The $\mathbb{Z}_2$-genus of Kuratowski minors[*]

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Abstract

A drawing of a graph on a surface is *independently even* if every pair of non-adjacent edges in the drawing crosses an even number of times. The $\mathbb{Z}_2$-*genus* of a graph G is the minimum g such that G has an independently even drawing on the orientable surface of genus g. An unpublished result by Robertson and Seymour implies that for every t, every graph of sufficiently large genus contains as a minor a projective $t \times t$ grid or one of the following so-called t-*Kuratowski graphs*: $K_{3,t}$, or t copies of K_5 or $K_{3,3}$ sharing at most 2 common vertices. We show that the $\mathbb{Z}_2$-genus of graphs in these families is unbounded in t; in fact, equal to their genus. Together, this implies that the genus of a graph is bounded from above by a function of its $\mathbb{Z}_2$-genus, solving a problem posed by Schaefer and Štefankovič, and giving an approximate version of the Hanani–Tutte theorem on orientable surfaces. We also obtain an analogous result for Euler genus and Euler $\mathbb{Z}_2$-genus of graphs.

1   Introduction

The *genus* $\mathrm{g}(G)$ of a graph G is the minimum g such that G has an embedding on the orientable surface M_g of genus g. We say that two edges in a graph are *independent* (also *nonadjacent*) if they do not share a vertex. The $\mathbb{Z}_2$-*genus* $\mathrm{g}_0(G)$ of a graph G is the minimum g such that G has a drawing on M_g with every pair of independent edges crossing an even number of times. Clearly, every graph G satisfies $\mathrm{g}_0(G) \le \mathrm{g}(G)$.

The Hanani–Tutte theorem [15, 35] states that $\mathrm{g}_0(G) = 0$ implies $\mathrm{g}(G) = 0$. The theorem is usually stated in the following form, with the optional adjective "strong".

Theorem 1 (The (strong) Hanani–Tutte theorem [15, 35]). *A graph is planar if it can be drawn in the plane so that no pair of independent edges crosses an odd number of times.*

Theorem 1 gives an interesting algebraic characterization of planar graphs that can be used to construct a simple polynomial algorithm for planarity testing [30, Section 1.4.2].

Pelsmajer, Schaefer and Stasi [23] extended the strong Hanani–Tutte theorem to the projective plane, using the list of minimal forbidden minors. Colin de Verdière et al. [9] recently provided an alternative proof, which does not rely on the list of forbidden minors.

Theorem 2 (The (strong) Hanani–Tutte theorem on the projective plane [9, 23]). *If a graph G has a drawing on the projective plane such that every pair of independent edges crosses an even number of times, then G has an embedding on the projective plane.*

Whether the strong Hanani–Tutte theorem can be extended to some other surface than the plane or the projective plane has been an open problem. Schaefer and Štefankovič [31] conjectured that $g_0(G) = g(G)$ for every graph G and showed that a minimal counterexample to the extension of the strong Hanani–Tutte theorem on any surface must be 2-connected. Recently, we have found a counterexample on the orientable surface of genus 4 [13].

Theorem 3 ([13]). *There is a graph G with $g(G) = 5$ and $g_0(G) \leq 4$. Consequently, for every positive integer k there is a graph G with $g(G) = 5k$ and $g_0(G) \leq 4k$.*

The *Euler genus* $eg(G)$ of G is the minimum g such that G has an embedding on a surface of Euler genus g. The *Euler $\mathbb{Z}_2$-genus* $eg_0(G)$ of G is the minimum g such that G has an independently even drawing on a surface of Euler genus g.

Schaefer and Štefankovič [31] conjectured that $eg_0(G) = eg(G)$ for every graph G; this is still an open question. They also posed the following natural "approximate" questions.

Problem 1 ([31]). *Is there a function f such that $g(G) \leq f(g_0(G))$ for every graph G? Is there a function f such that $eg(G) \leq f(eg_0(G))$ for every graph G?*

We give a positive answer to Problem 1 for several families of graphs, which we conjectured to be "unavoidable" as minors in graphs of large genus. Recently we have found that a similar Ramsey-type statement is a folklore unpublished result in the graph-minors community. Together, these results would imply a positive solution to Problem 1 for all graphs. We state the results in detail in Sections 3 and 4 after giving necessary definitions in Section 2.

2 Preliminaries

2.1 Graphs on surfaces

We refer to the monograph by Mohar and Thomassen [22] for a detailed introduction into surfaces and graph embeddings. By a *surface* we mean a connected compact 2-dimensional

topological manifold. Every surface is either *orientable* (has two sides) or *nonorientable* (has only one side). Every orientable surface S is obtained from the sphere by attaching $g \geq 0$ *handles*, and this number g is called the *genus* of S. Similarly, every nonorientable surface S is obtained from the sphere by attaching $g \geq 1$ *crosscaps*, and this number g is called the *(nonorientable) genus* of S. The simplest orientable surfaces are the sphere (with genus 0) and the torus (with genus 1). The simplest nonorientable surfaces are the projective plane (with genus 1) and the Klein bottle (with genus 2). We denote the orientable surface of genus g by M_g, and the nonorientable surface of genus g by N_g.

Let $G = (V, E)$ be a graph with no multiple edges and no loops, and let S be a surface. A *drawing* of G on S is a representation of G where every vertex is represented by a unique point in S and every edge e joining vertices u and v is represented by a simple curve in S joining the two points that represent u and v. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts. We require that in a drawing no edge passes through a vertex, no two edges touch, every edge has only finitely many intersection points with other edges and no three edges cross at the same inner point. In particular, every common point of two edges is either their common endpoint or a crossing.

A drawing of G on S is an *embedding* if no two edges cross. A *face* of an embedding of G on S is a connected component of the topological space obtained from S by removing all the edges and vertices of G. A 2-*cell embedding* is an embedding whose each face is homeomorphic to an open disc. The *facewidth* (also called *representativity*) fw$(\mathcal{E})$ of an embedding $\mathcal{E}$ on a surface S of positive genus is the smallest nonnegative integer k such that there is a closed noncontractible curve in S intersecting $\mathcal{E}$ in k vertices.

The *rotation* of a vertex v in a drawing of G on an orientable surface is the clockwise cyclic order of the edges incident to v. We will represent the rotation of v by the cyclic order of the other endpoints of the edges incident to v. The *rotation system* of a drawing is the set of rotations of all vertices.

The *Euler characteristic* of a surface S of genus g, denoted by $\chi(S)$, is defined as $\chi(S) = 2 - 2g$ if S is orientable, and $\chi(S) = 2 - g$ if S is nonorientable. Equivalently, if v, e and f denote the numbers of vertices, edges and faces, respectively, of a 2-cell embedding of a graph on S, then $\chi(S) = v - e + f$. The *Euler genus* eg(S) of S is defined as $2 - \chi(S)$. In other words, the Euler genus of S is equal to the genus of S if S is nonorientable, and to twice the genus of S if S is orientable. This implies the following inequalities for the different notions of genus of a graph G, defined in the introduction:

$$\mathrm{eg}(G) \leq 2\mathrm{g}(G) \quad \text{and} \quad \mathrm{eg}_0(G) \leq 2\mathrm{g}_0(G). \tag{1}$$

An edge in a drawing is *even* if it crosses every other edge an even number of times. A drawing of a graph is *even* if all its edges are even. A drawing of a graph is *independently even* if every pair of independent edges in the drawing crosses an even number of times. In the literature, the notion of $\mathbb{Z}_2$-*embedding* is used to denote both an even drawing [6] and an independently even drawing [31].

The *embedding scheme* of a drawing $\mathcal{D}$ on a surface S consists of the rotation system

and a signature $+1$ or -1 assigned to every edge, representing the parity of the number of crosscaps the edge is passing through. If S is orientable, the embedding scheme can be given just by the rotation system. The following weak analogue of the Hanani–Tutte theorem was proved by Cairns and Nikolayevsky [6] for orientable surfaces and then extended by Pelsmajer, Schaefer and Štefankovič [24] to nonorientable surfaces. Loebl and Masbaum [19, Theorem 5] obtained an alternative proof for orientable surfaces.

