Clustered planarity testing revisited

Radoslav Fulek, Jan Kynčl, Igor Malinović and Dömötör Pálvölgyi

Charles University, Prague

and

EPFL

Clustered planarity

Graph: $G = (V, E)$, V finite, $E \subseteq \binom{V}{2}$

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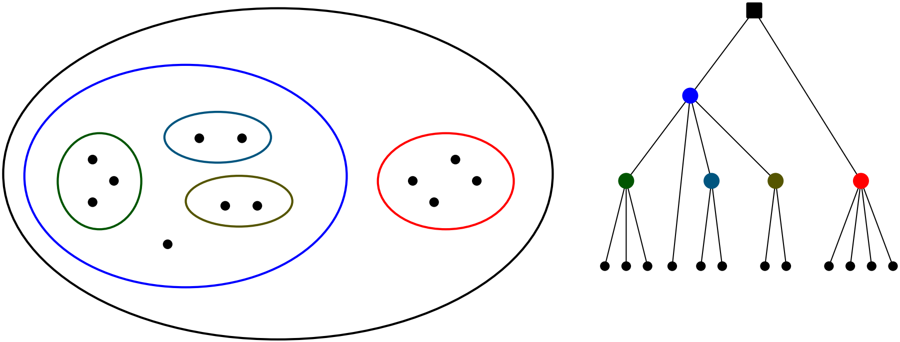
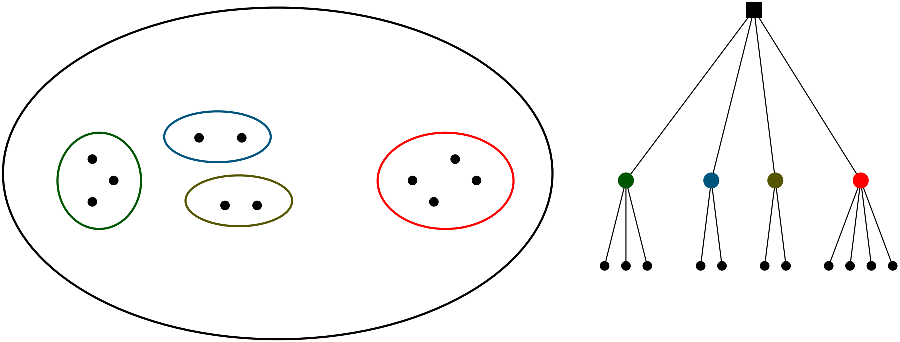
Clustered graph: (G, T) where T is a tree hierarchy of clusters



Clustered planarity

Flat clustered graph: nontrivial clusters form a partition of V



Clustered planarity

Clustered graph (G, T) is **clustered planar** if there is

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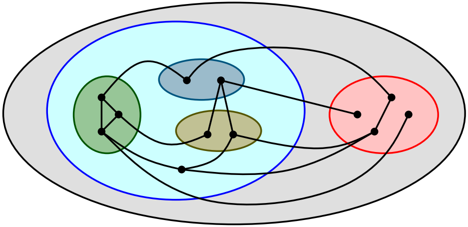
Clustered planarity

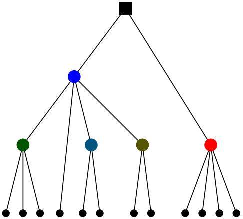
Clustered graph (G, T) is **clustered planar** if there is

- a plane embedding of G and
- a representation of the clusters as topological discs

such that

- disjoint clusters are drawn as disjoint discs,
- the containment among the clusters and vertices is preserved, and
- every edge of G crosses the boundary of each cluster at most once.





Such a representation is called a **clustered embedding** of (G, T).

Clustered planarity

introduced by Feng, Cohen and Eades (1995)
and also by Lengauer (1989) (“hierarchical planarity”)

Problem: Is there a polynomial algorithm for testing clustered planarity?

yes in special cases:

- c-connected clustered graphs (Lengauer, 1989; Feng, Cohen and Eades, 1995; Cortese et al., 2008)
- almost connected clustered graphs (Gutwenger et al., 2002)
- extrovert clustered graphs (Goodrich, Lueker and Sun, 2006)
- two clusters (Biedl, 1998; Gutwenger et al., 2002; Hong and Nagamochi, 2009)
- cycles, clusters form a cycle (Cortese et al., 2005)
- cycles, clusters form an embedded plane graph (Cortese et al., 2009)

