Monochromatic triangles in two-colored plane

Vít Jelínek
Jan Kynčl
Rudolf Stolař
Tomáš Valla

Euclidean Ramsey theory

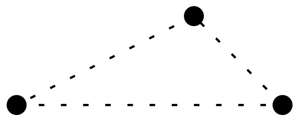
X ... a finite set of points in $\mathbb{E}^d$

c ... a finite number of colors

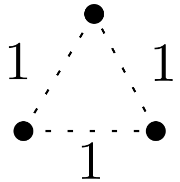
Question: Does every coloring of $\mathbb{E}^d$ with c colors contain a monochromatic copy of X?

copy of X = congruent copy of X = set obtained from X by translations and rotations

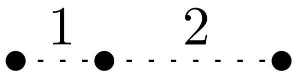
X' is monochromatic if all points of X' have the same color.



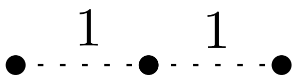
$\mathbb{E}^3$ 2 colors YES



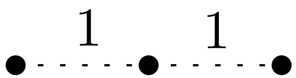
$\mathbb{E}^2$ 2 colors NO



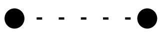
$\mathbb{E}^2$ 2 colors YES



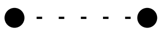
$\mathbb{E}^2$ 2 colors ???



$\mathbb{E}^n$ 4 colors NO

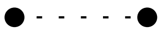


$\mathbb{E}^2$ 3 colors YES

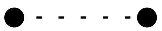


$\mathbb{E}^2$ 4 colors ???

$\mathbb{E}^2$ 5 colors ???



$\mathbb{E}^2$ 6 colors ???



$\mathbb{E}^2$ 7 colors NO

case $d = 2, c = 2, |X| = 3$

triangle ... a set of 3 points, including

degenerate triangle ... a set of 3 collinear points

(a, b, c)-triangle ... a triangle with sides of length a, b, c in counter-clockwise order

unit triangle ... a $(1, 1, 1)$ triangle

coloring ... a partition of $\mathbb{E}^2$ into two sets, $\mathcal{B}$ (black) and $\mathcal{W}$ (white)

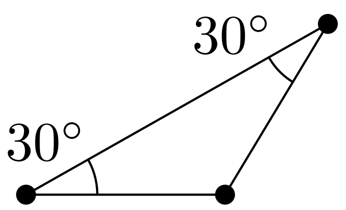
Coloring χ contains a triangle T if there is a monochromatic copy of T, otherwise χ avoids T.

Examples of known results

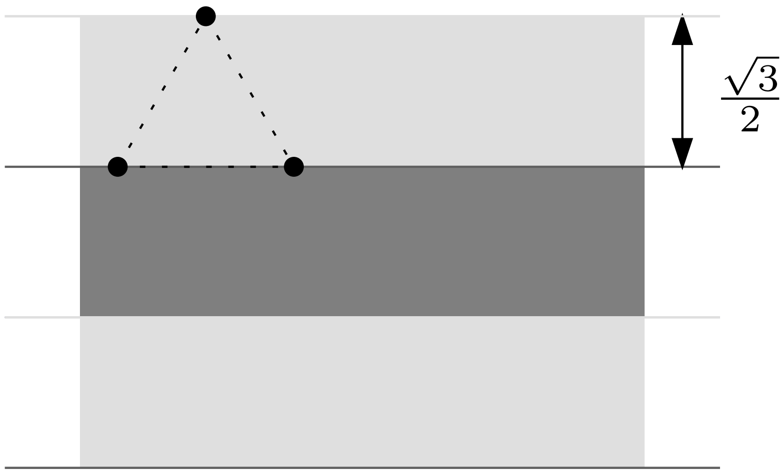
Theorem [Erdös et al., 1973; Shader, 1979]

Every coloring contains every

- triangle with a 30°, 90° or 150° angle
- triangle with a ratio between two sides equal to $2\sin 15^\circ$, $2\sin 36^\circ$, $2\sin 45^\circ$, $2\sin 60^\circ$ or $2\sin 75^\circ$
- $(a, 2a, 3a)$-triangle
- (a, b, c)-triangle satisfying $c^2 = a^2 + 2b^2$



Strip coloring avoiding a unit triangle:



Conjecture 1 [Erdös et al., 1973]

Every coloring contains every non-equilateral triangle.

