On the product of balanced sequences

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JM2010, Amiens, France, September 6-10, 2010

Balanced sequences

A infinite sequence v is *balanced* if for each letter a of the alphabet A and for all factors u and u' of v s.t. $|u| = |u'|$ we have that

$$||u|_a - |u'|_a| \leq 1$$

Example

- $w = abcadbcadbacbdacbd \cdots$ is a balanced sequence.
- $v = \mathbf{abcb}dbc\mathbf{cad}bacbdacbd \cdots$ is not a balanced sequence.

Remark

For a two-letter alphabet, being balanced is equivalent to being balanced with respect to one letter.

Binary case

- An infinite aperiodic sequence v is balanced if and only if v is a sturmian sequence.
- Sturmian sequences are defined as the infinite sequences having exactly $n+1$ distinct factors of length n.
- An infinite periodic sequence v^ω is balanced if and only if v is a conjugate of a standard word.

Example

Fibonacci words

$f_0 = b$
$f_1 = a$
$f_2 = ab$
$f_3 = aba$

$$f_0 = b \qquad f_1 = a$$
$$f_{n+1} = f_n f_{n-1} \ (n \geq 1)$$

The infinite Fibonacci word is the limit of the sequence of Fibonacci words.

Balanced words on larger alphabets

- If $|A| > 2$, the general structure of balanced words is not known.
- As a direct consequence of a result of Graham, one has that balanced sequences on a set of letters having different frequencies must be periodic.

Fraenkel's conjecture

Let $A_k = \{a_1, a_2, \ldots, a_k\}$. For each $k > 2$, there is only one circularly balanced word $F_k \in A_k^*$, having different frequencies. It is defined recursively as follow $F_1 = a_1$ and $F_k = F_{k-1} a_k F_{k-1}$ for all $k \geq 2$.

Direct product

Let us define a *direct product* of two infinite sequences $u = u_0 u_1 \cdots$ and $v = v_0 v_1 \cdots$ on $\Sigma = \{a, b\}$ as the sequence

$$u \otimes v =< u_0, v_0 >< u_1, v_1 > \cdots .$$

on $\Sigma \times \Sigma$.

u :	0	0	1	1
v :	0	1	0	1
w :	a	b	c	d

We define the *degree* of product, $deg(w)$, as the cardinality of the alphabet of the product itself.

The notion of product of two sequences has been introduced in [P. Salimov. On uniform recurrence of a direct product. In AutoMathA 2009], where the author studies the class of uniformly recurrent sequences such that the product of any of its members and each uniformly recurrent sequence is also uniformly recurrent.

Question

We ask us: when the product of two balanced sequences is balanced too?

Example

Consider the Fibonacci sequence f and the sturmian sequence s:

$$\begin{array}{cccccccccccccccccccccccc}
f: & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\
s: & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\
\hline
w: & a & c & b & a & c & a & d & a & a & d & a & a & c & b & c & a & a & d & a & c & b & \cdots
\end{array}$$

w is not a balanced sequence, because w has factors $u = aa$ and $v = cb$, for which $||u|_a - |v|_a| = 2$.

Example

Consider the two following sturmian sequences:

$$\begin{array}{cccccccccccccccccccccccc}
r: & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\
z: & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots \\
\hline
t: & a & d & a & b & c & a & b & a & d & a & b & a & c & b & a & d & a & b & a & c & b & \cdots
\end{array}$$

t is a balanced sequence.

On four-letters alphabets

Theorem

Let u, v be two binary balanced sequences. If $w = u \otimes v$ is balanced and $deg(w) = 4$ then w is (ultimately) periodic and is a suffix of one of the following sequences:

i) $(adacb)^t(adabc)^\omega$

ii) $(adabc)^t(adacb)^\omega$

iii) $(adabacb)^t(adabcab)^\omega$

iv) $(adabcab)^t(adabacb)^\omega$

where $t \in \mathbb{N}$.

On three-letters alphabets

Theorem

Any balanced sequence w on three letters can be obtained as the product of two binary balanced sequences u and v.

Example

The balanced sequence $w = abaadaabadaabaada\cdots$ is the product of two balanced sequences $u = 00001000010000010\cdots$ and $v = 01001001010010010\cdots$.

$$\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
\hline\hline
a & b & a & a & d & a & a & b & a & d & a & a & b & a & a & d & a & \cdots
\end{array}$$

On three-letters alphabets

Theorem

For any binary balanced sequence v, one can construct a binary balanced sequence u such that $w = u \otimes v$ is balanced and $deg(w) = 3$.

Example

If $v = 0010010010001001001 0\cdots$ then $u = 00000100000010000010\cdots$.

$$
\begin{array}{ccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\
\hline
a & a & b & a & a & d & a & a & b & a & a & a & d & a & a & b & a & a & d & a & \cdots
\end{array}
$$

And $w = u \otimes v = aabaadaabaaadaabaada\cdots$ is balanced.

