Planar L-Drawings of Directed Graphs*

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Abstract. We study planar drawings of directed graphs in the L-drawing standard. We provide necessary conditions for the existence of these drawings and show that testing for the existence of a planar L-drawing is an NP-complete problem. Motivated by this result, we focus on upward-planar L-drawings. We show that directed st-graphs admitting an upward-(resp. upward-rightward-) planar L-drawing are exactly those admitting a bitonic (resp. monotonically increasing) st-ordering. We give a linear-time algorithm that computes a bitonic (resp. monotonically increasing) st-ordering of a planar st-graph or reports that there exists none.

1 Introduction

In an L-drawing of a directed graph each vertex v is assigned a point in the plane with exclusive integer x- and y-coordinates, and each directed edge (u,v) consists of a vertical segment exiting u and of a horizontal segment entering v [1]. The drawings of two edges may cross and partially overlap, following the model of [18]. The ambiguity among crossings and bends is resolved by replacing bends with small rounded junctions. An L-drawing in which edges possibly overlap, but do not cross, is a planar L-drawing; see, e.g., Fig. 1b. A planar L-drawing is upward planar if its edges are y-monotone, and it is upward-rightward planar if its edges are simultaneously x-monotone and y-monotone.

Planar L-drawings correspond to drawings in the Kandinsky model [12] with exactly one bend per edge and with some restrictions on the angles around each

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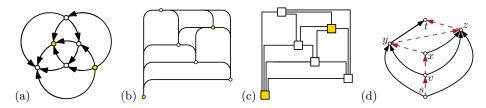


Fig. 1: (a) A bitonic st-orientation of the octahedron that admits an upward planar L-drawing (b). (c) The corresponding drawing in the Kandinsky model. (d) An upward planar st-graph U that does not admit an upward-planar L-drawing

vertex; see Fig. 1c. It is NP-complete [4] to decide whether a multigraph has a planar embedding that allows a Kandinsky drawing with at most one bend per edge [5]. On the other hand, every simple planar graph has a Kandinsky drawing with at most one bend per edge [5]. Bend-minimization in the Kandinsky-model is NP-complete [4] even if a planar embedding is given, but can be approximated by a factor of two [2,11]. Heuristics for drawings in the Kandinsky model with empty faces and few bends have been discussed by Bekos et al. [3].

Bitonic st-orderings were introduced by Gronemann for undirected planar graphs [14] as an alternative to canonical orderings. They were recently extended to directed plane graphs [16]. In a bitonic st-ordering the successors of any vertex must form an increasing and then a decreasing sequence in the given embedding. More precisely, a planar st-graph is a directed acyclic graph with a single source s and a single sink t that admits a planar embedding in which sand t lie on the boundary of the same face. A planar st-graph always admits an upward-planar straight-line drawing [7]. An st-ordering of a planar st-graph is an enumeration π of the vertices with distinct integers, such that $\pi(u) < \pi(v)$ for every edge $(u,v) \in E$. Given a plane st-graph, i.e., a planar st-graph with a fixed upward-planar embedding \mathcal{E} , consider the list $S(v) = \langle v_1, v_2, \dots, v_k \rangle$ of successors of v in the left-to-right order in which they appear around v. The list S(v) is monotonically decreasing with respect to an st-ordering π if $\pi(v_i) > 1$ $\pi(v_{i+1})$ for $i=1,\ldots,k-1$. It is bitonic with respect to π if there is a vertex v_h in S(v) such that $\pi(v_i) < \pi(v_{i+1}), i = 1, ..., h-1$ and $\pi(v_i) > \pi(v_{i+1}),$ $i = h, \ldots, k-1$. For an upward-planar embedding \mathcal{E} , an st-ordering π is bitonic or monotonically decreasing, respectively if the successor list of each vertex is bitonic or monotonically decreasing, respectively. Here, $\langle \mathcal{E}, \pi \rangle$ is called a bitonic pair or monotonically decreasing pair, respectively, of G.

Gronemann used bitonic st-orderings to obtain on the one hand upward-planar polyline grid drawings in quadratic area with at most |V|-3 bends in total [16] and on the other hand contact representations with upside-down oriented T-shapes [15]. A bitonic st-ordering for biconnected undirected planar graphs can be computed in linear time [14] and the existence of a bitonic st-ordering for plane (directed) st-graphs can also be decided in linear time [16]. However, in the variable embedding scenario no algorithm is known to decide whether an st-graph G admits a bitonic pair. Bitonic st-orderings turn out to

be strongly related to upward-planar L-drawings of st-graphs. In fact, the y-coordinates of an upward-planar L-drawing yield a bitonic st-ordering.

In this work, we initiate the investigation of planar and upward-planar L-drawings. In particular, our contributions are as follows. (i) We prove that deciding whether a directed planar graph admits a planar L-drawing is NP-complete. (ii) We characterize the planar st-graphs admitting an upward (upward-rightward, resp.) planar L-drawing as the st-graphs admitting a bitonic (monotonic decreasing, resp.) st-ordering. (iii) We provide a linear-time algorithm to compute an embedding, if any, of a planar st-graph that allows for a bitonic st-ordering. This result complements the analogous algorithm proposed by Gronemann for undirected graphs [14] and extends the algorithm proposed by Gronemann for planar st-graphs in the fixed embedding setting [16]. (iv) Finally, we show how to decide efficiently whether there is a planar L-drawing for a plane directed graph with a fixed assignment of the edges to the four ports of the vertices.

Due to space limitations, full proofs are provided in Appendix B.

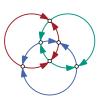
2 Preliminaries

We assume familiarity with basic graph drawing concepts and in particular with the notions of connectivity and SPQR-trees (see also [8] and Appendix A).

A (simple, finite) directed graph G = (V, E) consists of a finite set V of vertices and a finite set $E \subseteq \{(u, v) \in V \times V; u \neq v\}$ of ordered pairs of vertices. If (u, v) is an edge then v is a successor of u and u is a predecessor of v. A graph is planar if it admits a drawing in the plane without edge crossings. A plane graph is a planar graph with a fixed planar embedding, i.e., with fixed circular orderings of the edges incident to each vertex—determined by a planar drawing—and with a fixed outer face.

Given a planar embedding and a vertex v, a pair of consecutive edges incident to v is alternating if they are not both incoming or both outgoing. We say that v is k-modal if there exist exactly k alternating pairs of edges in the cyclic order around v. An embedding of a directed graph G is k-modal, if each vertex is at most k-modal. A 2-modal embedding is also called bimodal. An upward-planar drawing determines a bimodal embedding. However, the existence of a bimodal embedding is not a sufficient condition for the existence of an upward-planar drawing. Deciding whether a directed graph admits an upward-planar (straight-line) drawing is an NP-hard problem [13].

