Counterexample to a variant of the Hanani–Tutte theorem on the surface of genus 4

Radoslav Fulek and Jan Kynčl

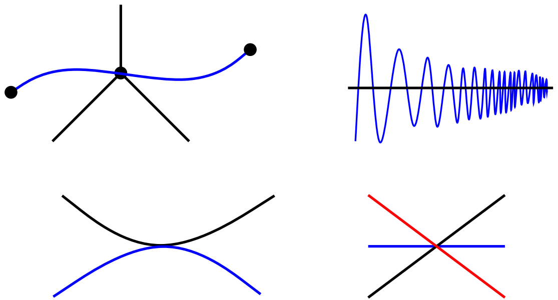
Charles University, Prague

Hanani–Tutte theorems

(Strong) Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

In a **drawing** the following situations are forbidden:



embedding = drawing with no crossings

Hanani–Tutte theorems

(Strong) Hanani–Tutte theorem: (Hanani, 1934; Tutte, 1970)

A graph is planar if and only if it has an **independently even** drawing in the plane; that is, every pair of non-adjacent edges crosses an even number of times.

Weak Hanani–Tutte theorem: (Cairns–Nikolayevsky, 2000; Pach–Tóth, 2000; Pelsmajer–Schaefer–Štefankovič, 2007)

If a graph G has an **even** drawing D in the plane (every pair of edges crosses an even number of times), then G is planar.
Moreover, G has a plane embedding with the same rotation system as D.

- recommended reading:

- M. Schaefer, Hanani-Tutte and related results (2011)
- Fulek et al., Hanani-Tutte, Monotone Drawings, and Level-Planarity (2012)
- M. Schaefer, Toward a theory of planarity: Hanani-Tutte and planarity variants (2013)

Hanani–Tutte theorems

Unified Hanani–Tutte theorem:
(Pelsmajer–Schaefer–Štefankovič, 2006; Fulek–K.–Pálvölgyi, 2016)

Let G be a graph and let W be a subset of vertices of G. Let $\mathcal{D}$ be an independently even drawing of G in the plane where, in addition, every pair of edges with a common endpoint in W crosses an even number of times.
Then G has a plane drawing where the rotations of vertices from W are the same as in $\mathcal{D}$.

- $W = \emptyset$: strong
- $W = V(G)$: weak

Hanani–Tutte theorems on surfaces

Weak Hanani–Tutte theorem on surfaces:
(Cairns–Nikolayevsky, 2000; Pelsmajer–Schaefer–Štefankovič, 2009)

If a graph G has an even drawing $\mathcal{D}$ on a surface S, then G has an embedding on S that preserves the embedding scheme of $\mathcal{D}$.

(Strong) Hanani–Tutte theorem on the projective plane:
(Pelsmajer–Schaefer–Stasi, 2009;
Colin de Verdière–Kaluža–Paták–Patáková–Tancer, 2016)

If a graph G has an independently even drawing on the projective plane, then G has an embedding on the projective plane.

Problem: Can the strong Hanani–Tutte theorem be extended to other surfaces?

Main result

- The strong Hanani–Tutte theorem does not generalize to the orientable surface of genus 4:

Theorem 1: There is a graph of genus 5 that has an independently even drawing on the orientable surface of genus 4.

(disproves a conjecture of Schaefer and Štefankovič, 2013)

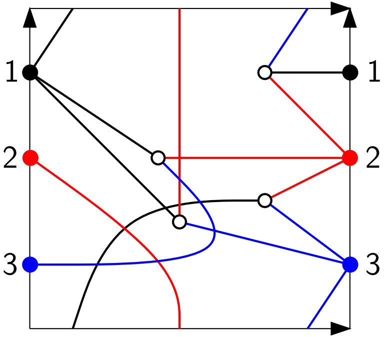
- Unified Hanani–Tutte theorem does not generalize to the torus:

Theorem 2: There is a graph G with the following two properties.

1) The graph G has an independently even drawing $\mathcal{D}$ on the torus, with a set W of four vertices such that every pair of edges with a common endpoint in W crosses an even number of times.
2) There is no embedding of G on the torus with the same rotations of the vertices of W as in $\mathcal{D}$.

Proof of Theorem 2

1)

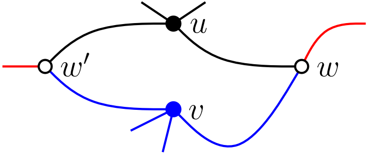


- $G = K_{3,4}$
- $W =$ the part with 4 vertices (empty circles)
- each vertex of W has rotation $(1, 2, 3)$

Proof of Theorem 2

2) Let $\mathcal{E}$ be an embedding of G on an orientable surface S of minimum genus such that the rotation of every vertex from W is $(1, 2, 3)$.

forbidden:



$\Rightarrow$ every facial walk has length at least 6, so $2e \geq 6f$

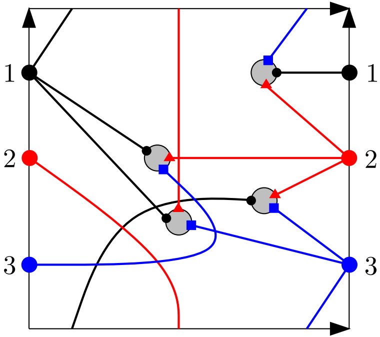
$\Rightarrow$ the Euler characteristic of S satisfies

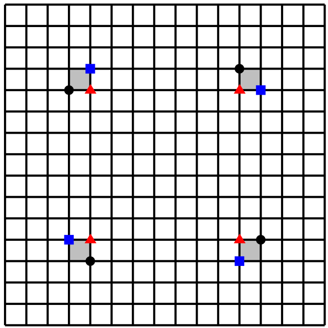
$$\chi(S) = v - e + f \leq \frac{1}{3}(3v - 2e) = \frac{1}{3}(21 - 24) = -1$$

$\Rightarrow$ the genus of S is at least $\lceil (2+1)/2 \rceil = 2$.

