The $\mathbb{Z}_2$-genus of Kuratowski minors

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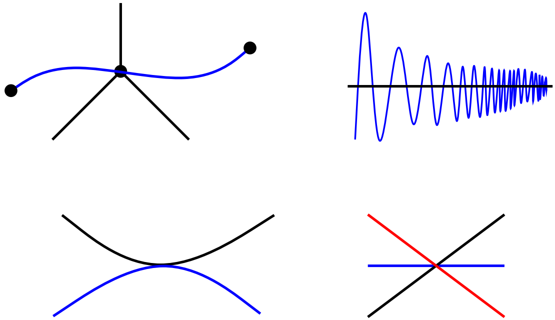
Drawings and embeddings

drawing of a graph:

vertices $\to$ points

edges $\to$ simple curves

forbidden:

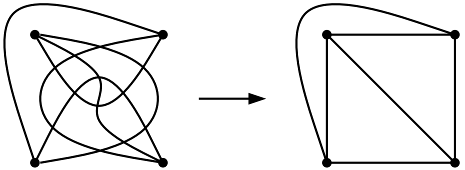


embedding = drawing with no crossings

Hanani–Tutte theorems

(Strong) Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.



application: polynomial-time algorithm for testing planarity

Hanani–Tutte theorems

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Weak Hanani–Tutte theorem: (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph G has an **even** drawing D in the plane (every pair of edges crosses an even number of times), then G is planar.
(Moreover, G has a plane embedding with the same rotation system as D.)

Hanani–Tutte theorems on surfaces

Weak Hanani–Tutte theorem on surfaces:
(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph G has an even drawing $\mathcal{D}$ on a surface S, then G has an embedding on S (that preserves the embedding scheme of $\mathcal{D}$).

(Strong) Hanani–Tutte theorem on the projective plane:
(Pelsmajer–Schaefer–Stasi, 2009;
Colin de Verdière–Kaluža–Paták–Patáková–Tancer, 2016)

If a graph G has an independently even drawing on the projective plane, then G has an embedding on the projective plane.

Problem: Can the strong Hanani–Tutte theorem be extended to other surfaces?

Partial answer: No to the orientable surface of genus 4 or larger
(Fulek–K., 2017)

Genus and $\mathbb{Z}_2$-genus

genus $\mathbf{g}(G)$ of a graph $G =$ minimum g such that G has an embedding on the orientable surface M_g of genus g.

$\mathbb{Z}_2$-genus $\mathbf{g}_0(G)$ of a graph G is the minimum g such that G has an independently even drawing on M_g.

Strong Hanani–Tutte theorem: $\mathbf{g}_0(G) = 0 \Rightarrow \mathbf{g}(G) = 0$.

Theorem: (Fulek–K., 2017)
There is a graph G with $\mathbf{g}(G) = 5$ and $\mathbf{g}_0(G) \leq 4$.
Consequently, for every positive integer k there is a graph G with $\mathbf{g}(G) = 5k$ and $\mathbf{g}_0(G) \leq 4k$.

Problem: (Schaefer–Štefankovič, 2013)
Is there a function f such that $\mathbf{g}(G) \leq f(\mathbf{g}_0(G))$ for every graph G?

Main result: YES, if a certain “folklore result” is true.

Bounding genus by $\mathbb{Z}_2$-genus—the plan

1) Ramsey-type statement:

If G has large genus $g = g(t)$, then G contains, as a minor, $G_1(t)$ or $G_2(t)$ or $\ldots$ or $G_r(t)$ of genus t.

2) Easier subproblem:

Show that the $\mathbb{Z}_2$-genus of each of $G_i(t)$ is unbounded in t.

Ramsey-type statements

Theorem: (Böhme–Kawarabayashi–Maharry–Mohar, 2008)
For every positive integer t, every sufficiently large 7-connected graph contains $K_{3,t}$ as a minor.