L-drawings. A planar L-drawing determines a 4-modal embedding. This implies that there exist planar directed graphs that do not admit planar L-drawings. A 6-wheel whose central vertex is incident to alternating incoming and outgoing edges is an example of a graph that does not admit any 4-modal embedding, and therefore any planar L-drawing.



On the other hand, the existence of a 4-modal embedding is not sufficient for the existence of a planar L-drawing. E.g., the octahedron depicted in the figure on the right does not admit a planar L-drawing. Since the octahedron is triconnected, it admits a unique combinatorial embedding (up to a flip). Each vertex is 4-modal. However, the rightmost vertex in a planar L-drawing must be 1-modal or 2-modal.

Any upward-planar L-drawing of an st-graph G can be modified to obtain an upward-planar drawing of G: Redraw each edge as a y-monotone curve arbitrarily close to the drawing of the corresponding 1-bend orthogonal polyline while avoiding crossings and edge-edge overlaps. However, not every upward-planar graph admits an upward-planar L-drawing. E.g., the graph in Fig. 1d contains a subgraph that does not admit a bitonic st-ordering [16]. In Section 4 (Theorem 3), we show that this means it does not admit an upward planar L-drawing.

The Kandinsky Model. In the Kandinsky model [12], vertices are drawn as squares of equal sizes on a grid and edges—usually undirected—are drawn as orthogonal polylines on a finer grid; see Fig. 1c. Two consecutive edges in the clockwise order around a vertex define a face and an angle in $\{0, \pi/2, \pi, 3\pi/2, 2\pi\}$ in that face. In order to avoid edges running through other vertices, the Kandinsky model requires the so called bend-or-end property: There is an assignment of bends to vertices with the following three properties. (a) Each bend is assigned to at most one vertex. (b) A bend may only be assigned to a vertex to which it is connected by a segment (i.e., it must be the first bend on an edge). (c) If e_1, e_2 are two consecutive edges in the clockwise order around a vertex v that form a 0 angle inside face f, then a bend of e_1 or e_2 forming a $3\pi/2$ angle inside f must be assigned to v. Further, the Kandinsky model requires that there are no empty faces.

Given a planar L-drawing, consider a vertex v and all edges incident to one of the four ports of v. By assigning to v all bends on these edges—except the bend furthest from v—we satisfy the bend-or-end property. This implies the following lemma, which is proven in Appendix B.

Lemma 1. A graph has a planar L-drawing if and only if it admits a drawing in the Kandinsky model with the following properties: (i) Each edge bends exactly once; (ii) at each vertex, the angle between any two outgoing (or between any two incoming) edges is 0 or π ; and (iii) at each vertex, the angle between any incoming edge and any outgoing edge is $\pi/2$ or $3\pi/2$.

3 General Planar L-Drawings

We consider the problem of deciding whether a graph admits a planar L-drawing. In Section 3.1, we show that the problem is NP-complete if no planar embedding is given. In the fixed embedding setting (Section 3.2) the problem can be described as an ILP. It is solvable in linear time if we also fix the ports.

3.1 Variable Embedding Setting

As a central building block for our hardness reduction we use a directed graph W that can be constructed starting from a 4-wheel with central vertex c and

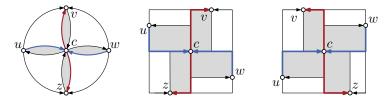


Fig. 2: 4-wheel graph W and two planar L-drawings of W.

rim (u, v, w, z). We orient the edges of W so that v and z (the V-ports of W) are sinks and u and w (the H-ports of W) are sources. Finally, we add directed edges (v, c), (z, c), (c, w), and (c, u); see Fig. 2. We now provide Lemma 2 which describes the key property of planar L-drawings of W.

Lemma 2. In any planar L-drawing of W with cycle (u, v, w, z) as the outer face the edges of the outer face form a rectangle (that contains vertex c).

We are now ready to give the main result of the section.

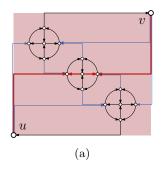
Theorem 1. It is NP-complete to decide whether a directed graph admits a planar L-drawing.

Sketch of proof. We reduce from the NP-complete problem of HV-rectilinear planarity testing [10]. In this problem, the input is a biconnected degree-4 planar graph G with edges labeled either H or V, and the goal is to decide whether G admits an HV-drawing, i.e., a planar drawing such that each H-edge (V-edge) is drawn as a horizontal (vertical) segment. Starting from G, we construct a graph G' by replacing: (i) vertices with 4-wheels as in Fig. 2; (ii) V-edges with the gadget shown in Fig. 3a; and (iii) H-edges with an appropriately rotated and re-oriented version of the V-edge gadget. If (u,v) is a V-edge, the two vertices labeled u and v of its gadget are identified with a V-port of the respective vertex gadgets. Otherwise, they are identified with an H-port. Figure 3b shows a vertex gadget with four incident edges. The proof that G' and G are equivalent is somewhat similar to Brückner's hardness proof in [5, Theorem 3] and exploits Lemma 2. Refer to Appendix B for the full details.

3.2 Fixed Embedding and Port Assignment

In this section, we show how to decide efficiently whether there is a planar L-drawing for a plane directed graph with a fixed assignment of the edges to the four ports of the vertices. Using Lemma 1 and the ILP formulation of Barth et al. [2], we first set up linear inequalities that describe whether a plane 4-modal graph has a planar L-drawing. Using these inequalities, we then transform our decision problem into a matching problem that can be solved in linear time.