Conjecture 2 [Erdös et al., 1973]

The strip coloring is the only coloring avoiding any triangle (up to scaling and modification of colors on the boundaries of the strips).

Our results

Theorem 1 Each coloring $\chi = (\mathcal{B}, \mathcal{W})$, where $\mathcal{B}$ is a closed set (and $\mathcal{W}$ is open), contains every triangle.

Theorem 2

- Each polygonal coloring contains every non-equilateral triangle.
- Characterization of all polygonal colorings avoiding an equilateral triangle
- Conjecture 2 is false.

Reduction to equilateral triangles

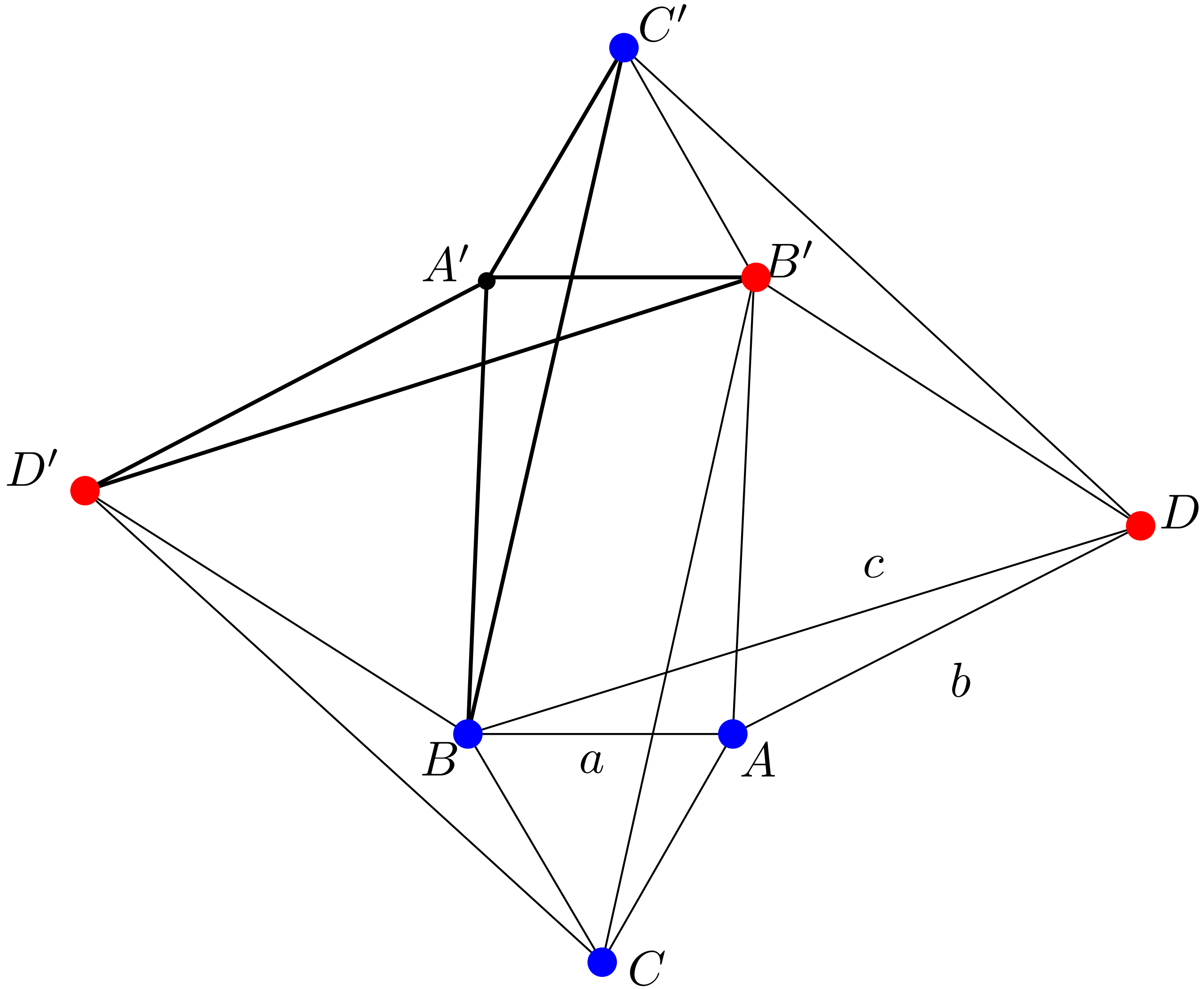
Lemma [Erdös et al., 1973]

Let χ be a coloring of the plane.

