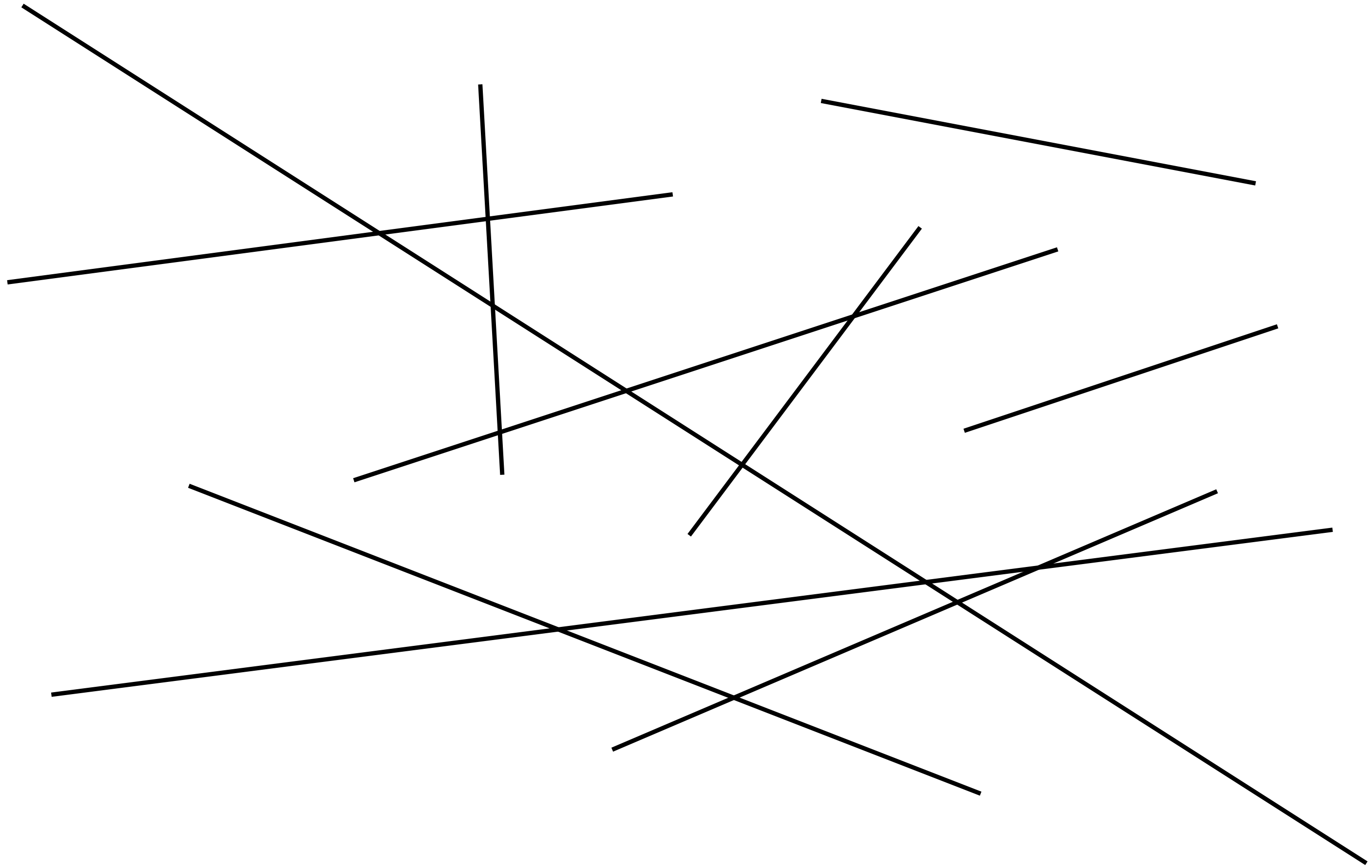
Ramsey-type constructions for arrangements of segments

Jan Kynčl

Charles University, Prague

Arrangement of segments:



set of straight-line segments in general position in the plane

Problem: What is the largest number $r(k)$ such that there exists an arrangement of $r(k)$ segments with at most k pairwise crossing and at most k pairwise disjoint segments?

Known results:

$r(k) \leq k^5$ [Larman et al., 1994]

$r(k) \geq k^{\log 5/\log 2} > k^{2.3219}$ [Larman et al., 1994]

$r(k) \geq k^{\log 27/\log 4} > k^{2.3774}$ [Károlyi et al., 1996]

Our improvement:

$r(k) \geq k^{\log 169/\log 8} > k^{2.4669}$

(for infinitely many values of k)