We call a vertex v an in/out-vertex on a face f if v is incident to both, an incoming edge and an outgoing edge on f. Let $x_{vf} \in \{0, 1, 2\}$ describe the angle



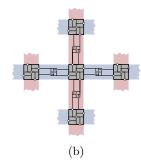


Fig. 3: (a) Edge gadget for a V-edge. (b) Connections among gadgets.

in a face f at a vertex v: the angle between two outgoing or two incoming edges is $x_{vf} \cdot \pi$ and the angle between an incoming and an outgoing edge is $x_{vf} \cdot \pi + \pi/2$. Let $x_{fe}^v \in \{0,1\}$ be 1 if there is a convex bend in face f on edge e assigned to a vertex v to fulfill the bend-or-end property. There is a planar L-drawing with these parameters if and only if the following four conditions are satisfied (see Appendix B.2 for details): (1) The angles around a vertex v sum to 2π . (2) Each edge has exactly one bend. (3) The number of convex angles minus the number of concave angles is 4 in each inner face and -4 in the outer face. (4) The bend-or-end property is fulfilled, i.e., for any two edges e_1 and e_2 that are consecutive around a vertex v and that are both incoming or both outgoing, and for the faces f_1 , f, and f_2 that are separated by e_1 and e_2 (in the cyclic order around v), it holds that $x_{vf} + x_{f_1e_1}^v + x_{f_2e_2}^v \ge 1$. Let e = (v, w) be incident to faces f and h, Condition (2) implies $-x_{he}^v - x_{he}^w = x_{fe}^v + x_{fe}^v - 1$. Hence, (3) yields

$$(3')\sum_{e=(v,w)\text{ incident to }f}(x_{fe}^v+x_{fe}^w)-\sum_{v\text{ on }f}x_{vf}=\pm 2+(\#\text{ in/out-vertices on }f-\deg f)/2.$$

Observe that the number of in/out-vertices on a face f is odd if and only if $\deg f$ is odd. Moreover, if we omit the bend-or-end property, we can formulate the remaining conditions as an uncapacitated network flow problem. The network has three types of nodes: one for each vertex, face, and edge of the graph. It has two types of edges: from vertices to incident faces and from faces to incident edges. The supplies are $\lceil \frac{4-k}{2} \rceil$ for the k-modal vertices, $\pm 2 + 1/2 \cdot (\# \text{in/out-vertices} - \deg f)$ for a face f, and -1 for the edges.

Theorem 2. Given a directed plane graph G and labels $out(e) \in \{\text{top}, \text{bottom}\}$ and $in(e) \in \{\text{right}, \text{left}\}$ for each edge e, it can be decided in linear time whether G admits a planar L-drawing in which each edge e leaves its tail at out(e) and enters its head at in(e).

Sketch of proof. First, we have to check whether the cyclic order of the edges around a vertex is compatible with the labels. The labels determine the bends and the angles around the vertices, i.e., $x_{fe}^v + x_{fe}^w$ for each edge e = (v, w) and each incident face f, and x_{vf} for each vertex v and each incidence to a face f.

We check whether these values fulfill Conditions 1, 2, and 3'. In order to also check Condition 4, we first assign for each port of a vertex v, all but the middle edges to v (where a *middle edge* of a port is the last edge in clockwise order bending to the left or the first edge bending to the right). We check whether we thereby assign an edge more than once. Assigning the middle edges can be reduced to a matching problem in a bipartite graph of maximum degree 2 where the nodes on one side are the ports with two middle edges and the nodes on the other side are the unassigned edges.

4 Upward- and Upward-Rightward Planar L-Drawings

In this section, we characterize (see Theorem 3) and construct (see Theorem 6) upward-planar and upward-rightward planar L-drawings.

4.1 A Characterization via Bitonic st-Orderings

Characterizing the plane directed graphs that admit an L-drawing is an elusive goal. However, we can characterize two natural subclasses of planar L-drawings via bitonic st-orderings.

Theorem 3. A planar st-graph admits an upward- (upward-rightward-) planar L-drawing if and only if it admits a bitonic (monotonically decreasing) pair.

Sketch of proof. " \Rightarrow ": Let G = (V, E) be an st-graph with n vertices. The y-coordinates of an upward- (upward-rightward-) planar L-drawing of G yield a bitonic (monotonically decreasing) st-ordering.

"\(\infty\)": Given a bitonic (monotonically decreasing) st-ordering π of G = (V, E), we construct an upward- (upward-rightward-) planar L-drawing of G using an idea of Gronemann [16]. For each vertex v, we use $\pi(v)$ as its y-coordinate.

For the x-coordinates we use a linear extension of a partial order \prec . Let v_1, \ldots, v_n be the vertices of G in the ordering given by π . Let G_i be the subgraph of G induced by $V_i = \{v_1, \ldots, v_i\}$. To construct \prec , we augment G_i to \overline{G}_i in such a way that the outer face $f_{\overline{G}_i}$ of \overline{G}_i is a simple cycle and all vertices on $f_{\overline{G}_i}$ are comparable: We start with a triangle on v_1 and two new vertices v_{-1} and v_{-2} , with y-coordinates -1 and -2, respectively, and set $v_{-2} \prec v_1 \prec v_{-1}$. For $i=2,\ldots,n$, let u_1,\ldots,u_k be the predecessors of v_i in ascending order with respect to \prec . If π is monotonically decreasing or if k=1, we add an edge e with head v_i . The tail of e is the right neighbor r of u_k or the left neighbor ℓ of u_1 on $f_{\overline{G}_i}$, respectively, if the maximum successor s_{\max} of u_1 is to the left (or equal to) or the right of v_i , respectively; see Fig. 4a. Now let u_1,\ldots,u_k be the predecessors of v_i in the possibly augmented graph; see Fig. 4b. We add the condition $u_{k-1} \prec v_i \prec u_k$.

Corollary 1. Any undirected planar graph can be oriented such that it admits an upward-planar L-drawing.

Proof. Triangulate the graph G and construct a bitonic st-ordering for undirected graphs [14]. Orient the edges from smaller to larger st-numbers.

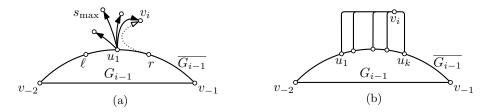


Fig. 4: How to turn a bitonic st-ordering into a planar L-drawing.

4.2 Bitonic st-Orderings in the Variable Embedding Setting

By Theorem 3, testing for the existence of an upward- (upward-rightward-) planar L-drawing of a planar st-graph G reduces to testing for the existence of a bitonic (monotonically decreasing) pair $\langle \mathcal{E}, \pi \rangle$ for G. In this section, we give a linear-time algorithm to test an st-graph for the existence of a bitonic pair $\langle \mathcal{E}, \pi \rangle$.

The following lemma is proved in Appendix B.4.

Lemma 3. Let G = (V, E) be a planar st-graph with source s, sink t, and $(s, t) \notin E$. Then there exists a supergraph G' = (V', E') of G, where $V' = V \cup \{s'\}$ and $E' = E \cup \{(s', s), (s', t)\}$, such that (i) G' is an st-graph with source s' and sink t, and (ii) G' admits a bitonic (resp., monotonically increasing) st-ordering if and only if G does.

By Lemma 3, in the following we assume that an st-graph G always contains edge (s,t). Hence, either G coincides with edge (s,t), which trivially admits a bitonic st-ordering, or it is biconnected.