Theorem 4 (The weak Hanani–Tutte theorem on surfaces [6, Lemma 3], [24, Theorem 3.2]). *If a graph G has an even drawing $\mathcal{D}$ on a surface S, then G has an embedding on S that preserves the embedding scheme of $\mathcal{D}$.*

A simple closed curve γ in a surface S is 1-*sided* if it has a small neighborhood homeomorphic to the Möbius strip, and 2-*sided* if it has a small neighborhood homeomorphic to the cylinder. We say that γ is *separating* in S if the complement $S \setminus \gamma$ has two components, and *nonseparating* if $S \setminus \gamma$ is connected. Note that on an orientable surface every simple closed curve is 2-sided, and every 1-sided simple closed curve (on a nonorientable surface) is nonseparating.

2.2 Special graphs

2.2.1 Projective grids and walls

For a positive integer n we denote the set $\{1, \ldots, n\}$ by $[n]$. Let $r, s \geq 3$. The *projective* $r \times s$ *grid* is the graph with vertex set $[r] \times [s]$ and edge set

$$\{\{(i,j),(i',j')\}; |i-i'| + |j-j'| = 1\} \ \cup \ \{\{(i,1),(r+1-i,s)\}; i \in [r]\}.$$

In other words, the projective $r \times s$ grid is obtained from the planar $r \times (s+1)$ grid by identifying pairs of opposite vertices and edges in its leftmost and rightmost column. See Figure 1, left. The projective $t \times t$ grid has an embedding on the projective plane with facewidth t. By the result of Robertson and Vitray [29], [22, p. 171], the embedding is unique if $t \geq 4$. Hence, for $t \geq 4$ the genus of the projective $t \times t$ grid is equal to $\lfloor t/2 \rfloor$ by the result of Fiedler, Huneke, Richter and Robertson [11], [22, Theorem 5.8.1].

Since grids have vertices of degree 4, it is more convenient for us to consider their subgraphs of maximum degree 3, called walls. For an odd $t \geq 3$, a *projective t-wall* is obtained from the projective $t \times (2t-1)$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for i odd and $1 \leq j \leq t-1$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for i even and $1 \leq j \leq t$. Similarly, for an even $t \geq 4$, a *projective t-wall* is obtained from the projective $t \times 2t$ grid by removing edges $\{(i,2j),(i+1,2j)\}$ for i odd and $1 \leq j \leq t$, and edges $\{(i,2j-1),(i+1,2j-1)\}$ for i even and $1 \leq j \leq t$. The projective t-wall has maximum degree 3 and can be embedded on the projective plane as a "twisted wall" with inner faces bounded by 6-cycles forming the "bricks", and with the "outer" face bounded by a $(4t-2)$-cycle for t odd and a $4t$-cycle for t even. See Figure 1, right. This embedding has facewidth t and so again, for $t \geq 4$ the projective t-wall has genus $\lfloor t/2 \rfloor$. It is easy to see that the projective 3-wall has genus 1 since it contains a subdivision of $K_{3,3}$ and embeds on the torus.
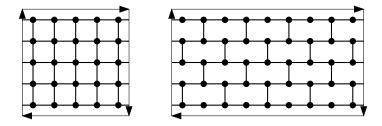
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Figure 1: Left: a projective 5×5 grid. Right: a projective 5-wall.

2.2.2 Kuratowski graphs

A graph is called a *t-Kuratowski* graph [32] if it is one of the following:

a) $K_{3,t}$,

b) a disjoint union of t copies of K_5,

c) a disjoint union of t copies of $K_{3,3}$,

d) a graph obtained from t copies of K_5 by identifying one vertex from each copy to a single common vertex,

e) a graph obtained from t copies of $K_{3,3}$ by identifying one vertex from each copy to a single common vertex,

f) a graph obtained from t copies of K_5 by identifying a pair of vertices from each copy to a common pair of vertices,

g) a graph obtained from t copies of $K_{3,3}$ by identifying a pair of adjacent vertices from each copy to a common pair of vertices,

h) a graph obtained from t copies of $K_{3,3}$ by identifying a pair of nonadjacent vertices from each copy to a common pair of vertices.

See Figure 2 for an illustration.

The genus of each of the t-Kuratowski graphs is known precisely. The genus of $K_{3,t}$ is $\lceil (t-2)/4 \rceil$ [4, 26], [22, Theorem 4.4.7], [14, Theorem 4.5.3], which coincides with the lower bound from Euler's formula. The genus of t copies of K_5 or $K_{3,3}$ sharing at most one vertex is t by the additivity of genus over blocks and connected components [1], [22, Theorem 4.4.2], [14, Theorem 3.5.3]. Finally, from a general formula by Decker, Glover and Huneke [10] it follows that the genus of t copies of K_5 or $K_{3,3}$ sharing a pair of adjacent or nonadjacent vertices is $\lceil t/2 \rceil$ if $t > 1$: cases f) and g) follow from their proof of Corollary 0.2, case h) follows from their Corollary 2.4 after realizing that $\mu(K_{3,3}) = 3$ if x, y are nonadjacent in $K_{3,3}$.
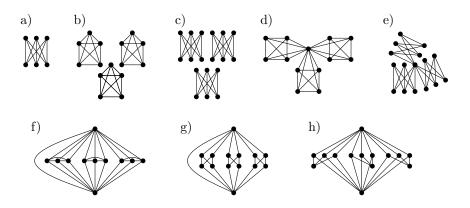
Figure 2: The eight 3-Kuratowski graphs.

The Euler genus of each of the t-Kuratowski graphs is also known precisely. The Euler genus of $K_{3,t}$ is $\lceil (t-2)/2 \rceil$ [4, 27]. The Euler genus of t copies of K_5 or $K_{3,3}$ sharing at most one vertex is t by the additivity of Euler genus over blocks [33, Corollary 2], [20, Theorem 1], [22, Theorem 4.4.3]. The additivity of Euler genus over connected components follows almost trivially: every embedding of a disconnected graph with components G_1, G_2 on a surface can be turned into an embedding of a connected graph on the same surface by adding an edge joining G_1 with G_2. Miller [20, Theorem 1] proved that Euler genus is also additive over edge-amalgamations, which implies that the Euler genus of t copies of K_5 or $K_{3,3}$ sharing a pair of adjacent vertices it t. Miller [20, Theorem 27] also proved a superadditivity of the Euler genus over 2-amalgamations. Richter [25, Theorem 1] proved a precise formula for the Euler genus of 2-amalgamations with respect to a pair of nonadjacent vertices. Since the graph obtained from $K_{3,3}$ by adding one edge has an embedding in the projective plane, Miller's [20] and Richter's [25] results also imply that the Euler genus of t copies of $K_{3,3}$ sharing a pair of nonadjacent vertices is t.

3   Ramsey-type results

The following Ramsey-type statement for graphs of large Euler genus is a folklore unpublished result.

Claim 5 (Robertson–Seymour [2, 32], unpublished). *There is a function g such that for every $t \geq 3$, every graph of Euler genus $g(t)$ contains a t-Kuratowski graph as a minor.*

For 7-connected graphs, Claim 5 follows from the result of Böhme, Kawarabayashi, Maharry and Mohar [2], stating that for every positive integer t, every sufficiently large 7-connected graph contains $K_{3,t}$ as a minor. Böhme et al. [3] later generalized this to graphs of larger connectivity and $K_{a,t}$ minors for every fixed $a > 3$. Fröhlich and Müller [12] gave an alternative proof of this generalized result.

Christian, Richter and Salazar [7] proved a similar statement for graph-like continua.

We obtain an analogous Ramsey-type statement for graphs of large genus as an almost direct consequence of Claim 5.

Theorem 6. *Claim 5 implies that there is a function h such that for every $t \geq 3$, every graph of genus $h(t)$ contains, as a minor, a t-Kuratowski graph or the projective t-wall.*

We give a detailed proof of Theorem 6 in Section 5.