Upper bound k^5

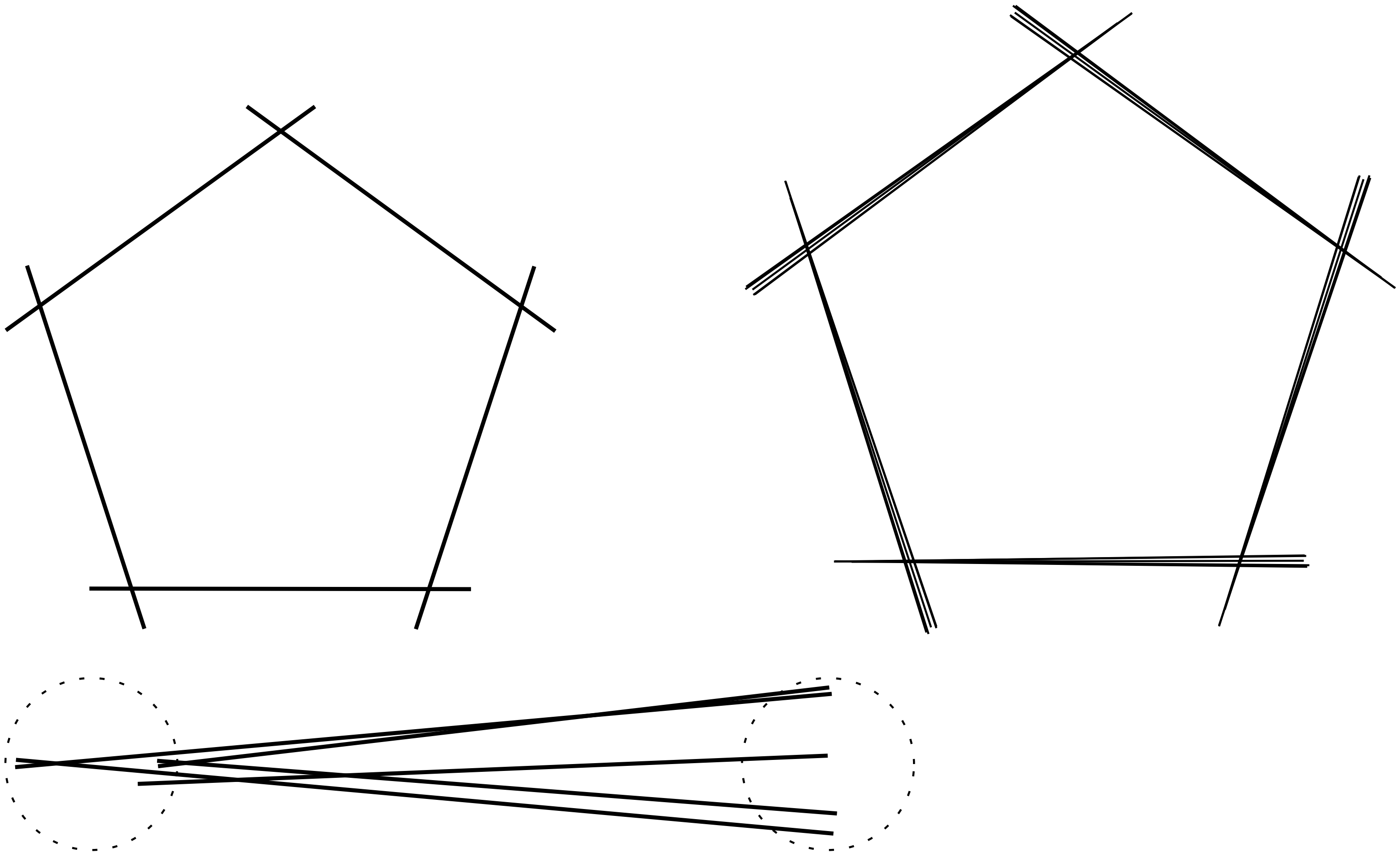
[Larman, Matoušek, Pach, Törőcsik, 1994]

$\prec_1$ $\prec_2$ $\prec_3$ $\prec_4$

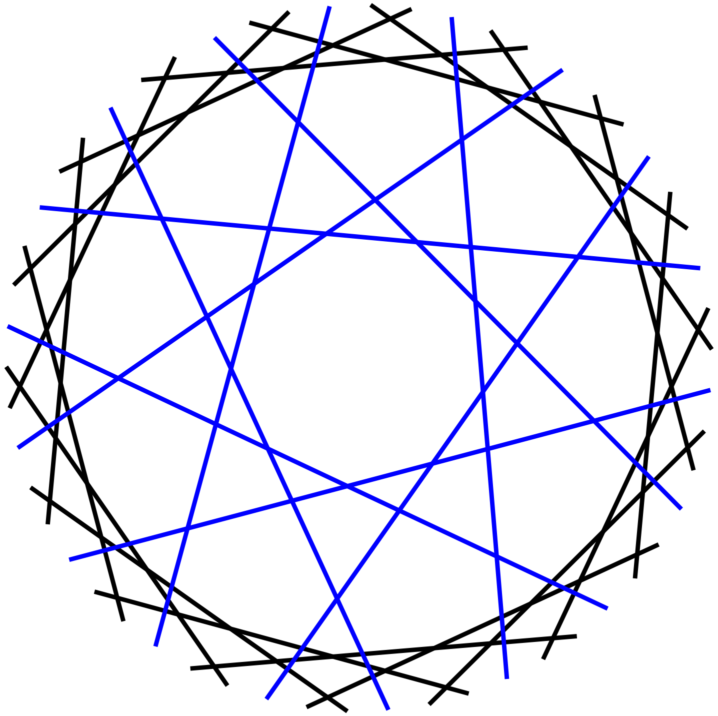
Dilworth's Theorem: A poset with $m \cdot n$ elements has a chain of size m or an anti-chain of size n

