and w are nonzero vectors then $\dfrac{v\cdot w}{\|v\|\|w\|} = \cos\theta$

Since $|\cos\theta| \leq 1$, this cosine formula gives two great inequalities:

Cauchy-Schwarz Inequality
For any vectors v and w, $|v\cdot w| \leq \|v\|\|w\|$.

Triangle Inequality
For any vectors v and w, $\|v+w\| \leq \|v\| + \|w\|$.

The triangle inequality comes directly from the Schwarz inequality:

$$\|v+w\|^2 = v^2 + 2v\cdot w + w^2 \leq v^2 + 2\|v\|\|w\| + w^2 = (\|v\| + \|w\|)^2$$

Take the square root of both sides to get $\|v+w\| \leq \|v\| + \|w\|$.

1.3 Row and Column Spaces

Column Space Think of the columns of A as vectors a_1, a_2, ..., a_n. The column space of A is the set of all possible combinations $Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n$.

Row Space The row space of A is the set of all possible combinations of the rows of A. The row space is the same as the column space of A^T.

1.4 Matrix Multiplication AB

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

We multiply A with each column of B, b_1 and b_2, to produce columns of AB. We can multiply Ab_1 the *row way* or the *column way*.

Row way, Dot products

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} row1 \cdot b_1 \\ row2 \cdot b_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 \\ 3 \cdot 5 + 4 \cdot 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Column way, Combine columns

$$Ab_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Four Ways to Multiply $AB = C$

Suppose we have A as a 3 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, B as a 2 by 4 matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$, then C is a 3 by 4 matrix.

1. Dot product way $\Rightarrow$ 12 numbers

$$(\text{Row } i \text{ of } A) \cdot (\text{Column } j \text{ of } B) = C_{ij}$$

$$AB = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 5 & 1 \cdot 2 + 2 \cdot 6 & 1 \cdot 3 + 2 \cdot 7 & 1 \cdot 4 + 2 \cdot 8 \\ 3 \cdot 1 + 4 \cdot 5 & 3 \cdot 2 + 4 \cdot 6 & 3 \cdot 3 + 4 \cdot 7 & 3 \cdot 4 + 4 \cdot 8 \\ 5 \cdot 1 + 6 \cdot 5 & 5 \cdot 2 + 6 \cdot 6 & 5 \cdot 3 + 6 \cdot 7 & 5 \cdot 4 + 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \end{bmatrix}$$

2. Column way $\Rightarrow$ 4 columns
For column j in C, we multiply A by B_{col_j}:

$$C_{\text{col}_j} = A \cdot \text{Column } j \text{ of } B$$

To calculate C_{col_1}:

$$C_{\text{col}1} = A \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \\ 35 \end{bmatrix}$$

The same goes for columns 2, 3, and 4 of C.
3. Row way $\Rightarrow$ 3 rows
For row i in C, we multiply A_{row_i} by B:

$$C_{\text{row}_i} = \text{Row } i \text{ of } A \cdot B$$

To calculate C_{row_1}:

$$C_{\text{row}_1} = \begin{bmatrix}1 & 2\end{bmatrix} \cdot B = \begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\end{bmatrix} = \begin{bmatrix}11 & 14 & 17 & 20\end{bmatrix}$$

The same goes for rows 2 and 3 of C.
4. Matrix way $\Rightarrow$ sum of 2 Rank 1 matrices

$$\begin{aligned}
C &= A_{\text{col}_1} \cdot B_{\text{row}_1} + A_{\text{col}_2} \cdot B_{\text{row}_2}\\
&= \begin{bmatrix}1\\3\\5\end{bmatrix} \cdot \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} + \begin{bmatrix}2\\4\\6\end{bmatrix} \cdot \begin{bmatrix}5 & 6 & 7 & 8\end{bmatrix}\\
&= \begin{bmatrix}1 & 2 & 3 & 4\\3 & 6 & 9 & 12\\5 & 10 & 15 & 20\end{bmatrix} + \begin{bmatrix}10 & 12 & 14 & 16\\20 & 24 & 28 & 32\\30 & 36 & 42 & 48\end{bmatrix}\\
&= \begin{bmatrix}11 & 14 & 17 & 20\\23 & 30 & 37 & 44\\35 & 46 & 57 & 68\end{bmatrix}
\end{aligned}$$

Associative & Distributive Law Both associative and distributive laws apply to matrix multiplication.