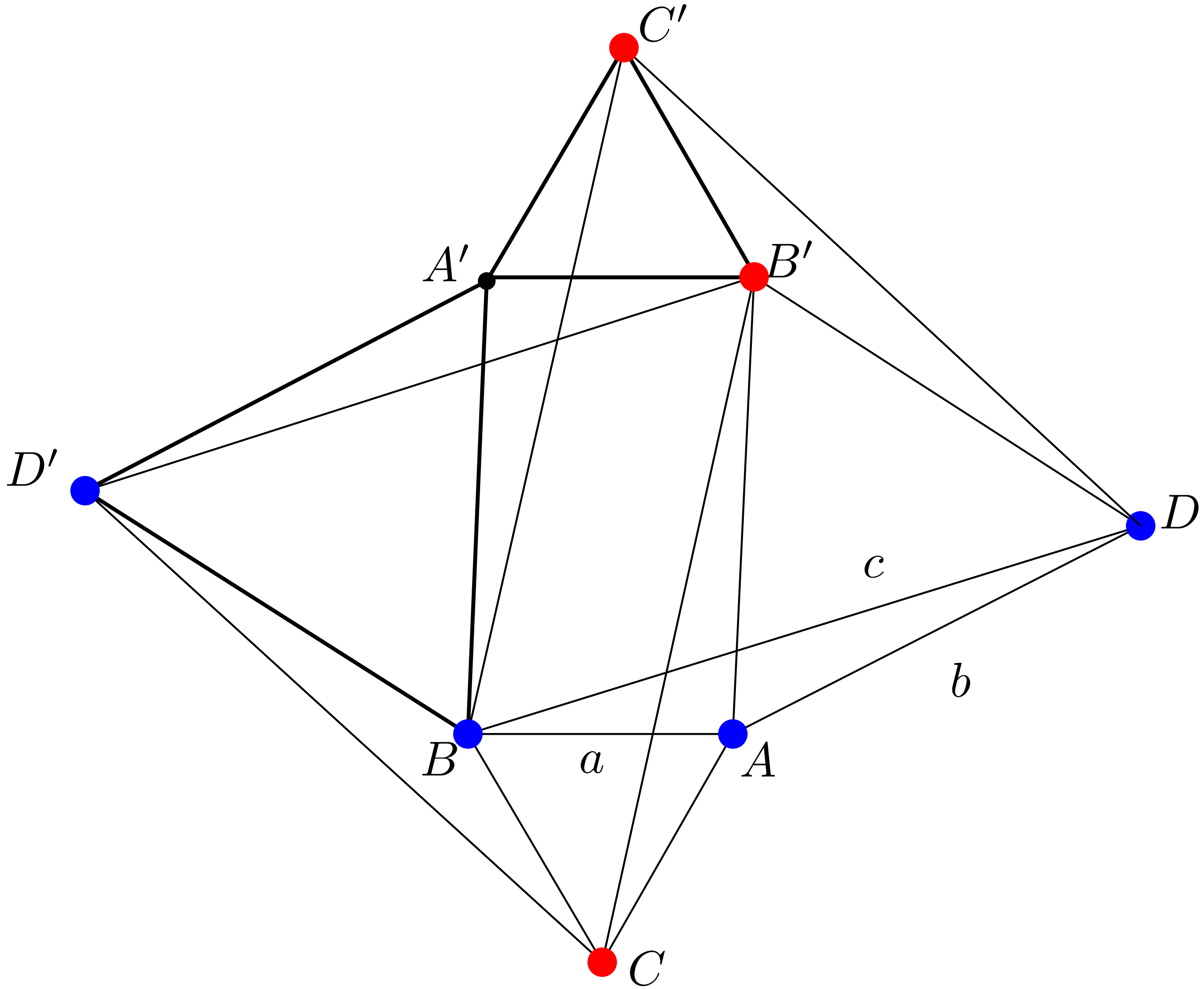
1. If χ contains an (a, a, a)-triangle for some $a > 0$, then χ contains any (a, b, c)-triangle, where $b, c > 0$ and a, b, c satisfy the (possibly degenerate) triangle inequality.