Previous constructions for the lower bound

[Larman, Matoušek, Pach, Törőcsik, 1994]



[Károlyi, Pach, Tóth, 1996]



27 segments, at most 4 pairwise crossing or pairwise disjoint

Lemma: Every **convex** arrangement can be flattened.

Limitations of convex arrangements

Theorem [Kostochka, 1988]

A **circle graph** G with $\alpha(G) \leq k$ and $\omega(G) \leq k$ has at most $(1 + o(1)) \cdot k^2 \log k$ vertices.

$\Rightarrow$ large convex arrangements can not give better lower bound for $r(k)$.

Our construction

A (k, l)-arrangement $=$ arrangement of segments with at most k pairwise crossing and at most l pairwise disjoint segments

base: an (8, 8)-arrangement of 169 segments
composed of
a (2, 4)-arrangement of 13 segments and
a (4, 2)-arrangement of 13 segments

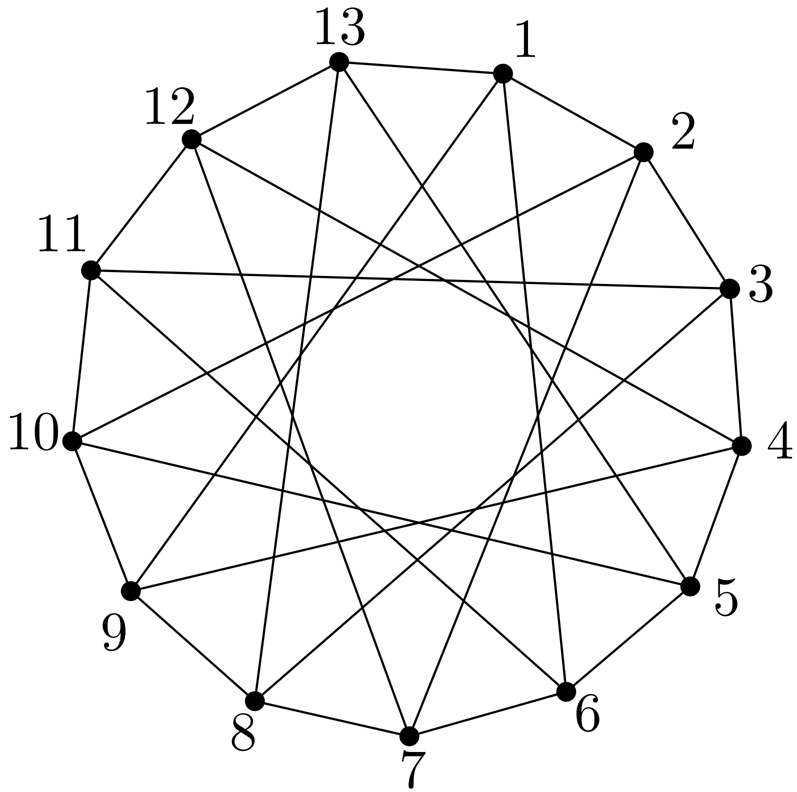
- Ramsey theorem for graphs: 13 is best possible

- Upper bound for convex case [Černý, 2008]:

convex (2, 4)-arrangement: at most 12 segments
convex (4, 2)-arrangement: at most 11 segments