A path p from u to v in a directed graph is monotonic increasing (monotonic decreasing) if it is exclusively composed of forward (backward) edges. A path p is monotonic if it is either monotonic increasing or monotonic decreasing. A path p with endpoints u and v is bitonic if it consists of a monotonic increasing path from u to w and of a monotonic decreasing path from w to v: if $u \neq w$ and $v \neq w$, then the path p is strictly bitonic and w is the apex of p. An st-graph G is v-monotonic, v-bitonic, or strictly v-bitonic if the subgraph of G induced by the successors of v is, after the removal of possible transitive edges, a monotonic, bitonic, or strictly-bitonic path p, respectively. The apex of p, if any, is also called the apex of v in G. If p is monotonic and it is directed from u to w, then vertices u and w are the first successor of v in G and the last successor of v in G, respectively. If p is strictly bitonic, then its endpoints are the first successors of v in G. If p consists of a single vertex, then such a vertex is both the first and the last successor of v in G. Let G be an st-graph and let G^* be an st-graph obtained by augmenting G with directed edges. We say that the pair $\langle G, G^* \rangle$ is v-monotonic, v-bitonic, or strictly v-bitonic if the subgraph of G^* induced by the successors of v in G is, after the removal of possible transitive edges, a monotonic, bitonic, or strictly-bitonic path, respectively.

Although Gronemann [16] didn't state this explicitly, the following theorem immediately follows from the proof of his Lemma 4.

Theorem 4 ([16]). A plane st-graph G = (V, E) admits a bitonic st-ordering if and only if it can be augmented with directed edges to a planar st-graph G^* such that, for each vertex $v \in V$, the pair $\langle G, G^* \rangle$ is v-bitonic. Further, any st-ordering of G^* is a bitonic st-ordering of G.

In the remainder of the section, we show how to test in linear-time whether it is possible to augment a biconnected st-graph G to an st-graph G^* in such a way that the pair $\langle G, G^* \rangle$ is v-bitonic, for any vertex v of G. By virtue of Theorem 4, this allows us to test the existence of a bitonic pair $\langle \mathcal{E}, \pi \rangle$ for G. We perform a bottom-up visit of the SPQR-tree T of G rooted at the reference edge (s,t) and show how to compute an augmentation for the pertinent graph of each node $\mu \in T$ together with an embedding of it, if any exists.

Note that each vertex in an st-graph is on a directed path from s to t. Further, by the choice of the reference edge, neither s nor t are internal vertices of the pertinent graph of any node of T. This leads to the next observation.

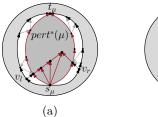
Observation 1 For each node $\mu \in T$ with poles u and v, the pertinent graph $pert(\mu)$ of μ is an st-graph whose source and sink are u and v, or vice versa.

Let e be a virtual edge of $skel(\mu)$ corresponding to a node ν whose pertinent graph is an st-graph with source s_{ν} and sink t_{ν} . By Observation 1, we say that e exits s_{ν} and enters t_{ν} .

The outline of the algorithm is as follows. Consider a node $\mu \in T$ and suppose that, for each child μ_i of μ , we have already computed a pair $\langle \operatorname{pert}^*(\mu_i), \mathcal{E}_i^* \rangle$ such that $\operatorname{pert}^*(\mu_i)$ is an augmentation of $\operatorname{pert}(\mu_i)$, \mathcal{E}_i^* is an embedding of $\operatorname{pert}^*(\mu_i)$, and $\langle \operatorname{pert}(\mu_i), \operatorname{pert}^*(\mu_i) \rangle$ is v-bitonic, for each vertex v of $\operatorname{pert}(\mu_i)$. We show how to compute a pair $\langle \operatorname{pert}^*(\mu), \mathcal{E}^* \rangle$ for node μ , such that (i) the pair $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is v-bitonic for each vertex v in $\operatorname{pert}(\mu)$, and (ii) the restriction of \mathcal{E}^* to $\operatorname{pert}^*(\mu_i)$ is \mathcal{E}_i^* , up to a flip. In the following, for the sake of clarity, we first describe an overall quadratic-time algorithm. We will refine this algorithm to run in linear time at the end of the section.

For a node $\mu \in T$, we say that the pair $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is of Type B if it is strictly s_{μ} -bitonic and it is of Type M if it is s_{μ} -monotonic. For simplicity, we also say that node μ is of Type B or of Type M when, during the traversal of T, we have constructed an augmentation $\operatorname{pert}^*(\mu)$ for μ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is of Type B or of Type M, respectively. Figure 5 shows an example where an augmentation G^* of G contains an augmentation $\operatorname{pert}^*(\mu)$ for μ which is replaced with an augmentation $\operatorname{pert}^+(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is of Type B, $\langle \operatorname{pert}(\mu), \operatorname{pert}^+(\mu) \rangle$ is of Type M, and G^* admits a bitonic st-ordering if and only if it still does after this replacement. The following lemma formally shows that this type of replacement is always possible.

Lemma 4. Let G be a biconnected st-graph and let G^* be an augmentation of G such that $\langle G, G^* \rangle$ is v-bitonic, for each vertex v of G. Consider a node μ of the SPQR-tree of G and let $\operatorname{pert}^*(\mu)$ be the subgraph of G^* induced by the vertices of $\operatorname{pert}(\mu)$. Suppose that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is of Type B and that $\operatorname{pert}(\mu)$ also admits an augmentation $\operatorname{pert}^+(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^+(\mu) \rangle$ is of Type M and



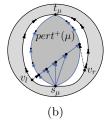


Fig. 5: Illustration for Lemma 4.

it is v-bitonic, for each vertex v of pert(μ). There exists an augmentation G^+ of G such that $\langle G, G^+ \rangle$ is v-bitonic, for each vertex v of G, and such that the subgraph of G^+ induced by the vertices of pert(μ) is pert⁺(μ).

Consider a node μ of the SPQR-tree T of G. We now show how to test the existence of a pair $\langle \operatorname{pert}^*(\mu), \mathcal{E}^* \rangle$ such that (i) μ is of Type M or, secondarily, of Type B, or report that no such a pair exists, and (ii) \mathcal{E}^* is a planar embedding of $\operatorname{pert}^*(\mu)$. In fact, by Lemma 4, an embedding of μ of Type M would always be preferable to an embedding of Type B.

In any planar embedding \mathcal{E} of pert(μ) in which the poles are on the outer face f_{out} of \mathcal{E} , we call *left path* (right path) of \mathcal{E} the path that consists of the edges encountered in a clockwise traversal (in a counter-clockwise traversal) of the outer face of \mathcal{E} from s_{μ} to t_{μ} .