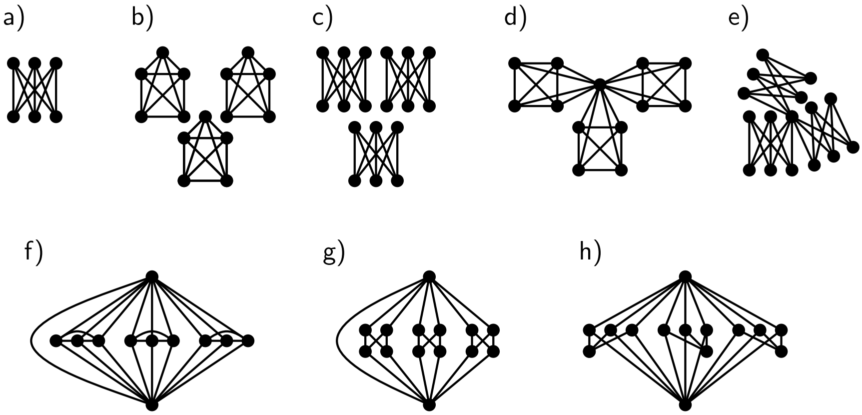
Generalization: (Böhme–Kawarabayashi–Maharry–Mohar, 2009)
- larger connectivity and $K_{a,t}$ minors for every fixed $a > 3$.

Unpublished "folklore" result: (Robertson–Seymour)
There is a function g such that for every $t \geq 3$, every graph of Euler genus $g(t)$ contains a ***t*-Kuratowski graph** as a minor.

***t*-Kuratowski graph**:

- $K_{3,t}$, or
- t copies of K_5 or $K_{3,3}$ sharing at most 2 common vertices

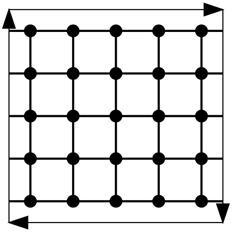
3-Kuratowski graphs



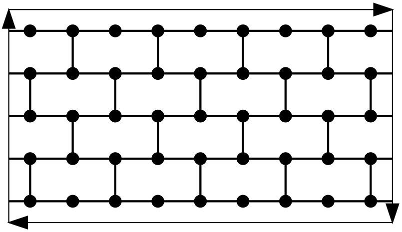
- what about graphs with large (orientable) genus and constant Euler genus?

Ramsey-type statement for genus

projective 5×5 **grid**



projective 5**-wall**



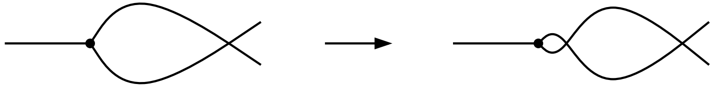
Theorem:
The "folklore result" implies that there is a function h such that for every $t \geq 3$, every graph of genus $h(t)$ contains, as a minor, a t-Kuratowski graph or the projective t-wall.

Lower bounds on the $\mathbb{Z}_2$-genus

Theorem: (Schaefer–Štefankovič, 2013)
If G consists of t copies of K_5 or $K_{3,3}$ sharing at most 1 vertex, then $\mathbf{g}_0(G) = \mathbf{g}(G) = t$.
(The $\mathbb{Z}_2$-genus is additive for disjoint unions and 1-amalgamations.)

Observation:
If G has maximum degree 3, then $\mathbf{g}_0(G) = \mathbf{g}(G)$. In particular, the $\mathbb{Z}_2$-genus of the projective t-wall is $\lfloor t/2 \rfloor$.

- “correct” the rotation of each vertex to obtain an even drawing



- use the weak Hanani–Tutte theorem for surfaces

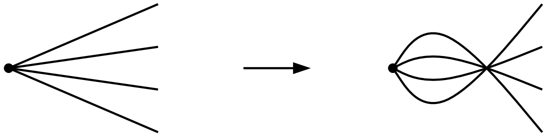
Theorem:
We have $\mathbf{g}_0(G) = \mathbf{g}(G)$ also for each of the remaining t-Kuratowski graphs G: $K_{3,t}$ and 2-amalgamations of t copies of K_5 or $K_{3,3}$.

Lower bounds on the $\mathbb{Z}_2$-genus of $K_{3,t}$

problem with independently even drawings:
- no faces, rotations "do not matter", no Euler's formula ...