4 Our results

As our main result we complete a proof that the $\mathbb{Z}_2$-genus of each t-Kuratowski graph and the projective t-wall grows to infinity with t; in fact, the $\mathbb{Z}_2$-genus of each of these graphs is equal to their genus. Analogously, we also show that the Euler $\mathbb{Z}_2$-genus of each t-Kuratowski graph is equal to its Euler genus. Schaefer and Štefankovič [31] proved this for those t-Kuratowski graphs that consist of t copies of K_5 or $K_{3,3}$ sharing at most one vertex. For the projective t-wall, the result follows directly from the weak Hanani–Tutte theorem on orientable surfaces [6, Lemma 3]: indeed, all vertices of the projective t-wall have degree at most 3, therefore pairs of adjacent edges crossing oddly in an independently even drawing can be redrawn in a small neighborhood of their common vertex so that they cross evenly, and the weak Hanani–Tutte theorem can be applied. Thus, the remaining cases are t-Kuratowski graphs of type a), f), g) and h).

Theorem 7. *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each t-Kuratowski graph of type a), f), g) and h) is equal to its genus, and also its Euler $\mathbb{Z}_2$-genus is equal to its Euler genus. In particular,*

a) $g_0(K_{3,t}) \geq \lceil (t-2)/4 \rceil$, $eg_0(K_{3,t}) \geq \lceil (t-2)/2 \rceil$, and

b) *if G consists of t copies of K_5 or $K_{3,3}$ sharing a pair of adjacent or nonadjacent vertices, then $g_0(G) \geq \lceil t/2 \rceil$ and $eg_0(G) \geq t$.*

Combining Theorem 7 with the result of Schaefer and Štefankovič [31] and the simple argument for the projective t-wall we obtain the following result.

Corollary 8. *For every $t \geq 3$, the $\mathbb{Z}_2$-genus of each t-Kuratowski graph and the projective t-wall is equal to its genus, and the Euler $\mathbb{Z}_2$-genus of each t-Kuratowski graph is equal to its Euler genus.*

Combining Corollary 8 with Theorem 6 we get the following implication.

Corollary 9. *Claim 5 implies a positive answer to both parts of Problem 1.*

5 Unavoidable minors of large genus

In this section we prove Theorem 6.

5.1 Tools and preparations

We will need the following classical result by Robertson and Seymour [28] about surface minors. *Surface minors* are defined for embeddings analogously as minors for graphs, by deleting and contracting edges on the underlying surface [22].

Theorem 10 ([28], [17, Theorem 3.5], [21, Theorem 5.2], [22, Theorem 5.9.2]). *For every surface S and every embedding $\mathcal{H}$ of a graph H on S there exists a constant $w(\mathcal{H}, S)$ such that every embedding of a graph on S with facewidth at least $w(\mathcal{H}, S)$ contains $\mathcal{H}$ as a surface minor.*

Let $\mathcal{W}_t$ be an embedding of the projective t-wall on the projective plane; see Figure 1, right. With a slight abuse of notation, for each nonorientable surface N_i with $i \geq 2$, we choose an embedding of the projective t-wall on N_i and denote it again by $\mathcal{W}_t$. Without loss of generality, we will assume that $w(\mathcal{W}_t, N_i)$ is nondecreasing in i; otherwise we inductively redefine $w(\mathcal{W}_t, N_i)$ as $\max\{w(\mathcal{W}_t, N_j); j \leq i\}$. For all integers k', i, k satisfying $0 \leq 2k' < i \leq k$, let

$$w(k', i, k, t) = i(i - 2k') \cdot (w(\mathcal{W}_t, N_i) + 2k).$$

This function will be used as a "potential function" in the proof of Proposition 12.

We will also use the following simple statement about the "continuity" of facewidth under the operation of removing all vertices of a face.

Proposition 11 ([22, Propositions 5.5.7 and 5.5.8]). *Let $\mathcal{E}$ be an embedding of a graph on a surface S with $\mathrm{fw}(\mathcal{E}) \geq 3$. Let f be a face of $\mathcal{E}$ and let $\mathcal{E}'$ be the embedding obtained from $\mathcal{E}$ by removing all vertices incident to f. Then $\mathrm{fw}(\mathcal{E}') \geq \mathrm{fw}(\mathcal{E}) - 2$.*

5.2 Proof of Theorem 6

Let $t \geq 3$ and let g be a sufficiently large integer, larger than $g(t)/2$ where $g(t)$ is the number from Claim 5. Let G be a graph of genus g. If the Euler genus of G is larger than $g(t)$, then G has a t-Kuratowski minor by Claim 5. For the rest of the proof we thus assume that the Euler genus of G is at most $k = g(t)$, and our goal is to find the projective t-wall as a minor in G. Since $2g > k$, this implies that G has an embedding $\mathcal{E}$ on N_k.

The operation of *gluing a pair of vertices* u, v in a graph G creates a graph with vertex set $V(G) \setminus \{u, v\} \cup \{w\}$, where $w \notin V(G)$, and edge set $E(G[V(G) \setminus \{u, v\}]) \cup \{\{w, x\}; \{u, x\} \in E(G)\}\} \cup \{\{w, x\}; \{v, x\} \in E(G)\}$. An inverse operation is called *splitting a vertex*; in general, this is not unique for a given graph and a vertex.

We show the following proposition by induction on i.

Proposition 12. *Let i, k, t be positive integers with $t \geq 3$ and $i \leq k$. Let G be a graph that has an embedding $\mathcal{E}$ on N_i, let F be a set of at most $k - i$ faces in $\mathcal{E}$, and let Z be the set of all vertices of $\mathcal{E}$ incident to at least one face in F. Then at least one of the following holds:*

1) $G - Z$ has a projective t-wall as a minor, or

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2) *there is an integer k' satisfying $0 \le 2k' < i$ such that G can be obtained from a graph H of genus at most k' by at most $w(k', i, k, t)$ consecutive operations of gluing a pair of vertices (shortly* gluings*).*

Proof. The main idea of the proof is to cut the surface recursively along "short" non-contractible curves until we obtain an embedding of large facewidth on a nonorientable surface, or until all the pieces are orientable.

We distinguish two cases according to the facewidth of $\mathcal{E}$.

1) $\mathrm{fw}(\mathcal{E}) \ge w(\mathcal{W}_t, N_i) + 2(k - i)$. By Proposition 11, the induced embedding $\mathcal{E}'$ of $G - Z$ in $\mathcal{E}$ has facewidth at least $w(\mathcal{W}_t, N_i)$. Thus, $\mathcal{W}_t$ is a surface minor of $\mathcal{E}'$ and so the projective t-wall is a minor of $G - Z$.

2) $\mathrm{fw}(\mathcal{E}) < w(\mathcal{W}_t, N_i) + 2(k - i)$. In this case there is a noncontractible closed curve γ on S intersecting $\mathcal{E}$ in less than $w(\mathcal{W}_t, N_i) + 2(k - i)$ points, all of which can be assumed to be vertices. Let W be the set of the vertices in $\mathcal{E} \cap \gamma$. We have three cases according to the type of γ: a) γ is 1-sided, b) γ is 2-sided but nonseparating in N_i, c) γ is 2-sided and separates N_i into two components.

In each case, we cut N_i along γ, obtaining a surface or a pair of surfaces with boundary, and fill the boundary cycles with discs. The resulting surfaces may be orientable or nonorientable. In case a) we obtain a surface S of Euler genus $i - 1$. In case b) we obtain a surface S of Euler genus $i - 2$. In case c) we obtain a pair of surfaces S_1 and S_2 with Euler genera i_1 and i_2, respectively, such that $i_1 + i_2 = i$ and $1 \le i_1, i_2 \le i - 1$.

While cutting the surface N_i along γ, we also obtain an embedding $\mathcal{E}'$ of a graph G' on S or a pair of embeddings $\mathcal{E}'_1$ and $\mathcal{E}'_2$ of G'_1 and G'_2 on S_1 and S_2, respectively, obtained from $\mathcal{E}$ by splitting each vertex in W into two copies, each copy keeping adjacent edges only from one side of γ. We now consider each of the three cases separately.