Associative $(AB)C = A(BC)$
Distributive $A(B + C) = AB + AC$

1.5 The Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or linearly independent columns in that matrix.
Rank-One Matrix: A matrix where all rows (or columns) are linearly dependent on one another. Example:

$$A = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix}$$

Rank-Two Matrix: A matrix where there are two linearly independent rows (or columns). Example:

$$A = \begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

The two columns of A are linearly independent. The three rows of A are not all linearly independent, specifically, $r_3 = -r_1 + 2r_2$. So the maximum number of linearly independent rows or columns is 2.

Row rank equals column rank for any matrix.

1.6 $A = CR$

C represents a matrix of linearly independent columns of A. R is a reduced matrix that expresses how the columns of A can be reconstructed using the columns of C.

C contains the First r Independent Columns of A Suppose we go from left to right, looking for independent columns of A:

- If column 1 of A is not all zero, put it into the matrix C
- If column 2 of A is not a multiple of column 1, put it into C
- If column 3 of A is not a combination of columns 1 and 2, put it into C. *Continue*

C will have r columns, where r is the rank of A and C. The r columns of C are linearly independent.

Part of R is the *Identity Matrix* This corresponds to the columns of A that went straight into C. The remaining columns of R contain the coefficients that express the dependent columns of A as a combination of the columns of C. **When a column of A goes into C, a column of I goes into R.**

Example 1.

$$A = \begin{bmatrix} a_1 & a_2 & 3a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix} = CR$$

Find R with Elimination The elimination process is called *row reduction.*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \end{bmatrix} = CR$$

First perform $row_1 - row_2 \times 2$ and $row_3 - row_1 \times 3$ to get B. Then perform $row_3 - row_2$ to get U.

$$A \to B = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & -2 & -6 \end{bmatrix} \to U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \text{upper triangular matrix } U$$

Eliminate upwards to produce more zeros in U. First divide row 2 by -2, then make the entry above the pivot element (row_1, col_2) equal to zero with $row_1 - 3 \times row_2$.

$$U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = R_0$$

Removing the zero row of R_0 gives R.

Chapter 2

Solving Linear Equations $Ax = b$

A systematic way to solve the equation $Ax = b$:

1. Apply elimination to $Ax = b$
2. Get upper triangular U
3. Solve $Ux = c$ by **back substitution**

2.1 The number of solutions to $Ax = b$

1. Exactly one solution A has independent columns. A has an inverse matrix A^{-1}. Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad \Rightarrow \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. No solution b is not in the column space of A. Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 15 \end{bmatrix} \quad \Rightarrow \quad 0 = 3$$

3. Infinitely many solutions A has dependent columns. Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \quad \Rightarrow \quad x = \begin{bmatrix} 3\alpha \\ 6 - 2\alpha \end{bmatrix} \text{ for any number } \alpha$$

2.2 Elimination Matrix E_{ij} & Permutation Matrix P

Elimination Matrix E_{ij} We move column by column from left to right. Typically, the first non-zero element in each column is chosen as the pivot (one row below the row of the pivot we just used). The pivots are used to eliminate the elements below them. The elimination matrix E_{ij} eliminates the element a_{ij} in the ith row and jth column.

Example:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix}, \quad b = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$$

Step 1 The first pivot is $a_{11} = 2$. Eliminate the elements below it.

Eliminate a_{21} $$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \quad E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 50 \end{bmatrix}$$

Eliminate a_{31} $$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix} \quad E_{31}E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 31 \end{bmatrix}$$

Step 2 The second pivot is $a_{22} = 5$. Eliminate the elements below it.

Eliminate a_{32} $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ $E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$ $E_{32}E_{31}E_{21}b = \begin{bmatrix} 19 \\ 17 \\ 7 \end{bmatrix}$

Step 3 Solve $Ux = c$ by back substitution.

