Beyond Outerplanarity*

Steven Chaplick¹, Myroslav Kryven¹, Giuseppe Liotta², Andre Löffler¹, and Alexander Wolff¹**

Lehrstuhl für Informatik I, Universität Würzburg, Germany.
 www1.informatik.uni-wuerzburg.de/en/staff
 Department of Engineering, University of Perugia, Italy.
 giuseppe.liotta@unipg.it

Abstract. We study straight-line drawings of graphs where the vertices are placed in convex position in the plane, i.e., convex drawings. We consider two families of graph classes with nice convex drawings: outer k-planar graphs, where each edge is crossed by at most k other edges; and, outer k-quasi-planar graphs where no k edges can mutually cross. We show that the outer k-planar graphs are $(\lfloor \sqrt{4k+1} \rfloor + 1)$ -degenerate, and consequently that every outer k-planar graph can be $(\lfloor \sqrt{4k+1} \rfloor + 2)$ -colored, and this bound is tight. We further show that every outer k-planar graph has a balanced separator of size at most 2k+3. For each fixed k, these small balanced separators allow us to test outer k-planarity in quasi-polynomial time, i.e., none of these recognition problems are NP-hard unless ETH fails.

For the outer k-quasi-planar graphs we discuss the edge-maximal graphs which have been considered previously under different names. We also construct planar 3-trees that are not outer 3-quasi-planar.

Finally, we restrict outer k-planar and outer k-quasi-planar drawings to closed drawings, where the vertex sequence on the boundary is a cycle in the graph. For each k, we express closed outer k-planarity and closed outer k-quasi-planarity in extended monadic second-order logic. Thus, since outer k-planar graphs have bounded treewidth, closed outer k-planarity is linear-time testable by Courcelle's Theorem.

1 Introduction

A drawing of a graph maps each vertex to a distinct point in the plane, each edge to a Jordan curve connecting the points of its incident vertices but not containing the point of any other vertex, and two such Jordan curves have at most one common point. In the last few years, the focus in graph drawing has shifted from exploiting structural properties of planar graphs to addressing the question of how to produce well-structured (understandable) drawings in the presence of edge crossings, i.e., to the topic of beyond-planar graph classes. The primary approach here has been to define and study graph classes which allow some edge crossings, but restrict the crossings in various ways. Two commonly studied such graph classes are:

^{*} In Proc. 25th Int. Symp. on Graph Drawing and Network Visualization (GD 2017).

^{**} ORCID: orcid.org/0000-0001-5872-718X

- 1. k-planar graphs, the graphs which can be drawn so that each edge (Jordan curve) is crossed by at most k other edges.
- 2. k-quasi-planar graphs, the graphs which can be drawn so that no k pairwise non-incident edges mutually cross.

Note that the 0-planar graphs and 2-quasi-planar graphs are precisely the planar graphs. Additionally, the 3-quasi-planar graphs are simply called *quasi-planar*.

In this paper we study these two families of classes of graphs by restricting the drawings so that the points are placed in convex position and edges mapped to line segments, i.e., we apply the above two generalizations of planar graphs to outerplanar graphs and study *outer k-planarity* and *outer k-quasi-planarity*. We consider balanced separators, treewidth, degeneracy (see paragraph "Concepts" below), coloring, edge density, and recognition for these classes.

Related work. Ringel [27] was the first to consider k-planar graphs by showing that 1-planar graphs are 7-colorable. This was later improved to 6-colorable by Borodin [8]. This is tight since K_6 is 1-planar. Many additional results on 1-planarity can be found in a recent survey paper [21]. Generally, each n-vertex k-planar graph has at most $4.108n\sqrt{k}$ edges [26] and treewidth $O(\sqrt{kn})$ [14].

Outer k-planar graphs have been considered mostly for $k \in \{0,1,2\}$. Of course, the outer 0-planar graphs are the classic outerplanar graphs which are well-known to be 2-degenerate and have treewidth at most 2. It was shown that essentially every graph property is testable on outerplanar graphs [5]. Outer 1-planar graphs are a simple subclass of planar graphs and can be recognized in linear time [4,18]. Full outer 2-planar graphs, which form a subclass of outer 2-planar graphs, can been recognized in linear time [19]. General outer k-planar graphs were considered by Binucci et al. [7], who (among other results) showed that, for every k, there is a 2-tree which is not outer k-planar. Wood and Telle [30] considered a slight generalization of outer k-planar graphs in their work and showed that these graphs have treewidth O(k).

The k-quasi-planar graphs have been heavily studied from the perspective of edge density. The goal here is to settle a conjecture of Pach et al. [25] stating that every n-vertex k-quasi-planar graph has at most $c_k n$ edges, where c_k is a constant depending only on k. This conjecture is true for k=3 [2] and k=4 [1]. The best known upper bound is $(n \log n)2^{\alpha(n)^{c_k}}$ [16], where α is the inverse of the Ackermann function. Edge density was also considered in the "outer" setting: Capoyleas and Pach [9] showed that any k-quasi-planar graph has at most $2(k-1)n-\binom{2k-1}{2}$ edges, and that there are k-quasi-planar graphs meeting this bound. More recently, it was shown that the semi-bar k-visibility graphs are outer (k+2)-quasi-planar [17]. However, the outer k-quasi-planar graph classes do not seem to have received much further attention.

The relationship between k-planar graphs and k-quasi-planar graphs was considered recently. While any k-planar graph is clearly (k+2)-quasi-planar, Angelini et al. [3] showed that any k-planar graph is even (k+1)-quasi-planar.

The *convex* (or 1-page book) crossing number of a graph [29] is the minimum number of crossings which occur in any convex drawing. This concept has been

introduced several times (see [29] for more details). The convex crossing number is NP-complete to compute [23]. However, recently Bannister and Eppstein [6] used treewidth-based techniques (via extended monadic second order logic) to show that it can be computed in linear FPT time, i.e., $O(f(c) \cdot n)$ time where c is the convex crossing number and f is a computable function. Thus, for any k, the outer k-crossing graphs can be recognized in time linear in n+m.

Concepts. We briefly define the key graph theoretic concepts that we will study. A graph is d-degenerate when every subgraph of it has a vertex of degree at most d. This concept was introduced as a way to provide easy coloring bounds [22]. Namely, a d-degenerate graph can be inductively d+1 colored by simply removing a vertex of degree at most d. A graph class is d-degenerate when every graph in the class is d-degenerate. Furthermore, a graph class which is d-degenerate when every graph in that class has a vertex of degree at most d. Note that outerplanar graphs are 2-degenerate, and planar graphs are 5-degenerate.

A separation of a graph G is pair A,B of subsets of V(G) such that $A \cup B = V(G)$, and no edge of G has one end in $A \setminus B$ and the other in $B \setminus A$. The set $A \cap B$ is called a separator and the size of the separation (A,B) is $|A \cap B|$. A separation (A,B) of a graph G on n vertices is balanced if $|A \setminus B| \leq \frac{2n}{3}$ and $|B \setminus A| \leq \frac{2n}{3}$. The separation number of a graph G is the smallest number s such that every subgraph of G has a balanced separation of size at most s. The treewidth of a graph was introduced by Robertson and Seymour [28]; it is closely related to the separation number. Namely, any graph with treewidth t has separation number at most t+1 and, as Dvořák and Norin [15] recently showed, any graph with separation number s has treewidth at most s foraphs with bounded treewidth are well-known due to Courcelle's Theorem (see Theorem 6) [10], i.e., having bounded treewidth means many problems can be solved efficiently.