In case a), the embedding $\mathcal{E}'$ has one new face f, containing the disc that was used to fill a boundary cycle while creating S. On the other hand, the faces of $\mathcal{E}$ intersecting γ are no longer faces of $\mathcal{E}'$, as they were cut and merged into f. Let F' be the union of $\{f\}$ and the subset of faces in F that are still faces of $\mathcal{E}'$. Clearly, we have $|F'| \le |F| + 1 \le k - i + 1 = k - \mathrm{eg}(S)$.

If S is orientable, then G' is a graph of genus at most $\mathrm{eg}(S)/2 = (i - 1)/2$ and G can be obtained from G' by less than $w(\mathcal{W}_t, N_i) + 2(k - i) \le w((i - 1)/2, i, k, t)$ gluings.

If S is nonorientable, we apply induction to the embedding $\mathcal{E}'$ and the set of faces F'. Let Z' be the set of vertices of $\mathcal{E}'$ incident with at least one face in F'. Observe that Z' contains all vertices in Z and also all new vertices created by splitting the vertices in W. Hence, $G' - Z'$ is a subgraph of $G - Z$. Therefore, if case 1) of the proposition occurs, we obtain a projective t-wall as a minor in both $G' - Z'$ and $G - Z$. In case 2) we obtain G' from a graph H of genus $k' < (i - 1)/2$ by at most $w(k', i - 1, k, t)$ gluings. Since G is obtained from G' by less than $w(\mathcal{W}_t, N_i) + 2(k - i)$ gluings, we can obtain G from H by less than $w(k', i, k, t)$ gluings.

In case b), $\mathcal{E}'$ has two new faces f_1 and f_2. Let F' be the union of $\{f_1, f_2\}$ and the subset of faces in F that are still faces of $\mathcal{E}'$. Clearly, we have $|F'| \le |F| + 2 \le k - i + 2 = k - \mathrm{eg}(S)$.

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If S is orientable, then G' is a graph of genus at most $\operatorname{eg}(S)/2 = (i-2)/2$ and G can be obtained from G' by less than $w(\mathcal{W}_t, N_i) + 2(k-i) \le w((i-2)/2, i, k, t)$ gluings.

If S is nonorientable, we apply induction to $\mathcal{E}'$ and F' and proceed analogously as in case a). If case 2) of the proposition occurs, we obtain G' from a graph H of genus $k' < (i-2)/2$ by at most $w(k', i-2, k, t)$ gluings, and thus we can again obtain G from H by less than $w(k', i, k, t)$ gluings.

In case c), $\mathcal{E}'_1$ has a new face f_1 and $\mathcal{E}'_2$ has a new face f_2. For $l \in \{1, 2\}$ we define F'_l as the union of $\{f_l\}$ and the the subset of faces in F that are still faces of $\mathcal{E}'_l$. Again, for each $l \in \{1, 2\}$ we have $|F'_l| \le |F| + 1 \le k - i + 1 \le k - \operatorname{eg}(S_l)$.

Notice that at least one of the surfaces S_1, S_2 is nonorientable, since N_i is their connected sum. Let $l \in \{1, 2\}$. If S_l is orientable, then G'_l is a graph of genus at most $\operatorname{eg}(S_l)/2 = i_l/2$. If S_l is nonorientable, we apply induction to $\mathcal{E}'_l$ and F'_l. Let Z'_l be the set of vertices of $\mathcal{E}'_l$ incident with at least one face in F'_l. Observe that Z'_l contains all vertices in $Z \cap V(G'_l)$ and all new vertices in G'_l created by splitting the vertices in W. Hence, $G'_l - Z'_l$ is a subgraph of $G - Z$. Therefore, if case 1) of the proposition occurs, we obtain a projective t-wall as a minor in both $G'_l - Z'_l$ and $G - Z$. In case 2) we obtain G'_l from a graph H_l of genus $k'_l < i_l/2$ by at most $w(k'_l, i_l, k, t)$ gluings.

If we have not obtained the projective t-wall as a minor in $G - Z$, then for each $l \in \{1, 2\}$, the graph G'_l is obtained from a graph H_l of genus $k'_l \le i_l/2$ by at most $w(k'_l, i_l, k, t)$ gluings (where $w(i_l/2, i_l, k, t) = 0$), and $k'_1 + k'_2 \le (i-1)/2$ since at least one of S_1, S_2 is nonorientable. Let H be the disjoint union of H_1 and H_2. Then H is a graph of genus at most $k' = k'_1 + k'_2 < i/2$, and G can be obtained from H by less than $w(k'_1, i_1, k, t) + w(k'_2, i_2, k, t) + w(\mathcal{W}_t, N_i) + 2(k-i)$ gluings. By the monotonicity of $w(\mathcal{W}_t, N_i)$, we have

$$\begin{aligned}
& w(k'_1, i_1, k, t) + w(k'_2, i_2, k, t) + w(\mathcal{W}_t, N_i) + 2(k-i) \\
& \le (i_1(i_1 - 2k'_1) + i_2(i_2 - 2k'_2) + 1) \cdot (w(\mathcal{W}_t, N_i) + 2k) \\
& \le (i(i - 2k') + 1 - i_1(i_2 - 2k'_2) - i_2(i_1 - 2k'_1)) \le w(k', i, k, t).
\end{aligned}$$

This finishes the proof of the proposition. $\qquad\square$

We apply Proposition 12 with $i = k$ and $F = \emptyset = Z$. If case 1) occurs, then G has the projective t-wall as a minor. If case 2) occurs, then there is an integer k' satisfying $0 \le 2k' < k$ such that G can be obtained from a graph H of genus at most k' by at most $w(k', k, k, t)$ gluings. Since every gluing increases the genus of a graph by at most 1, we conclude that the genus of G is at most $k' + w(k', k, k, t) \le k^2 \cdot (w(\mathcal{W}_t, N_k) + 2k)$. This will be a contradiction if $g > k^2 \cdot (w(\mathcal{W}_t, N_k) + 2k)$. Therefore, in Theorem 6 it is sufficient to take $h(t) = g^2(t) \cdot (w(\mathcal{W}_t, N_{g(t)}) + 2g(t))$ where $g(t)$ is the number from Claim 5.

6 Lower bounds on the $\mathbb{Z}_2$-genus and Euler $\mathbb{Z}_2$-genus

In this section we prove Theorem 7. By (1), the lower bounds on the Euler $\mathbb{Z}_2$-genus of the t-Kuratowski graphs in Theorem 7 imply the lower bounds on their $\mathbb{Z}_2$-genus; thus it

will be sufficient to prove the lower bounds on their Euler $\mathbb{Z}_2$-genus.

The fact that the (Euler) $\mathbb{Z}_2$-genus of $K_{3,t}$ or the other t-Kuratowski graphs is unbounded when t goes to infinity is not obvious at first sight. The traditional lower bound on the (Euler) genus of $K_{3,t}$ relies on Euler's formula and the notion of a face. However, there is no analogue of a "face" in an independently even drawing, and the rotations of vertices no longer "matter". We thus need different tools to compute the (Euler) $\mathbb{Z}_2$-genus.

6.1 $\mathbb{Z}_2$-homology of curves

We refer to Hatcher's textbook [16] for an excellent general introduction to homology theory. Unfortunately, except for the very short summary by Colin de Verdière [8, p. 14–15], we were unable to find a compact treatment of the homology theory for curves on surfaces in the literature, thus we sketch here the main aspects that are most important for us.

We will use the $\mathbb{Z}_2$-homology of closed curves on surfaces. That is, for a given surface S, we are interested in its first homology group with coefficients in $\mathbb{Z}_2$, denoted by $H_1(S; \mathbb{Z}_2)$. It is well-known that for each $g \geq 0$, the first homology group $H_1(M_g; \mathbb{Z}_2)$ of M_g is isomorphic to $\mathbb{Z}_2^{2g}$ [16, Example 2A.2. and Corollary 3A.6.(b)]. This fact was crucial in establishing the weak Hanani–Tutte theorem on M_g [6, Lemma 3]. Similarly, for each $g \geq 1$, the first homology group $H_1(N_g; \mathbb{Z}_2)$ of N_g is isomorphic to $\mathbb{Z}_2^g$ [16, Example 2.37 and Corollary 3A.6.(b)].

