The hamburger theorem

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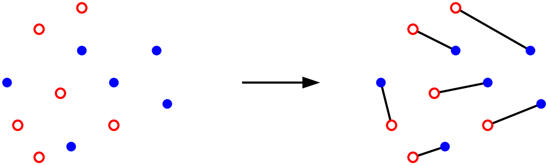
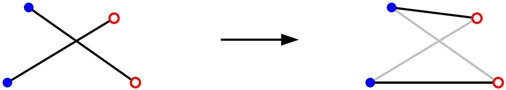
EPFL

Problem:

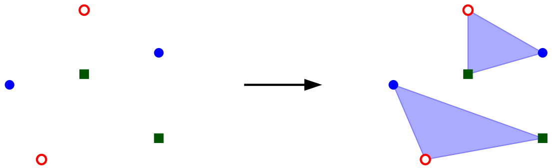
Given n red and n blue points in the plane in general position, draw n noncrossing red-blue segments.



Solution: take the shortest red-blue perfect matching.



Theorem: (Akiyama and Alon, 1989)
Given point sets $X_1, X_2, \dots, X_d \subset \mathbb{R}^d$ in general position, with $|X_1| = |X_2| = \cdots = |X_d| = n$, then there are n disjoint rainbow $(d-1)$-dimensional simplices.



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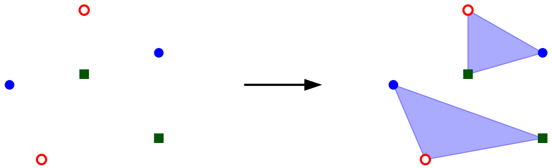

Proof: recursive cutting by hyperplanes, using the discrete ham-sandwich theorem.

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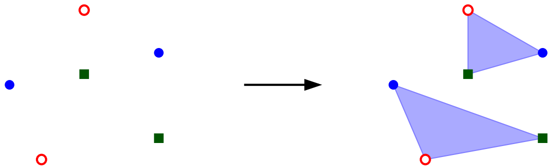
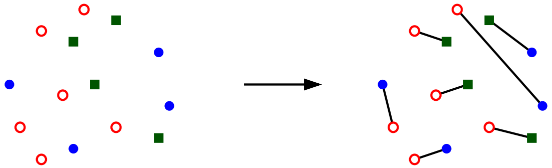
Given point sets $X_1, X_2, \ldots, X_d \subset \mathbb{R}^d$ in general position, with $|X_1| = |X_2| = \cdots = |X_d| = n$, then there are n disjoint rainbow $(d-1)$-dimensional simplices.



Proof: recursive cutting by hyperplanes, using the discrete ham-sandwich theorem.

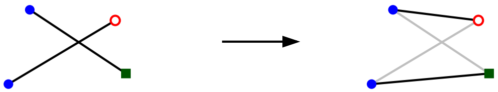
The discrete ham-sandwich theorem: (Stone and Tukey, 1942)
If $X_1, X_2, \ldots, X_d \subset \mathbb{R}^d$ are disjoint finite sets in general position, then there is a hyperplane that bisects each X_i exactly in half.

Theorem: (Kano, Suzuki and Uno, 2014)
Let $R, G, B \subset \mathbb{R}^2$ be sets of red, green and blue points in general position such that $|R| + |G| + |B| = 2n$ and $|R|, |G|, |B| \leq n$. Then there are n disjoint rainbow segments.



Proof: using a special result for partitioning colored sets on a line

Alternative solution: shortest rainbow perfect matching



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Corollary: Given point sets $X_1, X_2, \ldots, X_r \subset \mathbb{R}^2$ in general position such that $|X_1| + |X_2| + \cdots + |X_r| = 2n$ and $|X_i| \leq n$ for every $i \in [d+1]$, then there are n disjoint rainbow segments.

