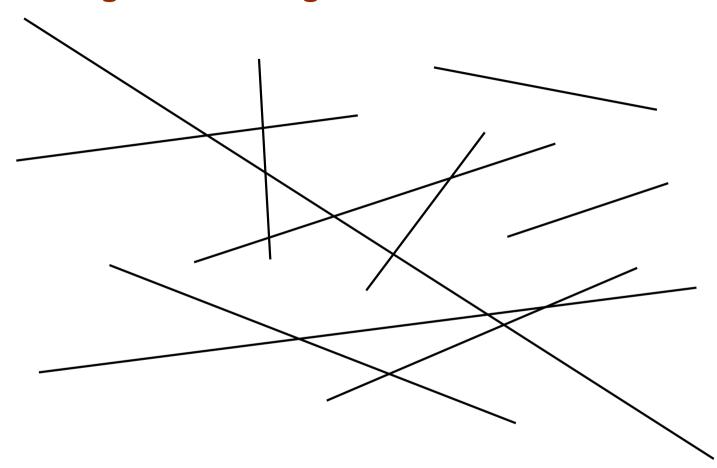
# Ramsey-type constructions for arrangements of segments

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## **Arrangement of segments:**



set of straight-line segments in general position in the plane

**Problem:** What is the largest number r(k) such that there exists an arrangement of r(k) segments with at most k pairwise crossing and at most k pairwise disjoint segments?

#### **Known results:**

$$r(k) \leq k^5$$
 [Larman et al., 1994]

$$r(k) \ge k^{\log 5/\log 2} > k^{2.3219}$$
 [Larman et al., 1994]

$$r(k) \geq k^{\log 27/\log 4} > k^{2.3774}$$
 [Károlyi et al., 1996]

#### **Our improvement:**

$$r(k) > k^{\log 169/\log 8} > k^{2.4669}$$

(for infinitely many values of k)

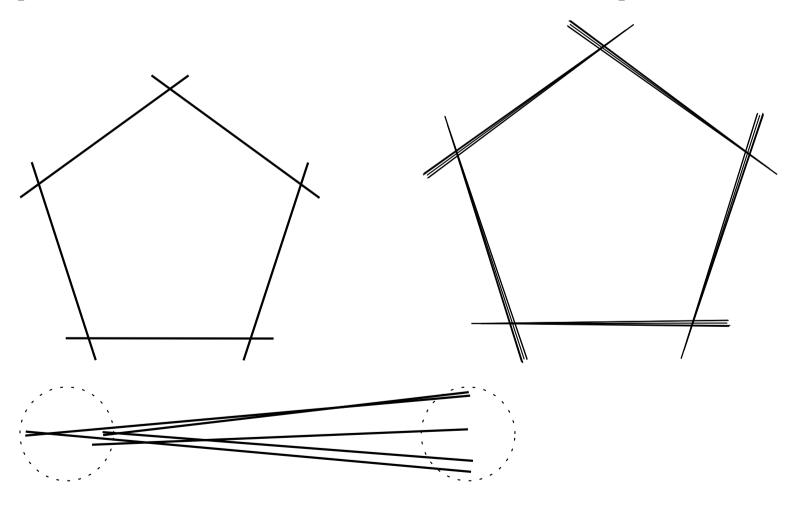
## Upper bound $k^5$

[Larman, Matoušek, Pach, Törőcsik, 1994]

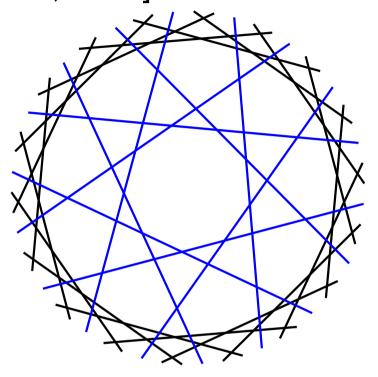
**Dilworth's Theorem:** A poset with  $m \cdot n$  elements has a chain of size m or an anti-chain of size n

### Previous constructions for the lower bound

[Larman, Matoušek, Pach, Törőcsik, 1994]



[Károlyi, Pach, Tóth, 1996]



27 segments, at most 4 pairwise crossing or pairwise disjoint

Lemma: Every convex arrangement can be flattened.

## **Limitations of convex arrangements**

Theorem [Kostochka, 1988]

A circle graph G with  $\alpha(G) \leq k$  and  $\omega(G) \leq k$  has at most  $(1+o(1)) \cdot k^2 \log k$  vertices.