To every closed curve γ in a surface S one can assign its homology class $[\gamma] \in H_1(S; \mathbb{Z}_2)$, and this assignment is invariant under continuous deformation (homotopy). In particular, the homology class of each contractible curve is 0. More generally, the homology class of each separating curve in S is 0 as well. Moreover, if γ is obtained by a composition of γ_1 and γ_2, the homology classes satisfy $[\gamma] = [\gamma_1] + [\gamma_2]$. The assignment of homology classes to closed curves is naturally extended to formal integer combinations of the closed curves, called *cycles*, and so $[\gamma]$ can be considered as a set of cycles. Since we are interested in homology with coefficients in $\mathbb{Z}_2$, it is sufficient to consider cycles with coefficients in $\mathbb{Z}_2$, which may also be regarded as finite sets of closed curves.

If γ_1 and γ_2 are cycles in S that cross in finitely many points and have no other points in common, we denote by $\mathrm{cr}(\gamma_1, \gamma_2)$ the number of their common crossings. We use the following well-known fact, which may be seen as a consequence of the Jordan curve theorem.

Fact 13. *Let $\gamma_1' \in [\gamma_1]$ and $\gamma_2' \in [\gamma_2]$ be a pair of cycles in a surface S such that the intersection number $\mathrm{cr}(\gamma_1', \gamma_2')$ is defined and is finite. Then*

$$\mathrm{cr}(\gamma_1', \gamma_2') \equiv \mathrm{cr}(\gamma_1, \gamma_2) \ (\mathrm{mod} \ 2).$$

Fact 13 allows us to define a group homomorphism (which is also a bilinear form)

$$\Omega_S : H_1(S; \mathbb{Z}_2) \times H_1(S; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

such that

$$\Omega_S([\gamma_1], [\gamma_2]) = \mathrm{cr}(\gamma_1, \gamma_2) \ \mathrm{mod} \ 2$$

whenever $\mathrm{cr}(\gamma_1, \gamma_2)$ is defined and is finite. Cairns and Nikolayevsky [6] call Ω_{M_g} the *intersection form* on M_g. Clearly, Ω_S is symmetric, and for every 2-sided simple closed curve γ we have $\Omega_S([\gamma], [\gamma]) = 0$. This implies that for every cycle γ in an orientable surface M_g we have $\Omega_{M_g}([\gamma], [\gamma]) = 0$, since all simple closed curves in M_g are 2-sided, and every closed curve with finitely many self-intersections is a composition of finitely many simple closed curves. On the other hand, if γ is a 1-sided simple closed curve in N_g, then $\Omega_{N_g}([\gamma], [\gamma]) = 1$.

We have the following simple observation about intersections of disjoint cycles in independently even drawings.

Observation 14 ([31, Lemma 1]). *Let $\mathcal{D}$ be an independently even drawing of a graph G on a surface S. Let C_1 and C_2 be vertex-disjoint cycles in G, and let γ_1 and γ_2 be the closed curves representing C_1 and C_2, respectively, in $\mathcal{D}$. Then $\mathrm{cr}(\gamma_1, \gamma_2) \equiv 0 \pmod 2$, which implies that $\Omega_S([\gamma_1], [\gamma_2]) = 0$.* $\square$

6.2 Combinatorial representation of the $\mathbb{Z}_2$-homology of drawings

Schaefer and Štefankovič [31] used the following combinatorial representation of drawings of graphs on M_g and N_g. First, every drawing of a graph on M_g can be considered as a drawing on the nonorientable surface N_{2g+1}, since M_g minus a point is homeomorphic to an open subset of N_{2g+1}. The surface N_h minus a point can be represented combinatorially as the plane with h *crosscaps*. A crosscap at a point x is a combinatorial representation of a Möbius strip whose boundary is identified with the boundary of a small circular hole centered in x. Informally, the main "objective" of a crosscap is to allow a set of curves intersect transversally at x without counting it as a crossing.

Every closed curve γ drawn in the plane with h crosscaps is assigned a vector $y_\gamma \in \{0, 1\}^h$ such that $(y_\gamma)_i = 1$ if and only if γ passes an odd number of times through the ith crosscap. When γ represents a 2-sided curve in a surface S, then y_γ has an even number of coordinates equal to 1. The vectors y_γ represent the elements of the homology group $H_1(S; \mathbb{Z}_2)$, and the value of the intersection form $\Omega_S([\gamma], [\gamma'])$ is equal to the scalar product $y_\gamma^\top y_{\gamma'}$ over $\mathbb{Z}_2$. Analogously, we assign a vector $y_e^{\mathcal{D}}$ (or simply y_e) to every curve representing an edge e in a drawing $\mathcal{D}$ of a graph in this model.

We use the following two lemmata by Schaefer and Štefankovič [31].

Lemma 15 ([31, Lemma 5]). *Let G be a graph that has an independently even drawing $\mathcal{D}$ on a surface S and let F be a forest in G. Let $h = 2g + 1$ if $S = M_g$ and $h = g$ if $S = N_g$. Then G has a drawing $\mathcal{E}$ in the plane with h crosscaps, such that*

1) *every pair of independent edges has an even number of common crossings outside the crosscaps, and*

2) *every edge f of F passes through each crosscap an even number of times; that is, $y_f^{\mathcal{E}} = 0$.*
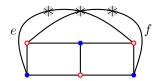
Figure 3: An embedding of $K_{3,3}$ on the torus represented as a drawing $\mathcal{D}$ in the plane with three crosscaps. The nonzero vectors assigned to the edges are $y_e^{\mathcal{D}} = (1,1,0)$ and $y_f^{\mathcal{D}} = (0,1,1)$.

Moreover, the drawing in S corresponding to $\mathcal{E}$ can be obtained from $\mathcal{D}$ by a sequence of continuous deformations of edges and neighborhoods of vertices, so the homology classes of all cycles are preserved between the two drawings.

Lemma 16 ([31, Lemma 4]). *Let G be a graph that has a drawing $\mathcal{D}$ in the plane with finitely many crosscaps with every pair of independent edges having an even number of common crossings outside the crosscaps. Let d be the dimension of the vector space generated by the set $\{y_e^{\mathcal{D}}; e \in E(G)\}$. Then G has an independently even drawing on a surface of Euler genus d.*

Lemma 15 and Lemma 16 imply the following corollary generalizing the strong Hanani–Tutte theorem.

Corollary 17. *Let G be a connected graph with an independently even drawing on a surface S such that each cycle in the drawing is homologically zero (that is, the homology class of the corresponding closed curve is 0). Then G is planar.*

Proof. Let F be a spanning tree of G and let $\mathcal{E}$ be a drawing obtained from Lemma 15. The cycle space of G is generated by the fundamental cycles with respect to F. Every edge $e \in E(G) \setminus E(F)$ determines a unique fundamental cycle $C_e \subseteq F \cup \{e\}$. Since $y_f^{\mathcal{E}} = 0$ for every edge f of F, the homology class of C_e in $\mathcal{E}$ is represented by $y_e^{\mathcal{E}}$. Therefore, under the assumption that the homology classes of all cycles are zero, we have $y_e^{\mathcal{E}} = 0$ for every edge e of G. Lemma 16 then implies that G has an independently even drawing in the plane. Finally, G is planar by the strong Hanani–Tutte theorem (Theorem 1). $\square$

Corollary 17 can be further strengthened using Lemma 15 as follows.

Lemma 18. *Let G be a connected graph with an independently even drawing $\mathcal{D}$ on a surface S. Let F be a spanning tree of G. If G is nonplanar, then there are independent edges $e, f \in E(G) \setminus E(F)$ such that the closed curves γ_e and γ_f representing the fundamental cycles of e and f, respectively, satisfy $\Omega_S([\gamma_e], [\gamma_f]) = 1$.*

Proof. Let $\mathcal{E}$ be a drawing of G from Lemma 15. By the strong Hanani–Tutte theorem, there are two independent edges e and f in G that cross an odd number of times in $\mathcal{E}$. Moreover, conditions 1) and 2) of Lemma 15 imply that none of the edges e and f is in F and so e and f cross an odd number of times in the crosscaps. This means that $y_e^\top y_f = 1$, which is equivalent to $\Omega_S([\gamma_e], [\gamma_f]) = 1$. $\square$

6.3 Proof of Theorem 7a)

We will show three lower bounds on $g_0(K_{3,t})$ and $eg_0(K_{3,t})$, in the order of increasing strength and complexity of their proof.