The Exponential Time Hypothesis (ETH) [20] is a complexity theoretic assumption defined as follows. For $k \geq 3$, let $s_k = \inf\{\delta : \text{there is an } O(2^{\delta n})\text{-time}$ algorithm to solve k-SAT}. ETH states that for $k \geq 3$, $s_k > 0$, e.g., there is no quasi-polynomial time³ algorithm that solves 3-SAT. So, finding a problem that can be solved in quasi-polynomial time and is also NP-complete, would contradict ETH. In recent years, ETH has become a standard assumption from which many conditional lower bounds have been proven [12].

Contribution. In Section 2, we consider outer k-planar graphs. We show that they are $(\lfloor \sqrt{4k+1}\rfloor + 1)$ -degenerate, and observe that the largest outer k-planar clique has size $(\lfloor \sqrt{4k+1}\rfloor + 2)$, i.e., implying each outer k-planar graph can be $(\lfloor \sqrt{4k+1}\rfloor + 2)$ -colored and this is tight. We further show that every outer k-planar graph has separation number at most 2k+3. For each fixed k, we use these balanced separators to obtain a quasi-polynomial time algorithm to test outer k-planarity, i.e., these recognition problems are not NP-hard unless ETH fails.

³ i.e., with a runtime of the form $2^{\text{poly}(\log n)}$.

In Section 3, we consider outer k-quasi-planar graphs. Specifically, we discuss the edge-maximal graphs which have been considered previously under different names [9,13,24]. We also relate outer k-quasi-planar graphs to planar graphs.

Finally, in Section 4, we restrict outer k-planar and outer k-quasi-planar drawings to closed drawings, where the sequence of vertices on the outer boundary is a cycle. For each k, we express both closed outer k-planarity and closed outer k-planarity in extended monadic second-order logic. Thus, closed outer k-planarity is testable in $O(f(k) \cdot n)$ time, for a computable function f.

2 Outer k-Planar Graphs

In this section we show that every outer k-planar graph is $O(\sqrt{k})$ -degenerate and has separation number O(k). This provides tight bounds on the chromatic number, and allows for testing outer k-planarity in quasi-polynomial time.

Degeneracy. We show that every outer k-planar graph has a vertex of degree at most $\sqrt{4k+1}+1$. First we note the size of the largest outer k-planar clique and then we prove that each outer k-planar graph has a vertex matching the clique's degree. This also tightly bounds the chromatic number in terms of k, i.e., Theorem 1 follows from Lemma 1 (proven in Appendix B.1) and Lemma 2.

Lemma 1. Every outer k-planar clique has at most $|\sqrt{4k+1}| + 2$ vertices.

Lemma 2. An outer k-planar graph can have maximum minimum degree at most $\sqrt{4k+1}+1$ and this bound is tight.

Proof. Assume that the outer k-planar graph has maximum minimum degree δ . Since we can create a clique with $\lfloor \sqrt{4k+1} \rfloor + 2$ vertices (see Lemma 1), $\delta \geq \lfloor \sqrt{4k+1} \rfloor + 1$. Let us show that δ cannot be larger than $\sqrt{4k+1} + 1$.

Consider an edge ab that cuts $l \in \mathbb{N}$ vertices of the graph to one side (not counting a and b), then there are at least $\delta l - l(l+1)$ edges crossing the edge ab. We will now show by induction that if there existed an outer k-planar graph with minimum degree $\delta \geq \sqrt{4k+1}+2$, it would be too small to accommodate such a minimum degree vertex.

Any edge ab that cuts l vertices is crossed by at least $\delta l - l(l+1)$ edges. Therefore, if $\delta \geq \sqrt{4k+1}+2$, there is l^* such that ab cannot cut $l^* \geq \frac{1}{2}(\delta-1-\sqrt{(\delta-1)^2-4(k+1)})$ vertices because then it is crossed by $\delta l^*-l^*(l^*+1) \geq k+1$ edges. Take the smallest such l^* and let us show that there also cannot be an edge ab that cuts more than l^* vertices. As the induction hypothesis, assume that no edge ab cuts between l^* and l vertices inclusive. Thus, the minimum number of edges that cross ab is: $\delta l - l(l+1) + 2(\sum_{j=1}^{l-l^*} j) > k$, where the last term accounts for the absent edges that cut more than $l-l^*$ vertices. Now, if ab cuts l+1 vertices, it is crossed by

$$\geq \delta l - l(l+1) + 2(\sum_{j=1}^{l-l^*} j) + \delta - 2(l+1) + 2(l-l^*+1) > k + \delta - 2(l+1) + 2(l-l^*+1) > k$$

edges if $\delta > 2l^*$.

Since for $\delta > \sqrt{4k+1}+2$ the inequality is always satisfied, there cannot be an edge that cuts more then $l^* < \sqrt{4k+1}/2$ vertices in any outer k-planar graph with the maximum minimum degree $\delta \geq \sqrt{4k+1}+2$. But then, such a graph can have at most $2l^* < \sqrt{4k+1}$ vertices, which is not enough to accommodate the minimum degree vertex required; a contradiction.

Theorem 1. Each outer k-planar graph is $\sqrt{4k+1}+2$ colorable. This is tight.

Quasi-polynomial time recognition via balanced separators. We show that outer k-planar graphs have separation number at most 2k+3 (Theorem 2). Via a result of Dvořák and Norin [15], this implies they have O(k) treewidth. However, Proposition 8.5 of [30] implies that every outer k-planar graph has treewidth at most 3k+11, i.e., a better bound on the treewidth than applying the result of Dvořák and Norin to our separators. The treewidth 3k+11 bound also implies a separation number of 3k+12, but our bound is better. Our separators also allow outer k-planarity testing in quasi-polynomial time (Theorem 3).

Theorem 2. Each outer k-planar graph has separation number at most 2k + 3.

Proof. Consider an outer k-planar drawing. If the graph has an edge that cuts $\left[\frac{n}{3}, \frac{2n}{3}\right]$ vertices to one side, we can use this edge to obtain a balanced separator of size at most k+2, i.e., by choosing the endpoints of this edge and a vertex cover of the edges crossing it. So, suppose no such edge exists. Consider a pair of vertices (a, b) such that the line ab divides the drawing into left and right sides having an almost equal number of vertices (with a difference at most one). If the edges which cross the line ab also mutually cross each other, there can be at most k of them. Thus, we again have a balanced separator of size at most k+2. So, it remains to consider the case when we have a pair of edges that cross the line ab, but do not cross each other. We call such a pair of edges parallel. We now pick a pair of parallel edges in a specific way. Starting from b, let b_l be the first vertex along the boundary in clockwise direction such that there is an edge $b_l b'_l$ that crosses the line ab. Symmetrically, starting from a, let a_r be the first vertex along the boundary in clockwise direction such that there is an edge $a_r a'_r$ that crosses the line ab; see Fig. 1 (left). Note that the edges $a_r a'_r$ and $b_l b'_l$ are either identical or parallel. In the former case, we see that all other edges crossing the line ab must also cross the edge $a_r a'_r = b_l b'_l$, and as such there are again at most k edges crossing the line ab. In the latter case, there are two subcases that we treat below. For two vertices u and v, let [u,v] be the set of vertices that starts with u and, going clockwise, ends with v. Let $(u, v) = [u, v] \setminus \{u, v\}$.

Case 1. The edge b_lb_l' cuts $\mu \leq \frac{n}{3}$ vertices to the top; see Fig. 1 (center). In this case, either $[b_l', b]$ or $[b, b_l]$ has $[\frac{n}{3}, \frac{n}{2}]$ vertices. We claim that neither the line bb_l nor the line bb_l' can be crossed more than k times. Namely, each edge that crosses the line bb_l also crosses the edge b_lb_l' . Similarly, each edge that crosses the line bb_l' also crosses the edge b_lb_l' . Thus, we have a separator of size at most k+2, regardless of whether we choose bb_l or bb_l' to separate the graph. As we observed above, one of them is balanced.