First lower bound: $\mathbf{g}_0(K_{3,t}) \geq \Omega(\log\log\log t)$

- "correct" the rotation of each degree-3 vertex so that incident edges cross evenly
- use Ramsey's theorem for each degree-t vertex so that incident edges cross with the same parity
- if the parity is odd for some vertex v, "flip" a neighborhood of v



- use the weak Hanani–Tutte theorem for surfaces
- use the fact $\mathbf{g}(K_{3,n}) = \lceil (n-2)/4 \rceil$

Lower bounds on the $\mathbb{Z}_2$-genus of $K_{3,t}$

Second lower bound: $\mathbf{g}_0(K_{3,t}) \geq \Omega(\log\ t)$

- $\mathbb{Z}_2$-homology of closed curves on M_g and pigeon-hole principle

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Third lower bound: $\mathbf{g}_0(K_{3,t}) = \mathbf{g}(K_{3,t}) = \lceil (t-2)/4 \rceil$

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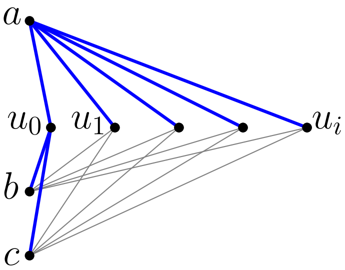

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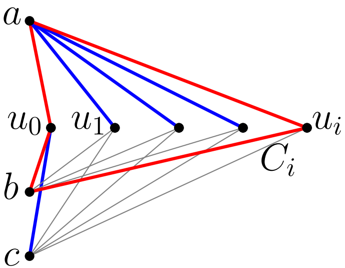

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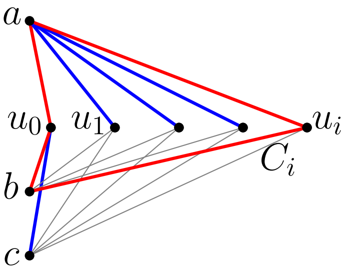

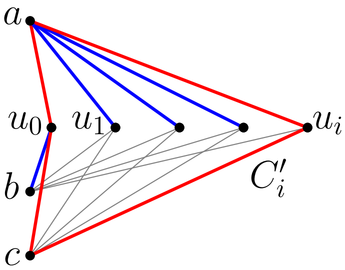

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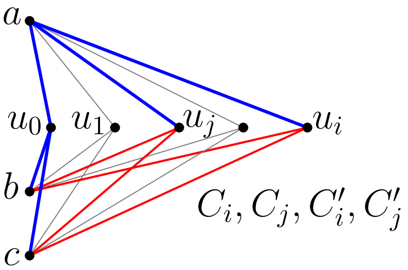
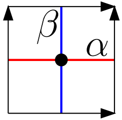


$\mathbb{Z}_2$-homology of closed curves on M_g

Fact: $H_1(M_g; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2^{2g}$.



$$H_1(M_1; \mathbf{Z}_2) = \langle \alpha, \beta \rangle$$

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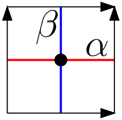

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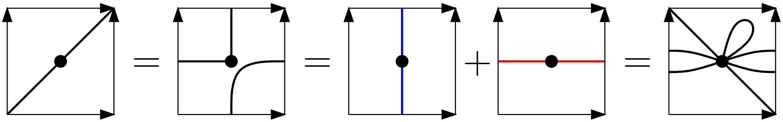

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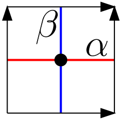

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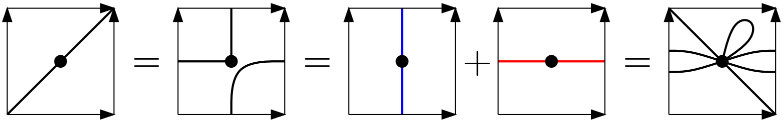

intersection form (Cairns–Nikolayevsky, 2000) (symmetric, bilinear)

$$\mathrm{cr} : H_1(M_g; \mathbb{Z}_2) \times H_1(M_g; \mathbb{Z}_2) \to \mathbb{Z}_2$$

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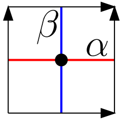

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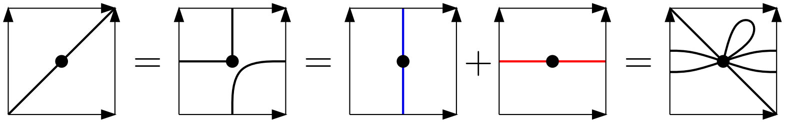

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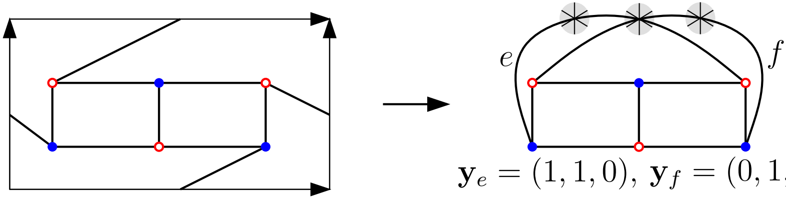
for $g = 1$:

$$\mathrm{cr}(\alpha, \alpha) = 0 \qquad \mathrm{cr}(\beta, \beta) = 0 \qquad \mathrm{cr}(\alpha, \beta) = 1$$