We will adopt the following notation for the vertices of $K_{3,t}$. The vertices of degree t forming one part of the bipartition are denoted by a, b, c, and the remaining vertices by $u_0, u_1, \ldots, u_{t-1}$. Let $U = \{u_0, u_1, \ldots, u_{t-1}\}$. For each $i \in [t-1]$, let C_i be the cycle au_ibu_0 and C'_i the cycle au_icu_0.

The first lower bound follows from Ramsey's theorem and the weak Hanani–Tutte theorem on surfaces.

Proposition 19. *We have* $2g_0(K_{3,t}) \geq eg_0(K_{3,t}) \geq \Omega(\log \log \log t)$.

Proof. Let $t \geq 3, g \geq 0$ and let $\mathcal{D}$ be an independently even drawing of $K_{3,t}$ on a surface S of Euler genus g. By Ramsey's theorem, there is a subset $U_a \subseteq U$ of size $\Omega(\log t)$ such that all the edges between a and U_a cross each other an odd number of times, or all the edges between a and U_a cross each other an even number of times. Repeating the same argument with vertices b and c, we find a subset $U_b \subseteq U_a$ of size $\Omega(\log \log t)$, and a subset $U_c \subseteq U_b$ of size $\Omega(\log \log \log t)$ such that the number of crossings of each pair of edges between b and U_b has the same parity, and the number of crossings of each pair of edges between c and U_c has the same parity. If the parity is odd for some of the vertices a, b, c, we modify the drawing locally around this vertex by introducing one more crossing for each pair of incident edges; see [6, Fig. 4]. Finally, we modify the drawing locally around each vertex u in U_c so that again, every pair of the three edges incident to u crosses an even number of times. After these modifications we obtain an even drawing of the complete bipartite graph induced by $\{a, b, c\} \cup U_c$. By the weak Hanani–Tutte theorem for surfaces (Theorem 4), the graph $K_{3,|U_c|}$ has an embedding on S and so $g \geq \lfloor (|U_c|-2)/2 \rfloor$. It follows that $g \geq \Omega(\log \log \log t)$. □

The second lower bound is based on the pigeonhole principle and Corollary 17 from the previous subsection.

Proposition 20. *We have* $2g_0(K_{3,t}) \geq eg_0(K_{3,t}) \geq \Omega(\log t)$.

Proof. Let $\mathcal{D}$ be an independently even drawing of $K_{3,t}$ on a surface S of Euler genus g. By the pigeonhole principle, there is a subset $I_b \subseteq [t-1]$ of size at least $(t-1)/2^g$ such that all the cycles C_i with $i \in I_b$ have the same homology class in $\mathcal{D}$. Analogously, there is a subset $I_c \subseteq I_b$ of size at least $|I_b|/2^g$ such that all the cycles C'_i with $i \in I_c$ have the same homology class in $\mathcal{D}$. Suppose that $t \geq 2 \cdot 4^g + 2$. Then $|I_b| \geq 2 \cdot 2^g + 1$ and $|I_c| \geq 3$. Let $i, j, k \in I_c$ be three distinct integers. We now consider the subgraph H of $K_{3,t}$ induced by the vertices a, b, c, u_i, u_j, u_k, isomorphic to $K_{3,3}$, and show that all its cycles are homologically zero. Indeed, the cycle space of H is generated by the four cycles au_ibu_j, au_ibu_k, au_icu_j and au_icu_k, and each of them is the sum (mod 2) of two cycles of the same homology class: $au_ibu_j = C_i + C_j$, $au_ibu_k = C_i + C_k$, $au_icu_j = C'_i + C'_j$ and $au_icu_k = C'_i + C'_k$. Corollary 17 now implies that H is planar, but this is a contradiction. Therefore $t \leq 2 \cdot 4^g + 1$. □
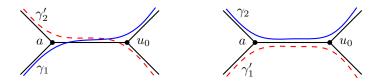
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Figure 4: The curves $\gamma_1, \gamma_2', \gamma_1', \gamma_2$ after deformation in the neighborhood of their common edge au_0.

To prove the lower bound in Theorem 7a), we use the same general idea as in the previous proof. However, we will need the following more precise lemma about drawings of $K_{3,3}$, strengthening Corollary 17 and Lemma 18. We also replace the pigeonhole principle with a linear-algebraic trick.

Lemma 21. *Let $\mathcal{D}$ be an independently even drawing of $K_{3,3}$ on a surface S. For $i \in \{1, 2\}$, let γ_i and γ_i' be the closed curves representing the cycles C_i and C_i', respectively, in $\mathcal{D}$. The intersection numbers of their homology classes satisfy*

$$\Omega_S([\gamma_1], [\gamma_2']) + \Omega_S([\gamma_1'], [\gamma_2]) = 1.$$

Lemma 21 is a consequence of Corollary 27. Here we include a direct proof using a different method.

Proof. Since the maximum degree of $K_{3,3}$ is 3, we may assume that the drawing $\mathcal{D}$ is even: if some adjacent edges cross oddly, we may modify the drawing locally around their common vertex so that they cross evenly, without changing the values of the intersection form.

Cairns and Nikolayevsky [6, Lemma 1] formulated a special case of an identity expressing the intersection form Ω_S as the sum of a "combinatorial" crossing number of cycles and the number of crossings of their edges. We use an analogous identity for the drawing $\mathcal{D}$, and also include its derivation to make the proof self-contained.

The cycles C_1 and C_2' share only the vertices a and v_0 and the edge av_0, and the same is true for the cycles C_1' and C_2. Let O be a small neighborhood of the curve representing the edge av_0 in $\mathcal{D}$. Deform the curves $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ within O so that they cross each other at most once in O; see Figure 4. Assume without loss of generality that the rotation of a in $\mathcal{D}$ is (u_0, u_1, u_2) and the rotation of u_0 in $\mathcal{D}$ is (a, b, c). Then the curves obtained by deforming γ_1 and γ_2' cross exactly once in O, and the curves obtained by deforming γ_1' and γ_2 do not intersect in O. All the other crossings between these closed curves coincide with the crossings between edges in $\mathcal{D}$. Since $\mathcal{D}$ is an even drawing, the value of the intersection form is determined by the parity of the number of crossings inside O. In particular, we have $\Omega_S([\gamma_1], [\gamma_2']) = 1$ and $\Omega_S([\gamma_1'], [\gamma_2]) = 0$. $\square$

Proposition 22. *We have $g_0(K_{3,t}) \geq \lceil (t-2)/4 \rceil$ and $eg_0(K_{3,t}) \geq \lceil (t-2)/2 \rceil$.*

Proof. Let $\mathcal{D}$ be an independently even drawing of $K_{3,t}$ on a surface S of Euler genus g. For every $i \in [t-1]$, let γ_i and γ_i' be the closed curves representing the cycles C_i and C_i', respectively, in $\mathcal{D}$. For every $i, j \in [t-1]$, $i < j$, we apply Lemma 21 to the drawing of $K_{3,3}$ induced by the vertices a, b, c, u_0, u_i, u_j in $\mathcal{D}$. Let A be the $(t-1) \times (t-1)$ matrix with entries

$$A_{i,j} = \Omega_S([\gamma_i], [\gamma_j']).$$

Lemma 21 implies that $A_{i,j} + A_{j,i} = 1$ whenever $i \neq j$; in other words, A is a *tournament matrix* [5]. Repeating the argument by de Caen [5], it follows that $A + A^\top$, with the addition mod 2, is the matrix with zeros on the diagonal and 1-entries elsewhere. This implies that the rank of A over $\mathbb{Z}_2$ is at least $(t-2)/2$. Hence, the rank of Ω_S is at least $(t-2)/2$, which implies $g \geq (t-2)/2$. □

6.4   Proof of Theorem 7b)

Before proving Theorem 7b) we first show an asymptotic $\Omega(\log t)$ lower bound on the (Euler) $\mathbb{Z}_2$-genus for a more general class of graphs that includes the t-Kuratowski graphs of types f), g) and h).