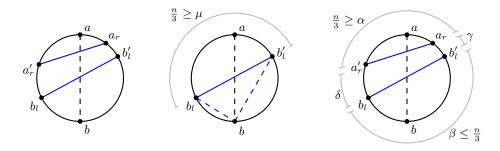


Fig. 1: Left: the pair of parallel edges $b_l b'_l$ and $a_r a'_r$; center: case 1; right: case 2

Case 1'. The edge $a_r a'_r$ cuts at most $\frac{n}{3}$ vertices to the bottom. This is symmetric to case 1.

Case 2. The edge $b_lb'_l$ cuts at most $\frac{n}{3}$ vertices to the bottom, and the edge $a_ra'_r$ cuts at most $\frac{n}{3}$ vertices to the top; see Fig. 1 (right).

We show that we can always find a pair of parallel edges such that one cuts at most $\frac{n}{3}$ vertices to the bottom and the other cuts at most $\frac{n}{3}$ vertices to the top, and no edge between them is parallel to either of them. We call such a pair close. If there is an edge e between $b_lb'_l$ and $a_ra'_r$, we form a new pair by using e and $a_ra'_r$ if e cuts at most $\frac{n}{3}$ vertices to the bottom or by using e and $b_lb'_l$ if e cuts at most $\frac{n}{3}$ vertices to the top. By repeating this procedure, we always find a close pair. Hence, we can assume that $b_lb'_l$ and $a_ra'_r$ actually form a close pair. Let $\alpha = |(a'_r, a_r)|, \beta = |(b'_l, b_l)|, \gamma = |(a_r, b'_l)|, \text{ and } \delta = |(b_l, a'_r)|$; see Fig. 1 (right).

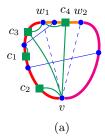
Suppose that $a'_r = b_l$ or $a_r = b'_l$. We can now use both edges $b_l b'_l$ and $a_r a'_r$ (together with any edges crossing them) to obtain a separator of size at most 2k+3. The separator is balanced since $\alpha+\beta \leq \frac{2n}{3}$ and $\gamma+\delta \leq \frac{2n}{3}$. So, now a_r, a'_r, b_l, b'_l are all distinct. Note that $\gamma, \delta \leq \frac{n}{2}$ since each side of

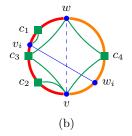
So, now a_r, a'_r, b_l, b'_l are all distinct. Note that $\gamma, \delta \leq \frac{n}{2}$ since each side of the line ab has at most $\frac{n}{2}$ vertices. We separate the graph along the line $b_l a_r$. Namely, all the edges that cross this line must also cross $b_l b'_l$ or $a'_r a_r$. Therefore, we obtain a separator of size at most 2k + 2.

To see that the separator is balanced, we consider two cases. If $\delta \geq \frac{n}{3}$ (or $\gamma \geq \frac{n}{3}$), then $\alpha + \beta + \gamma \leq \frac{2n}{3}$ (or $\alpha + \beta + \delta \leq \frac{2n}{3}$). Otherwise $\delta < \frac{n}{3}$ and $\gamma < \frac{n}{3}$. In this case $\delta + \alpha \leq \frac{2n}{3}$ and $\gamma + \beta \leq \frac{2n}{3}$. In both cases the separator is balanced.

Theorem 3. For fixed k, testing the outer k-planarity of an n-vertex graph takes $O(2^{\text{polylog }n})$ time.

Proof. Our approach is to leverage the structure of the balanced separators as described in the proof of Theorem 2. Namely, we enumerate the sets which could correspond to such a separator, pick an appropriate outer k-planar drawing of these vertices and their edges, partition the components arising from this separator into regions, and recursively test the outer k-planarity of the regions.





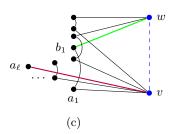


Fig. 2: Shapes of separators, special separator S in blue, regions in different colors (red, orange, and pink), components connected to blue vertices in green: (a) closest-parallels case; (b) single-edge case; (c) special case for single-edge separators.

To obtain quasi-polynomial runtime, we need to limit the number of components on which we branch. To do so, we group them into regions defined by special edges of the separators.

By the proof of Theorem 2, if our input graph has an outer k-planar drawing, there must be a separator which has one of the two shapes depicted in Fig. 2 (a) and (b). Here we are not only interested in the up to 2k + 3 vertices of the balanced separator, but actually the set S of up to 4k + 3 vertices one obtains by taking both endpoints of the edges used to find the separator. Note: S is also a balanced separator. We use a brute force approach to find such an S. Namely, we first enumerate vertex sets of size up to 4k + 3. We then consider two possibilities, i.e., whether this set can be drawn similar to one of the two shapes from Fig. 2. So, we now fix this set S. Note that since S has O(k) vertices, the subgraph G_S induced by S can have at most a function of k different outer k-planar drawings. Thus, we further fix a particular drawing of G_S .

We now consider the two different shapes separately. In the first case, in S, we have three special vertices v, w_1 and w_2 and in the second case we will have two special vertices v and v. These vertices will be called boundary vertices and all other vertices in S will be called regional vertices. Note that, since we have a fixed drawing of G_S , the regional vertices are partitioned into regions by the specially chosen boundary vertices. Now, from the structure of the separator which is guaranteed by the proof of Theorem 2, no component of S0 can be adjacent to regional vertices which live in different regions with respect to the boundary vertices.

We first discuss the case of using G_S as depicted in Fig. 2 (a). Here, we start by picking the three special vertices v, w_1 and w_2 from S to take the role as shown in Fig. 2 (a). The following arguments regarding this shape of separator are symmetric with respect to the pair of opposing regions.

Notice that if there is a component connected to regional vertices of different regions, we can reject this configuration. From the proof of Theorem 2, we further observe that no component can be adjacent to all three boundary vertices. Namely, this would contradict the closeness of the parallel edges or it would contradict the members of the separator, i.e., it would imply an edge connecting distinct regions. We now consider the four possible different types of components c_1, c_2, c_3 and c_4 in Fig. 2 (a) that can occur in a region neighboring w_1 . Components of type c_1 are connected to (possibly many) regional vertices of the same region and may be connected to boundary vertices as well. In any valid drawing, they will end up in the same region as their regional vertices. Components of type c_2 are not connected to any regional vertices and only connected to one of the three boundary vertices. Since they are not connected to regional vertices, they can not interfere with other parts of the drawing, so we can arbitrarily assign them to an adjacent region of their boundary vertex. Components that are connected to two boundary vertices appear at first to have two possible placements, e.g., as c_3 or c_4 in Fig. 2 (a). However, c_4 is not a valid placement for this type of component since it would contradict the fact that this separator arose from two close parallel edges as argued in the proof of Theorem 2. From the above discussion, we see that from a fixed configuration (i.e., set S, drawing of G_S , and triple of boundary vertices), if the drawing of G_S has the shape depicted in Fig. 2 (a), we can either reject the current configuration (based on having bad components), or we see that every component of $G \setminus S$ is either attached to exactly one boundary vertex or it has a well-defined placement into the regions defined by the boundary vertices. For those components which are attached to exactly one boundary vertex, we observe that it suffices to recursively produce a drawing of that component together with its boundary vertex and to place this drawing next to the boundary vertex. For the other components, we partition them into their regions and recurse on the regions. This covers all cases for this separator shape.

