Popularity and Similarity in SAT

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SAT

Set of variables $\Sigma = \{a, b, c, \dots\}$
Literals: variables either affirmed a or negated $\neg a$
Clause: disjunction of literals $a \vee \neg b \vee \neg c$
Formula: conjunction of clauses $\{a \vee b \ , \ \neg a \vee c\}$
SAT: find an assignment $I : \Sigma \to \{0, 1\}$ such that
at least one literal of **every** clause is set to 1.
Note that $I(a) + I(\neg a) = 1$

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$$
\begin{array}{l}
a \vee \neg b \vee \neg c \\
\neg a \vee b \\
a \vee c \\
\neg a \vee c
\end{array}
$$

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$$a \vee \neg b \vee \neg c$$
$$\neg a \vee b$$
$$a \vee c$$
$$\neg a \vee c$$

$$a \to 0$$
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$$c \to 1$$

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$$a \vee \boxed{\mathrm{c}}$$
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Solving SAT in the 60's

$$a \vee \neg b \vee \neg c$$
$$\neg a \vee b$$
$$a \vee c$$
$$\neg a \vee c$$

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$a \vee c$

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$c = 0$ decision

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$\downarrow$ $a = 1$ unit propagation

$$a \vee \neg b \vee \neg c$$
$$\neg a \vee b$$
$$a \vee c$$
$\neg a \vee c \leftarrow$ Empty clause

Solving SAT in the 60's

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$$a \vee c$$
$$\neg a \vee c$$

$c = 0$ decision | $c = 1$ backtracking

$c = 0$	$c = 1$
$a \vee \neg b \vee \neg c$	$a \vee \neg b \vee \neg c$
$\neg a \vee b$	$\neg a \vee b$
$a \vee c$	$a \vee c$
$\neg a \vee c$	$\neg a \vee c$

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- SAT is NP-complete (whether P=NP is one of the Millennium Prize Problems)
- P=NP iff there exists a proof-system that can prove every tautology in polynomial size

  Resolution:

  $$\frac{\begin{array}{c} x \vee A \\ \neg x \vee B \end{array}}{A \vee B}$$

  The pigeon-hole principle require exponential resolution proofs
- Random SAT formulas require exponential tree-like refutations

SAT in Theory

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SAT in Practice

- Despite negative theoretical results...
  SAT solvers are able to solve industrial instances with thousands of variables and millions of clauses in few seconds
  ...not so random instances
- Every year we celebrate a SAT solver competition and we have hundreds of real-world SAT instances coming from industrial applications:

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  - Hardware verification
  - Software verification
  - Planning
  - ...

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Objectives

- Study the structural properties of real-world SAT instances
- Propose new models of random formulas
- Exploit this knowledge to improve SAT solvers specialized in those kind of formulas

"Modern" SAT Solvers at a Glance

- Learning: Every time we find a conflict, generate a learnt clause. This clause is redundant. We try to detect the conflict with less decisions next time
- VSID heuristics: Decide on those variables more frequently involved in recent conflicts
- Clause deletion: From time to time, remove some learnt clauses according to their "quality"
- Restarts: From time to time, forget the partially computed assignment keeping only learnt clauses

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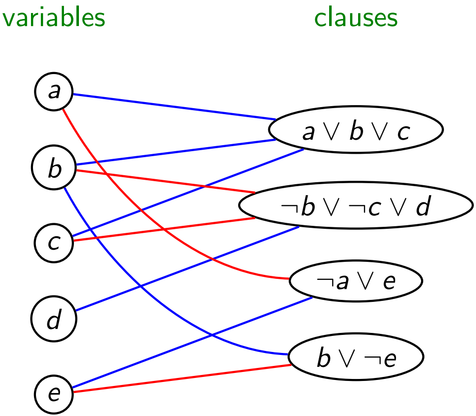
SAT Instances as Networks (or Graphs)



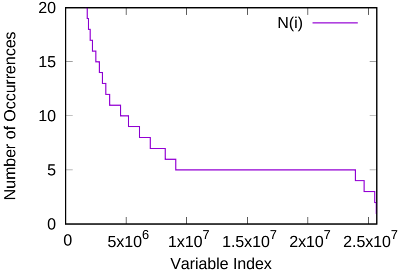
Analysis of Industrial Formulas

Consider all variables of the SAT'08 Competition and sort them

$N(i)$ = number of occurrences of i-th most frequent variable

Most have 5 occurrences, although the average is 13.6

A few have millions of occurrences!!!