2. If χ contains an (a, b, c)-triangle, then χ contains an (a, a, a), (b, b, b), or (c, c, c)-triangle.

proof: (1)



proof: (2)



Corollary:

1. χ contains every triangle if and only if χ contains every equilateral triangle.

2. χ contains every non-equilateral triangle if and only if there exists an $a > 0$ such that χ contains all equilateral triangles except of the (a, a, a)-triangle.

3. χ contains an (a, b, c)-triangle if and only if χ contains a (b, a, c)-triangle.

Coloring by open and closed sets

(proof of Theorem 1)

- it satisfies to find a monochromatic unit triangle

ε-almost unit triangle ...an (a, b, c)-triangle whose edge-lengths satisfy $1 - \varepsilon \leq a, b, c \leq 1 + \varepsilon$

$Q(a)$... a square $[-a, a] \times [-a, a]$

Proposition Let $Q(3) = \mathcal{B} \cup \mathcal{R}$ be an arbitrary coloring of the square $Q(3)$ avoiding the unit triangle. Then for every $\varepsilon > 0$ both $\mathcal{B}$ and $\mathcal{R}$ contain an ε-almost unit triangle.

(if $\mathcal{B}$ is closed, then $\mathcal{B}^3$ is a compact set containing a sequence of $\frac{1}{n}$-almost unit triangles...)

proof of the proposition:

- given $\varepsilon > 0$ and a coloring $\chi = (\mathcal{B}, \mathcal{R})$ of $Q(3)$

- assume that χ avoids the unit triangle and that $\mathcal{R}$ does not contain any ε-almost unit triangle

- in $Q(1)$, find a red point R and a blue point S, such that $|R - S| < \varepsilon$

- construct a circle $\mathcal{C}$ with the center S and radius 1

- denote $K(\alpha) = S + (\cos\alpha, \sin\alpha)$

- $K(\alpha)$ and $K(\alpha + \frac{\pi}{3})$ must have different color

- for every blue $K(\alpha)$, each $K(\beta)$, $|K(\beta) - K(\alpha)| < \varepsilon$, is also blue.