The other shape of our separator can be seen in Fig. 2 (b). Note that we now have two boundary vertices v and w and thus only have two regions. Again we see the two component types c_1 and c_2 and can handle them as above. We also have components connected to both v and w but no regional vertices. These components now truly have two different placement options c_3 , c_4 . If we have an edge $v_i w_i$ (as in Fig. 2 (b)) of the separator that is not vw, we now observe that there cannot be more than k such components. Namely, in any drawing, for each component, there will be an edge connecting this component to either v or w which crosses $v_i w_i$. Thus, we now enumerate all the different placements of these components as type c_3 or c_4 and recurse accordingly.

However, the separator may be exactly the pair (v, w). Note that there are no components of type c_1 and the components of type c_2 can be handled as before. We will now argue that we can have at most a function of k different components of type c_3 or c_4 in a valid drawing. Consider the components of type c_3 (the components of type c_4 can be counted similarly). In a valid drawing, each type c_3 component defines a sub-interval of the left region spanning from its highest to its lowest vertex such that these vertices are adjacent to one of v or v. Two such intervals relate in one of three ways: They overlap, they are disjoint, or one is contained in the other. We group components with either overlapping or disjoint intervals into layers. We depict this situation in Fig. 2 (c) where, for

simplicity, for every component we only draw its highest vertex and its lowest vertex and they are connected by one edge.

Let a_1b_1 be the bottommost component of type c_3 (i.e., a_1 is the clockwisefirst vertex from v in a component of type c_3). The first layer is defined as the component a_1b_1 together with every component whose interval either overlaps or is disjoint from the interval of a_1b_1 . Now consider the green edge b_1w (see Fig. 2 (c)), note we may have that this edge connects a_1 to w instead. Now, for every component of this layer which is disjoint from the interval of a_1b_1 , this edge is crossed by at least one edge connecting it to v. Furthermore, for every component of this layer which overlaps the interval of a_1b_1 , there is an edge connecting b_1 to either v or w which is crossed by at least one edge within that component. So in total, there can only be O(k) components in this first layer. New layers are defined by considering components whose intervals are contained in a_1b_1 . To limit the total number of layers, let a_ℓ be the bottommost vertex of the first component of the deepest layer and consider the purple edge va_{ℓ} . This edge is crossed by some edge of every layer above it and as any edge can only have k crossings, there can only be O(k) different levels in total. This leaves us with a total of at most $O(k^2)$ components per region and again we can enumerate their placements and recurse accordingly.

The above algorithm provides the following recurrence regarding its runtime. Namely, we let T(n) denote the runtime of our algorithm, and we can see that the following expression generously upper bounds its value. Here f(s) denotes the number of different outer k-planar drawings of a graph with s vertices.

$$T(n) \le \begin{cases} n^{O(k)} \cdot f(4k+3) \cdot n^3 \cdot n \cdot T(\frac{2n}{3}) & \text{for } n > 5k \\ f(n) & \text{otherwise} \end{cases}$$

Thus, the algorithm runs in quasi-polynomial time, i.e., $2^{\text{poly}(\log n)}$.

3 Outer k-Quasi-Planar Graphs

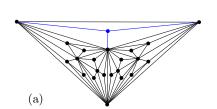
In this section we consider outer k-quasi-planar graphs. We first describe some classes of graphs which are outer 3-quasi-planar. We then discuss edge-maximal outer k-quasi-planar drawings.

Note, all sub-Hamiltonian planar graphs are outer 3-quasi-planar. One can also see which complete and bipartite complete graphs are outer 3-quasi-planar.

Proposition 1. The following graphs are outer 3-quasi-planar: (a) $K_{4,4}$; (b) K_5 ; (c) planar 3-tree with three complete levels; (d) square-grids of any size.

Proof. (a) and (b) are easily observed. (c) was experimentally verified by constructing a Boolean expression and using MiniSat to check it for satisfiability; see Appendix A. (d) follows from square-grids being sub-Hamiltonian. \Box

Correspondingly, we note complete and complete bipartite graphs which are not outer-quasi planar. Furthermore, not all planar graphs are outer quasi-planar,



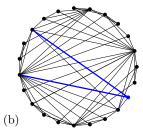


Fig. 3: A vertex-minimal 23-vertex planar 3-tree which is not outer quasi-planar:
(a) planar drawing; (b) deleting the blue vertex makes the drawing outer quasi-planar

e.g., the vertex-minimal planar 3-tree in Fig. 3 (a) is not outer quasi-planar, this was verified checking for satisfiability the corresponding Boolean expression; see Section A. A drawing of the graph in Fig. 3 (b) was constructed by removing the blue vertex and drawing the remaining graph in an outer quasi-planar way.

Proposition 2. The following graphs are not outer 3-quasi-planar: (a) $K_{p,q}$, $p \geq 3$, $q \geq 5$; (b) K_n , $n \geq 6$; (c) planar 3-tree with four complete levels.

Together, Propositions 1 and 2 immediately yield the following.

Theorem 4. Planar graphs and outer 3-quasi-planar graphs are incomparable under containment.

Remark 1. For outer k-quasi-planar graphs (k > 3) containment questions become more intricate. Every planar graph is outer 5-quasi-planar because planar graphs have page number 4 [31]. We also know a planar graph that is not outer 3-quasi-planar. It is open whether every planar graph is outer 4-quasi-planar.

Maximal outer k-quasi-planar graphs. A drawing of an outer k-quasi-planar graph is called maximal if adding any edge to it destroys the outer k-quasi-planarity. We call an outer k-quasi-planar graph maximal if it has a maximal outer k-quasi-planar drawing. Recall that Capoyleas and Pach [9] showed the following upper bound on the edge density of outer k-quasi-planar graphs on n vertices: $|E| \leq 2(k-1)n - {2k-1 \choose 2}$.

We prove (see Appendix B.2) that each maximal outer k-quasi-planar graph meets this bound. Our proof builds on the ideas of Capoyleas and Pach [9] and directly shows the result via an inductive argument. However, while preparing the camera-ready version of this paper, we learned of two other proofs of this result in the literature [13,24]. We thank David Wood for pointing us to these results. Both papers prove a slightly stronger theorem (concerning edge flips) as their main result. Namely, for a drawing G = (V, E), an edge flip produces a new drawing G^* by replacing an edge $e \in E$ with a new edge $e^* \in \binom{n}{2} \setminus E$. They [13,24] show that, for every two maximal outer k-quasi-planar drawings G = (V, E) and G' = (V, E'), there is a sequence of edge flips producing drawings $G = G_1, G_2, \ldots, G_t = G'$ such that each G_i is a maximal k-quasi-planar drawing.

Together with the tight example of Capoyleas and Pach [9], this implies the next theorem, and makes our proof fairly redundant.

Theorem 5 ([13,24]). Each maximal outer k-quasi-planar drawing G = (V, E) has:

$$|E| = \begin{cases} \binom{|V|}{2} & \text{if } |V| \le 2k - 1, \\ 2(k - 1)|V| - \binom{2k - 1}{2} & \text{if } |V| \ge 2k - 1. \end{cases}$$

4 Closed Convex Drawings in MSO₂

Here we express graph properties in extended monadic second-order logic (MSO_2) . This subset of second-order logic is built from the following primitives.

- variables for vertices, edges, sets of vertices, and sets of edges;
- binary relations for: equality (=), membership in a set (\in), subset of a set (\subseteq), and edge-vertex incidence (I);
- standard propositional logic operators: \neg , \wedge , \vee , \rightarrow .
- standard quantifiers (\forall, \exists) which can be applied to all types of variables.

For a graph G and an MSO₂ formula ϕ , we use $G \models \phi$ to indicate that ϕ can be satisfied by G in the obvious way. Properties expressed in this logic allow us to use the powerful algorithmic result of Courcelle stated next.