The definition of gluing a pair of vertices from Subsection 5.2 can be extended in a straightforward way to *gluing* an arbitrary finite *set of vertices*. Let H be a 2-connected graph and let x, y be two nonadjacent vertices of H. Let t be a positive integer. The *2-amalgamation* of t copies of H (with respect to x and y), denoted by $\amalg_{x,y} tH$, is the graph obtained from t disjoint copies of H by gluing all t copies of x into a single vertex and gluing all t copies of y into a single vertex. The two vertices obtained by gluing are again denoted by x and y.

An *xy-wing* is a 2-connected graph H with two nonadjacent vertices x and y such that the subgraph $H - x - y$ is connected, and the graph obtained from H by adding the edge xy is nonplanar. Clearly, the graphs $K_5 - e$ and $K_{3,3} - e$, where $e = xy$, are *xy*-wings, and similarly $K_{3,3}$, with nonadjacent vertices x and y, is an *xy*-wing. The t-Kuratowski graphs of types f) and g) are obtained from $\amalg_{x,y} t(K_5 - e)$ and $\amalg_{x,y} t(K_{3,3} - e)$, respectively, by adding the edge xy, whereas the t-Kuratowski graph of type h) is exactly the 2-amalgamation $\amalg_{x,y} t(K_{3,3})$. See Figure 5 for an illustration of 2-amalgamations of two *xy*-wings.

Let H be an *xy*-wing. We will use the following notation. Let w be a vertex of H adjacent to y and let F' be a spanning tree of $H - x - y$. Let F be a spanning tree of $H - y$ extending F'. In the 2-amalgamation $\amalg_{x,y} tH$ we distinguish the ith copy of H, its vertices, edges, and subgraphs, by the superscript $i \in \{0, 1, \ldots, t-1\}$. In particular, for every $i \in \{0, 1, \ldots, t-1\}$, H^i is an induced subgraph of $\amalg_{x,y} tH$, F^i is a spanning tree of $H^i - y$ and x is a leaf of F^i. For a given t, let

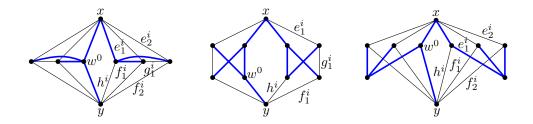$$T = yw^0 + \bigcup_{i=0}^{t-1} F_i$$

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Figure 5: 2-amalgamations of two Kuratowski xy-wings. The spanning tree T is drawn bold.

be a spanning tree of $\amalg_{x,y}tH$. For every edge $e \in E(\amalg_{x,y}tH) \setminus E(T)$, let C_e be the fundamental cycle of e with respect to T; that is, the unique cycle in $T + e$.

Enumerate the edges of $E(H) \setminus E(F)$ incident to x as $e_1, \ldots, e_k$, the edges of $E(H) \setminus E(F) \setminus \{yw\}$ incident to y as $f_1, \ldots, f_l$, and the edges of $E(H - x - y) \setminus E(F)$ as $g_1, \ldots, g_m$. Let h be the edge yw. Thus, for every $i \in [t-1]$, we have $E(H^i) \setminus E(T) = \{e_1^i, \ldots, e_k^i\} \cup \{f_1^i, \ldots, f_l^i\} \cup \{g_1^i, \ldots, g_m^i\} \cup \{h^i\}$.

If C and C' are cycles in $\amalg_{x,y}tH$, we denote by $C + C'$ the element of the cycle space of $\amalg_{x,y}tH$ obtained by adding C and C' mod 2. We also regard $C + C'$ as a subgraph of $\amalg_{x,y}tH$ with no isolated vertices. Note that if C and C' are fundamental cycles sharing at least one edge then $C + C'$ is again a cycle.

Observation 23. *Let $i \in [t-1]$.*

a) *For every $j \in [k]$, the cycle $C_{e_j^i}$ is a subgraph of $H^i - y$.*

b) *For every $j \in [l]$, the cycle $C_{f_j^i} + C_{h^i}$ is a subgraph of $H^i - x$.*

c) *For every $j \in [m]$, the cycle $C_{g_j^i}$ is a subgraph of $H^i - x - y$.*

The cycles $C_{e_j^i}$ with $j \in [k]$, $C_{f_j^i} + C_{h^i}$ with $j \in [l]$, and $C_{g_j^i}$ with $j \in [m]$ generate the cycle space of H^i; in particular, they are the fundamental cycles of H^i with respect to the spanning tree $F^i + yw^i$. $\square$

Corollary 24. *Let $i, i' \in [t-1]$ be distinct indices. Then the following pairs of cycles are vertex-disjoint, for all possible pairs of indices j, j':*

a) $C_{e_j^i}$ *and* $C_{f_{j'}^{i'}} + C_{h^{i'}}$,

b) $C_{f_j^i} + C_{h^i}$ *and* $C_{g_{j'}^{i'}}$,

c) $C_{e_j^i}$ *and* $C_{g_{j'}^{i'}}$,
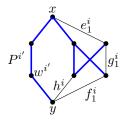
d) $C_{g_j^i}$ *and* $C_{g_{j'}^{i'}}$.

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Figure 6: An example of the graph $H^{i,i'}$. Its spanning tree $F^{i,i'}$ is drawn bold.

Our first lower bound on the (Euler) $\mathbb{Z}_2$-genus of 2-amalgamations of xy-wings is similar to Proposition 20, and combines the pigeonhole principle and Lemma 18.

Proposition 25. *Let H be an xy-wing. Then $2\mathrm{g}_0(\amalg_{x,y} tH) \geq \mathrm{eg}_0(\amalg_{x,y} tH) \geq \Omega(\log t)$.*

Proof. Let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y} tH$ on a surface S of Euler genus g. For every $i \in [t-1]$ and $e \in E(H) \setminus E(F)$, let $\gamma(e^i)$ be the closed curve representing C_{e^i} in $\mathcal{D}$.

The homology class $[\gamma(e^i)]$ has one of 2^g possible values in $H_1(S; \mathbb{Z}_2)$. Thus, if $t \geq 2^{g(k+l+m+1)} + 2$, then there are distinct indices $i, i' \in [t-1]$ such that for every $e \in E(H) \setminus E(F)$ we have $[\gamma(e^i)] = [\gamma(e^{i'})]$. Using this, we can compute the intersection form for certain pairs of cycles by replacing them with vertex-disjoint pairs; this gives the first equality in the following formulas. The second equality follows from Corollary 24 and Observation 14. In particular, for all possible pairs of indices j, j', we have

$$\Omega_S([\gamma(e^i_j)], [\gamma(f^i_{j'})] + [\gamma(h^i)]) = \Omega_S([\gamma(e^i_j)], [\gamma(f^{i'}_{j'})] + [\gamma(h^{i'})]) = 0, \tag{2}$$

$$\Omega_S([\gamma(f^i_j)] + [\gamma(h^i)], [\gamma(g^i_{j'})]) = \Omega_S([\gamma(f^i_j)] + [\gamma(h^i)], [\gamma(g^{i'}_{j'})]) = 0, \tag{3}$$

$$\Omega_S([\gamma(e^i_j)], [\gamma(g^i_{j'})]) = \Omega_S([\gamma(e^i_j)], [\gamma(g^{i'}_{j'})]) = 0, \tag{4}$$

$$\Omega_S([\gamma(g^i_j)], [\gamma(g^i_{j'})]) = \Omega_S([\gamma(g^i_j)], [\gamma(g^{i'}_{j'})]) = 0. \tag{5}$$

Let $H^{i,i'}$ be the union of the graph H^i with the unique xy-path $P^{i'}$ in $F^{i'} + yw^{i'}$; see Figure 6. Since H is an xy-wing, the graph $H^{i,i'}$ is nonplanar. The graph $F^{i,i'} = F^i \cup P^{i'}$ is a spanning tree of $H^{i,i'}$, and $E(H^{i,i'}) \setminus E(F^{i,i'}) = E(H^i) \setminus E(T)$.