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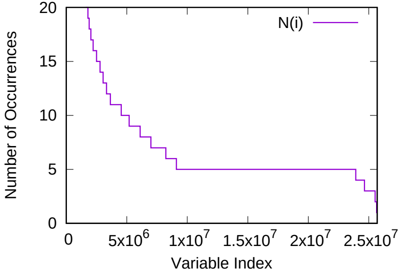
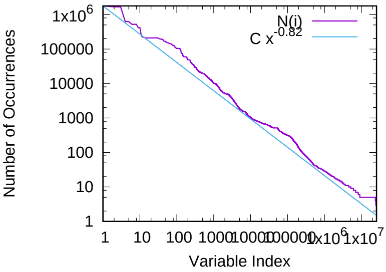


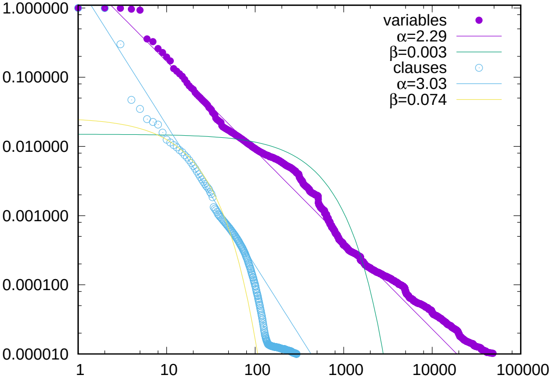


Expected number of occurrences of i-th most frequent variable and degree distribution

$$N(i) \sim i^{-0.82} \qquad P(k) \sim k^{-2.22}$$

Analysis of Industrial Formulas

Seen as a graph, industrial SAT formulas are scale-free on variables and on clauses degree distributions

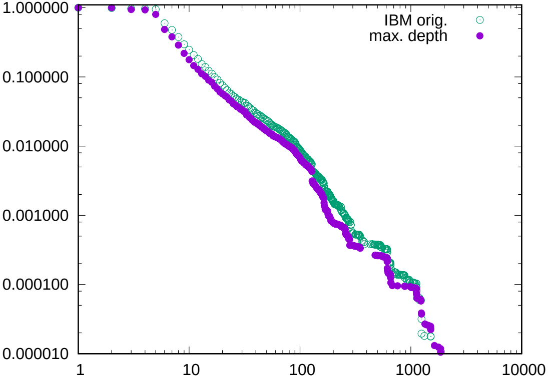


“Destroying” (Solving) SAT Formulas

- Instantiating 5% (even 1%) of most frequent variables randomly make most formulas unsatisfiable.
  SAT solvers do not instantiate so many
- Scale-free structure is not affected
  Notice that after instantiating we remove clauses and variables

"Destroying" (Solving) SAT Formulas

The effect of SAT solvers on an industrial formula:



Instantiation does not "destroy" scale-free structure.

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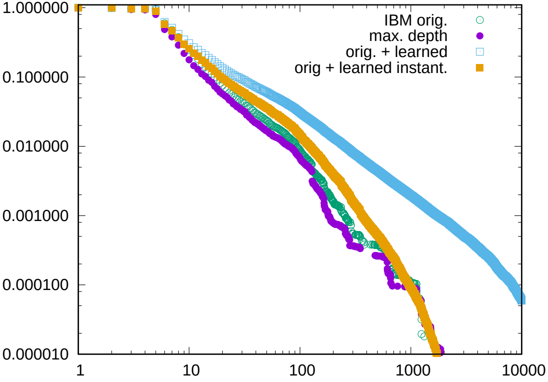
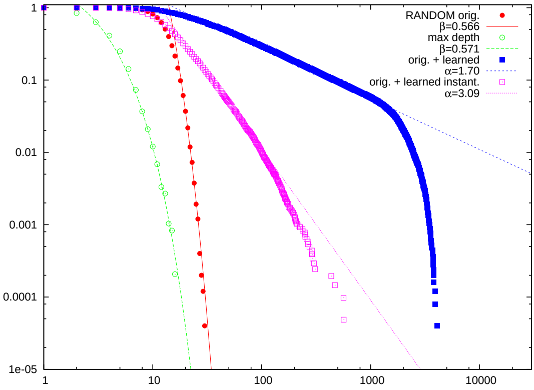


Instantiation does not “destroy” scale-free structure.
Learning seems to reinforce scale-free structure.

"Destroying" (Solving) SAT Formulas

The effect on a random (Erdös-Rényi) formula:



Learning acts as preferential attachment!