Theorem 6 ([10,11]). For any integer $t \geq 0$ and any MSO_2 formula ϕ of length ℓ , an algorithm can be constructed which takes a graph G with treewidth at most t and decides in $O(f(t,\ell) \cdot (n+m))$ time whether $G \models \phi$ where the function f from this time bound is a computable function of t and ℓ .

Outer k-planar graphs are known to have treewidth O(k) (see Proposition 8.5 of [30]). So, expressing outer k-planarity by an MSO₂ formula whose size is a function of k would mean that outer k-planarity could be tested in linear time. However, this task might be out of the scope of MSO₂. The challenge in expressing outer k-planarity in MSO₂ is that MSO₂ does not allow quantification over sets of pairs of vertices which involve non-edges. Namely, it is unclear how to express a set of pairs that forms the circular order of vertices on the boundary of our convex drawing. However, if this circular order forms a Hamiltonian cycle in our graph, then we can indeed express this in MSO₂. With the edge set of a Hamiltonian cycle of our graph in hand, we can then ask that this cycle was chosen in such a way that the other edges satisfy either k-planarity or k-quasi-planarity. With this motivation in mind, we define the classes closed outer k-planar and closed outer k-quasi-planar, where closed means that there is an appropriate convex drawing where the circular order forms a Hamiltonian cycle. Our main result here is stated next.

Theorem 7. Closed outer k-planarity and closed outer k-quasi-planarity can be expressed in MSO_2 . Thus, closed outer k-planarity can be tested in linear time.

The formulas for our graph properties are built using formulas for Hamiltonicity (Hamiltonian), partitioning of vertices into disjoint subsets (Vertex-Partition) and connected induced subgraphs on sets of vertices using only a subset of the edges (Connected). They can be found in Appendix C.

For a closed outer k-planar or closed outer k-quasi-planar graph G, we want to express that two edges e and e_i cross. To this end, we assume that there is a Hamiltonian cycle E^* of G that defines the outer face. We partition the vertices of G into three subsets C, A, and B, as follows: C is the set containing the endpoints of e, whereas A and B are connected subgraphs on the remaining vertices that use only edges of E^* . In this way, we partition the vertices of G into two sets, one left and the other one right of e. For such a partition, e_i must cross e whenever e_i has one endpoint in A and one in B.

CROSSING
$$(E^*, e, e_i) \equiv (\forall A, B, C) [(VERTEX-PARTITION(A, B, C) $\land (I(e, x) \rightarrow x \in C) \land CONNECTED(A, E^*) \land CONNECTED(B, E^*))$
 $\rightarrow (\exists a \in A)(\exists b \in B)[I(e_i, a) \land I(e_i, b)]]$$$

Now we can describe the crossing patterns for closed outer k-planarity and closed outer k-quasi-planarity as follows:

CLOSED OUTER
$$k$$
-Planar $_G \equiv (\exists E^*) \Big[\text{Hamiltonian}(E^*) \land (\forall e) \Big[(\forall e_1, \dots, e_{k+1}) \Big[\Big(\bigwedge_{i=1}^{k+1} e_i \neq e \land \bigwedge_{i \neq j} e_i \neq e_j \Big) \rightarrow \bigvee_{i=1}^{k+1} \neg \text{Crossing}(E^*, e, e_i) \Big] \Big] \Big]$

Here we insist that G is Hamiltonian and that, for every edge e and any set of k+1 distinct other edges, at least one among them does not cross e.

Closed Outer
$$k$$
-Quasi-Planar $_G \equiv (\exists E^*) \Big[\text{Hamiltonian}(E^*) \land (\forall e_1, \dots, e_k) \Big[\Big(\bigwedge_{i \neq j} e_i \neq e_j \Big) \rightarrow \bigvee_{i \neq j} \neg \text{Crossing}(E^*, e_i, e_j) \Big] \Big]$

Again, we insist that G is Hamiltonian and further that, for any set of k distinct edges, there is at least one pair among them that does not cross.

We conclude this section by mentioning an intermediate concept between closed outer k-planarity and outer k-planarity, i.e., full outer k-planarity [19]. The full outer k-planar graphs are defined as having a convex drawing which is k-planar and additionally there is no crossing on the outer boundary of the drawing. Hong and Nagamochi [19] gave a linear-time recognition algorithm for full outer 2-planar graphs. Clearly, the closed 2-planar graphs are a subclass of the full 2-planar graphs. So, one open question is whether one can generalize our MSO₂ expressions of closed outer k-planarity and closed outer k-quasi-planarity to the full versions. If yes, this would provide linear-time recognition of full outer k-planar graphs for every k, including the full outer 2-planar case.

Acknowledgement. We acknowledge Alexander Ravsky, Thomas van Dijk, Fabian Lipp, and Johannes Blum for their comments and preliminary discussion. We also thank David Wood for pointing us to references [13,24,30].

References

- Ackerman, E.: On the maximum number of edges in topological graphs with no four pairwise crossing edges. Discrete Comput. Geom. 41(3), 365–375 (2009), https://doi.org/10.1007/s00454-009-9143-9
- Ackerman, E., Tardos, G.: On the maximum number of edges in quasi-planar graphs. J. Combin. Theory Ser. A 114(3), 563-571 (2007), https://doi.org/10. 1016/j.jcta.2006.08.002
- Angelini, P., Bekos, M.A., Brandenburg, F.J., Lozzo, G.D., Battista, G.D., Didimo, W., Liotta, G., Montecchiani, F., Rutter, I.: On the relationship between k-planar and k-quasi planar graphs. In: Bodlaender, H. (ed.) WG 2017. LNCS, vol. XXXX, p. ... (2017), http://arxiv.org/abs/1702.08716
- Auer, C., Bachmaier, C., Brandenburg, F.J., Gleißner, A., Hanauer, K., Neuwirth, D., Reislhuber, J.: Outer 1-planar graphs. Algorithmica 74(4), 1293–1320 (2016), https://doi.org/10.1007/s00453-015-0002-1
- Babu, J., Khoury, A., Newman, I.: Every property of outerplanar graphs is testable. In: Jansen, K., Mathieu, C., Rolim, J.D.P., Umans, C. (eds.) APPROX/RANDOM 2016. LIPIcs, vol. 60, pp. 21:1-21:19. Schloss Dagstuhl, Leibniz-Zentrum für Informatik, Dagstuhl (2016), https://doi.org/10.4230/LIPIcs.APPROX-RANDOM. 2016.21
- Bannister, M.J., Eppstein, D.: Crossing minimization for 1-page and 2-page drawings of graphs with bounded treewidth. In: Duncan, C., Symvonis, A. (eds.) GD 2014, LNCS, vol. 8871, pp. 210–221. Springer-Verlag (2014), https://doi.org/10.1007/978-3-662-45803-7_18
- 7. Binucci, C., Di Giacomo, E., Hossain, M.I., Liotta, G.: 1-page and 2-page drawings with bounded number of crossings per edge. In: Lipták, Z., Smyth, W.F. (eds.) IWOCA 2015, LNCS, vol. 9538, pp. 38–51. Springer-Verlag (2016), https://doi.org/10.1007/978-3-319-29516-9_4
- 8. Borodin, O.V.: Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs. Metody Diskret. Analiz. 41, 12–26, 108 (1984)
- Capoyleas, V., Pach, J.: A Turán-type theorem on chords of a convex polygon. J. Combin. Theory Ser. B 56(1), 9-15 (1992), https://doi.org/10.1016/0095-8956(92)90003-G
- Courcelle, B.: The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. Inform. Comput. 85(1), 12–75 (1990), https://doi.org/10.1016/ 0890-5401(90)90043-H
- Courcelle, B., Engelfriet, J.: Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach. Cambridge University Press (2012)
- 12. Cygan, M., Fomin, F.V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized Algorithms, chap. Lower Bounds Based on the Exponential-Time Hypothesis, pp. 467–521. Springer-Verlag (2015), https://doi.org/10.1007/978-3-319-21275-3_14
- 13. Dress, A.W.M., Koolen, J.H., Moulton, V.: On line arrangements in the hyperbolic plane. Eur. J. Comb. 23(5), 549–557 (2002), https://doi.org/10.1006/eujc.2002.0582

