Improved enumeration of simple topological graphs

Jan Kynčl

Charles University, Prague

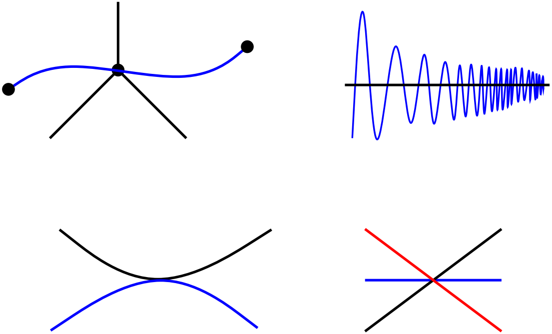
Graph: $G = (V, E)$, V finite, $E \subseteq \binom{V}{2}$

Topological graph: drawing of an (abstract) graph in the plane

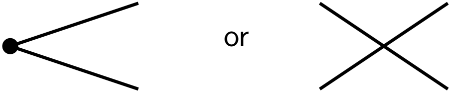
vertices $=$ points

edges $=$ simple curves

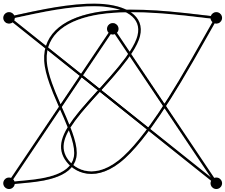
forbidden:



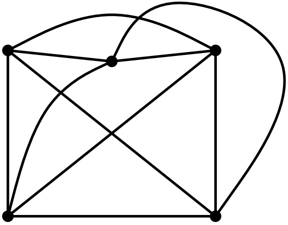
simple: any two edges have at most one common point



complete: $E = \binom{V}{2}$



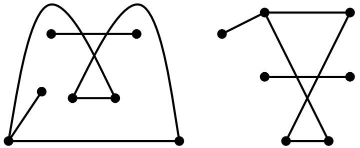
topological graph



simple complete topological graph

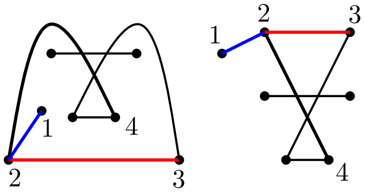
Topological graphs G, H are

- **isomorphic** if there exists a homeomorphism (of the sphere) which maps G onto H
- **weakly isomorphic** if the same pairs of edges cross in G and H



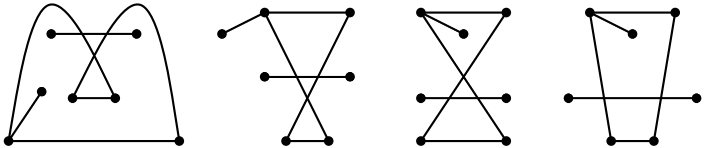
Topological graphs G, H are

- **isomorphic** if there exists a homeomorphism (of the sphere) which maps G onto H
- **weakly isomorphic** if the same pairs of edges cross in G and H



Topological graphs G, H are

- **isomorphic** if there exists a homeomorphism (of the sphere) which maps G onto H
- **weakly isomorphic** if the same pairs of edges cross in G and H



Weak isomorphism classes

$T_{\mathrm{w}}(G) =$ number of weak isomorphism classes of simple topological graphs that realize G

Complete graphs

Theorem: (J. Pach and G. Tóth, 2006)

$$2^{\Omega(n^2)} \leq T_{\mathrm{w}}(K_n) \leq 2^{O(n^2 \log n)}$$

Weak isomorphism classes

$T_{\mathrm{w}}(G)$ $=$ number of weak isomorphism classes of simple topological graphs that realize G

Complete graphs

Theorem: (J. Pach and G. Tóth, 2006)

$$2^{\Omega(n^2)} \leq T_{\mathrm{w}}(K_n) \leq 2^{O(n^2 \log n)}$$

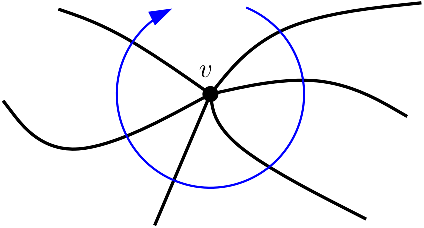
Main Theorem 1:

$$T_{\mathrm{w}}(K_n) \leq 2^{n^2 \cdot \alpha(n)^{O(1)}}$$

Weak isomorphism classes, complete graphs

tools:

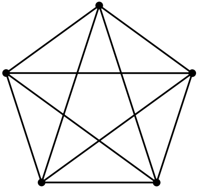
- weak isomorphism class $\leftrightarrow$ a rotation system



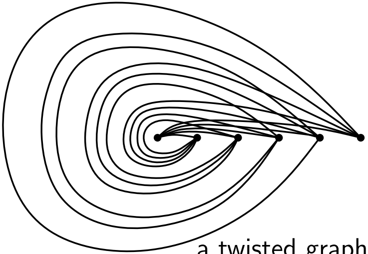
Weak isomorphism classes, complete graphs

tools:

- weak isomorphism class $\leftrightarrow$ a rotation system
- (J. Pach, J. Solymosi and G. Tóth, 2003)
  Every simple complete topological graph with 4^{30^4} vertices contains one of the following subgraphs:



a convex graph C_5



a twisted graph T_6

- an upper bound on the size of a set of permutations with bounded VC-dimension (J. Cibulka and JK, 2012)

General graphs

Main Theorem 2: Let G be a graph with n vertices and m edges. Then

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \leq 2^{O(mn^{1/2} \log n)}.$$

General graphs

Main Theorem 2: Let G be a graph with n vertices and m edges. Then

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \leq 2^{O(mn^{1/2} \log n)}.$$

