Linear Algebra Lecture Notes

October 17, 2024

Contents

Chapter 1

Vectors and Matrices

1.1 Vectors and Linear Combination

Linear Combination Vectors v and w are both 2D vectors. The linear combination of v and w are the vectors $cv + dw$ for any scalars c and d.:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The linear combinations $c \begin{bmatrix} 2 \\ 4 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + 1d \\ 4c + 3d \end{bmatrix}$ form xy plane.

v and w are **linearly independent**. There is exactly one solution b_1, b_2.

The 2 by 2 matrix $A = \begin{bmatrix} v & w \end{bmatrix}$ is **invertible**.

Column Way, Row Way, Matrix Way
Column way, Linear combination:

$$c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + d \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Row way, Two equations for c and d:

$$v_1 c + w_1 d = b_1, \quad v_2 c + w_2 d = b_2$$

Matrix way, 2 by 2 matrix:

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vectors in 3D We need *three* independent vectors to span 3D space R^3.

Identity Matrix I: denoted by I_n for an nxn identity matrix, where n is the number of rows or columns.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying any matrix by I leaves the matrix unchanged. I is the matrix

1.2 Length and Angles from Dot Products

Dot Product The dot product of two vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is $v \cdot w = v_1 w_1 + v_2 w_2 = w \cdot v$.

Unit Vector A unit vector is a vector with length 1. The unit vector in the direction of v is $\dfrac{v}{\|v\|}$.

Perpendicular Vectors Two vectors v and w are perpendicular if $v \cdot w = 0$.

$$\|v+w\|^2 = (v+w)\cdot(v+w) = v\cdot v + 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$

$$\|v-w\|^2 = (v-w)\cdot(v-w) = v\cdot v - 2v\cdot w + w\cdot w = \|v\|^2 + \|w\|^2$$

Angle between Vectors The angle between two vectors v and w is $\theta = \cos^{-1}\left(\frac{v\cdot w}{\|v\|\|w\|}\right)$.

Example 1. The unit vectors $v = (\cos\alpha, \sin\alpha)$ and $w = (\cos\beta, \sin\beta)$ have $v \cdot w = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. In trigonometry, this is the formula for $\cos(\alpha-\beta)$ or $\cos(\beta-\alpha)$.

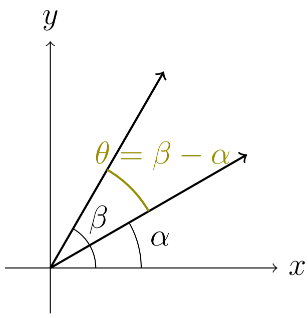

Figure 1.1: Visualization of $\cos(\beta-\alpha) = \cos(\theta)$ in the unit circle.

For unit vectors, $\cos\theta = v \cdot w$. When v and w are not unit vectors, divide by their length to get $u = v/\|v\|$ and $U = u/\|u\|$ and turn them into unit vectors.