- Dujmović, V., Eppstein, D., Wood, D.R.: Structure of graphs with locally restricted crossings. SIAM J. Discrete Math. 31(2), 805–824 (2017)
- 15. Dvořák, Z., Norin, S.: Treewidth of graphs with balanced separations. ArXiv (2014), http://arxiv.org/abs/1408.3869
- Fox, J., Pach, J., Suk, A.: The number of edges in k-quasi-planar graphs. SIAM J. Discrete Math. 27(1), 550–561 (2013), https://doi.org/10.1137/110858586
- 17. Geneson, J., Khovanova, T., Tidor, J.: Convex geometric (k + 2)-quasiplanar representations of semi-bar k-visibility graphs. Discrete Math. 331, 83–88 (2014), https://doi.org/10.1016/j.disc.2014.05.001
- Hong, S.H., Eades, P., Katoh, N., Liotta, G., Schweitzer, P., Suzuki, Y.: A linear-time algorithm for testing outer-1-planarity. Algorithmica 72(4), 1033–1054 (2015), https://doi.org/10.1007/s00453-014-9890-8
- Hong, S.H., Nagamochi, H.: Testing full outer-2-planarity in linear time. In: Mayr, E.W. (ed.) WG 2016, LNCS, vol. 9224, pp. 406–421. Springer-Verlag (2016), https://doi.org/10.1007/978-3-662-53174-7_29
- 20. Impagliazzo, R., Paturi, R.: On the complexity of k-SAT. J. Comput. Syst. Sci. 62(2), 367–375 (2001), https://doi.org/10.1006/jcss.2000.1727
- 21. Kobourov, S.G., Liotta, G., Montecchiani, F.: An annotated bibliography on 1-planarity. ArXiv (2017), http://arxiv.org/abs/1703.02261
- 22. Lick, D.R., White, A.T.: k-degenerate graphs. Canadian J. Math. 22, 1082–1096 (1970), http://dx.doi.org/10.4153/CJM-1970-125-1
- Masuda, S., Kashiwabara, T., Nakajima, K., Fujisawa, T.: On the NP-completeness of a computer network layout problem. In: Proc. IEEE Int. Symp. Circuits and Systems. pp. 292–295 (1987)
- Nakamigawa, T.: A generalization of diagonal flips in a convex polygon. Theor. Comput. Sci. 235(2), 271–282 (2000), https://doi.org/10.1016/S0304-3975(99) 00199-1
- 25. Pach, J., Shahrokhi, F., Szegedy, M.: Applications of the crossing number. Algorithmica 16(1), 111–117 (1996), https://doi.org/10.1007/BF02086610
- 26. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. Combinatorica 17(3), 427–439 (1997), https://doi.org/10.1007/BF01215922
- 27. Ringel, G.: Ein Sechsfarbenproblem auf der Kugel. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 29(1), 107–117 (1965), https://doi.org/10.1007/BF02996313
- 28. Robertson, N., Seymour, P.D.: Graph minors. III. Planar tree-width. J. Combin. Theory Ser. B 36(1), 49-64 (1984), https://doi.org/10.1016/0095-8956(84)
- 29. Schaefer, M.: The graph crossing number and its variants: A survey. Electronic J. Combin. DS21, 100 pages (2013 and 2014), http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS21
- 30. Wood, D.R., Telle, J.A.: Planar decompositions and the crossing number of graphs with an excluded minor. New York J. Math. 13, 117–146 (2007)
- 31. Yannakakis, M.: Embedding planar graphs in four pages. J. Comput. Syst. Sci. 38(1), 36–67 (1989), https://doi.org/10.1016/0022-0000(89)90032-9

A SAT Experiments

In this section, we describe a logical formula for testing whether a given graph is outer quasi-planar. We present the formula in first-order logic. After transformation to Boolean logic, we solve the formula using MiniSat http://minisat.se/.

A quasi outer-planar embedding corresponds to a circular order of the vertices. If we cut a circular order at some vertex to turn the circular into a linear order, the edge crossing pattern remains the same. Therefore, we look for a linear order. For any pair of vertices $u \neq v \in V$ and any pair of edges $e \neq e' \in E$, we introduce 0–1 variables $x_{u,v}$ and $y_{e,e'}$, respectively. The intended meaning of $x_{u,v} = 1$ is that vertex u comes before v. The intended meaning of $y_{e,e'} = 1$ is that edge e crosses edge e'. Now we list the clauses of our SAT formula.

$$x_{u,v} \wedge x_{v,w} \Rightarrow x_{u,w}$$
 for each $u \neq v \neq w \in V$; (1)

$$x_{u,v} \Leftrightarrow \neg x_{v,u}$$
 for each $u \neq v \in V$; (2)

$$x_{u,u'} \wedge x_{u',v} \wedge x_{v,v'} \Rightarrow y_{e,e'}$$
 for each $e = (u,v) \neq e' = (u',v') \in E;$ (3)

$$\neg(y_{e_1,e_2} \land y_{e_1,e_3} \land y_{e_2,e_3}) \qquad \text{for each } e_1,e_2,e_3 \in E \text{ with different endpoints.}$$
(4)

The first two sets of clauses describe the linear order. Clause (1) ensures transitivity, and clause (2) anti-symmetry. Clause (3) realizes the intended meaning of variable $y_{e,e'}$. Finally, clause (4) ensures that no three edges pairwise cross.

B Omitted Proofs

B.1 Proof of Lemma 1

Proof. Let the largest outer k-planar clique have n vertices. An edge ab is crossed by the number of vertices on one side times the number of vertices on the other side of ab. Therefore, the edge that is crossed the most has almost equal number of vertices on both sides, so $\left(\frac{n-2}{2}\right)^2 \leq k$ if n is even and $\frac{(n-3)(n-1)}{4} \leq k$ if n is odd. Therefore, n is at most $\lfloor \sqrt{4k+1} \rfloor + 2$.

B.2 Proof of Theorem 5

For an outer k-quasi-planar drawing of graph G we call an edge \overline{ab} a long edge, if a and b are separated along the outer face of G by at least (k-1) vertices on both sides. In the following, long edges will always be drawn vertically with a on top, dividing the graph into regions left and right of a. Further, we call all edges intersecting the long edge $crossing\ edges$. All vertices incident to crossing edges will be called $crossing\ vertices$ and for illustration we will label them in a top-to-bottom fashion like depicted in Fig. 4 (a). On the left side of \overline{ab} , vertices will be labeled v_1, \ldots, v_g and w_1, \ldots, w_h on the right side, respectively $(g, h \ge k-1)$. In the following, we will count the number of crossing edges and give an inductive argument for the maximality of the considered graph.

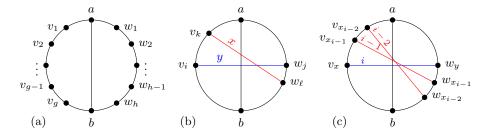


Fig. 4: (a) Labeling scheme for left and right side vertices according to long edge \overline{ab} ; (b) illustration of property 1: $v_k w_\ell$ is crossing $v_i w_j$ from above; (c) property 2: blue edge in level i and the first two edges of a possible certificate.

To do so, we are following the idea given by Capoyleas and Pach: we will construct (k-2) hierarchical levels and define a replacement-operation that will use these levels to split the original graph into two subgraphs on less vertices. The hierarchical levels will be built greedily as in Algorithm 1.