Now we have $U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$ and $c = \begin{bmatrix} 19 \\ 17 \\ 7 \end{bmatrix}$. We can go from the bottom up to solve for x.

An easy way to come up with elimination matrix

To go from $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix}$ to $B = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 5 & 13 \end{bmatrix}$, we need to come up with E_{31}:

1. For the first row in E_{31}:

A_{row_1} is $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$, B_{row1} is $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$. We take $1 \times A_{row_1}$, $0 \times A_{row_2}$, and $0 \times A_{row_3}$. So the first row of E_{31} is $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

2. For the second row in E_{31}:

A_{row_2} is $\begin{bmatrix} 0 & 5 & 6 \end{bmatrix}$, B_{row2} is $\begin{bmatrix} 0 & 5 & 6 \end{bmatrix}$. We take $0 \times A_{row_1}$, $1 \times A_{row_2}$, and $0 \times A_{row_3}$. So the second row of E_{31} is $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

3. For the third row in E_{31}:

A_{row_3} is $\begin{bmatrix} 2 & 8 & 17 \end{bmatrix}$, B_{row3} is $\begin{bmatrix} 0 & 5 & 13 \end{bmatrix}$. We take $-1 \times A_{row_1}$, $0 \times A_{row_2}$, and $1 \times A_{row_3}$. So the third row of E_{31} is $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$.

4. Put it togeter:

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Permutation Matrix P When zero appears in a pivot position, we can exange rows to bring a nonzero element to that position.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 5 & 13 \end{bmatrix} = B$$

Exchange row 2 with row 3:

$$PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 5 & 13 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 13 \\ 0 & 0 & 6 \end{bmatrix} = U$$

Properties of Permutation Matrix P:

- P has a 1 in each row and a 1 in each column. All other entries are 0
- If A is invertible, then $PA = LU$
- $P^T = P^{-1}$

Row Permutation P and Column Permutation Q Start with a 3 by 3 matrix A. Reorder its rows 1, 2, 3 by P:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad PA = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Then reorder its columns by Q in the order 3, 2, 1:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad PAQ = \begin{bmatrix} a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Augmented Matrix $[A \ b]$ To make sure the operations on the matrix A are also applied to the vector b, we can combine them into an augmented matrix $[A\ b]$.

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 19 \\ 4 & 11 & 14 & 55 \\ 2 & 8 & 17 & 50 \end{bmatrix} \xrightarrow{E} \begin{bmatrix} 2 & 3 & 4 & 19 \\ 0 & 5 & 6 & 17 \\ 0 & 0 & 7 & 7 \end{bmatrix} = \begin{bmatrix} U & c \end{bmatrix}$$

2.3 Zero on the Diagonal of U^* and Dependence

$$U^* = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \mathbf{0} & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

The entry 0 in row 3, column 3 indicates:

- The first three columns are dependent.
- The last two rows are dependent.

2.4 Inverse Matrix A^{-1}

A is a square matrix. It needs n independent columns to have inverse matrix A^{-1}.

2 by 2 Inverse $\qquad A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The number $ad - bc$ is called the **determinant** of the matrix.

Suppose there is a nonzero vector x such that $Ax = 0$, then A has dependent columns. It cannot have an inverse.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To solve $Ax = 0$, we get $x_1 = -2x_2$. That means for any nonzero x_2, we can solve for x_1.

Inverse of a Product

$$(AB)^{-1} = B^{-1}A^{-1}$$

Similarly:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

2.5 Gause-Jordan Elimination

Solve $AX = I \rightarrow X = A^{-1}$. Slower than solving $Ax = b$. We have a matrix $[A\ I]$ and apply elimination to get $[I\ A^{-1}]$.