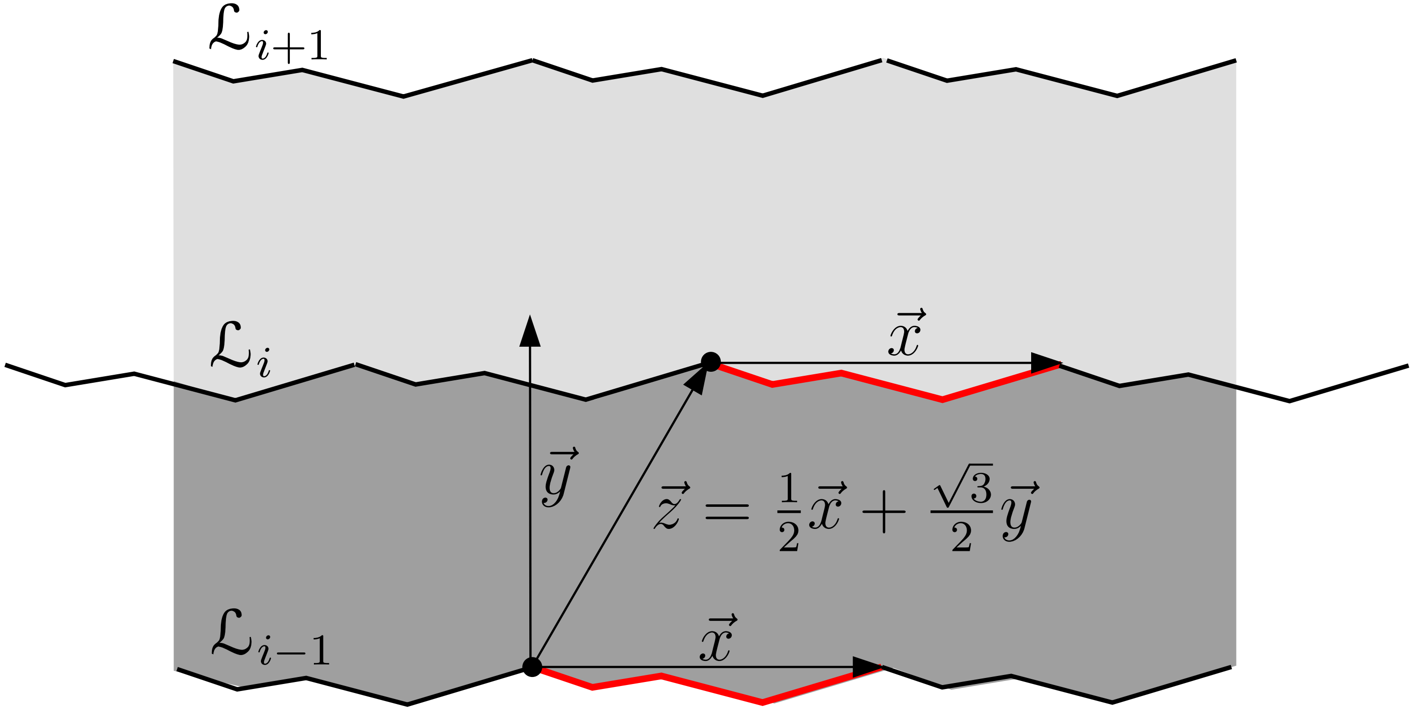
$\Rightarrow$ whole $\mathcal{C}$ is blue, a contradiction. $\square$

Polygonal colorings

A coloring $\chi = (\mathcal{B}, \mathcal{W})$ is polygonal, if

- each of the two sets $\mathcal{B}$ and $\mathcal{W}$ is contained in the closure of its interior
- The boundary of χ (a common boundary of $\mathcal{B}$ and $\mathcal{W}$), is a union of straight line segments (called boundary segments), which can intersect only at their endpoints (boundary vertices) .
- Every bounded region of the plane is intersected by only finitely many boundary segments.

Theorem 2 Polygonal coloring χ avoids a unit triangle if and only if χ is zebra-like (up to modification of the colors on the boundary).