A (2, 4)-arrangement

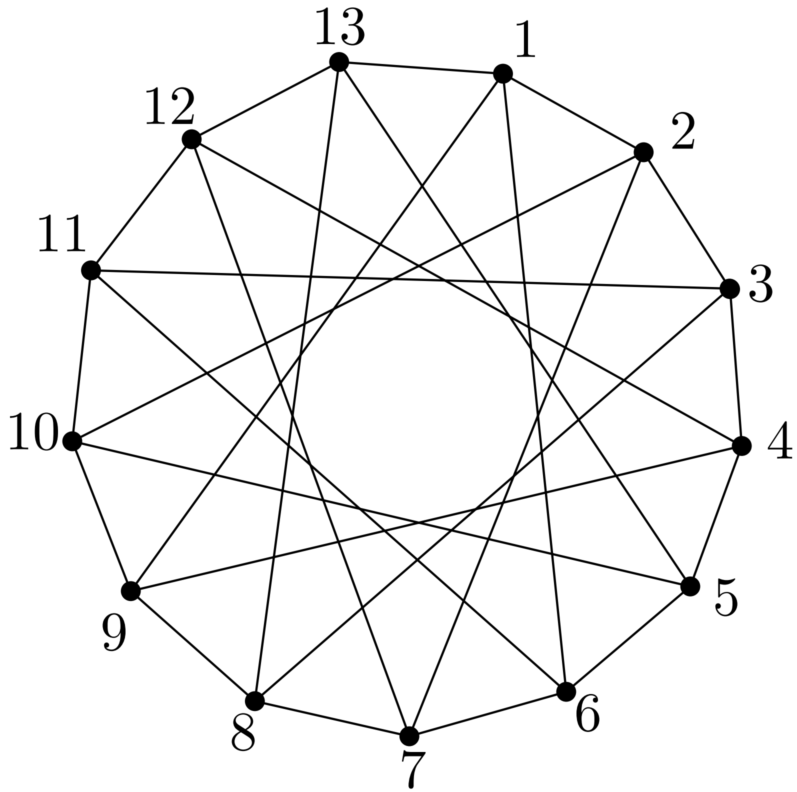
intersection graph: $\mathbf{Cay}(\mathbb{Z}_{13}; 1, 5)$

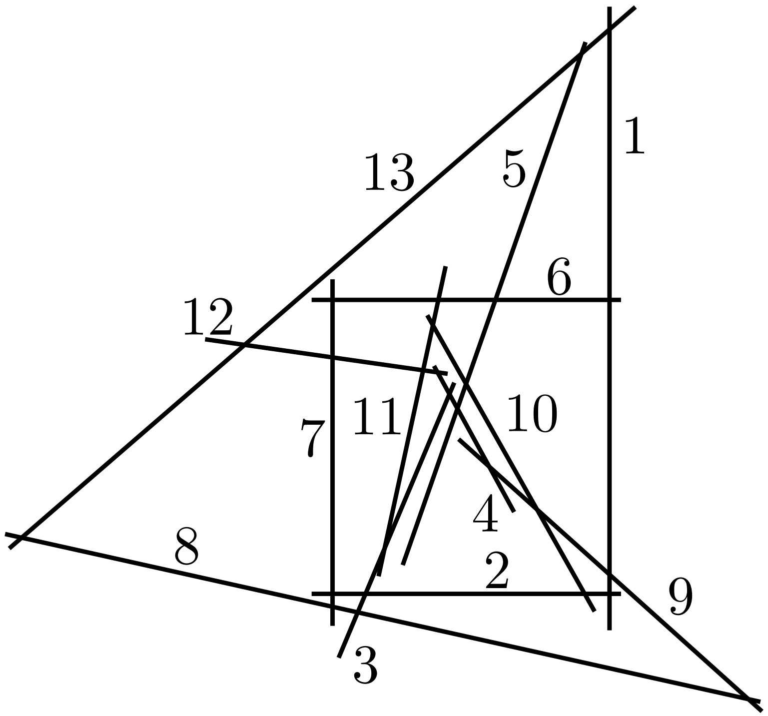


- has no clique of size 3 and no independent set of size 5

A (2, 4)-arrangement

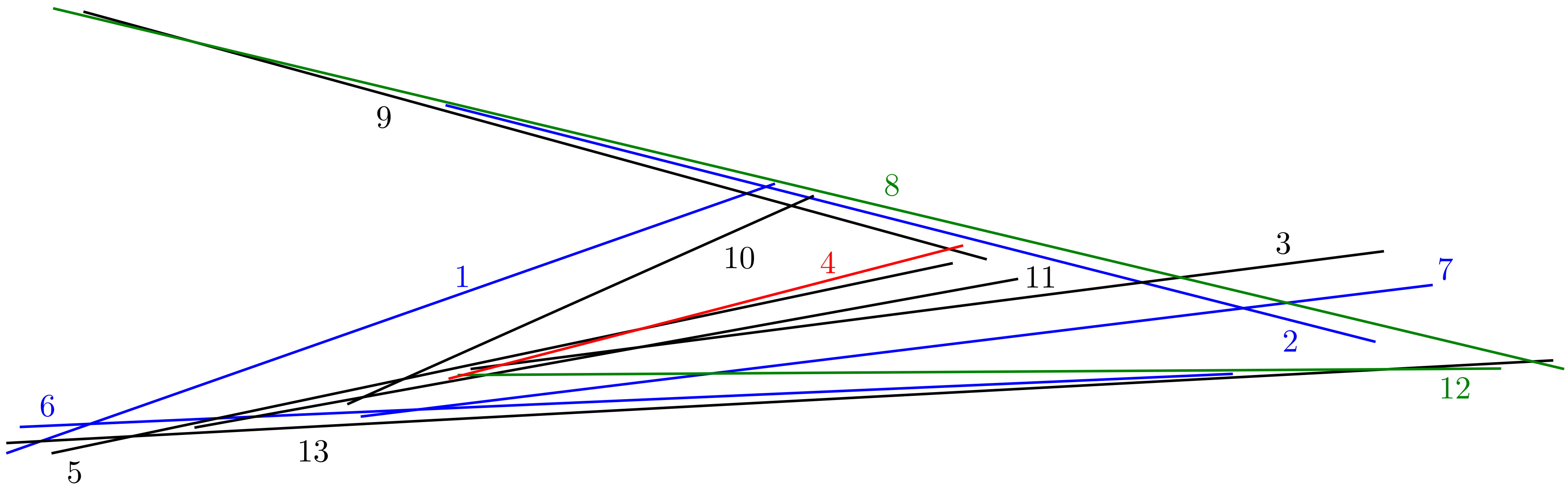
intersection graph: $\mathbf{Cay}(\mathbb{Z}_{13}; 1, 5)$





