The Complexity of Several Realizability Problems for Abstract Topological Graphs

Jan Kynčl

Charles University, Prague

Graph: $G = (V, E)$, $|V| < \infty$, $E \subseteq \binom{V}{2}$

Topological graph: a drawing of a graph in the plane

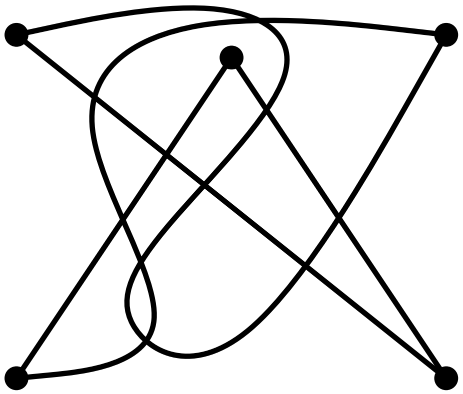
vertices $=$ points

edges $=$ simple curves

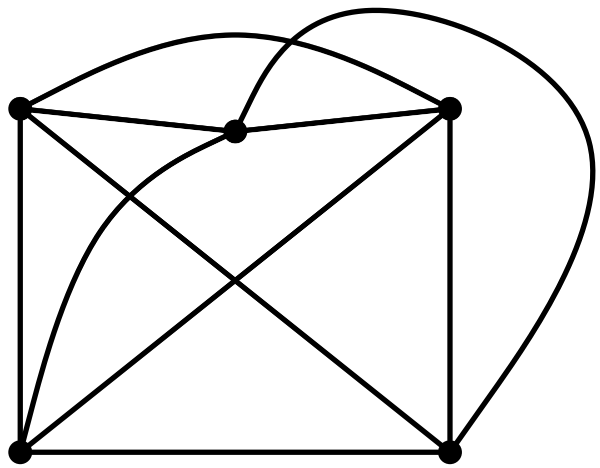
- edges do not pass through any vertices other than their end-points
- any two edges have only finitely many common points
- any intersection point of two edges is either a common end-point or a **crossing** (no touching allowed)
- at most two edges can intersect in one crossing

simple: any two edges have at most one common point

complete: $E = \binom{V}{2}$



topological graph



simple complete topological graph

Abstract topological graph (AT-graph):

$A = (G, R)$; $G = (V, E)$ is a graph, $R \subseteq \binom{E}{2}$

in a topological graph T ... $R_T =$ set of crossing pairs of edges

AT-graph A is

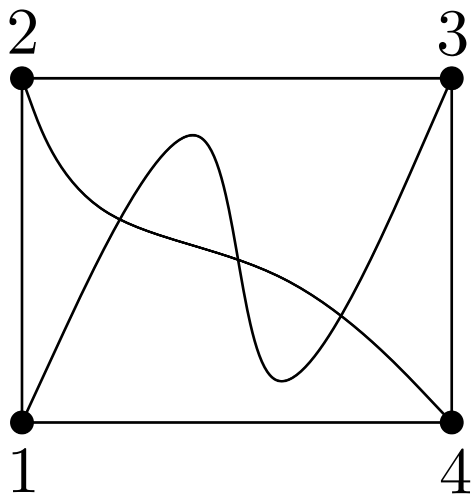
realizable if there exists a topological graph T which is a drawing of G and $R_T = R$.

simply realizable ... T is simple

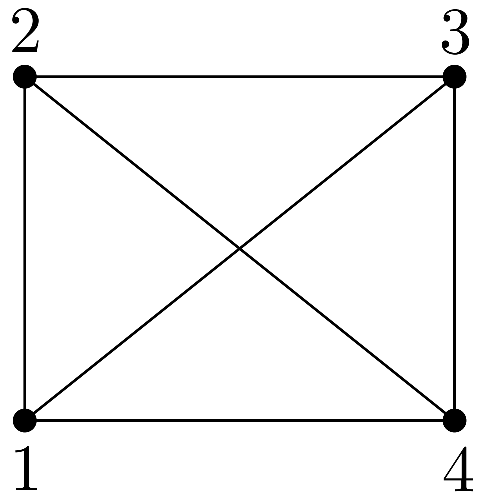
rectilinearly realizable ... edges of T are straight-line segments