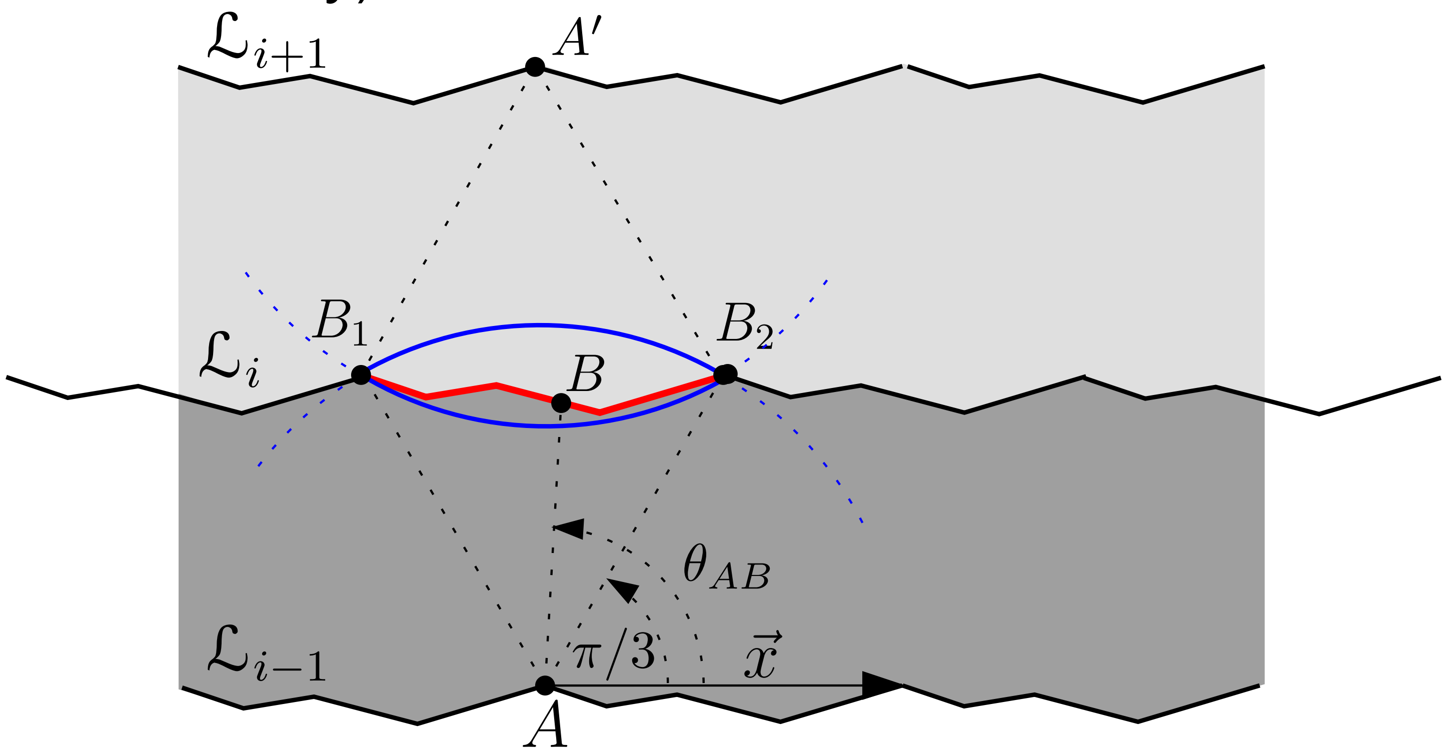


$$|\vec{x}| = |\vec{y}| = 1, \vec{x} \perp \vec{y}$$

$$L_i = L_i + \vec{x}$$

$$L_{i+1} = L_i + \vec{z}$$

Theorem 2 Polygonal coloring χ avoids a unit triangle if and only if χ is zebra-like (up to modification of the colors on the boundary).



$$|AB| < 1 \Leftrightarrow \theta_{AB} \in (\pi/3, 2\pi/3)$$

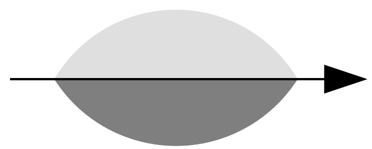
proof of '$\Rightarrow$' (outline):

- given a polygonal coloring $\chi = (\mathcal{B}, \mathcal{W})$ with a boundary Δ avoiding the unit triangle.

- $\mathcal{C}(A)$ ... a unit circle centered at A

- a boundary point A is feasible, if it is not a boundary vertex and $\mathcal{C}(A)$ does not contain any boundary vertex. Other boundary points are called infeasible.