- has no clique of size 3 and no independent set of size 5

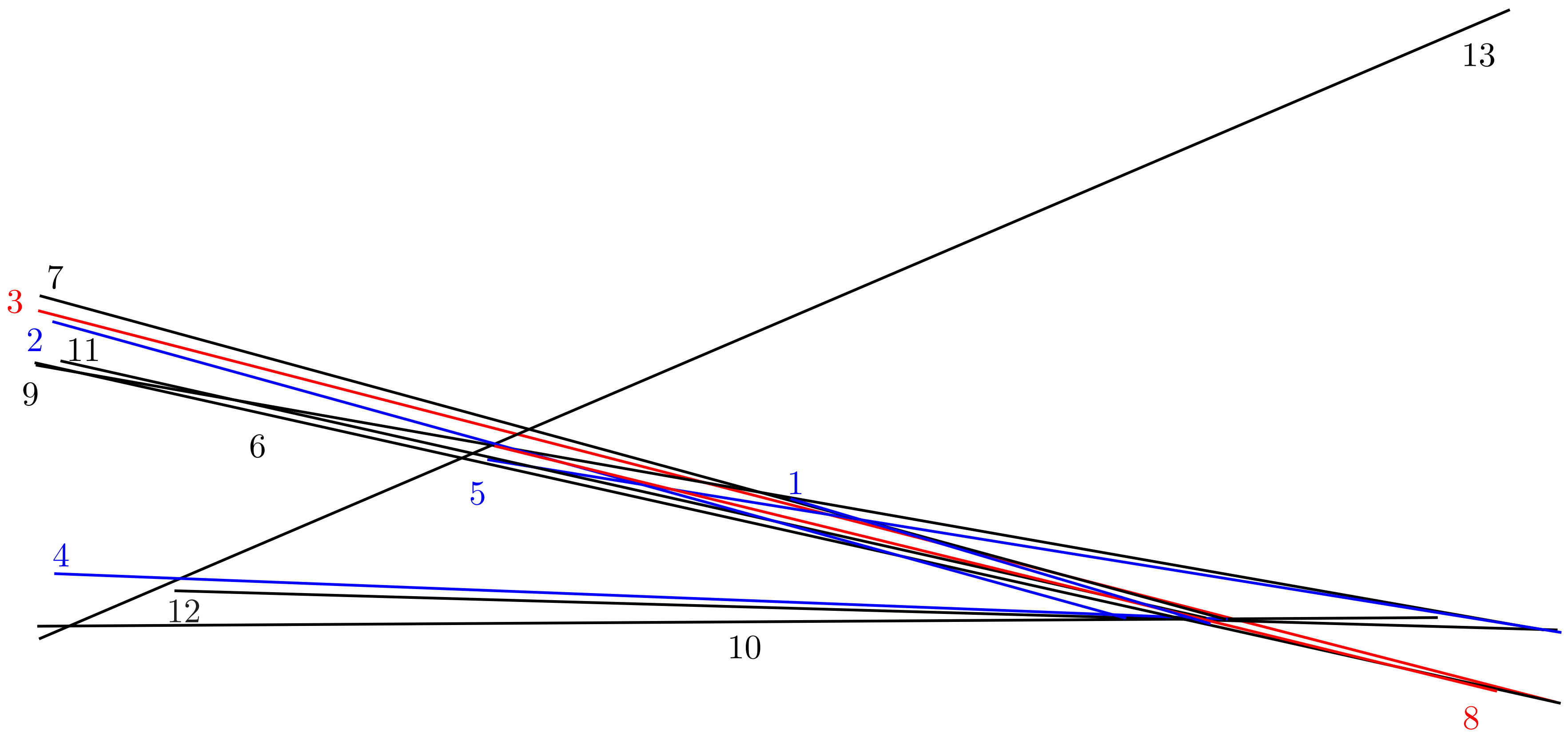
A partially flattened (2, 4)-arrangement