Conclusions and further works

- We have proved that:
  - All balanced (periodic or aperiodic) sequences on an alphabet with three letters are obtained by the product of two binary balanced sequences.
  - There exist only finitely many balanced sequences on four letters that can be obtained as product of two binary balanced sequences. Moreover they are ultimately periodic.

Conclusions and further works

Given two integer k and h, one could determine the maximum degree of the product $w = u \otimes v$, such that u, v are balanced sequences, $deg(u) = k$ and $deg(v) = h$:

$$m(k,h) = max\{deg(w) \text{ s.t. } w = u \otimes v, u, v \in \mathcal{B}, deg(u) = k, deg(v) = h\}$$

where $\mathcal{B}$ denotes the set of the balanced sequences.

Example

$$\begin{array}{rcccccccccccccccccccl}
u: & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots & deg(u) = 2 \\
v: & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & \cdots & deg(v) = 3 \\
\hline \hline
w: & a & d & b & c & a & d & b & c & d & a & c & b & d & a & c & b & d & c & \cdots & deg(w) = 4
\end{array}$$

Several experiments suggest that it is not possible to obtain a balanced sequence w with $deg(w) = 5$ or $deg(w) = 6$ as product of two balanced sequences u and v, where $deg(u) = 2$ and $deg(v) = 3$.

Conclusions and further works

Example

u :	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	$\cdots$	$deg(u) = 2$
v :	0	1	2	0	1	2	0	2	1	0	2	1	0	2	1	2	$\cdots$	$deg(v) = 3$
w :	a	b	d	a	b	c	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w) = 4$

u, v, and w are balanced sequences.

Example

u' :	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	$\cdots$	$deg(u') = 2$
v' :	0	1	2	0	2	1	0	2	1	0	2	1	0	2	1	2	$\cdots$	$deg(v') = 3$
w' :	a	b	d	a	c	b	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w') = 4$

u', w' are two balanced sequences, but v' is not balanced sequence.

Conclusions and further works

And on five letters alphabets ...

Example

$$
\begin{array}{lcccccccccccccccccccccccccccc}
u: & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \color{red}1 & \color{red}0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & \cdots \\
v: & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & \color{red}2 & \color{red}1 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & \cdots \\
\hline \hline
w: & a & b & c & a & d & b & c & a & e & b & c & a & \color{red}d & \color{red}b & a & c & b & e & a & c & b & d & a & c & b & e & \cdots
\end{array}
$$

u, v, and w are balanced sequences, where $deg(u) = 3, deg(v) = 3, deg(w) = 5$.

Example

$$
\begin{array}{lcccccccccccccccccccccccccccc}
u': & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & \color{red}1 & \color{red}2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & \cdots \\
v': & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \color{red}0 & \color{red}1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & \cdots \\
\hline \hline
w': & a & b & c & a & d & b & c & a & e & b & c & a & \color{red}b & \color{red}d & a & c & b & e & a & c & b & d & a & c & b & e & \cdots
\end{array}
$$

$deg(u) = 3, deg(v) = 3, deg(w) = 5$. w' is a balanced sequence.

But u' and v' are not balanced sequences.

Conclusions and further works

Given k, is it possible to classify the balanced sequences $w = u \otimes v$, with $degree(w) = k$ according to $deg(u)$ and $deg(v)$?

Example

On a four-letter alphabet:

- There exist only finitely many balanced sequences on four letters that can be obtained as product of two binary balanced sequences. Moreover they are ultimately periodic.
- The balanced sequence $w = u \otimes v = adbcadbcdacbdacbdc \cdots$ is obtained as product of two balanced sequences u and v, where $deg(u) = 2$ and $deg(v) = 3$ (the previous example).
- Can all remaining balanced sequences w on four letters be obtained as product $u \otimes v$, where $deg(u) = 2$ and $deg(v) = 3$?

Conclusions and further works

- Clearly, a balanced sequence over k letters can always be obtained by the product of $k-1$ sequences.

Example

$$
\begin{array}{cccccccccccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\hline \hline
a & d & b & e & a & c & b & d & a & e & b & c & a & d & b & e & c & \cdots
\end{array}
$$

$$
\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\hline \hline
a & d & b & e & a & c & b & d & a & e & b & c & a & d & b & e & c & \cdots
\end{array}
$$

Is it possible to obtain the sequence as product of 3 binary balanced sequences?

Conclusions and further works

- To determine the smallest value of h such that a balanced sequence over a k-letters alphabet is obtained as product of h binary balanced sequences.

$$g(k) = min\{h \text{ s.t. } w = u_1 \otimes u_2 \otimes \cdots \otimes u_h, deg(w) = k, u_i \in \mathcal{B}, deg(u_i) = 2, \text{ for each } i\}$$

- Is it possible to classify the balanced sequences according to the different value of h?

Thank you

for your attention!