Session 20: Abstract Formulation II (with Solutions)

Q1 (Transportation Planning)

Your software company has launched a new Analytics product. As sales manager, you are planning to promote the product by sending salesforce to software conventions running concurrently in Los Angeles, Saint Louis, and Detroit.

You have 6 representatives available at each of your Little Rock and Urbana Branches. You would like to send at least 2 to the Los Angeles convention, 5 to the Saint Louis convention, and at least 4 to the Detroit convention.

Roundtrip airfare between the locations are as follows:

	Los Angeles	St. Louis	Detroit
Little Rock	250	150	200
Urbana	300	200	150

Formulate an optimization problem to allocate your sales force so as to minimize total airfare, then generalize the formulation to be able to handle arbitrary number of office branches, conventions cities, availability of representatives, requirement for conventions, and transportation cost.

Step 1. Identify the decision, objective, and constraints in English

Decision: How many representative to send from each branch to each convention.
Objective: Minimize total airfare.
Constraints:

- (Supply) Not use more representatives than available at each branch.
- (Demand) Send enough representives to each convention.

Step 2. Formulate the optimization as linear expressions of decision variables

Denote Little Rock and Urbana by 1 and 2 respectively, and the three cities by L, S, and D respectively.
Decision variables: Let X_{ij} denote how many representatives to send from branch $i \in \{1,2\}$ to convention $j \in \{L,S,D\}$. (Integer)
Objective and constraints:

$$\text{Min} \quad 250X_{1L} + 150X_{1S} + 200X_{1D} + 300X_{2L} + 200X_{2S} + 150X_{2D}$$

$$\text{s.t.}$$

$$\text{(Supply Little Rock)} \quad X_{1L} + X_{1S} + X_{1D} \leq 6$$
$$\text{(Supply Urbana)} \quad X_{2L} + X_{2S} + X_{2D} \leq 6$$
$$\text{(Demand LA)} \quad X_{1L} + X_{2L} \geq 2$$
$$\text{(Demand St. Louis)} \quad X_{1S} + X_{2S} \geq 5$$
$$\text{(Demand Detroit)} \quad X_{1D} + X_{2D} \geq 4$$
$$\text{(Non-negativity)} \quad X_{ij} \geq 0 \quad \text{for all } i \text{ and } j.$$

Step 3a. Use variables to encode all input data

Data:

- I: the set of office branches.
- J: the set of conventions.
- s_i: the number of available representatives at office branch $i \in I$.
- d_j: the number of representatives needed at convention $j \in J$.
- c_{ij}: the roundtrip airfare from branch i to convention j.

Step 3b. Formulate the LP/MIP in terms of only data and decision variables

Decision variables: Let X_{ij} denote how many representatives to send from branch $i \in I$ to convention $j \in J$. (Integer)
Objective and constraints:

$$\text{Minimize} \quad \sum_{i \in I, j \in J} c_{ij} x_{ij}$$

$$\text{s.t.}$$

$$\text{(Supply)} \quad \sum_{j \in J} x_{ij} \leq s_i \quad \text{for each branch } i \in I,$$

$$\text{(Demand)} \quad \sum_{i \in I} x_{ij} \geq d_j \quad \text{for each convention } j \in J,$$

$$\text{(Non-negativity)} \quad x_{ij} \geq 0 \quad \text{for each } i \in I, j \in J.$$

Condensed Notation

The following is also correct, if we write in words before hand that every summation over i is taken over the set I and every summation over j is taken over J.

$$\text{Minimize} \quad \sum_{i,j} c_{ij} x_{ij}$$

$$\text{s.t.}$$

$$\text{(Supply)} \quad \sum_{j} x_{ij} \leq s_i \quad \text{for each branch } i,$$

$$\text{(Demand)} \quad \sum_{i} x_{ij} \geq d_j \quad \text{for each convention } j,$$

$$\text{(Non-negativity)} \quad x_{ij} \geq 0 \quad \text{for each } i \text{ and } j.$$