A flattened (2, 4)-arrangement

	left x	left y	right x	right y
$\mathbf{1}$	$-\varepsilon$	0	$1-2\varepsilon$	$2\varepsilon^2+2\varepsilon^6$
$\mathbf{2}$	ε^2	$\varepsilon-\varepsilon^3$	$1-\varepsilon^2$	ε^3
$\mathbf{3}$	0	$\varepsilon^4+\varepsilon^6$	1	$\varepsilon^3+3\varepsilon^4$
$\mathbf{4}$	0	$\varepsilon^4-\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2-\varepsilon^6$
$\mathbf{5}$	$-\varepsilon+\varepsilon^2$	0	$1-2\varepsilon^2$	$2\varepsilon^3-2\varepsilon^4$
$\mathbf{6}$	$-\varepsilon$	$2\varepsilon^6$	$1-\varepsilon$	$2\varepsilon^6$
$\mathbf{7}$	0	ε^6	1	$\varepsilon^3+2\varepsilon^4$
$\mathbf{8}$	0	ε	$1+\varepsilon^3$	0
$\mathbf{9}$	0	ε	$1-2\varepsilon^2$	$2\varepsilon^3-\varepsilon^4$
$\mathbf{10}$	$-\varepsilon^2+3\varepsilon^3$	$3\varepsilon^6$	$1-2\varepsilon$	$2\varepsilon^2+\varepsilon^6$
$\mathbf{11}$	$-\varepsilon^2$	ε^6	$1-2\varepsilon^2$	$2\varepsilon^3-3\varepsilon^4$
$\mathbf{12}$	0	ε^4	1	0
$\mathbf{13}$	$-\varepsilon$	0	$1+\varepsilon$	0