The fundamental cycle C'_{h^i} of h^i in $H^{i,i'}$ with respect to $F^{i,i'}$ is equal to $C_{h^i} + C_{h^{i'}}$. Since $[\gamma(h^i)] = [\gamma(h^{i'})]$, the cycle C'_{h^i} is homologically zero.

For every $j \in [k]$, the fundamental cycle of e^i_j in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{e^i_j}$ and its homology class in $\mathcal{D}$ is $[\gamma(e^i_j)]$.

For every $j \in [l]$, the fundamental cycle of f^i_j in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{f^i_j} + C_{h^{i'}}$ and its homology class is $[\gamma(f^i_{j'})] + [\gamma(h^{i'})] = [\gamma(f^i_{j'})] + [\gamma(h^i)]$.

For every $j \in [m]$, the fundamental cycle of g^i_j in $H^{i,i'}$ with respect to $F^{i,i'}$ is $C_{g^i_j}$ and its homology class in $\mathcal{D}$ is $[\gamma(g^i_j)]$.

By (2)–(5), for every pair of independent edges in $E(H^{i,i'}) \setminus E(F^{i,i'})$, the homology classes of their fundamental cycles are orthogonal with respect to Ω_S. This is a

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contradiction with Lemma 18 applied to $H^{i,i'}$ and the spanning tree $F^{i,i'}$. Therefore, $t \leq 2^{g(k+l+m+1)} + 1$. $\qquad\square$

To prove the lower bound in Theorem 7b), we follow the idea of the previous proof and again replace the pigeonhole principle with a linear-algebraic argument. We will also need the following stronger variant of the Hanani–Tutte theorem and Lemma 18 for the graphs K_5 and $K_{3,3}$.

Lemma 26 (Kleitman [18]). *In every drawing of K_5 and $K_{3,3}$ in the plane the total number of pairs of independent edges crossing an odd number of times is odd.*

A very detailed proof of Lemma 26 was given by Székely [34, Sections 7 and 8].

Corollary 27. *Let $G = K_5$ or $G = K_{3,3}$. Let F be a forest in G. Let $\mathcal{E}$ be a drawing of G from Lemma 15. Then there are an odd number of pairs of independent edges e, f in $E(G) \setminus E(F)$ such that $y_e^\top y_f = 1$.* $\qquad\square$

The following simple fact is a key ingredient in the proof of Lemma 26.

Observation 28. *The graph obtained from each of K_5 and $K_{3,3}$ by removing an arbitrary pair of adjacent vertices is a cycle; in particular, all of its vertices have an even degree.* $\qquad\square$

An xy-wing H is called a *Kuratowski xy-wing* if H is one of the graphs $K_5 - e$ where $e = xy$, $K_{3,3} - e$ where $e = xy$, or $K_{3,3}$; see Figure 5. Observation 28 implies the following important property of Kuratowski xy-wings.

Observation 29. *Let H be a Kuratowski xy-wing and let u be a vertex adjacent to x in H. Then $H - x - u$ is a cycle; in particular, y is incident to exactly two edges in $H - x - u$.* $\qquad\square$

In the following key lemma we keep using the notation for the 2-amalgamation $\amalg_{x,y} t H$ established earlier in this subsection.

Lemma 30. *Let $t \geq 2$, let H be a Kuratowski xy-wing and let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y} t H$ on a surface S. Then for every $i \in [0, t-1]$ the graph H^i has two cycles C_1^i and C_2^i such that*

- *(C_1^i is a subgraph of $H^i - x$ and C_2^i is a subgraph of $H^i - y$) or C_2^i is a subgraph of $H^i - x - y$, and*

- *the closed curves γ_1^i and γ_2^i representing C_1^i and C_2^i, respectively, in $\mathcal{D}$ satisfy $\Omega_S([\gamma_1^i], [\gamma_2^i]) = 1$.*

Proof. For every $i \in [t-1]$, let $H^{i,0}$ be the union of the graph H^i with the unique xy-path P^0 in $F^0 + yw^0$. The graph $F^{i,0} = F^i \cup P^0$ is a spanning tree of $H^{i,0}$, and $E(H^{i,0}) \setminus E(F^{i,0}) = E(H^i) \setminus E(T)$.

Let $\mathcal{E}$ be a drawing of $\amalg_{x,y} t H$ from Lemma 15. If $H = K_{3,3}$, we apply Corollary 27 to $G = H^i$ and $F = F^i$. If $H = K_5 - e$ or $H = K_{3,3} - e$ where $e = xy$, we apply Corollary 27

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to $G = H^i + e$, $F = F^i + e$, and the drawing of $H^i + e$ where e is drawn along the path P^0 in $\mathcal{E}$ (with self-crossings removed if necessary). In each of the three cases, we have an odd number of pairs of independent edges e, f in $E(H_i) \setminus E(T)$ such that $y_e^\top y_f = 1$. By Observation 29, for each $j \in [k]$, there are exactly two edges in $E(H^i) \setminus E(T)$ incident with y and independent from e_j^i; see also Figure 5. Therefore, considering all possible pairs of independent edges in $E(H_i) \setminus E(T)$, at least one of the following alternatives occurs:

1) $y_{h^i}^\top y_{g_1^i} = 1$,

2) $y_{e_j^i}^\top (y_{f_{j'}^i} + y_{h^i}) = 1$ for some $j \in [k]$ and $j' \in [l]$.

In further arguments, we no longer use the fact that the pairs of edges involved in the scalar products are independent.

To finish the proof of the lemma for $i \in [t-1]$, we use Observation 23. In particular, in case 1) we choose $C_1^i = C_{h^i}$ and $C_2^i = C_{g_1^i}$, and in case 2) we choose $C_1^i = C_{f_{j'}^i} + C_{h^i}$ and $C_2^i = C_{e_j^i}$.

Finally, by exchanging the roles of H^1 and H^0 in $\amalg_{x,y} tH$ in the proof, we also obtain cycles C_1^0 and C_2^0 with the required properties. $\qquad\square$

We are now ready to finish the proof of Theorem 7b).

Proposition 31. *Let $t \geq 2$ and let H be a Kuratowski xy-wing. Then $g_0(\amalg_{x,y} tH) \geq \lceil t/2 \rceil$ and $\mathrm{eg}_0(\amalg_{x,y} tH) \geq t$.*

Proof. Let $\mathcal{D}$ be an independently even drawing of $\amalg_{x,y} tH$ on a surface S of Euler genus g. For every $i \in [0, t-1]$, let C_1^i and C_2^i be the cycles from Lemma 30 and let γ_1^i and γ_2^i, respectively, be the closed curves representing them in $\mathcal{D}$.

Without loss of generality, we assume that there is an $s \in [0, t-1]$ such that

- for every $i \in [0, s]$, C_1^i is a subgraph of $H^i - x$ and C_2^i is a subgraph of $H^i - y$, and

- for every $i \in [s+1, t-1]$, the cycle C_2^i is a subgraph of $H^i - x - y$.

It follows that for distinct $i, i' \in [0, t]$, the cycles C_1^i and $C_2^{i'}$ are vertex-disjoint whenever $i, i' \in [0, s]$, $i, i' \in [s+1, t-1]$, or $i \leq s < i'$.

Let A be the $t \times t$ matrix with entries

$$A_{i,i'} = \Omega_S([\gamma_1^i], [\gamma_2^{i'}]).$$

By Lemma 30, Observation 14 and the previous discussion, the matrix A has 1-entries on the diagonal and 0-entries above the diagonal; see Figure 7. Thus, the rank of A over $\mathbb{Z}_2$ is t. Hence, the rank of Ω_S is at least t, which implies $g \geq t$. $\qquad\square$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$

Figure 7: An example of the matrix A with $t = 4$ and $s = 2$. The entries marked with $*$ may be equal to 0 or 1; the remaining entries are determined uniquely.

7   Acknowledgements

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