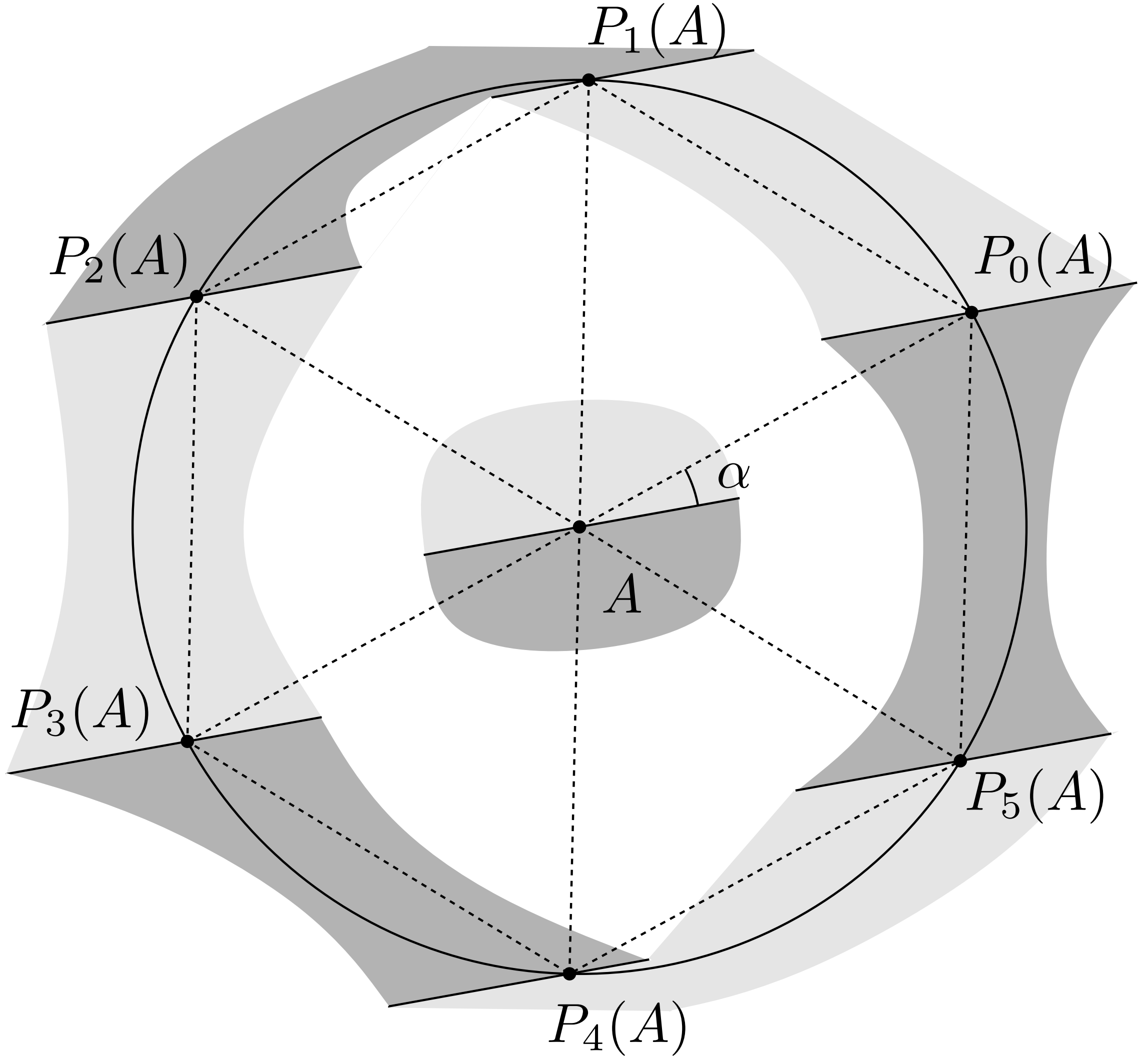
- orientation of boundary segments (white region on the left)



Local properties of χ

Lemma: Let s be a (horizontal) boundary segment containing a feasible point A, let $P(\alpha)$ denote the point $A + (\cos\alpha, \sin\alpha)$ on $\mathcal{C}(A)$. Let $B = P(\beta) \in \Delta$ and let t be a segment passing through B. Then

- s and t are parallel
- AB is not perpendicular to t, i.e, $\beta \notin \{-\frac{\pi}{2}, \frac{\pi}{2}\}$.
- $P(\alpha) \in \Delta$ if and only if $P(\alpha + \frac{\pi}{3}) \in \Delta$.
- If $\beta \in (\frac{\pi}{6}, \frac{5\pi}{6})$ or $\beta \in (\frac{7\pi}{6}, \frac{11\pi}{6})$, then s and t have opposite orientation. If $|\beta| < \frac{\pi}{6}$ or $|\beta - \pi| < \frac{\pi}{6}$, then s and t have the same orientation.
- For every θ there is exactly one value of $\alpha \in [\theta, \theta + \frac{\pi}{3})$ such that $P(\alpha) \in \Delta$.



$P_1(A)$
$P_2(A)$
$P_0(A)$
α
A
$P_3(A)$
$P_5(A)$
$P_4(A)$

Global properties of χ

Lemma: The size of the convex angle formed by two segments sharing an endpoint is greater than $\frac{2\pi}{3}$.

$\Rightarrow$ No three boundary segments share a common endpoint.

$\Rightarrow$ Every boundary component is a piecewise linear curve (closed or unbounded).