A partially flattened (4, 2)-arrangement

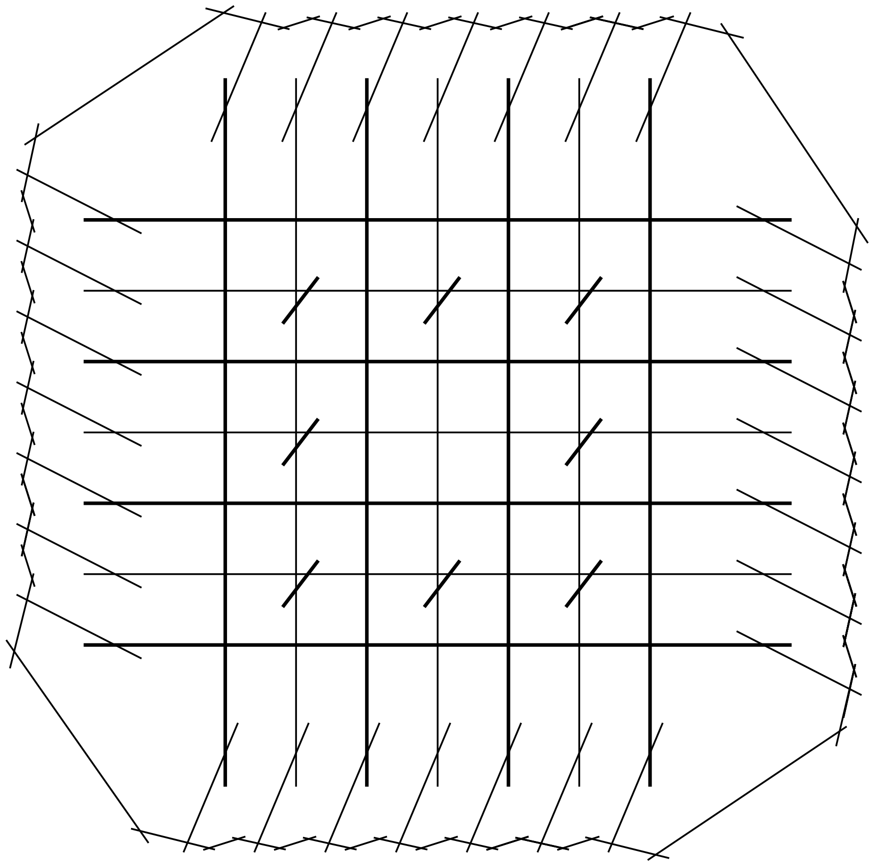


A flattened (4, 2)-arrangement

	left $\boldsymbol{x}$	left $\boldsymbol{y}$	right $\boldsymbol{x}$	right $\boldsymbol{y}$
1	ε	$\varepsilon^2-\varepsilon^3+\varepsilon^4-2\varepsilon^5$	$1+\varepsilon^2$	$-\varepsilon^4+\varepsilon^6$
2	0	$\varepsilon^2+3\varepsilon^5$	$1-\varepsilon^3$	ε^7
3	0	$\varepsilon^2+4\varepsilon^5$	$1+\varepsilon$	$-\varepsilon^3$
4	0	$2\varepsilon^3$	$1+3\varepsilon^4$	$-\varepsilon^8$
5	$\varepsilon-\varepsilon^2+\varepsilon^3$	$\varepsilon^2-\varepsilon^3+\varepsilon^4-\varepsilon^8$	$1+\varepsilon$	$-\varepsilon^4$
6	0	$\varepsilon^2+\varepsilon^5$	$1+\varepsilon$	$-\varepsilon^3$
7	0	$\varepsilon^2+5\varepsilon^5$	$1+3\varepsilon^4$	$-3\varepsilon^7$
8	$\varepsilon-\varepsilon^2+\varepsilon^3+\varepsilon^4+2\varepsilon^5$	$\varepsilon^2-\varepsilon^3+\varepsilon^4+\varepsilon^5+\varepsilon^6$	$1+\varepsilon-\varepsilon^4$	$-\varepsilon^3$
9	0	ε^2	$1+\varepsilon$	$-\varepsilon^4$
10	0	0	$1+5\varepsilon^3$	0
11	0	$\varepsilon^2+2\varepsilon^5$	$1+3\varepsilon^4-2\varepsilon^5$	ε^8
12	$\varepsilon-\varepsilon^3$	$\varepsilon^3-\varepsilon^4$	$1+\varepsilon$	$-\varepsilon^4$
13	0	0	1	ε

Can every arrangement be flattened?

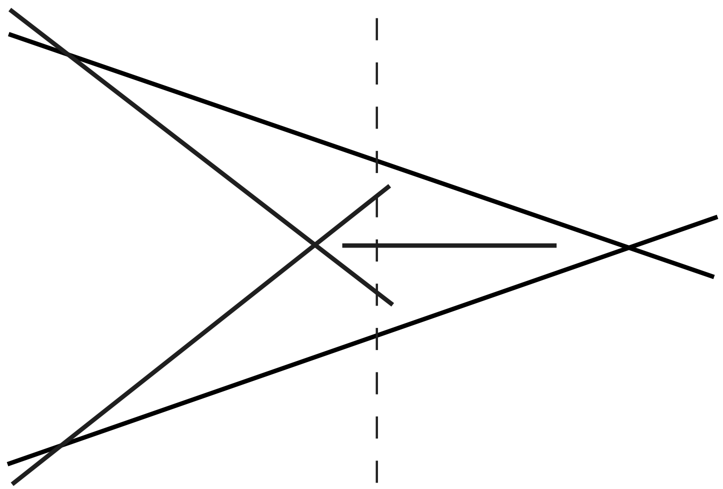
NO



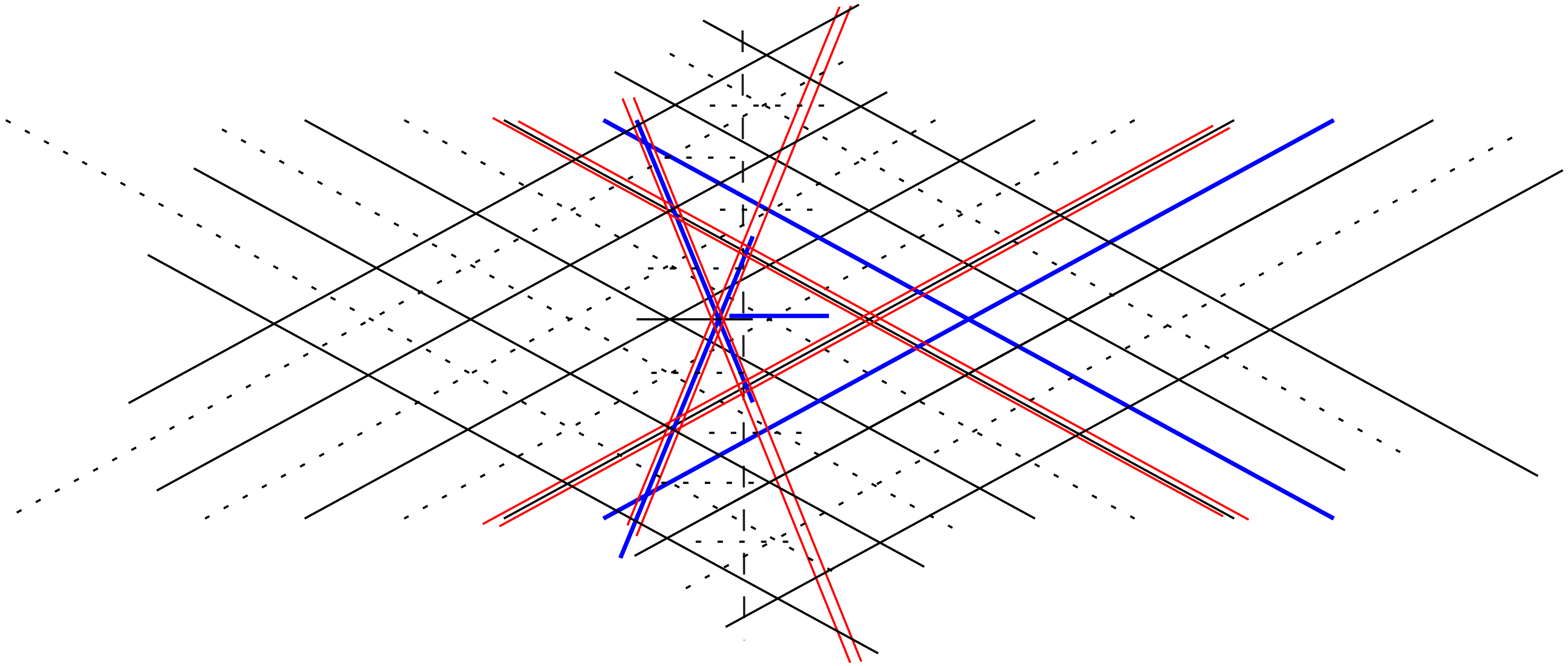
(not even topologically)