Proof: merging the smallest sets:

$$(4, 4, 3, 2, 1) \to (4, 4, 3, 3) \to (4, 4, 6)$$

(also shortest rainbow perfect matching)

Conjecture: (Kano and Suzuki) Let $r \geq d \geq 3$ and $n \geq 1$. Given point sets $X_1, X_2, \ldots, X_r \subset \mathbb{R}^d$ in general position such that $|X_1| + |X_2| + \cdots + |X_r| = dn$ and $|X_i| \leq n$ for every $i \in [r]$, then there are n disjoint rainbow $(d-1)$-dimensional simplices.

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Theorem: The conjecture is true for $d \geq 2$ and $r = d + 1$.

Plan of proof: recursive cutting by a hyperplane, into a pair of **balanced** subsets; that is, for each of the two halfspaces H, we want

$$|H \cap X_i| \leq \frac{1}{d} \sum_{j=1}^{r} |H \cap X_j|.$$

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$\rightarrow$ we need a generalization of the discrete ham-sandwich theorem to $d + 1$ sets in $\mathbb{R}^d$.
First we show a continuous version, then discretize it.

Definition:

Let $r \geq d$ and let $\mu_1, \mu_2, \ldots, \mu_r$ be finite Borel measures on $\mathbb{R}^d$. We say that $\mu_1, \mu_2, \ldots, \mu_r$ are **balanced** in a subset $X \subseteq \mathbb{R}^d$ if for every $i \in [r]$, we have

$$\mu_i(X) \leq \frac{1}{d} \cdot \sum_{j=1}^{r} \mu_j(X).$$

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Then there exists a hyperplane h such that for each open halfspace H defined by h, the measures $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are balanced in H and

$$\sum_{j=1}^{d+1} \mu_j(H) \geq \min\left(\frac{1}{2}, 1 - d\omega\right) \geq \frac{1}{d+1}.$$

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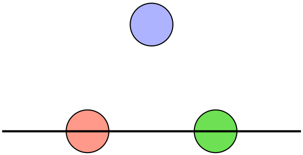
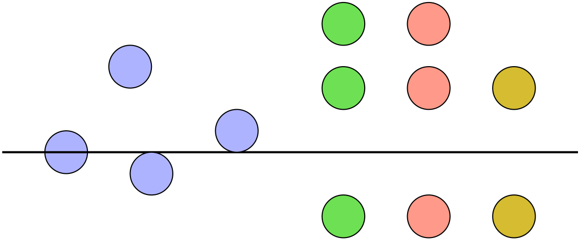
For $\omega = 0$ we get exactly the ham-sandwich theorem.

- The lower bound $1/(d+1)$ is tight:



- discretization is nontrivial:



Proof of the hamburger theorem

analogous to the proof of the ham-sandwich theorem:

- parametrize half-spaces in $\mathbb{R}^d$ by the points of $S^d = \{\mathbf{u} = (u_0, u_1, \ldots, u_d) \in \mathbb{R}^{d+1}; u_0^2 + u_1^2 + \cdots + u_d^2 = 1\}$:
  if $|u_0| < 1$, then

$$H^-(\mathbf{u}) := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; u_1x_1 + u_2x_2 + \cdots + u_dx_d < u_0\},$$

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and

$$H^-(1,0,0,\ldots,0) := \mathbb{R}^d, \qquad H^+(1,0,0,\ldots,0) := \emptyset,$$
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We have $H^-(\mathbf{u}) = H^+(-\mathbf{u})$ for every $\mathbf{u} \in S^d$.