The following observation will prove useful to construct embedding \mathcal{E}^* .

Observation 2 Let $\langle \operatorname{pert}^*(\mu), \mathcal{E}^* \rangle$ be a pair such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic and \mathcal{E}^* is a planar embedding of $\operatorname{pert}^*(\mu)$ in which s_{μ} and t_{μ} lie on the external face. We have that:

- (i) If μ is of Type M, then the first and the last successors of s_{μ} in pert*(μ) lie one on the left path and the other on the right path of \mathcal{E}^* . In particular, if the first and the last successor of μ are the same vertex, then such a vertex belongs to both the left path and the right path of \mathcal{E}^* .
- (ii) If μ is of Type B, then the two first successors of s_{μ} in pert*(μ) lie one on the left path and the other on the right path of \mathcal{E}^* .

We distinguish four cases based on whether node μ is an S-, P-, Q-, or R-node. Q-node. Here, $\langle \operatorname{pert}(\mu), \operatorname{pert}(\mu) \rangle$ is trivially of Type M, i.e., $\operatorname{pert}^*(\mu) = \operatorname{pert}(\mu)$. S-node. Let e_1, \ldots, e_k be the virtual edges of $\operatorname{skel}(\mu)$ in the order in which they appear from the source s_μ to the target t_μ of $\operatorname{skel}(\mu)$, and let μ_1, \ldots, μ_k be the corresponding children of μ , respectively. We obtain $\operatorname{pert}^*(\mu)$ by replacing each virtual edge e_i in $\operatorname{skel}(\mu)$ with $\operatorname{pert}^*(\mu_i)$. Also, we obtain the embedding \mathcal{E}^* by arbitrarily selecting a flip for each embedding \mathcal{E}^*_i of $\operatorname{pert}^*(\mu_i)$. Clearly, node μ is of Type M if and only if μ_1 is of Type M and it is of Type B, otherwise.

P-node. Let e_1, \ldots, e_k be the virtual edges of $skel(\mu)$ and let μ_1, \ldots, μ_k be the corresponding children of μ , respectively.

First, observe that if there exists more than one child of μ that is of Type B, then node μ does not admit an augmentation $\operatorname{pert}^*(\mu)$ where $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic. In fact, if there exist two such nodes μ_i and μ_j , then both the subgraphs of $\operatorname{pert}^*(\mu_i)$ and $\operatorname{pert}^*(\mu_j)$ induced by the successors of s_{μ} in $\operatorname{pert}(\mu_i)$ and in $\operatorname{pert}(\mu_j)$, respectively, contain an apex vertex. This implies that s_{μ} would have more than one apex.

Second, observe that if there exists a child μ_i of μ of Type B and the edge (s_{μ}, t_{μ}) belongs to $\operatorname{pert}(\mu)$, then node μ does not admit an augmentation $\operatorname{pert}^*(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic. In fact, $\operatorname{pert}^*(\mu_i)$ contains a apex of s_{μ} different from t_{μ} ; this is due to the fact that edge $(s_{\mu}, t_{\mu}) \notin \operatorname{pert}^*(\mu_i)$. Also, vertex t_{μ} must be an apex of s_{μ} in any augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is v-bitonic, for each vertex v of $\operatorname{pert}(\mu)$. Namely, any augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ yields an st-graph with source s_{μ} and sink t_{μ} and, as such, no directed path exits from t_{μ} in $\operatorname{pert}^*(\mu)$. As for the observation in the previous paragraph, this implies that s_{μ} would have more than one apex.

We construct $\operatorname{pert}^*(\mu)$ as follows. We embed $\operatorname{skel}(\mu)$ in such a way that the edge (s_{μ}, t_{μ}) , if any, or the virtual edge corresponding to the unique child of μ that is of Type B, if any, is the right-most virtual edge in the embedding. Let e_1, \ldots, e_k be the virtual edges of $\operatorname{skel}(\mu)$ in the order in which they appear clockwise around s_{μ} in $\operatorname{skel}(\mu)$. Then, for each child μ_i of μ , we choose a flip of embedding \mathcal{E}_i^* such that a first successor of s_{μ} in $\operatorname{pert}^*(\mu_i)$ lies along the left path of \mathcal{E}_i^* . Now, for $i=1,\ldots,k-2$, we add an edge connecting the last successor of s_{μ} in $\operatorname{pert}^*(\mu_i)$ and the first successor of s_{μ} in $\operatorname{pert}^*(\mu_{i+1})$. Finally, we possibly add an edge connecting the last successor v_l of s_{μ} in $\operatorname{pert}^*(\mu_{k-1})$ and a suitable vertex in $\operatorname{pert}^*(\mu_k)$. Namely, if a node μ_k is of Type B, then we add an edge between v_l and the first successor of s_{μ} in $\operatorname{pert}^*(\mu_k)$ that lies along the left path of \mathcal{E}_k^* . If μ_k is of Type M and it is not a Q-node, then we add an edge between v_l and the first successor of s_{μ} in $\operatorname{pert}^*(\mu_k)$. Otherwise $\operatorname{pert}^*(\mu_k) = (s_{\mu}, t_{\mu})$ and we add the edge (v_l, t_{μ}) if no such an edge belongs to $\operatorname{pert}^*(\mu_{k-1})$.

Observe that, the added edges do not introduce any directed cycle as there exists no directed path from a vertex in $\operatorname{pert}^*(\mu_{i+1})$ to a vertex in $\operatorname{pert}^*(\mu_i)$. Further, by Observation 2 the added edges do not disrupt planarity. Therefore, the obtained augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ is, in fact, a planar st-graph.

Finally, we have that node μ is of Type M if and only if μ_k is of Type M.

R-node. The case of an R-node μ is detailed in Appendix B.4. For each node v of $\mathrm{skel}(\mu)$, we have to consider the virtual edges e_1,\ldots,e_k of $\mathrm{skel}(\mu)$ exiting v and the corresponding children μ_1,\ldots,μ_k of μ , respectively. Similarly to the P-node case, we pursue an augmentation of $\mathrm{pert}(\mu)$ by inserting edges that connect $\mathrm{pert}(\mu_i)$ with $\mathrm{pert}(\mu_{i+1})$, with $i=1,\ldots,k-1$. Differently from the P-node case, however, more than one $\mathrm{pert}(\mu_i)$ may contain an edge between the poles of μ_i . Further, also the faces of $\mathrm{skel}(\mu)$ may play a role, introducing additional constraints on the existence and the choice of the augmentation.