- cycles and 3-connected graphs, clusters of size at most 3 (Jelínková et al., 2009)
- at most 4 outgoing edges (Jelínek et al., 2009a)
- at most 5 outgoing edges (Bläsius and Rutter, 2014)
- each cluster and its complement have at most two components (Bläsius and Rutter, 2014)
- embedded graphs, each cluster has at most 2 components (Jelínek et al., 2009b)
- embedded graphs with at most 5 vertices per face (Di Battista and Frati, 2007)
- embedded graphs with at most 2 vertices per face and cluster (Chimani et al., 2014)

Main goal of our project

- improve our theoretical insight into clustered planarity
- obtain alternative, simpler algorithms

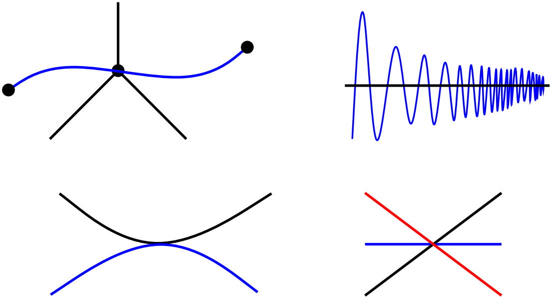
We do NOT aim for optimizing the running time.

Our main tool

Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every two non-adjacent edges cross an even number of times.

In a **drawing** the following situations are forbidden:



embedding = drawing with no crossings

Our main tool

Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every two non-adjacent edges cross an even number of times.

Weak Hanani–Tutte theorem: (Cairns and Nikolayevsky, 2000; Pach and Tóth, 2000; Pelsmajer, Schaefer and Štefankovič, 2007)

If a graph G has an **even** drawing D in the plane (every two edges cross an even number of times), then G is planar. Moreover, G has a plane embedding with the same rotation system as D.

recommended reading:

- M. Schaefer, Hanani-Tutte and related results (2011)
- Fulek et al., Hanani-Tutte, Monotone Drawings, and Level-Planarity (2012)
- M. Schaefer, Toward a theory of planarity: Hanani-Tutte and planarity variants (2013)

Algebraic planarity algorithm

(Tutte, 1970; Wu, 1985; Schaefer, 2011)

given a graph G

- draw an arbitrary drawing D of G
- for every pair of independent edges e, f, define $x^D_{e,f} = 1$ if e and f cross oddly and $x^D_{e,f} = 0$ if e and f cross evenly.
- by the Hanani–Tutte theorem, G is planar if and only if there is a drawing D' such that all $x^{D'}_{e,f} = 0$.

Algebraic planarity algorithm

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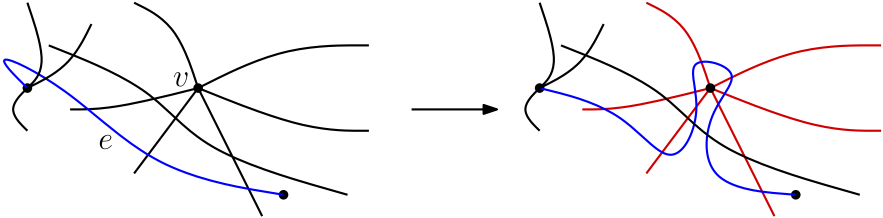
- during a continuous deformation, the vector $\mathbf{x}^D$ changes only when an edge passes over a vertex



Algebraic planarity algorithm

(Tutte, 1970; Wu, 1985; Schaefer, 2011)

given a graph G

- draw an arbitrary drawing D of G
- for every pair of independent edges e, f, define $x_{e,f}^D = 1$ if e and f cross oddly and $x_{e,f}^D = 0$ if e and f cross evenly.
- by the Hanani–Tutte theorem, G is planar if and only if there is a drawing D' such that all $x_{e,f}^{D'} = 0$.
- during a continuous deformation, the vector $\mathbf{x}^D$ changes only when an edge passes over a vertex
- the **edge-vertex switch** is represented by a vector $\mathbf{y}^{(e,v)}$ over $\mathbb{Z}_2$

Algebraic planarity algorithm

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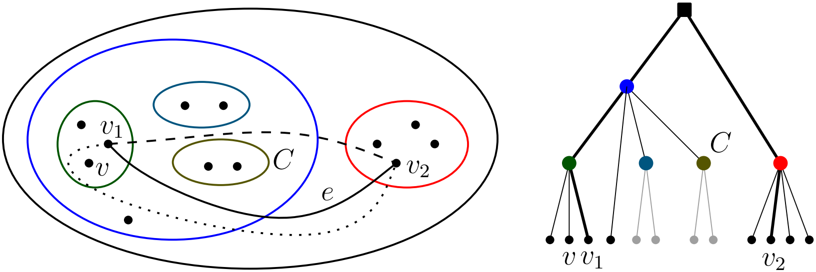
- for every pair of independent edges e, f, define $x^D_{e,f} = 1$ if e and f cross oddly and $x^D_{e,f} = 0$ if e and f cross evenly.
- by the Hanani–Tutte theorem, G is planar if and only if there is a drawing D' such that all $x^{D'}_{e,f} = 0$.
- during a continuous deformation, the vector $\mathbf{x}^D$ changes only when an edge passes over a vertex
- the **edge-vertex switch** is represented by a vector $\mathbf{y}^{(e,v)}$ over $\mathbb{Z}_2$
- G is planar if and only if $\mathbf{x}^D$ is a linear combination of the vectors $\mathbf{y}^{(e,v)}$
- solve the linear system!