Let $\varepsilon > 0$. If G is a graph with no isolated vertices and satisfies $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$, then

$$T_{\mathrm{w}}(G) \geq 2^{\Omega(\max(m, n \log n))}.$$

General graphs

Main Theorem 2: Let G be a graph with n vertices and m edges. Then

$$T_{\mathrm{w}}(G) \leq 2^{O(n^2 \log(m/n))}.$$

If $m < n^{3/2}$, then

$$T_{\mathrm{w}}(G) \leq 2^{O(mn^{1/2} \log n)}.$$

Let $\varepsilon > 0$. If G is a graph with no isolated vertices and satisfies $m > (1+\varepsilon)n$ or $\Delta(G) < (1-\varepsilon)n$, then

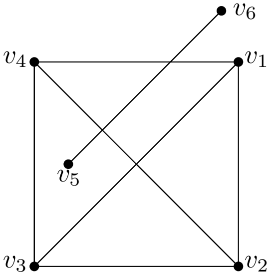
$$T_{\mathrm{w}}(G) \geq 2^{\Omega(\max(m, n \log n))}.$$

Corollary: There are at most $2^{O(n^{3/2} \log n)}$ intersection graphs of n pseudosegments.

Weak isomorphism classes, general graphs

tools for upper bounds:

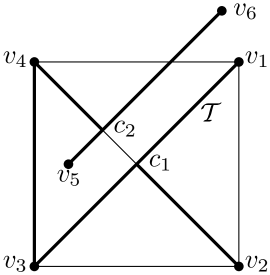
- topological spanning tree $\mathcal{T}$



Weak isomorphism classes, general graphs

tools for upper bounds:

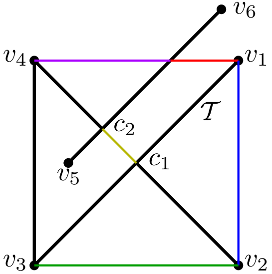
- topological spanning tree $\mathcal{T}$

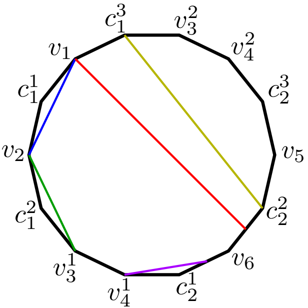


Weak isomorphism classes, general graphs

tools for upper bounds:

- topological spanning tree $\mathcal{T}$ $\qquad$ $\mathcal{T}$-representation





- types of (pseudo)chords $\Rightarrow$ pairs of crossing edges
- bounded number of crossings of the pseudochords

Isomorphism classes

$T(G)$ $=$ number of isomorphism classes
of simple topological graphs that realize G

Complete graphs

Theorem: (JK, 2009)

$$2^{\Omega(n^4)} \leq T(K_n) \leq 2^{(1/12+o(1))(n^4)}$$

General graphs

Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{1 \cdot m^2 + O(mn)}.$$

General graphs

Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{m^2+11.51mn+O(n\log n)} \leq 2^{23.118m^2} + c, \text{ and}$$

$$T(G) \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+3.443)+O(n\log n)} \leq 2^{11.265m^2} + c.$$

General graphs

Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{m^2+11.51mn+O(n\log n)} \leq 2^{23.118m^2} + c, \quad \text{and}$$

$$T(G) \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+3.443)+O(n\log n)} \leq 2^{11.265m^2} + c.$$

Let $\varepsilon > 0$. For graphs G with $m > (6+\varepsilon)n$ we have

$$T(G) \geq 2^{\Omega(m^2)}.$$

General graphs

Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{m^2+11.51mn+O(n\log n)} \leq 2^{23.118m^2} + c, \text{ and}$$
$$T(G) \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+3.443)+O(n\log n)} \leq 2^{11.265m^2} + c.$$

Let $\varepsilon > 0$. For graphs G with $m > (6+\varepsilon)n$ we have

$$T(G) \geq 2^{\Omega(m^2)}.$$

For graphs G with $m > \omega(n)$ we have

$$T(G) \geq 2^{m^2/60} - c.$$

General graphs

Theorem: Let G be a graph with n vertices, m edges and no isolated vertices. Then

$$T(G) \leq 2^{m^2+11.51mn+O(n\log n)} \leq 2^{23.118m^2}+c, \text{ and}$$
$$T(G) \leq 2^{m^2+2mn(\log(1+\frac{m}{4n})+3.443)+O(n\log n)} \leq 2^{11.265m^2}+c.$$

Let $\varepsilon > 0$. For graphs G with $m > (6+\varepsilon)n$ we have

$$T(G) \geq 2^{\Omega(m^2)}.$$

For graphs G with $m > \omega(n)$ we have

$$T(G) \geq 2^{m^2/60} - c.$$

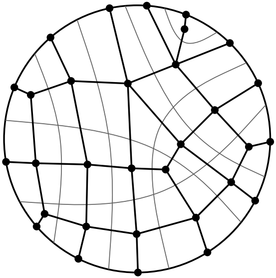
"very sparse" graphs $\to$ rooted connected planar loopless maps (T.R.S. Walsh and A. B. Lehman, 1975)

$$T(G) \leq 2^{(\log_2(256/27)+o(1))m^2} \leq 2^{3.246m^2}+c$$

Isomorphism classes, general graphs

tools for upper bounds:

- topological spanning tree and $\mathcal{T}$-representation
- simple quadrangulations of a disc
  (R. C. Mullin and P. J. Schellenberg, 1968)



- chord diagrams (R. C. Read, 1979) and arrangements of pseudochords with fixed boundary

Open questions

- Is $T_w(G)$ maximal for $G = K_n$, among graphs G with n vertices?
- Or, more generally, is $T_w(H) \leq T_w(G)$ for $H \subseteq G$?
- What is the number of weak isomorphism classes of drawings of G where every two edges have at most two common points?