Theorem There exists a non-flattenable arrangement of segments such that all segments cross a common line (so it can be flattened topologically).

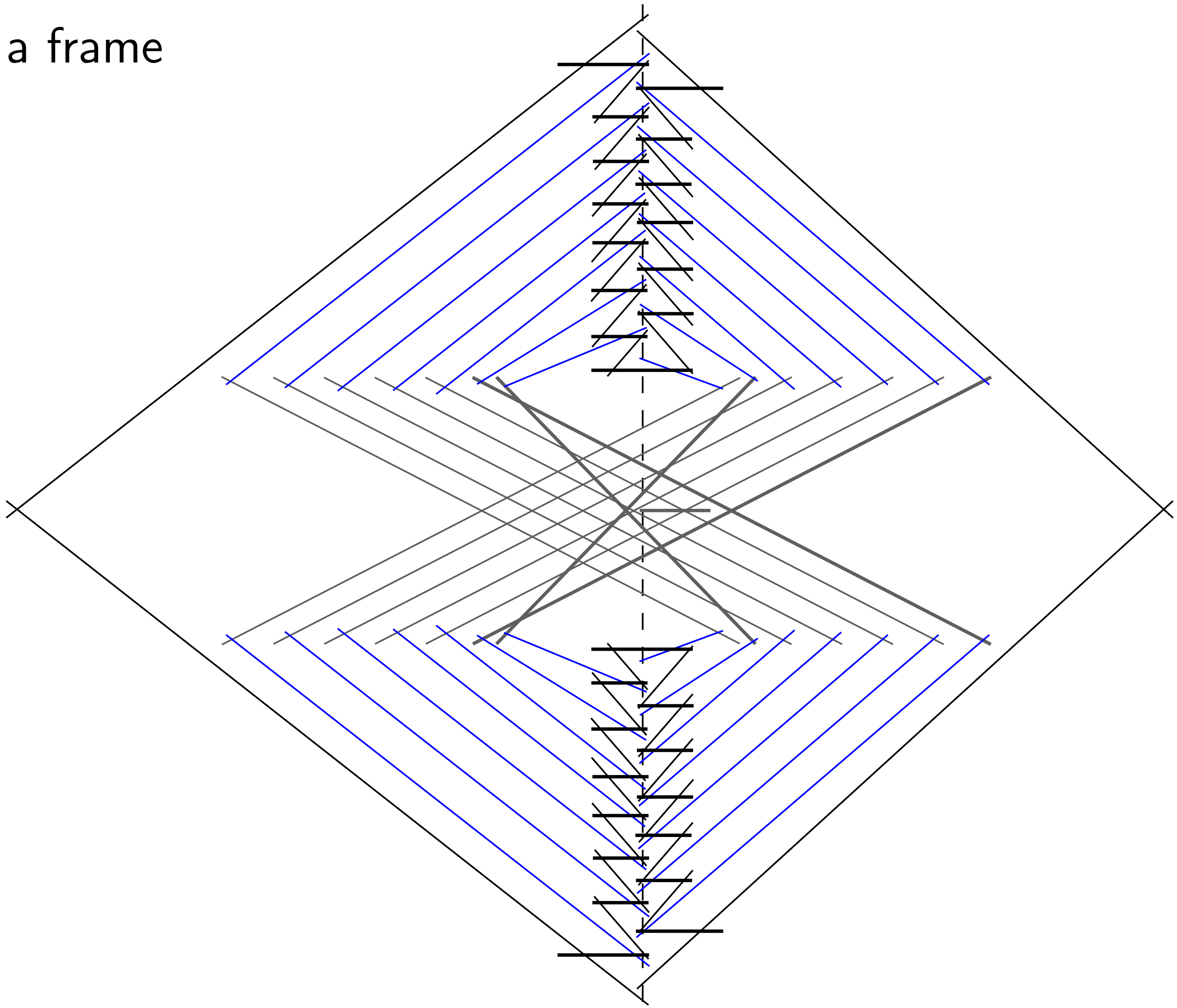
core:



a grid of supporting segments



a frame



Open problems

- better upper and lower bound for $r(k)$
- upper bound for pseudosegments
- upper bound for curves (string graphs)