 $\Rightarrow$  large convex arrangements can not give better lower bound for r(k).

#### **Our construction**

A (k, l)-arrangement = arrangement of segments with at most k pairwise crossing and at most l pairwise disjoint segments

**base**: an (8, 8)-arrangement of 169 segments composed of

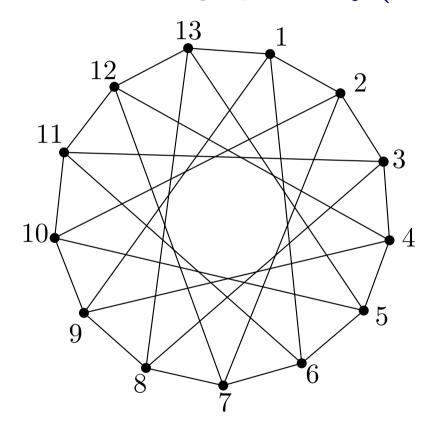
- a (2, 4)-arrangement of 13 segments and
- a (4, 2)-arrangement of 13 segments
- Ramsey theorem for graphs: 13 is best possible
- Upper bound for convex case [Černý, 2008]:

convex (2, 4)-arrangement: at most 12 segments

convex (4, 2)-arrangement: at most 11 segments

## A (2, 4)-arrangement

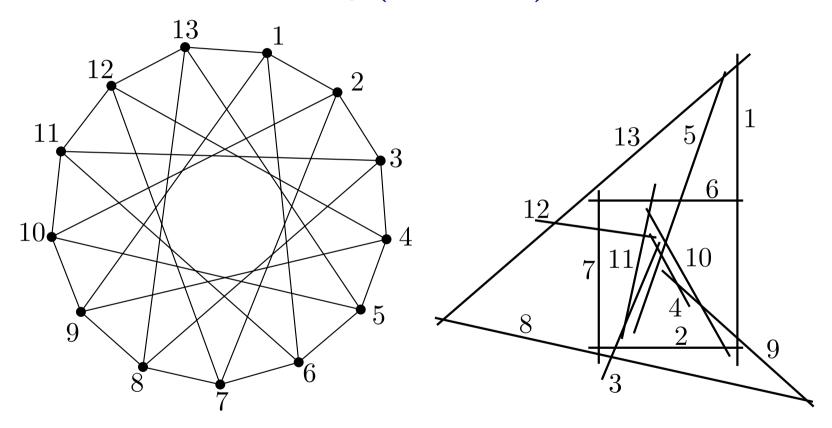
intersection graph:  $\mathbf{Cay}(\mathbb{Z}_{13}; 1, 5)$ 