Random Scale-Free Formulas

- Theoretical SAT researchers: We were wrong! Classical random SAT formulas (Erdös-Rényi model) are not appropriate, use (simple) random scale-free formulas:
  Sample variable x_i with $P(x_i) \sim i^{-\beta}$, to get $P(k) \sim k^{1+1/\beta}$
- Random formulas exhibit a linear SAT-UNSAT phase transition
  In scale-free formulas (for big β) this is sub-linear
  Proofs are small!
- Since formulas are scale-free, the best variable branching heuristics is assigning most frequent variables
  We tried it in the past, BUT we have better heuristics
- Scale-free formulas are too easy on practice
- Fortunately... I met Dmitri (thanks Dmitri)... and he told me about similarity

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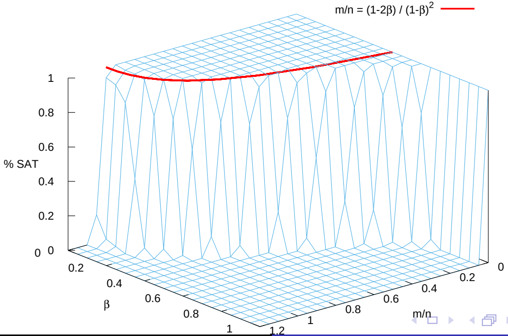

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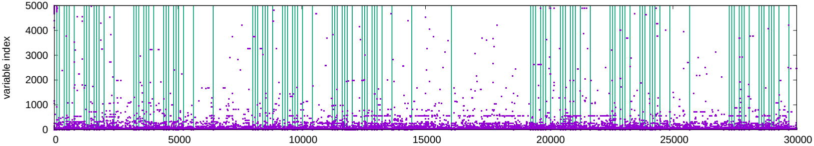
Random Formulas with Popularity and Similarity

$\beta = 0.8$ $T = 1.5$ (vars. ordered by popularity)



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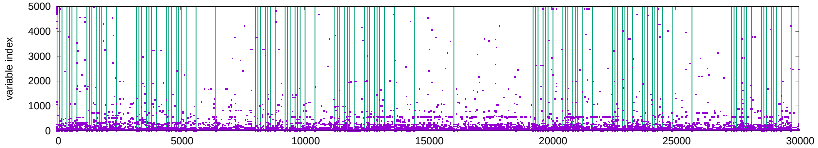
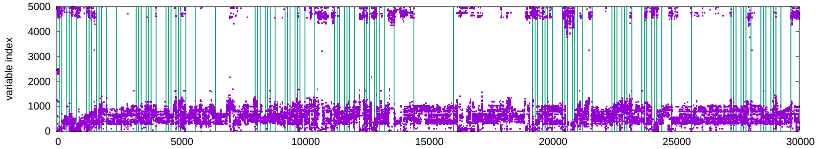


$\beta = 0.1$ $T = 0.75$ (vars. ordered by similarity)



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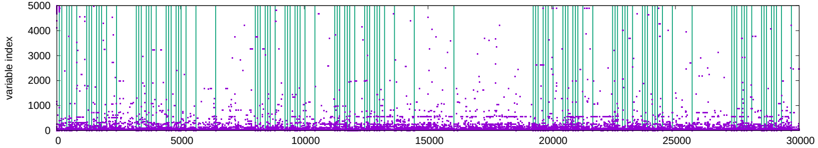

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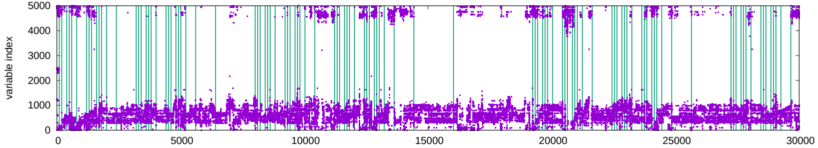
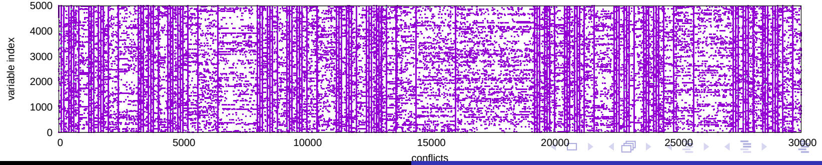


$T = 100$ ($\beta = 0.1$ and vars. ordered by similarity)



Conclusions and Further Work

- An equilibrium between the forces of popularity and similarity defines the structure of industrial SAT instances
- Modern SAT solvers exploit both properties
- Explicit computation of variable coordinates may lead to better branching heuristics
- Analysis of the temperature of formulas may characterize their difficulty