Let $A \in \Delta$. For $t \in \mathbb{R}$ let $A(t)$ be a point on the same boundary component as A, such that the directed length of the boundary curve between A and $A(t)$ is t.

Let $p_i(t) = P_i(A(t))$ (for feasible $A(t)$).

Clearly, $A(t)$ is a continuous function of t.

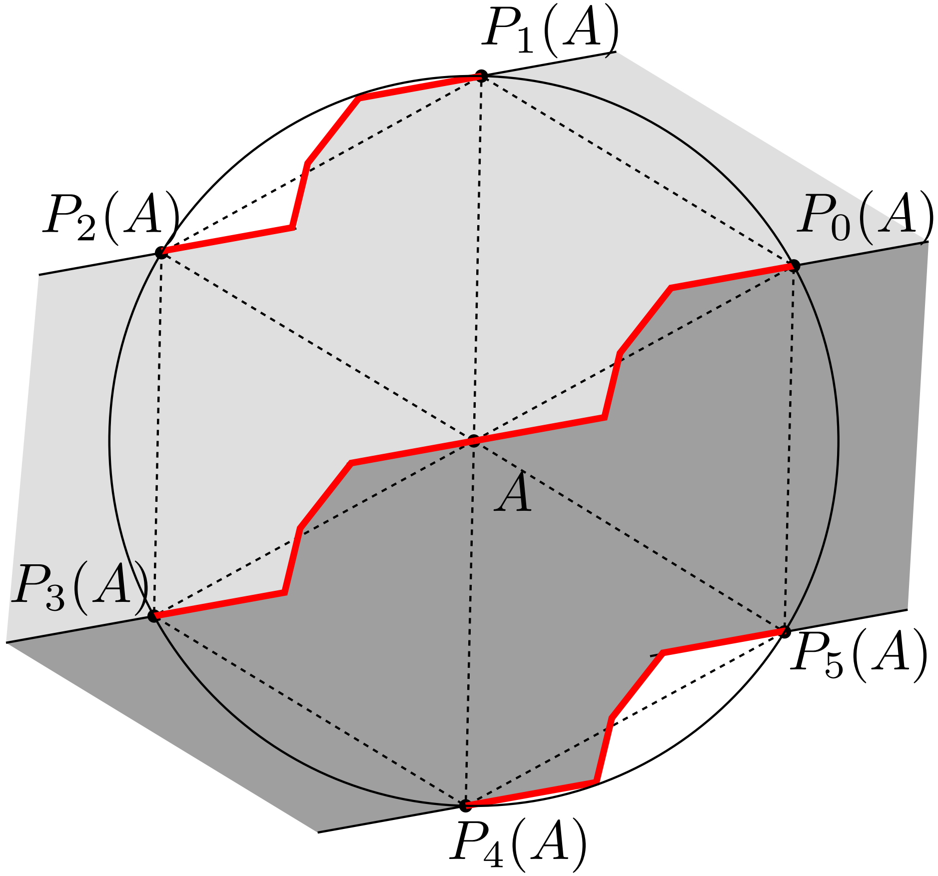
Lemma: The functions $p_i(t)$ can be extended to continuous functions by defining $P_i(A(t))$ for infeasible points $A(t)$.

Lemma: Let $A \in \Delta$ be an arbitrary boundary point. For each $i = 0, 1, \ldots, 5$, all the unit segments of the form $A(t)p_i(t)$ have the same slope, independently of the choice of t.

$\Rightarrow$ The translation by vector $P_i(A) - A$ is Δ-invariant.

Lemma: Infeasible boundary points A have similar local properties as feasible points (the circle $\mathcal{C}(A)$ can touch the boundary at points different from $P_i(A)$).

Lemma: Let $A \in \Delta$ be an arbitrary boundary point. Then inside $\mathcal{C}(A)$, $P_1(A)$ is connected with $P_2(A)$, P_0 with A, A with P_3, and P_4 with P_5.



$\Rightarrow \chi$ **is zebra-like!**

Open problems

- monochromatic triangles in colorings by regions with curved boundary
- monochromatic triangles in measurable colorings
- polygonal chromatic number of the plane
  (lower bound is 6 [Woodall, 1973])
- measurable chromatic number of the plane
  (lower bound is 5 [Falconer, 1981])