- has no clique of size 3 and no independent set of size 5

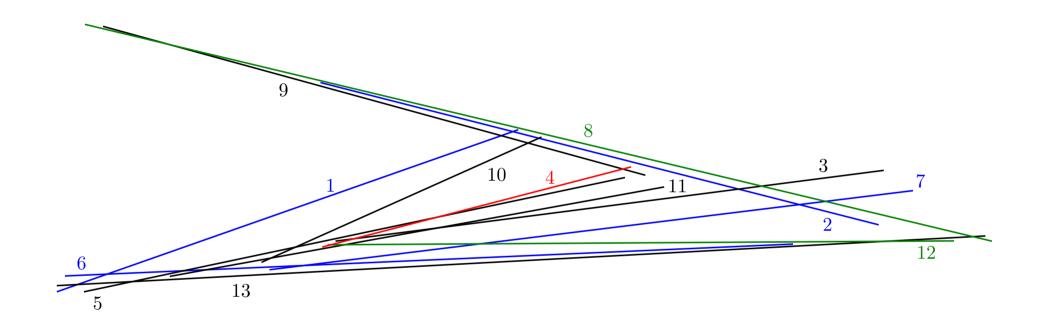
## A (2, 4)-arrangement

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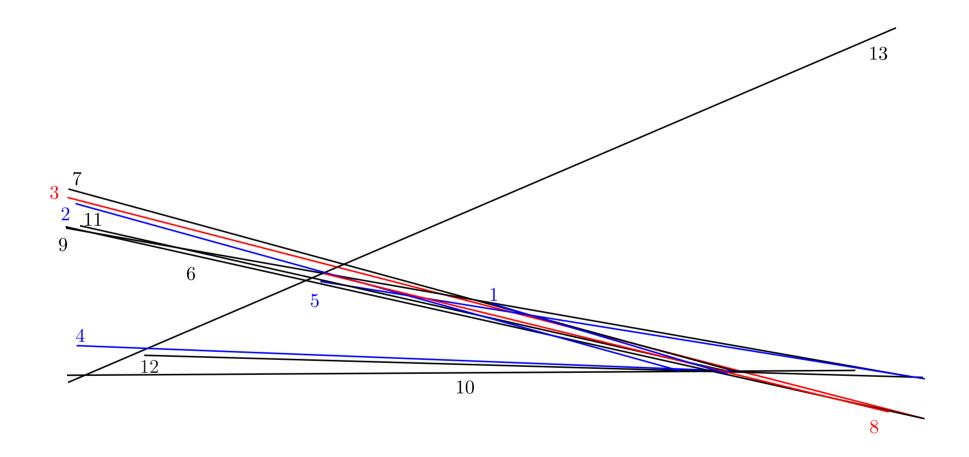
# A partially flattened (2, 4)-arrangement



# A flattened (2, 4)-arrangement

	l			l
	left $oldsymbol{x}$	left $oldsymbol{y}$	right $oldsymbol{x}$	right $oldsymbol{y}$
1	$-\varepsilon$	0	1-2arepsilon	$2arepsilon^2 + 2arepsilon^6$
2	$arepsilon^2$	$arepsilon - arepsilon^3$	$1-\varepsilon^2$	$arepsilon^3$
3	0	$arepsilon^4 + arepsilon^6$	1	$arepsilon^3 + 3arepsilon^4$
4	0	$arepsilon^4 - arepsilon^6$	1-2arepsilon	$2arepsilon^2-arepsilon^6$
<b>5</b>	$-\varepsilon+\varepsilon^2$	0	$1-2arepsilon^2$	$2arepsilon^3-2arepsilon^4$
6	-arepsilon	$2\varepsilon^6$	1-arepsilon	$2\varepsilon^6$
7	0	$arepsilon^6$	1	$arepsilon^3 + 2arepsilon^4$
8	0	$oldsymbol{arepsilon}$	$1+arepsilon^3$	0
9	0	$oldsymbol{arepsilon}$	$1-2arepsilon^2$	$2arepsilon^3-arepsilon^4$
<b>10</b>	$-arepsilon^2+3arepsilon^3$	$3arepsilon^6$	1-2arepsilon	$2arepsilon^2+arepsilon^6$
11	$-arepsilon^2$	$arepsilon^6$	$1-2arepsilon^2$	$2arepsilon^3 - 3arepsilon^4$
12	0	$arepsilon^4$	1	0
<b>13</b>	-arepsilon	0	1+arepsilon	0

# A partially flattened (4, 2)-arrangement

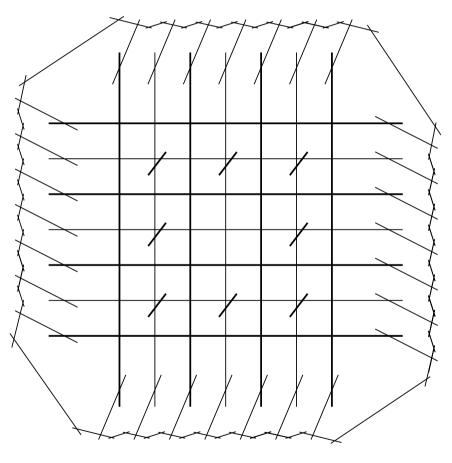


# A flattened (4, 2)-arrangement

	left $oldsymbol{x}$	left $oldsymbol{y}$	right $oldsymbol{x}$	right $oldsymbol{y}$
1	ε	$arepsilon^2 - arepsilon^3 + arepsilon^4 - 2arepsilon^5$	$1+arepsilon^2$	$-arepsilon^4 + arepsilon^6$
<b>2</b>	0	$arepsilon^2 + 3arepsilon^5$	$1-\varepsilon^3$	$arepsilon^7$
3	0	$arepsilon^2 + 4arepsilon^5$	1+arepsilon	$-arepsilon^3$
4	0	$2arepsilon^3$	$1+3arepsilon^4$	$-arepsilon^8$
5	$\varepsilon - \varepsilon^2 + \varepsilon^3$	$arepsilon^2 - arepsilon^3 + arepsilon^4 - arepsilon^8$	1+arepsilon	$-arepsilon^4$
6	0	$arepsilon^2 + arepsilon^5$	1+arepsilon	$-arepsilon^3$
7	0	$arepsilon^2 + 5arepsilon^5$	$1+3arepsilon^4$	$-3arepsilon^7$
8	$arepsilon - arepsilon^2 + arepsilon^3 + arepsilon^4 + 2arepsilon^5$	$arepsilon^2 - arepsilon^3 + arepsilon^4 + arepsilon^5 + arepsilon^6$	$1+arepsilon-arepsilon^4$	$-arepsilon^3$
9	0	$arepsilon^2$	1+arepsilon	$-arepsilon^4$
10	0	0	$1+5arepsilon^3$	0
11	0	$arepsilon^2 + 2arepsilon^5$	$1+3arepsilon^4-2arepsilon^5$	$arepsilon^8$
12	$arepsilon - arepsilon^3$	$arepsilon^3 - arepsilon^4$	1+arepsilon	$-arepsilon^4$
13	0	0	1	ε