Algorithm 1: BUILDLEVELS(Outer k-quasi-planar Graph G, long edge \overline{ab}) $\{v_1, \ldots, v_g\} \leftarrow$ vertices left of \overline{ab} ; $S = \{S_1, \ldots, S_{k-2}\} \leftarrow$ empty level sets for i = 1 to k - 2 do

for j = 1 to g do $S' \leftarrow$ edges of v_j not crossing edges in S_i ; $S_i \leftarrow S_i \cup S'$ return S

Lemma 3. For a given outer k-quasi-planar graph G, Algorithm 1 generates k-2 hierarchical levels – subsets of the crossing edges of G that form maximal crossing-free connected subgraphs of G.

Proof. We need to argue that the algorithm creates k-2 levels that cover all crossing edges with respect to \overline{ab} and that every level is connected. We order the levels by order of construction so we say level y is after level x or $x \prec y$, if y is constructed after x. To do so, we first state two important properties:

- (P1) If an edge (v_i, w_j) of level y is crossed by an edge (v_k, w_ℓ) of level x with $x \prec y$, then it must cross from above: i > k and $j < \ell$; see Fig. 4 (b).
- (P2) For any edge e of level i, there are edges of previous levels e_1, \ldots, e_{i-1} such that $\{e, e_1, \ldots, e_{i-1}\}$ is a set of i pairwise crossing edges; see Fig. 4 (c).

Property (P1) follows from the construction of level x: If there where no edge of level x crossing edge (v_i, w_j) , then (v_i, w_j) would also belong to level x.

Property (P2) holds by induction: For the first level there is no previous level. Edges of level two are crossed by edges of the first level due to (P1). Any edge (v_x, w_y) of level i must be crossed by some edge $(v_{x_{i-1}}, w_{y_{i-1}})$ of level i-1. Inductively we know that e_{i-1} is crossed by an edge of every previous level.

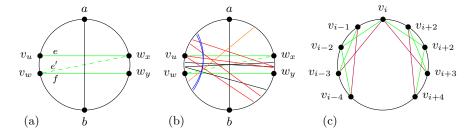


Fig. 5: Connectivity of hierarchical level i: (a) disconnected level i in green, missing edge e'; (b) set of edges blocking e' using locally left edges; (c) frame of a maximal outer k-quasi-planar graph (black for any k, green for k = 3, purple for k = 4).

Together, they form a chain of pairwise crossings from above, and we get the following index patterns:

$$x > x_{i-1} > x_{i-2} > \dots > x_1$$
 and $y < y_{i-1} < y_{i-2} < \dots < y < y_1$.

These patterns indicate that in fact all the considered edges are pairwise crossing. For a given edge e of level i, we call any set following the description in property (P2) a *certificate* for e to be in i. In fact, any edge of level i-1 crossing e can be extended to some certificate for e.

Let t be the last level and consider the last edge e taken from that level. Suppose the algorithm created too many levels, so t > k - 2. By property (P2), the certificate of e belonging to t together with e and \overline{ab} forms a set of $t+1 \geq k$ pairwise crossing edges. This contradicts that the graph from which e is taken is outer k-quasi-planar and so we never create more than k-2 levels.

Note, the maximality of each level comes from the way it is constructed.

To argue about the connectivity of the levels we carefully consider the way they will be generated in a maximal graph and make sure that greedily picking edges never disconnects the edge set of any level. Consider Fig. 5 (a): Assume we already generated levels 1 to i-1 and in level i we pick the edges e and f but not e' and thus would end up disconnecting i. The edge e' is going upward with respect to the other edges of level i and thus cannot cross them from above due to (P1). Thus, if e' is present in our graph, it would belong to level i and i would be connected. So further assume that e' is not present. As the graph is supposed to be maximal, there must be a prevention set of k-1 pairwise crossing edges that prevent its existance. Due to the edge ab, the prevention set contains at most k-2 crossing edges. By existence of the edges e and f we know that the prevention set also cannot consist of edges only locally on one side. In this context, \overline{ab} is considered to be both locally left and locally right. This implies two possible prevention sets: crossing edges together with locally left or right edges – see Fig. 5 (b). For illustration we color the edges of the prevention set differently: black edges start between v_u and v_w , appearing between e and fbut not belonging to the same level as e; red edges cross e' from above ending strictly below w_x ; orange edges cross e' from below; blue edges are locally on

one side crossing all edges of other colors. By verifying the following claim, we finish the proof.

Claim. A prevention set \mathcal{P} for e' can be transformed into a prevention set for f (or e) – contradicting \mathcal{P} 's existence.

For the transformation, we keep all the orange and blue edges of \mathcal{P} , as they pairwise cross and also cross f. For the remaining edges, we need to make sure that, if we replace them, the new edges must also cross the blue and orange ones. Other edges in the set belong to previous levels because of the following:

- Red edges all cross e' from above (P1).
- The black edges divide into two cases: crossing f or completely between e and f. Edges crossing f do so from above (P1). Edges between e and f can not be crossed by edges of level i, so they must have been picked before.

In fact we now see that the prevention set for e' consists of a certificate for e' they all pairwise cross (P2) – together with blue and orange edges. We now consider the edges of the certificate for e' in the order of the level they belong to starting at i-1. To see that for any such level, there is an edge crossing f whose left vertex is covered by all blue edges consider the following: Suppose the edge e_k of level k < i does not cross f. In order for an edge e^\times of level k to cross f, its right vertex must be below the right vertex of e_k . Thus its left vertex can either be the left vertex of e_k or some other vertex below it but still above v_w ; in both cases, the new left vertex will be covered by all blue edges. After doing one replacement, in the worst case every remaining edge of the certificate for e' does not cross e^\times because it is below them. This we can fix be using a certificate for e^\times to be in level k; notice that every edge of this new certificate must be below its corresponding counterpart in the old certificate and thus still all starting vertices are covered by blue edges.

Thus, a prevention set for e' using locally left edges is not possible since it would imply a prevention set for f as well. A similar argument can be made using locally right edges by again taking a prevention set for e', transforming the certificate for e' into one for e and observing that the endpoints of the edges of the certificate are again still covered by the locally right edges.

The construction of Pach and Capoyleas [9, proof of Claim 4] yields the following:

Remark 2. In an outer k-quasi-planar graph with n vertices labeled according to their cyclic order along the outer face $(v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, every vertex v_i can be adjacent to v_ℓ with $\ell \in [i - (k-1), i + (k-1)]$. These frame edges are present in any maximal outer k-planar graph; see Fig. 5 (c).

Using our hierarchical levels, we now describe a replacement operation, which is used to split our graph into two smaller parts. Let G be a maximal outer k-quasi-planar graph with a long edge \overline{ab} and hierarchical levels created by Algorithm 1. Let L_i and R_i be the vertices of G incident to edges of level i left and right of \overline{ab} , respectively. We make two graphs G_1, G_2 from G. For G_1 (G_2),

delete all vertices left (right) of \overline{ab} and for every level add a new *level*-vertex v_i connected to all vertices in R_i (L_i), e.g., see Fig. 6 (a) and (b). Finally, add any absent frame edges to G_1 (G_2) to make it maximal.

Lemma 4. After applying the replacement operation to a maximal outer k-quasi-planar graph G, the following relations among G, G_1 and G_2 hold:

(i)
$$|V(G)| = |V(G_1)| + |V(G_2)| - 2k + 2$$
 and
(ii) $|E(G)| = |E(G_1)| + |E(G_2)| - (|E_1'| + |E_2'|) + |E'| - 1$,

where E' is the set of edges of G crossing \overline{ab} and E'_1, E'_2 are the sets of crossing edges added to G_1, G_2 by the replacement operation.

