Polynomial-time Computation of Exact Correlated Equilibrium in Compact Games

Albert Xin Jiang Kevin Leyton-Brown
Department of Computer Science
University of British Columbia

Outline

Correlated Equilibrium

- correlated equilibrium (CE) [Aumann, 1974; Aumann, 1987]

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  - n players
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- a CE is a distribution x over S:
  - a trusted intermediary draws a strategy profile s from this distribution
  - announce to each player p (privately) her own component s_p
  - p will have no incentive to choose another strategy, assuming others follow suggestions

LP formulation

- incentive constraints: for all players p and all $i, j \in S_p$:

$$\sum_{s \in S_{-p}} [u^p_{is} - u^p_{js}] x_{is} \geq 0$$

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- m^n variables, nm^2 constraints
- polynomial in the size of normal form

Computing CE for Compact Game Representations

Representations for games with structured utility functions

- symmetric games / anonymous games
- graphical games [Kearns, Littman & Singh, 2001]
- congestion games [Rosenthal, 1973]
- action-graph games [Jiang, Leyton-Brown & Bhat, 2011]

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Goal: computing a sample CE in time polynomial in the size of representation

- LP would have exponential number of variables (m^n)
- writing a solution (i.e. CE) explicitly requires m^n numbers

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Papadimitriou and Roughgarden [2008]

- Polynomial-time algorithm for computing a CE when the representation satisfies:
  - polynomial type: # of players and # of actions for each player are bounded by polynomials in the size of the representation.
  - polynomial expectation property: poly-time algorithm for computing expected utility under any product distribution
    - x is a product distribution when each player p is randomizing independently over her actions according to some distribution x^p, i.e. $x_s = \prod_p x^p_{s_p}$.

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- computes CEs that are mixtures of polynomial number of product distributions

Existence Proof

The algorithm is based on proofs of the existence of CE via LP duality [Hart & Shmeidler 1989], [Nau & Mcardle 1990], [Myerson 1997]

- consider the linear program (P):

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- (P) either has $x = 0$ as its optimal solution or is unbounded. In the latter case the game has a correlated equilibrium.
- can prove the existence of CE by showing the infeasibility of its dual (D):

$$U^T y \leq -1$$
$$y \geq 0$$

Infeasibility of the Dual

$$U^T y \leq -1$$
$$y \geq 0$$

Lemma ([Papadimitriou & Roughgarden, 2008])

For every dual vector $y \geq 0$, there is a product distribution x such that $xU^T y = 0$.

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Lemma ([Papadimitriou & Roughgarden, 2008])

For every dual vector $y \geq 0$, there is a product distribution x such that $xU^T y = 0$.

- The lemma implies that the dual program (D) is infeasible (and therefore a CE must exist).
  - This is because $xU^T y$ is a convex combination of the left hand sides of the rows of (D), and for any feasible y the result must be less than or equal to -1.

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- there is a polynomial-time algorithm that computes such an x given y.

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- run the ellipsoid algorithm on (D), with the following Product Separation Oracle:
  - given a vector $y^{(i)}$, the corresponding product distribution $x^{(i)}$ is generated according to the Lemma, and $[x^{(i)} U^T] y \leq -1$ is given to the ellipsoid algorithm as a cutting plane.

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- The ellipsoid algorithm will stop after a polynomial number of steps and determine that the program is infeasible.

Ellipsoid Against Hope (cont'd)

- Let X be the matrix whose rows are the generated product distributions $x^{(1)}, \ldots, x^{(L)}$. Consider the linear program (D'):

$$[XU^T]y \leq -1, \qquad y \geq 0$$

  If we apply the same ellipsoid method, with a separation oracle that returns the cut $x^{(i)}U^T y \leq -1$ given query $y^{(i)}$, it would go through the same sequence of queries $y^{(i)}$ and return infeasible.
- Therefore (D') is infeasible (presuming that numerical problems do not arise).

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- Therefore (D') is infeasible (presuming that numerical problems do not arise).
- This implies that its dual program (P'):

$$[UX^T]\alpha \geq 0, \qquad \alpha \geq 0$$

  is unbounded and has polynomial size. Given such a nonzero α vector, scaled to be a distribution, $X^T\alpha$ satisfies the incentive constraints and is therefore a correlated equilibrium.

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- On the other hand, entries of XU^T are differences of expected utilities under product distributions, thus can be computed in polynomial time.