## Can every arrangement be flattened?

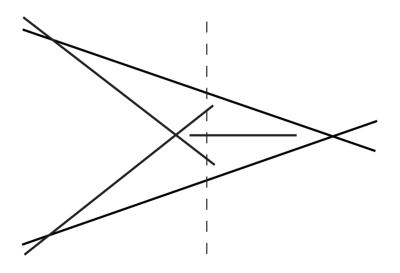
NO



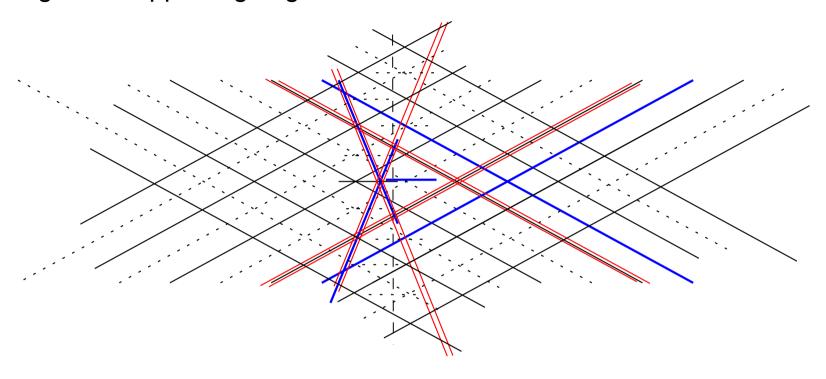
(not even topologically)

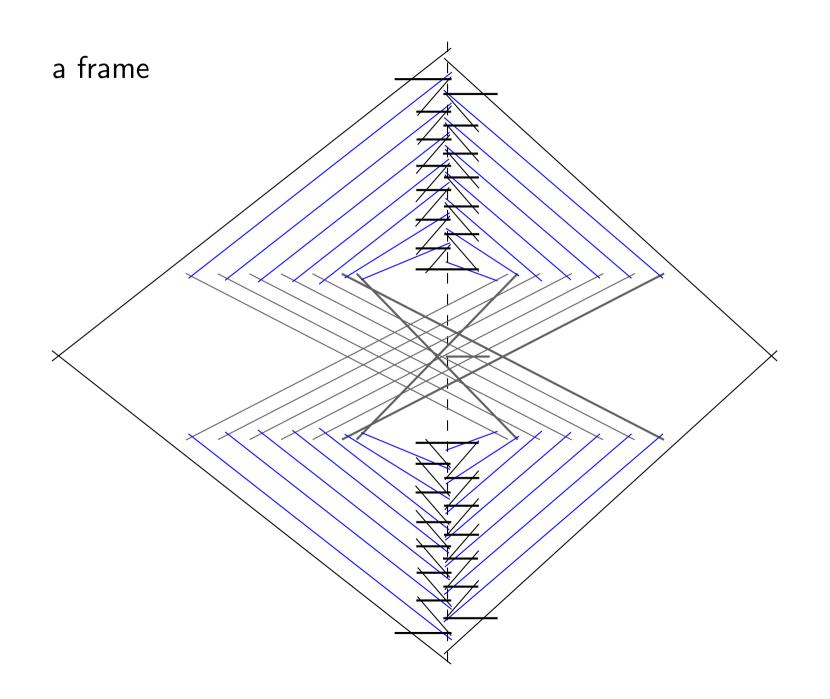
Theorem There exists a non-flattenable arrangement of segments such that all segments cross a common line (so it can be flattened topologically).

#### core:



### a grid of supporting segments





## **Open problems**

- ullet better upper and lower bound for r(k)
- upper bound for pseudosegments
- upper bound for curves (string graphs)