We have the following theorem.

Theorem 5. It is possible to decide in linear time whether a planar st-graph G admits a bitonic pair $\langle \mathcal{E}, \pi \rangle$.

Proof. Let ρ be the root of the SPQR-tree of G. The algorithm described above computes a pair $\langle \operatorname{pert}^*(\rho), \mathcal{E}^* \rangle$ for G, if any exists, such that (i) the st-graph $\operatorname{pert}^*(\rho)$ is an augmentation of G, (ii) for any vertex v of G, $\langle \operatorname{pert}(\rho), \operatorname{pert}^*(\rho) \rangle$ is v-bitonic, and (iii) \mathcal{E}^* is a planar embedding of $\operatorname{pert}^*(\rho)$. Let \mathcal{E} be the restriction of \mathcal{E}^* to G. By Theorem 4, any st-ordering π of $\operatorname{pert}^*(\rho)$ is a bitonic st-ordering of G with respect to \mathcal{E} . Hence, $\langle \mathcal{E}, \pi \rangle$ is a bitonic pair of G.

We first show that the described algorithm has a quadratic running time. Then, we show how to refine it in order to run in linear time. For each node μ of T, the algorithm stores a pair $\langle \operatorname{pert}^*(\mu), \mathcal{E} \rangle$. Processing a node takes $O(|\operatorname{pert}^*(\mu)|)$ time. Since $|\operatorname{pert}^*(\mu)| \in O(|\operatorname{pert}(\mu)|)$, the overall running time is $O(|G|^2)$.

To achieve a linear running time, observe that we do not need to compute the embeddings of the augmented pertinent graphs $\operatorname{pert}^*(\mu)$, for each node μ of T, during the bottom-up traversal of T. In fact, any embedding \mathcal{E}^* of $\operatorname{pert}^*(\rho)$ yields an embedding \mathcal{E} of G such that π is bitonic with respect to \mathcal{E} . To determine the endpoints of the augmenting edges, we only need to associate a constant amount of information with the nodes of T. Namely, for each node μ in T, we maintain (i) whether μ is of Type B or of Type M, (ii) if μ is of Type M, the first successor and the last successor of s_{μ} in $\operatorname{pert}^*(\mu)$, and (iii) if μ is of Type B, the two first successors of s_{μ} in $\operatorname{pert}^*(\mu)$. Therefore, processing a node takes $O(|\operatorname{skel}(\mu)|)$ time. Since the sum of the sizes of the skeletons of the nodes in T is linear in the size of G [6], the overall running time is linear.

Corollary 2. It is possible to decide in linear time whether a planar st-graph G admits a monotonically decreasing pair $\langle \mathcal{E}, \pi \rangle$.

Proof. The statement immediately follows from the fact that, in the algorithm described in this section, when computing a pair $\langle \operatorname{pert}^*(\mu), \mathcal{E}^* \rangle$ for each node μ in T, a pair $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ of Type M is built whenever possible. Therefore, rejecting instances for which a pair $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ of Type B is needed yields the desired algorithm.

In conclusion, we have the following main result.

Theorem 6. It can be tested in linear time whether a planar st-graph admits an upward- (upward-rightward-) planar L-drawing, and if so, such a drawing can be constructed in linear time.

Proof. We first test in linear time whether a planar st-graph admits a bitonic pair (Theorem 5) or a monotonically decreasing pair (Corollary 2). Then, Theorem 3 shows how to construct in linear time an upward- (upward-rightward-) planar L-drawing from a bitonic (monotonically decreasing) pair.

5 Open Problems

Several interesting questions are left open: Can we efficiently test whether a directed plane graph admits a planar L-drawing? Can we efficiently recognize the directed graphs that are edge maximal subject to having a planar L-drawing

(they have at most 4n-6 edges where n is the number of vertices—see Appendix B.5)? Does every upward-planar graph have a (not necessarily upward-) planar L-drawing? Can we extend the algorithm for computing a bitonic pair in the variable embedding setting to single-source multi-sink di-graphs? Does every bimodal graph have a planar L-drawing?

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Appendix A SPQR Trees

In this appendix we describe SPQR-trees, a data structure introduced by Di Battista and Tamassia (see, e.g., [8]) which allows to handle the planar embeddings of an st-biconnectible planar graph.

A graph is st-biconnectible if adding the edge (s,t) yields a biconnected graph. Let G be an st-biconnectible graph. A separation pair of G is a pair of vertices whose removal disconnects the graph. A split pair of G is either a separation pair or a pair of adjacent vertices. A maximal split component of G with respect to a split pair $\{u,v\}$ (or, simply, a maximal split component of $\{u,v\}$) is either an edge (u,v) or a maximal subgraph G' of G such that G' contains u and v, and $\{u,v\}$ is not a split pair of G'. A vertex $w \neq u,v$ belongs to exactly one maximal split component of $\{u,v\}$. We call split component of $\{u,v\}$ the union of any number of maximal split components of $\{u,v\}$.

In this paper, we will assume that any SPQR-tree of a graph G is rooted at one edge of G, called *reference edge*.

The rooted SPQR-tree \mathcal{T} of a biconnected graph G, with respect to a reference edge e, describes a recursive decomposition of G induced by its split pairs. The nodes of \mathcal{T} are of four types: S, P, Q, and R. Their connections are called arcs, in order to distinguish them from the edges of G.

Each node μ of \mathcal{T} has an associated st-biconnectible multigraph, called the skeleton of μ and denoted by $\text{skel}(\mu)$. Skeleton $\text{skel}(\mu)$ shows how the children of μ , represented by "virtual edges", are arranged into μ . The virtual edge in $\text{skel}(\mu)$ associated with a child node ν , is called the virtual edge of ν in $\text{skel}(\mu)$.

For each virtual edge e_i of $skel(\mu)$, recursively replace e_i with the skeleton $skel(\mu_i)$ of its corresponding child μ_i . The subgraph of G that is obtained in this way is the *pertinent graph* of μ and is denoted by $G(\mu)$.

Given a biconnected graph G and a reference edge e = (u', v'), the SPQR-tree \mathcal{T} is recursively defined as follows. At each step, a split component G^* , a pair of vertices $\{u, v\}$, and a node ν in \mathcal{T} are given. A node μ corresponding to G^* is introduced in \mathcal{T} and attached to its parent ν . Vertices u and v are the poles of μ and denoted by $u(\mu)$ and $v(\mu)$, respectively. The decomposition possibly recurs on some split components of G^* . At the beginning of the decomposition $G^* = G - \{e\}$, $\{u, v\} = \{u', v'\}$, and ν is a Q-node corresponding to e.