"Crosscap vectors" of closed curves on M_g

Fact: $M_g - \{x\}$ is homeomorphic to a subset of N_{2g+1}
or, to a subset of the sphere with $2g+1$ **crosscaps**



$\mathbf{y}_e = (1,1,0),\ \mathbf{y}_f = (0,1,1)$

- a cycle $C \to$ a **crosscap vector** $\mathbf{y}^C = (y_1^C, y_2^C, \ldots, y_{2g+1}^C)$ where $y_i =$ number of passes of C through the ith crosscap mod 2.
- a drawing on $M_g \leftrightarrow$ a drawing with $2g+1$ crosscaps where the crosscap vector of each cycle has an even number of 1's.
- homology class of $C \leftrightarrow$ crosscap vector $\mathbf{y}^C$

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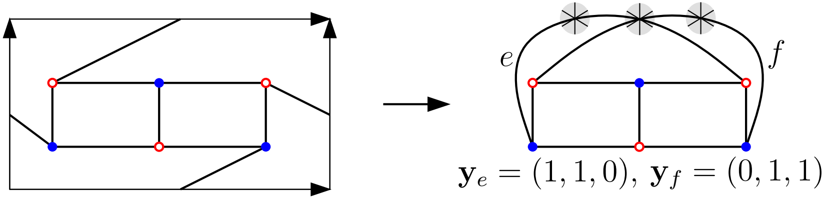

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- homology class of $C \leftrightarrow$ crosscap vector $\mathbf{y}^C$
- intersection form $\mathrm{cr}(C, D) \leftrightarrow$ scalar product $\mathbf{y}^C \cdot \mathbf{y}^D$

Third lower bound on the $\mathbb{Z}_2$-genus of $K_{3,t}$

Lemma: In every independently even drawing of $K_{3,3}$ (induced by $\{a, b, c, u_0, u_1, u_2\}$ from $K_{3,t}$) on M_g, we have

$$\mathrm{cr}(C_1, C_2') + \mathrm{cr}(C_1', C_2) = 1.$$

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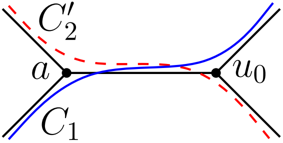

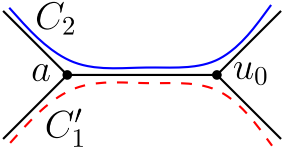

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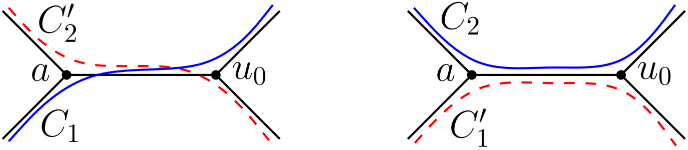

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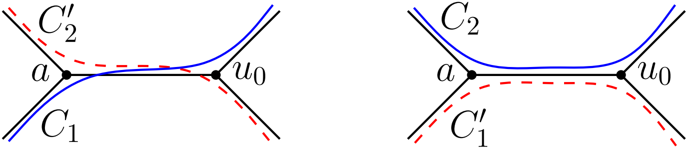

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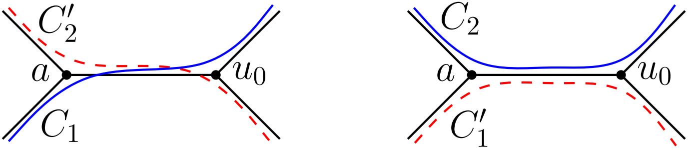

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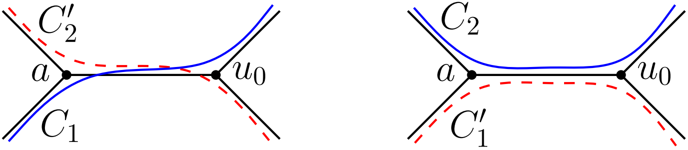

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- the rank of the intersection form is at least $(t-2)/2$ and so $2g \geq (t-2)/2$.