Proof. (i) In G_1 and G_2 , we only modify the right or left side of G, leaving the other side unmodified. The modification adds k-2 level-vertices to both graphs. Notice that \overline{ab} is present in both G_1 and G_2 , so subtracting the level-vertices and one copy of a and b, yields

$$|V(G)| = |V(G_1)| - (k-2) + |V(G_2)| - (k-2) - 2 = |V(G_1)| + |V(G_2)| - 2k + 2.$$

(ii) To count the edges added to both G_1 and G_2 and compare them to the number of edges removed by splitting G, we consider the structure of our hierarchical levels. As before, let L_i and R_i be the sets of vertices incident to edges of level i left (right) of \overline{ab} . Every level is a connected and non-crossing set of edges, so there are exactly $|L_i| + |R_i| - 1$ edges in level i. So the total number of crossing edges is

$$|E'| = \sum_{i=1}^{k-2} (|L_i| + |R_i| - 1).$$

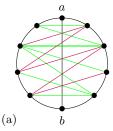
The sets of edges added to G_1 and G_2 consist of two different subsets: the edges added for the level-vertices and the missing frame edges added to make G_1 and G_2 maximal. The first can be expressed using the sets L_i and R_i with $1 \le i \le k-2$ as follows:

$$\sum_{i=1}^{k-2} |R_i| + |L_i|.$$

Vertices and edges on the unmodified sides of our subgraph remain maximal by maximality of G. But as can be seen in Fig. 6 (b), e.g. vertex v of G_1 seems to be missing an edge. The way we generate our hierarchical levels, in level i we take all remaining edges of right-side vertex $w_{h-(i-1)}$ – for the first level, we take all edges of the last right-side vertex and so on. The total number of edges missing this way in both subgraphs is

$$2 \cdot \sum_{i=1}^{k-2} (i-1).$$

Note that the frame edges on the modified side of each subgraph together with a and b form a clique of size k-2+2. So the exact total amount of edges



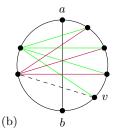


Fig. 6: Replacement operation according to hierarchical levels: (a) an outer 4-quasiplanar graph with 2 hierarchical levels (green and purple); (b) the resulting replaced graph G_1 , missing edge of v drawn dashed.

we add to G_1 and G_2 this way is $2({k \choose 2} - 1)$, not counting \overline{ab} . The total number of new edges in G_1 and G_2 then is

$$|E'_1| + |E'_2| = 2\left(\binom{k}{2} - 1\right) + \sum_{i=1}^{k-2} R_i + L_i + 2(i-1).$$

Notice that we did not account for \overline{ab} in any of the subgraphs yet, so we subtract one copy of it. The remaining parts of G_1 and G_2 are the unmodified sides of G and one copy of \overline{ab} .

To complete our inductive argument, we need to do two things: prove the existence of a long edge in any maximal outer k-quasi-planar graph and consider the base cases – those maximal outer k-quasi-planar graphs with the minimum number of vertices.

Lemma 5. Any maximal outer k-quasi-planar graph G = (V, E) either (i) is a clique of size $|V| \le 2k - 1$, or (ii) has a long edge.

Proof. We consider each case individually.

- (i) $n \leq 2k-1$: By maximality of G we know there are frame edges. For any vertex v these edges connect it to its k-1 neighbors both left and right of it. Thus every vertex is connected to all other vertices and we have a clique on n vertices.
- (ii) n > 2k 1: Given the frame, by a simple counting argument, for every vertex v_i $(i \in 1, ..., n)$, there is at least one other vertex w_i that it is not connected to via a frame edge. Consider the pair v_i, w_i for some i. If it were connected by an edge, this would be the long edge we are looking for. So suppose the edge v_i, w_i is missing. As we choose G to be maximal, there must be some set S of edges preventing the existence. As edges of S can not be part of the frame, they must span more than k-1 vertices on both sides. Then one of these edges is our long edge.

Finally, we combine the ingredients above to complete the proof of Theorem 5.

Proof (of Theorem 5). By Lemma 5 we can either always find a long edge to split by (ii) or we end up with a subgraph that is a clique (i). By splitting the graph G along a long edge, building the hierarchical levels and performing the replacement operation described above, we get two subgraphs G_1 and G_2 of smaller size. By Lemma 4 we get the relationships between G and the subgraphs. We recursively split these subgraphs until we encounter cliques and know that the number of edges matches the bound on total edge number.

Given a maximal outer k-quasi-planar graph G, we can recursively decompose it into pieces that are again maximal outer k-quasi-planar graphs. Considering the relationship among the edge sets of Lemma 4 (ii) and the maximality of the subgraphs, we get the following:

$$|E(G)| = |E(G_1)| + |E(G_2)| - (|E_1'| + |E_2'|) + |E'| - 1$$

$$= |E(G_1)| + |E(G_2)| - 2k^2 + 5k - 3$$

$$= 2(k-1)(n_1 + n_2) - 2\binom{2k-1}{2} - 2k^2 + 5k - 3$$

From the maximality of G and using Lemma 4 (i), we now have:

$$2(k-1)(-2k+2) = -\binom{2k-1}{2} - 2k^2 + 5k - 3$$
$$-4k^2 + 8k - 4 = -4k^2 + 8k - 4$$

The equation balances, completing the proof.

C Formulas Used to Build MSO₂-Expressions

Any formula presented here assumes that a graph G is given and uses edges, vertices and incidences of G. In the following, e, f are edges, F is a set of edges, u, v are vertices and U a set of vertices (also including sub- and superscripted variants). In addition to the quantifiers above we also use a logical shorthand $\exists^{=x}$ for the existence of exactly x elements satisfying the property, that all are unequal and that no x+1 such elements exist.

The following two formulas allow us to describe disconnectedness of subgraphs induced by either an edge set or a vertex set together with a set of edges.

DISCONNECTED-EDGES $(F) \equiv$

$$(\exists e, f)[e \in F \land f \notin F] \land \neg(\exists v)(\exists e, f)[I(e, v) \land I(f, v) \land e \in F \land f \in F]$$

DISCONNECTED-VERTICES $(U, F) \equiv$

$$(\exists u, v)[u \in U \land v \not\in U] \land \neg (\exists e \in F)(\exists u, v)[I(e, u) \land I(e, v) \land u \in U \land v \not\in U]$$

The following formula implies that the vertices of set U form a connected subgraph using only edges of set F:

Connected
$$(U, F) \equiv \neg \text{Disconnected-Vertices}(U, F)$$

These formulas are used to describe Hamiltonicity of G. CYCLE-SET implies that the edges of F form cycles, CYCLE implies maximality of the cycle and SPAN forces the cycle to have an edge incident to every vertex of G.

$$\begin{aligned} \text{Cycle-Set}(F) \equiv \\ & (\forall e) \Big[e \in F \to (\exists^{=2} f) \big[f \in F \land e \neq f \land (\exists v) [I(e,v) \land I(f,v)] \big] \Big] \\ \text{Cycle}(F) \equiv \text{Cycle-Set}(F) \land \neg \text{Disconnected-Edges}(F) \\ \text{Span}(F) \equiv (\forall v) (\exists e) [e \in F \land I(e,v)] \\ \text{Hamiltonian}(F) \equiv [\text{Cycle}(F) \land \text{Span}(F)] \end{aligned}$$

VERTEX-PARTITION implies the existence of a partition of the vertices of G into k disjoint subsets.

Vertex-Partition
$$(U_1, \dots, U_k) \equiv$$

$$(\forall v) \left[\left(\bigvee_{i=1}^k v \in U_k \right) \wedge \left(\bigwedge_{i \neq j} \neg (v \in U_i \wedge v \in U_j) \right) \right]$$