Base Case: If G^* consists of exactly one edge between u and v, then μ is a Q-node whose skeleton is G^* itself.

Parallel Case: If G^* is composed of at least two maximal split components G_1, \ldots, G_k $(k \geq 2)$ of G with respect to $\{u, v\}$, then μ is a P-node. The graph $\operatorname{skel}(\mu)$ consists of k parallel virtual edges between u and v, denoted by e_1, \ldots, e_k and corresponding to G_1, \ldots, G_k , respectively. The decomposition recurs on G_1, \ldots, G_k , with $\{u, v\}$ as pair of vertices for every graph, and with μ as parent node.

Series Case: If G^* is composed of exactly one maximal split component of G with respect to $\{u, v\}$ and if G^* has cut vertices c_1, \ldots, c_{k-1} $(k \geq 2)$, appearing in this order on a path from u to v, then μ is an S-node. Graph

skel(μ) is the path e_1, \ldots, e_k , where virtual edge e_i connects c_{i-1} with c_i ($i = 2, \ldots, k-1$), e_1 connects u with c_1 , and e_k connects c_{k-1} with v. The decomposition recurs on the split components corresponding to each of $e_1, e_2, \ldots, e_{k-1}, e_k$ with μ as parent node, and with $\{u, c_1\}, \{c_1, c_2\}, \ldots, \{c_{k-2}, c_{k-1}\}, \{c_{k-1}, v\}$ as pair of vertices, respectively.

Rigid Case: If none of the above cases applies, the purpose of the decomposition step is that of partitioning G^* into the minimum number of split components and recurring on each of them. We need some further definition. Given a maximal split component G' of a split pair $\{s,t\}$ of G^* , a vertex $w \in G'$ properly belongs to G' if $w \neq s,t$. Given a split pair $\{s,t\}$ of G^* , a maximal split component G' of $\{s,t\}$ is internal if neither u nor v (the poles of G^*) properly belongs to G', external otherwise. A maximal split pair $\{s, t\}$ of G^* is a split pair of G^* that is not contained in an internal maximal split component of any other split pair $\{s',t'\}$ of G^* . Let $\{u_1,v_1\},\ldots,\{u_k,v_k\}$ be the maximal split pairs of G^* $(k \geq 1)$ and, for i = 1, ..., k, let G_i be the union of all the internal maximal split components of $\{u_i, v_i\}$. Observe that each vertex of G^* either properly belongs to exactly one G_i or belongs to some maximal split pair $\{u_i, v_i\}$. The node μ is an R-node. The graph $skel(\mu)$ is the graph obtained from G^* by replacing each subgraph G_i with the virtual edge e_i between u_i and v_i . The decomposition recurs on each G_i with μ as parent node and with $\{u_i, v_i\}$ as pair of vertices.

For each node μ of \mathcal{T} with poles u and v, the construction of $\text{skel}(\mu)$ is completed by adding a virtual edge (u, v) representing the rest of the graph, that is, the graph obtained from G by removing all the vertices of $G(\mu)$, except for its poles, together with their incident edges.

The SPQR-tree \mathcal{T} of a graph G with n vertices and m edges has m Q-nodes and O(n) S-, P-, and R-nodes. Also, the total number of vertices of the skeletons stored at the nodes of \mathcal{T} is O(n). Finally, SPQR-trees can be constructed and handled efficiently. Namely, given a biconnected planar graph G, the SPQR-tree \mathcal{T} of G can be computed in linear time [6,9,17].

Appendix B Omitted Proofs

In this appendix we give full versions of sketched or omitted proofs.

Appendix B.1 Omitted Proofs of Section 3.1

As a central building block for our hardness reduction we use a graph W that can be constructed starting from a 4-wheel with central vertex c and rim (u, v, w, z) such that vertices v and z are sinks and u and w are sources, by adding edges (v, c), (z, c), (c, w), and (c, u); see Fig. 2. Note that the edges incident to c come in pairs of both directions. We denote the vertices v and v as v-ports of v and the vertices v and v as v-ports. We first study the properties of planar L-drawings of v.

Lemma 2. In any planar L-drawing of W with cycle (u, v, w, z) as the outer face the edges of the outer face form a rectangle (that contains vertex c).

Proof. In any orthogonal drawing of W, the outer cycle (u, v, w, z) forms an orthogonal polygon P with at least four convex corners. Since any two consecutive edges on the outer cycle have the same direction with respect to their common vertex $r \in \{u, v, w, z\}$, i.e., they are either both incoming or outgoing at r, they must use the same port or two opposite ports of r. In fact, if they would use the same port, they would form an angle of 2π in the outer face and force the edge (r, c) to use the very same port. This, however, would imply that all three edges incident to r have the same direction, which is a contradiction. Hence each of the four outer vertices has an angle of π in the outer face and cannot form a convex corner of P.

Since there are four edges on the outer cycle, each of which has exactly one bend, this immediately implies that P is a rectangle whose corners are formed by the bends of the four edges of the outer face and each of the four vertices of the outer face must lie on one of the rectangle sides. The remaining edges to c use the port inside P, consistently bend once (left or right) from the perspective of c, and then connect to c from all four sides. Figure 2 shows an example. \Box

Theorem 1. It is NP-complete to decide whether a directed graph admits a planar L-drawing.

Proof. We reduce from HV-rectilinear planarity testing, which is NP-hard even for biconnected graphs [10]. An instance of this problem is a degree-4 planar graph G where each edge is labeled either H or V. The task is to decide whether G admits a planar orthogonal drawing (without bends) such that H-edges are drawn horizontally and V-edges are drawn vertically. We call such a drawing a planar HV-drawing.

Given a biconnected HV-graph G, we construct an instance G' of planar L-drawing by replacing each vertex by a 4-wheel as in Fig. 2, each edge labeled V (V-edge) with the gadget shown in Fig. 3a and each edge labeled H (H-edge) with the gadget shown in Fig. 6. For a V-edge (u,v), the two vertices of the edge gadget labeled u and v are identified with a V-port of the respective vertex gadgets and for an H-edge with an H-port of the vertex gadgets. Obviously, this reduction is polynomial in the size of G.

Our high-level construction is somewhat similar to Brückner's NP-completeness proof for 1-EMBEDDABILITY in the Kandinsky model [5, Theorem 3] in that we define gadgets that have a very limited flexibility in terms of their embeddings to realize horizontal and vertical edges. Yet the internals of the gadgets themselves and the reduction are quite different.