Algebraic algorithm for clustered planarity

modifications:

- start with a **clustered** drawing (with edge crossings)
- assume a Hanani–Tutte theorem for the corresponding variant of clustered planarity
- for every edge $e = v_1 v_2$, we allow only those edge-vertex switches (e, v) and **edge-cluster switches** (e, C) such that v and C are children of some vertices of the shortest path between v_1 and v_2 in T.

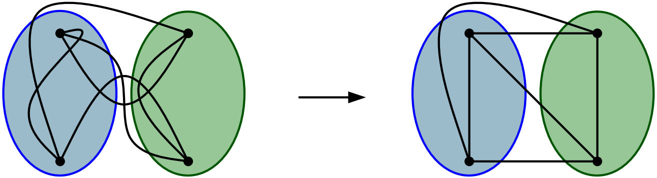


a different algorithm: Gutwenger, Mutzel and Schaefer (2014)

Main result

Theorem: (Hanani–Tutte for two clusters)

Let $\mathcal{G} = (G, (A, B))$ be a flat clustered graph with two clusters A, B forming a partition of the vertex set. If $\mathcal{G}$ has an independently even clustered drawing in the plane, then $\mathcal{G}$ is clustered planar.

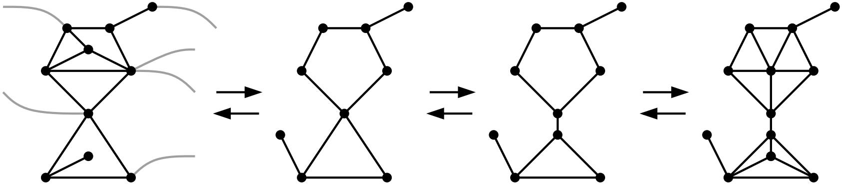


- Hanani–Tutte for c-connected clustered graphs
- weak Hanani–Tutte for two clusters
- generalization: weak Hanani–Tutte for strip planarity (Fulek, 2014)

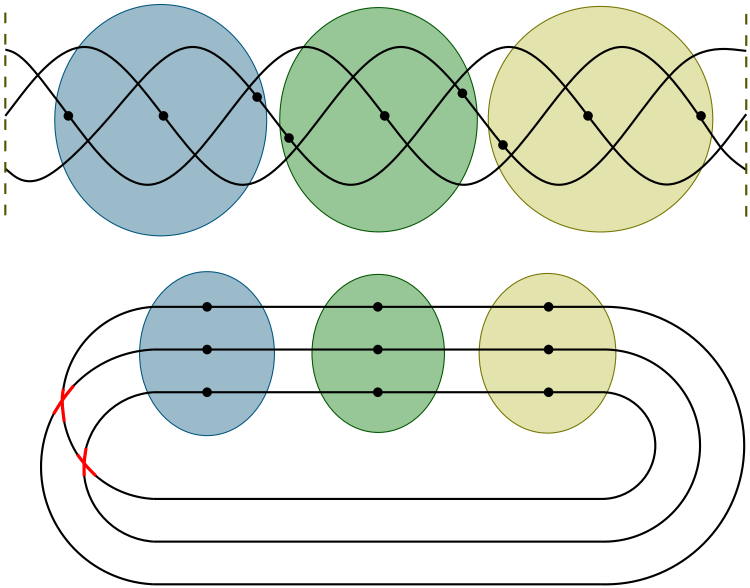
Sketch of the proof

given an independently even clustered embedding D of $\mathcal{G} = (G, A, B)$

- modify $\mathcal{G}$ and D:
  - create a **cactus** from each component of $G[A]$ and $G[B]$
  - make all cycles in $G[A]$ and $G[B]$ vertex disjoint by splitting vertices (edge decontractions)
  - fill all cycles with **wheels**
- apply the Hanani–Tutte theorem to the modified drawing
- flip all you can to the outer face
- remove the interiors of the wheels, contract the new edges, and draw the rest of G
- draw two disjoint discs around A and B



What about three clusters?



Are there other counterexamples???