Proof of Theorem 1

independently even drawing of a graph K on the orientable surface of genus 4:





- drill holes around the vertices of W in the drawing from Theorem 2, split the vertices of W
- glue the resulting drawing (left) with a sufficiently large grid (right)

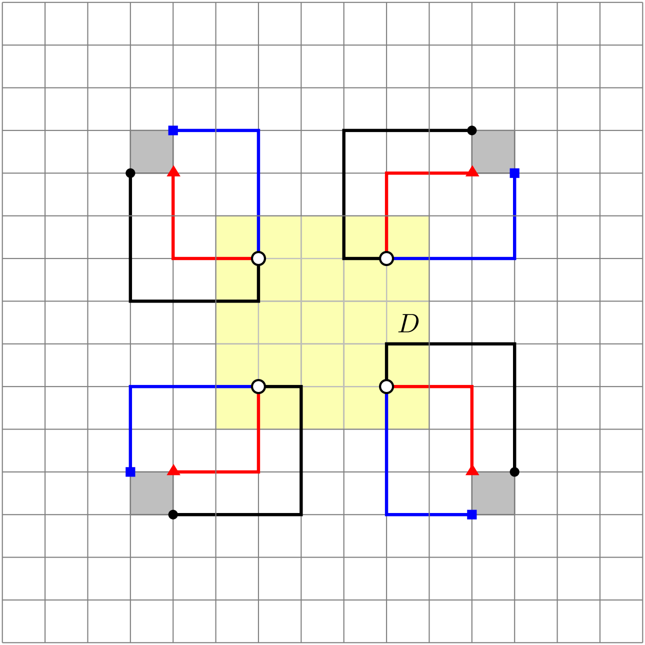
idea: the grid will fix the cyclic orders on the boundaries of the holes

Proof of Theorem 1

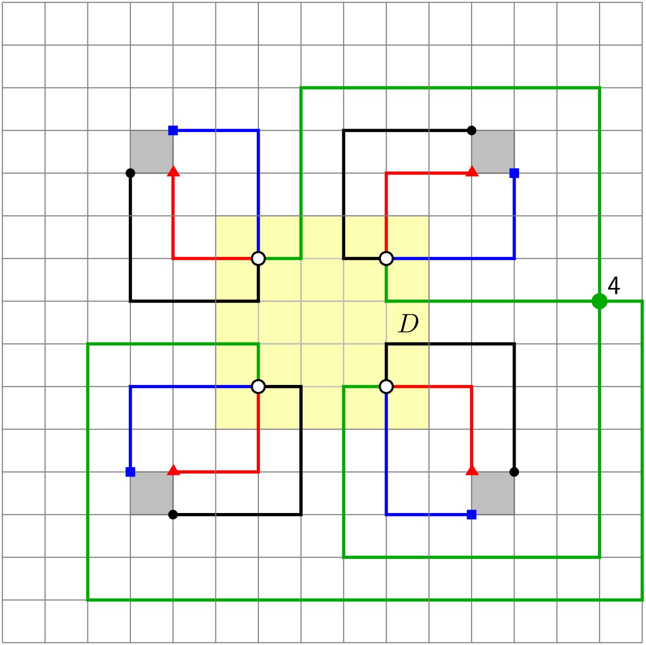
lower bound on the genus of K:

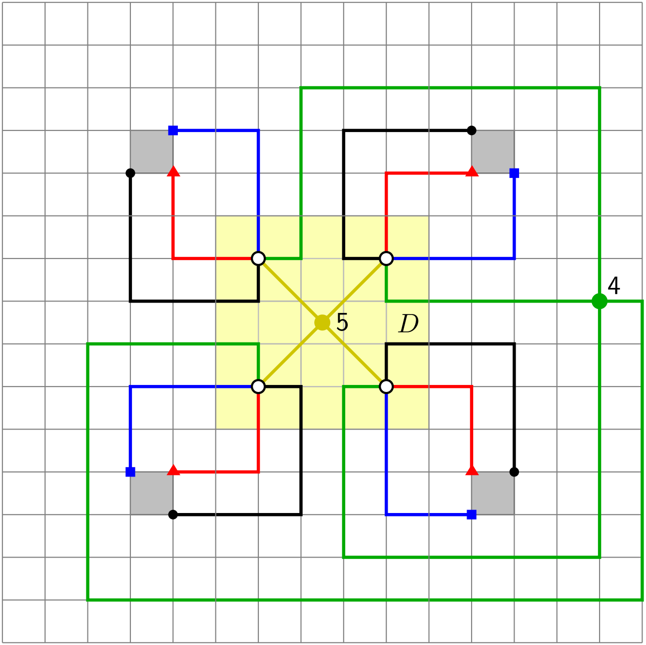
Lemma: (Geelen–Richter–Salazar, 2004; Thomassen, 1997; Mohar, 1992; Robertson–Seymour, 1990)

In every embedding of a large grid on a surface of fixed genus, a large portion of the grid is embedded in a planar way.



D


4
D


4
5
D