weakly realizable ... $R_T \subseteq R$

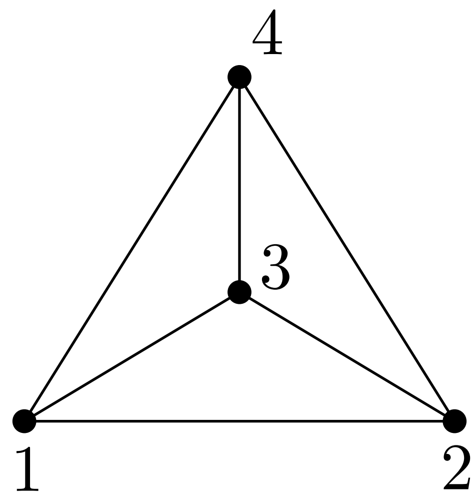
Example: $A = (K_4, \{\{\{1,3\},\{2,4\}\}\})$



realization



simple realization



weak realization

(simple, weak, rectilinear) realizability:

instance: AT-graf A
question: is A (simply, weakly, rectilinearly) realizable?

Theorem: [J. Kratochvíl, 1991]
The realizability and the weak realizability are NP-hard.

Theorem: [J. Pach, G. Tóth, 2002;
M. Schaefer, D. Štefankovič, 2004]
The realizability and the weak realizability are decidable.

Theorem: [M. Schaefer, E. Sedgwick, D. Štefankovič, 2004]
The realizability and the weak realizability are in NP.

Main results

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability		
weak r.		
simple r.		
weak simple r.		
weak rectilin. r.		

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	
weak r.	NP-complete [K91, SSŠ04]	
simple r.		
weak simple r.		
weak rectilin. r.		

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	
weak r.	NP-complete [K91, SSŠ04]	
simple r.	NP-complete	
weak simple r.	NP-complete	
weak rectilin. r.		

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	
weak r.	NP-complete [K91, SSŠ04]	
simple r.	NP-complete	
weak simple r.	NP-complete	
weak rectilin. r.	NP-hard	

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	NP-complete
weak r.	NP-complete [K91, SSŠ04]	NP-complete
simple r.	NP-complete	
weak simple r.	NP-complete	
weak rectilin. r.	NP-hard	

Main results

Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	NP-complete
weak r.	NP-complete [K91, SSŠ04]	NP-complete
simple r.	NP-complete	
weak simple r.	NP-complete	NP-complete
weak rectilin. r.	NP-hard	NP-hard

Main results

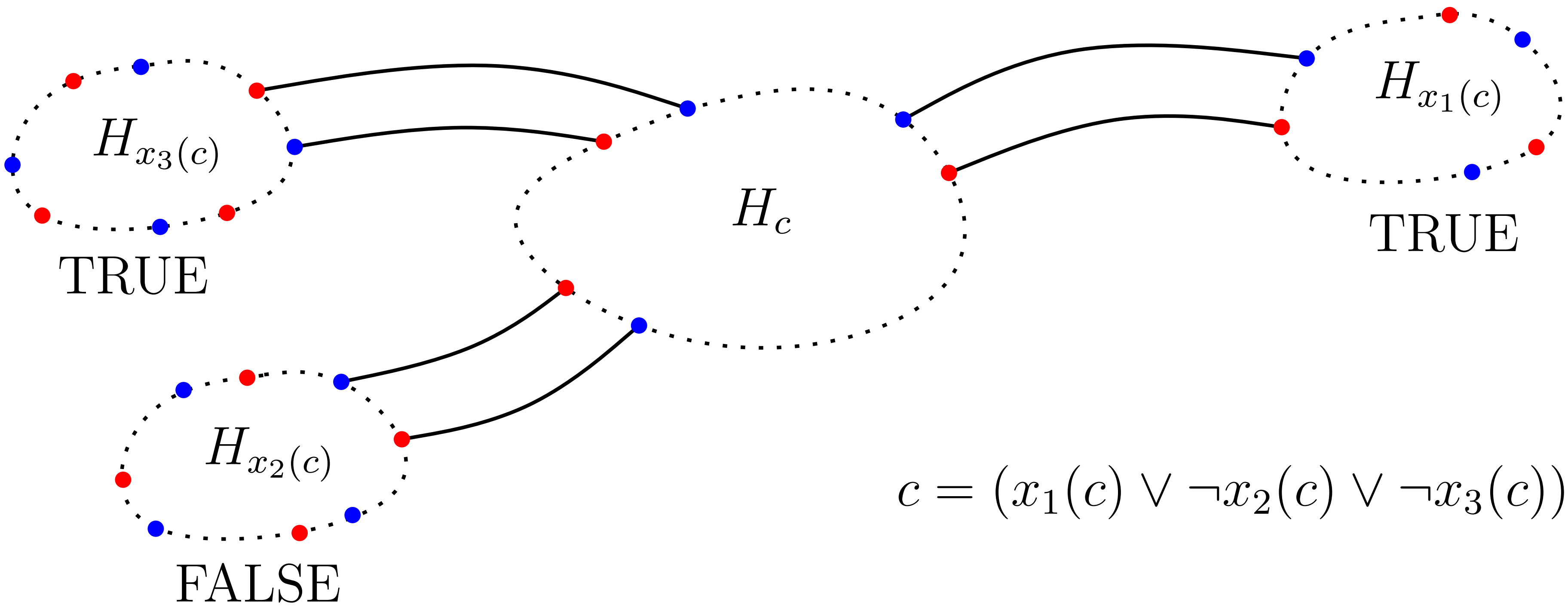
Theorem:

	AT-graphs	complete AT-graphs
realizability	NP-complete [K91, SSŠ04]	NP-complete
weak r.	NP-complete [K91, SSŠ04]	NP-complete
simple r.	NP-complete	P
weak simple r.	NP-complete	NP-complete
weak rectilin. r.	NP-hard	NP-hard

NP-hard problems

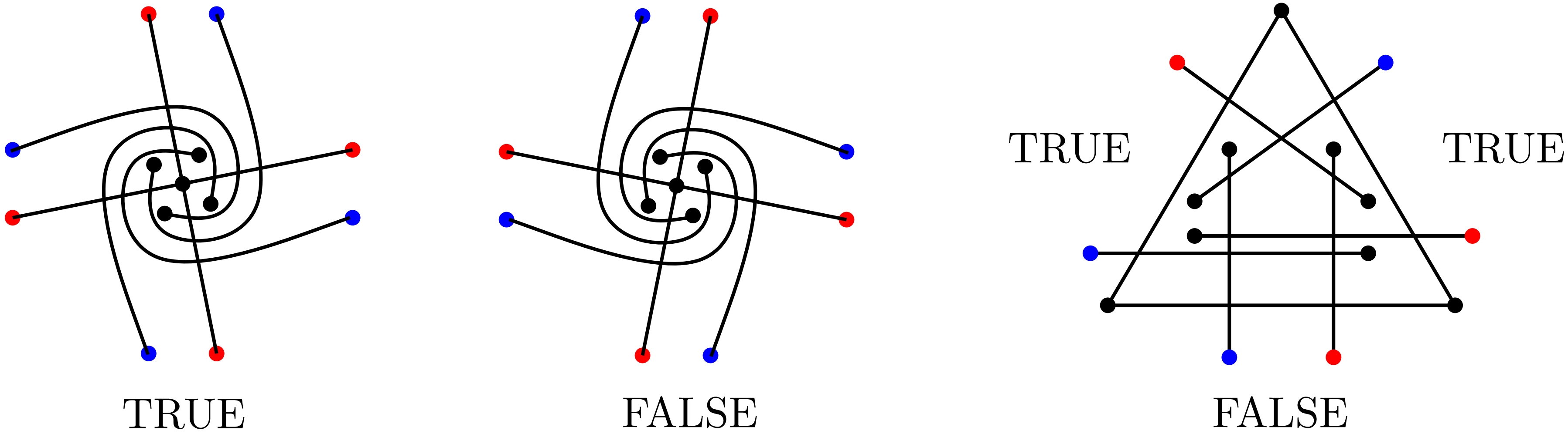
main idea of the proof:

reduction from the
planar 3-connected 3-SAT [J. Kratochvíl, 1991]
which is an NP-complete problem [J. Kratochvíl, 1994]



$$c = (x_1(c) \vee \neg x_2(c) \vee \neg x_3(c))$$

example of variable and clause gadgets for the simple realizability:



Simple realizability of complete AT-graphs

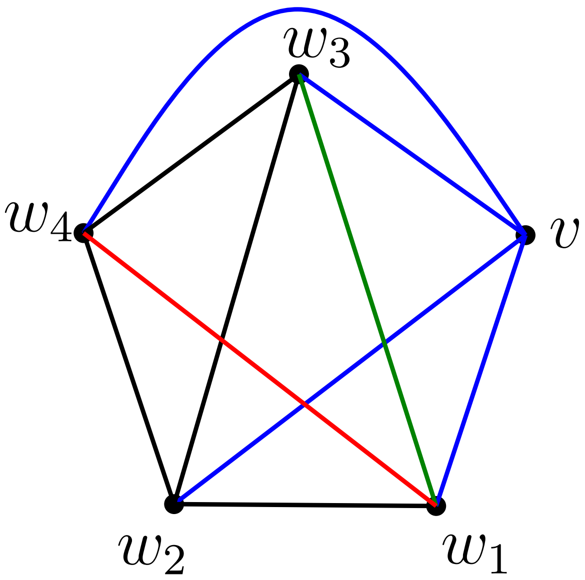
Proposition:

(1) A complete AT-graph determines the extended rotation system of its simple realization (up to inversion).

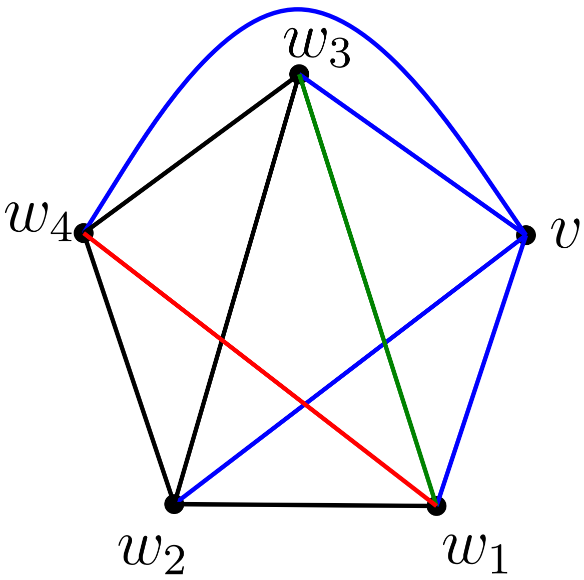
(2) For every edge e of a simple complete topological graph T and for each pair of edges $f, f' \in E(G)$ that have a common end-point and cross e, the AT-graph of T uniquely determines the order of crossings of e with the edges f and f'.

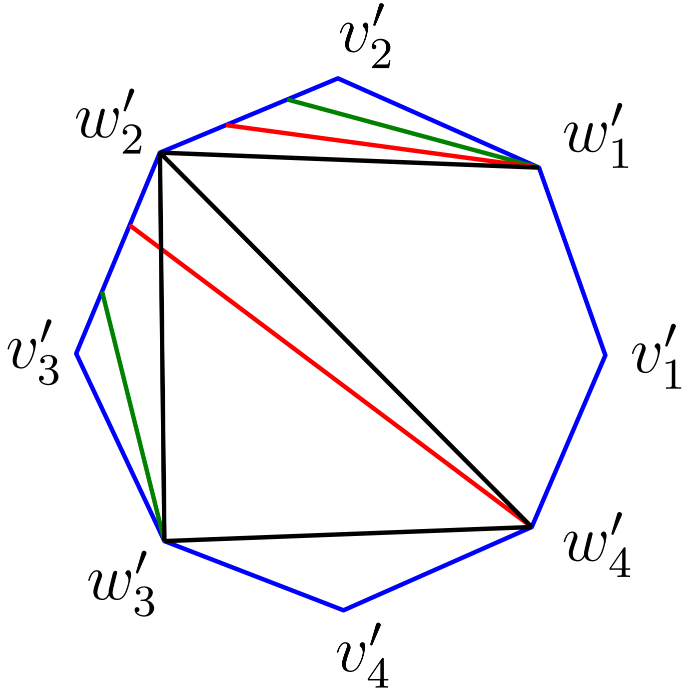
star-cut representation:

star-cut representation:



star-cut representation:





Algorithm:

- for each induced subgraph on 5 vertices: the rotation system
- the extended rotation system of the whole graph
- for a chosen vertex v, for each non-incident edge e: the order in which e crosses the edges of the star $S(v)$
- a (partial) star-cut representation
- the order of the end-points of the pseudochords on the perimeter minimizing the total number of crossings
- the order of crossings of pseudochords with other pseudochords