- define $f = (f_1, \ldots, f_{d+1}) : S^d \to \mathbb{R}^{d+1}$ by

$$f_i(\mathbf{u}) := \mu_i(H^-(\mathbf{u})).$$

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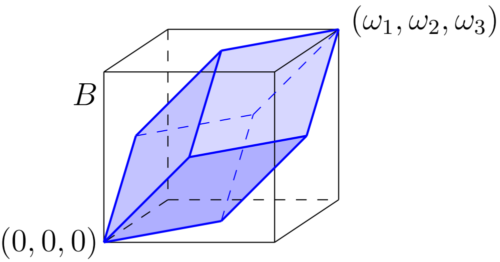
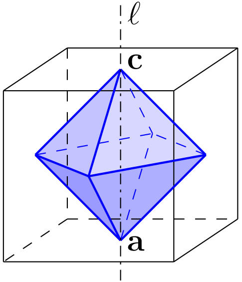
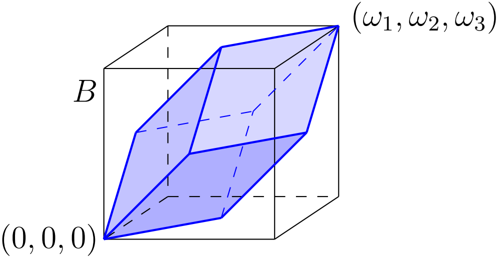
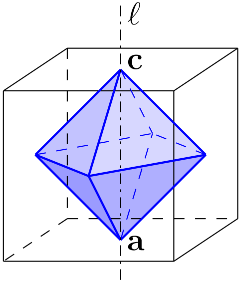
- our goal is to show that the image of f intersects the **target polytope**, determined by the conditions "balanced" and "nontrivial"

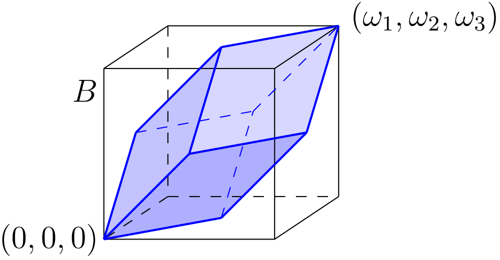


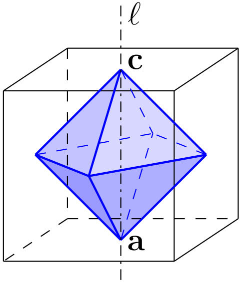






- assume $\omega = \omega_{d+1}$
- $t := \min\left(\frac{1}{2d}, \frac{1}{d} - \omega\right)$





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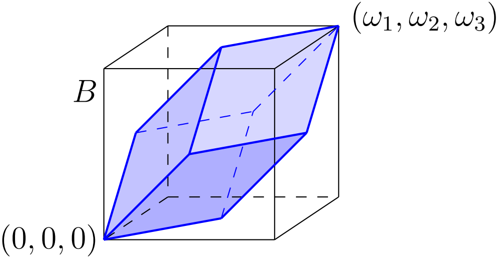
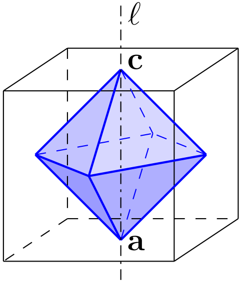
- $t := \min\left(\frac{1}{2d}, \frac{1}{d} - \omega\right)$
- $\mathbf{a} := (t, t, \ldots, t, 0)$, $\mathbf{b} := (\omega_1/2, \omega_2/2, \ldots, \omega_{d+1}/2)$,
- $\mathbf{c} := (\omega_1 - t, \omega_2 - t, \ldots, \omega_d - t, \omega_{d+1})$





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- $\mathbf{c} := \left(\omega_1 - t, \omega_2 - t, \ldots, \omega_d - t, \omega_{d+1}\right)$
- $\ell :=$ line $\mathbf{ac}$
- $\pi_\ell :=$ projection to a hyperplane orthogonal to ℓ
- $g(\mathbf{u}) := \pi_\ell(f(\mathbf{u}) - \mathbf{b})$

g is antipodal map from S^d to $\mathbb{R}^d$. By the Borsuk–Ulam theorem, there exists $\mathbf{u} \in S^d$ such that $g(\mathbf{u}) = \mathbf{0}$, which means that $f(\mathbf{u}) \in \ell$.