Q2 (Optimal Debt Payments)

Paris has come to you because she needs help paying off her credit card bills. Her statement at the beginning of month 1 shows the following balances:

Credit Card	Balance	Monthly Rate
Saks Fifth Avenue	$20,000	0.5%
Bloomingdale's	$50,000	1.0%
Macy's	$40,000	1.5%

Paris has agreed not to shop at any of these stores anymore, and she is willing to allocate up to 5,000 dollars per month to pay off these credit cards. All cards must be paid off within 36 months (meaning that her statement at the beginning of month 37 must be zero for all card). Paris' goal is to minimize the total of all her payments.

For this problem, assume that the interest for the month is applied after the payment for that month. For example, suppose Paris pays 5,000 on Saks for month 1. Then her Saks balance at

the beginning of month 2 is $(1.005) \times (20000 - 5000) = 15075$.

Help Paris solve her problem by formulating it into a linear program, then generalize it to be able to handle arbitrary number of credit cards, balances, monthly rates, monthly budget, and time required for full payment.

Step 1. Identify the decision, objective, and constraints in English

Decision: How much to pay for each card in each month.
Objective: Minimize total payment over all 36 months.
Constraints:

- Total amount paid in each month no more than 5000.
- Paying off all cards by month 37.
- Cashflow equation governing the relationship between the balance this month, payment this month, and balance next month.

Step 2. Formulate the optimization as linear expressions of decision variables

Decision variables:

- Let x_{it} denote how much to pay card $i \in \{S, B, M\}$ in month $t \in \{1, 2, \cdots, 36\}$.
- Let y_{it} denote the balance for card i at the beginning of month $t \in \{1, 2, \cdots, 37\}$.

Objective and constraints:

$$\text{Minimize: } \sum_{it} x_{it}$$

$$x_{St} + x_{Bt} + x_{Mt} \leq 5000 \quad \text{for each month } 1 \leq t \leq 36.$$

$$y_{S1} = 20000$$

$$y_{B1} = 50000$$

$$y_{M1} = 40000$$

$$y_{S(37)}, y_{B(37)}, y_{M(37)} = 0$$

$$(1 + 0.005)(y_{S1} - x_{S1}) = y_{S2}$$

$$(1 + 0.005)(y_{S2} - x_{S2}) = y_{S3}$$

$$\cdots$$

$$(1 + 0.01)(y_{B1} - x_{B1}) = y_{B2}$$

$$\cdots$$

$$(1 + 0.015)(y_{M1} - x_{M1}) = y_{M2}$$

$$\cdots$$

$$x_{it}, y_{it} \geq 0 \quad \text{for each } i \text{ and } t.$$

Step 3a. Use variables to encode all input data

Data:

- I: set of credit cards.
- n: number of months in which to pay off all cards.
- $T = \{1, \cdots, n\}$: the set of months.
- B_i: the balance of card i in month 1.
- C: the total cash available to pay the credit cards each month.
- r_i: monthly interest rate for each card i.

Step 3b. Formulate the LP/MIP in terms of only data and decision variables

Decision variable:

- Let x_{it} denote how much to pay card $i \in I$ in this month $t \in T$ (continuous);
- let y_{it} denote the balance for card $i \in I$ in the beginning of month $t \in \{1, \cdots, n+1\}$ (continuous).

Objective and constraints:

$$\text{Minimize:} \quad \sum_{i \in I, t \in T} x_{it}$$

subject to:

(Limit on payments)	$\sum_{i \in I} x_{it} \leq C$	for each month $t \in T$.
(Beginning balance)	$y_{i1} = B_i$	for each card $i \in I$.
(Paying off at the end)	$y_{in+1} = 0$	for each card $i \in I$.
(Credit card accounting)	$(1 + r_i)(y_{it} - x_{it}) = y_{it+1}$	for each card $i \in I$ and each month $t \in T$.
(Non-negativivity)	$x_{it}, y_{it} \geq 0$	for each card $i \in I$ and each month $t \in T$.