Suppose we have $A = \begin{bmatrix} 1 & 9 & 3 \\ 0 & 3 & 6 \\ 2 & 6 & 9 \end{bmatrix}$, then $\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 9 & 3 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 & 1 & 0 \\ 2 & 6 & 9 & 0 & 0 & 1 \end{array}\right]$

Step 1: Make zeros below the leading 1 in column 1

$$row_3 = row_3 - 2 \times row_1 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 9 & 3 & 1 & 0 & 0 \\ 0 & 3 & 6 & 0 & 1 & 0 \\ 0 & -12 & 3 & -2 & 0 & 1 \end{array}\right]$$

Step 2: Make leading 1 in column 2

$$row_2 = \frac{1}{3} row_2 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 9 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & -12 & 3 & -2 & 0 & 1 \end{array}\right]$$

Step 3: Make zeros above and below leading 1 in column 2

$$row_1 = row_1 - 9 \times row_2 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -15 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & -12 & 3 & -2 & 0 & 1 \end{array}\right]$$

$$row_3 = row_3 + 12 \times row_2 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -15 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & 0 & 27 & -2 & 4 & 1 \end{array}\right]$$

Step 4: Make leading 1 in column 3

$$row_3 = 1/27 \times row_3 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -15 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -2/27 & 4/27 & 1/27 \end{array}\right]$$

Step 5: Make zeros above leading 1 in column 3

$$row_1 = row_1 + 15 \times row_3 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & -7/9 & 5/9 \\ 0 & 1 & 2 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -2/27 & 4/27 & 1/27 \end{array}\right]$$

$$row_2 = row_2 - 2 \times row_3 \quad \rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & -7/9 & 5/9 \\ 0 & 1 & 0 & 4/27 & 1/27 & -2/27 \\ 0 & 0 & 1 & -2/27 & 4/27 & 1/27 \end{array}\right]$$

Then we get $A^{-1} = \begin{bmatrix} -1/9 & -7/9 & 5/9 \\ 4/27 & 1/27 & -2/27 \\ -2/27 & 4/27 & 1/27 \end{bmatrix}$.

2.6 Lower Triangular Matrix L

L is the Inverse of E

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ (l_{32}l_{21} - l_{31}) & -l_{32} & 1 \end{bmatrix}$$

Reverse order looks more beautiful:

$$E^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} = L$$

All the multiplier l_{ij} appear in their correct positions in L. This remains true for all matrix sizes.

To prove $A = LU$:

$$EA = U \rightarrow E^{-1}EA = E^{-1}U \rightarrow IA = E^{-1}U \rightarrow A = E^{-1}U = LU$$

2.7 Transpose Matrix A^T

Sum	The transpose of $A + B$ is	$A^T + B^T$
Product	The transpose of AB is	$B^T A^T$
Inverse	The transpose of A^{-1} is	$(A^T)^{-1}$

To prove $(AB)^T = B^T A^T$:

- Start with $(Ax)^T = x^T A^T$ when B is just a vector. Ax **combines the columns of** A **while** $x^T A^T$ **combines the rows of** A^T. So the transpose of the column Ax is the row $x^T A^T$.

- Suppose B has columns x_1, x_2, …, the columns of AB are Ax_1, Ax_2, …, $AB = \begin{bmatrix} Ax_1 & Ax_2 & \ldots \end{bmatrix}$. Transposing AB gives $\begin{bmatrix} x_1^T A^T \\ x_2^T A^T \\ \vdots \end{bmatrix}$ which is $B^T A^T$.

To prove $(A^{-1})^T = (A^T)^{-1}$:

- Start with $A^{-1}A = I$.
- $(A^{-1}A)^T = I^T = I \quad \to \quad A^T(A^{-1})^T = I \quad \to \quad (A^{-1})^T = (A^T)^{-1}$.

Transpose & Inner product The dot product of x and y is the sum of numbers $x_i y_j$. We can write it using matrix notation $x \cdot y = x^T y$ (both x and y are written as $n \times 1$ matrices).

$$(Ax)^T y = x^T(A^T y) \quad \text{Inner product of } Ax \text{ with } y = \text{Inner product of } x \text{ with } A^T y$$

2.8 Symmetric Matrix S

A symmetric matrix is equal to its transpose: $S = S^T$.

For any matrix A, the product of $A^T A$ is a **square symmetric** matrix: $(A^T A)^T = A^T (A^T)^T = A^T A$.

Symmetric matrices in elimination $S = LDL^T$ D is the diagonal matrix of pivots. L is the lower triangular matrix of multipliers.

$$S = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = LU$$
$$S = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^T$$