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  - initial ball with radius R
  - volume lower bound v
- correct values of R and v depend on the max encoding size of a constraint of the LP

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- infeasibility of (D') is not guaranteed
- Papadimitriou and Roughgarden [2008] proposed a method to overcome this issue
- Stein, Parrilo & Ozdaglar [2010] showed that it is insufficient to compute an exact CE.
  - a slightly modified version computes approximate CE in time polynomial in $\log \frac{1}{\epsilon}$ and representation size

What about exact CE?

- Stein *et al.* [2010] showed that if an algorithm
  1. outputs a rational solution
  2. outputs a convex combination of product distributions
  3. outputs a convex combination of symmetric product distributions when the game is symmetric

  then there is a symmetric game such that the algorithm fails to find an exact CE.

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  then there is a symmetric game such that the algorithm fails to find an exact CE.
- The Product Separation Oracle returns a symmetric product distribution given symmetric game and symmetric y.
- On the other hand, there always exists an exact rational CE
  - each vertex of the polytope of the set of CE is rational (correspond to basic feasible solutions)
  - such a CE has $O(nm^2)$ non-zero entries, i.e. polynomial-sized support

Outline

Our Results

A variant of the Ellipsoid Against Hope algorithm of [Papadimitriou & Roughgarden, 2008] that

- computes an exact, rational CE in polynomial time given a representation satisfying polynomial type and polynomial expectation property;
- outputs a CE that is a vertex of the set of CE, which has polynomial-sized support.

Overview of Our Approach

- We replace the Product Separation Oracle with a modified version (Purified Separation Oracle) that generates cuts corresponding to pure strategy profiles: given $y \geq 0$, output $(U_s)^T y \leq -1$ that is violated at y.

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- We replace the Product Separation Oracle with a modified version (Purified Separation Oracle) that generates cuts corresponding to pure strategy profiles: given $y \geq 0$, output $(U_s)^T y \leq -1$ that is violated at y.
- Now each constraint of (D') is one of the original constraints of (D).
  - any run of ellipsoid method that is valid for (D) is also valid for (D')
  - no longer requiring special mechanism to deal with numerical issues
- A solution of (P') is a mixture of polynomial number of pure-strategy profiles.
- Get vertex by applying a standard algorithm for finding basic feasible solutions given a feasible solution.

Purified Separation Oracle: Existence

Lemma

Given any dual vector $y \geq 0$, there exists a pure strategy profile s such that $(U_s)^T y \geq 0$.

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Proof.

- we know there exists a product distribution x such that $xU^T y = 0$.
- $x[U^T y]$ is the expected value of $(U_s)^T y$ under distribution x, which we denote $E_{s \sim x}[(U_s)^T y]$
- there must exist s such that $(U_s)^T y \geq xU^T y = 0$.

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- not efficiently constructive
- sampling from x yields approximate cutting planes

Purified Separation Oracle: Derandomization

Derandomize using the method of conditional probabilities

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   - pick $s_p \in S_p$ such that the conditional expectation

$$E_{s\sim x}[(U_s)^T y | s_1, \ldots, s_p] \geq 0.$$

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Can return asymmetric cuts even for symmetric games and symmetric y.

Conclusion

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  - polynomial-sized support
- Our derandomization approach can be applied to other related algorithms, e.g. Huang and Von Stengel [2008] polynomial-time algorithm for extensive-form correlated equilibria.
- Practical computation of CE by replacing the ellipsoid method with a cutting-plane method

Aumann, R. (1974). Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, *1*(1), 67–96.

Aumann, R. (1987). Correlated equilibrium as an expression of Bayesian rationality. *Econometrica: Journal of the Econometric Society*, 1–18.

Huang, W., & Von Stengel, B. (2008). Computing an extensive-form correlated equilibrium in polynomial time. *WINE: Proceedings of the Workshop on Internet and Network Economics* (pp. 506–513).

Papadimitriou, C., & Roughgarden, T. (2008). Computing correlated equilibria in multi-player games. *Journal of the ACM*, *55*(3), 14.

Rosenthal, R. (1973). A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory*, *2*, 65–67.

Stein, N. D., Parrilo, P. A., & Ozdaglar, A. (2010). Exchangeable equilibria contradict exactness of the Papadimitriou-Roughgarden algorithm. `http://arxiv.org/abs/1010.2871v1`.