We claim that G' has a planar L-drawing if and only if G has a planar HV-drawing. So first assume that G' admits a planar L-drawing Γ' . We transform Γ' into a planar HV-drawing. In a first step, we draw each vertex v of G at the position of the central vertex of the vertex gadget for v. Due to Lemma 2, the edge gadgets attach to the bounding boxes of the vertex gadgets. Hence, for each edge (u, v) of G, we can draw an orthogonal path from u to v by tracing

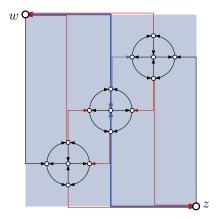


Fig. 6: Edge gadget for an H-edge.

the thick edges (red for a V-edge, blue for an H-edge) in its edge gadget and the two incident vertex gadgets (see Fig. 2 and 6). This intermediate drawing as a subdrawing of Γ' is a planar orthogonal drawing of G, where each edge is an 8-bend orthogonal staircase path with total rotation of 0. Using Tamassia's network flow model for orthogonal graph drawings [19], we can argue that an edge with rotation 0 is equivalent to a rectilinear edge without bends. In fact, the flow corresponding to the eight bends is cyclic and can be reduced to a flow of value 0, which implies no bends. We refer to Brückner [5, Lemma 7] for more details of this argument.

Now, conversely, assume that G admits a planar HV-drawing Γ . In order to show that Γ can be transformed into an L-drawing of G' we first "thicken" Γ by inflating vertices at grid points to squares and edges to corresponding rectangles, see Fig. 3b. This can easily be done without introducing any crossings of overlapping features by refining the grid on which Γ is drawn. Since each vertex gadget in G' can be drawn in a square (Fig. 2) and each edge gadget in a rectangle (Figs. 3a and 6), we can insert their drawings into the thickened drawing of G as illustrated in Fig. 3b. This produces an L-drawing of G'.

To see that the problem is in NP, we note that for an embedding of a graph and a given orthogonal representation (see Tamassia [19]) of that embedding, one can check whether all edges are represented as valid L-shapes in polynomial time.

We remark that the graph G' that we construct in our reduction is a simple directed graph. With the exception of the four spoke edges of the wheel graph W (see Fig. 2) each underlying undirected would not have multi-edges. It is not difficult to extend our reduction so that the red and blue edges in Fig. 2 are removed from the gadget and the entire graph G' becomes an oriented graph, i.e., a graph without 2-cycles. In that case, however, when we construct the intermediate staircase paths for the edges of the HV-drawing, we still use the

removed "mirrored" L-shape for the first and last two segments of each edge path, which is always possible without crossings in any L-drawing of W.

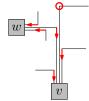
Appendix B.2 Omitted Proofs of Section 3.2 Including the Relation with the Kandinsky Model

Lemma 1. A graph has a planar L-drawing if and only if it admits a drawing in the Kandinsky model with the following properties

- 1. Each edge bends exactly once.
- 2. At each vertex, the angle between two outgoing (or between two incoming) edges is 0 or π .
- 3. At each vertex, the angle between an incoming edge and an outgoing edge is $\pi/2 \ or \ 3\pi/2.$

Proof. Given a drawing in the Kandinsky model that meets Conditions 1-3, we can bundle the edges on the finer grid to lie on the coarser grid. It remains to perturb the coordinates such that the x- and y-coordinates, respectively, of the vertices are distinct: Assume, two vertices v and w have the same y-coordinate. Let $\delta > 0$ be the minimum difference in y-coordinates between v and any vertex or segment above v. Since all edges have one bend, we can shift v upward by $\delta/2$ —changing only the drawing of edges incident to v. Doing this iteratively yields a planar L-drawing—or a rotation of $\pi/2$ of it.

Given a planar L-drawing, we can distribute the edges on the finer grid maintaining the embedding. Since all vertices have distinct x- and y-coordinates, there are no empty faces. It remains to assign the bends to the vertices in order to fulfill the bend-orend property: For each port (top, right, bottom, left) of a vertex v, we assign all bends of incident edges,



but the furthest to v (see the figure on the right—the furthest bend of top of vis encircled). Observe that if the bend on an edge $\{v, w\}$ is not a furthest bend for v then it is a furthest bend for w. Thus, no bend will be assigned to two vertices.

ILP formulation for the proof of Theorem 2

- 1. The angles around a vertex v sum to 2π : $\sum_{f \text{ incident } v} x_{vf} = \begin{cases} 2 \text{ if } v \text{ 1-modal} \\ 1 \text{ if } v \text{ 2-modal} \\ 0 \text{ if } v \text{ 4-modal} \end{cases}$ 2. All edges are bent exactly once i.e. for each odge v.
- 2. All edges are bent exactly once, i.e., for each edge $e = \{v, w\}$ separating the
- faces f and h, we have $x_{fe}^v + x_{he}^v + x_{fe}^w + x_{he}^w = 1$. 3. The number of convex angles minus the number of concave angles is 4 in each inner face and -4 in the outer face, i.e., for each face f, we have

$$\sum_{\substack{e = \{v, w\} \\ \text{separating} \\ f \text{ and } h}} (x_{fe}^v - x_{he}^v + x_{fe}^w - x_{he}^w) + \sum_{\substack{v \text{ on } f \\ \text{not in/out}}} (2 - 2x_{vf}) + \sum_{\substack{v \text{ on } f \\ \text{in/out}}} (2 - (2x_{vf} + 1)) = \pm 4.$$

4. The bend-or-end property is fulfilled, i.e., for any two edges e_1 and e_2 that are consecutive around a vertex v and that are both incoming or both outgoing, and for the faces f_1 , f, and f_2 that are separated by e_1 and e_2 (in the cyclic order around v), it holds that $x_{vf} + x_{f_1e_1}^v + x_{f_2e_2}^v \ge 1$.

Observe that (2) implies $-x_{he}^v - x_{he}^w = x_{fe}^v + x_{fe}^w - 1$. Hence, (3) yields

3'.
$$\sum_{\substack{e=\{v,w\}\\\text{incident }f}} (x_{fe}^v + x_{fe}^w) - \sum_{v \text{ on } f} x_{vf} = \pm 2 + (\# \text{ in/out-vertices on } f - \deg f)/2.$$

Theorem 2. Given a directed plane graph G and labels $out(e) \in \{\text{top}, \text{bottom}\}$ and $in(e) \in \{\text{right}, \text{left}\}$ for each edge e, it can be decided in linear time whether G admits a planar L-drawing in which each edge e leaves its tail at out(e) and enters its head at in(e).

Proof. Observe that the labeling determines the bends, i.e., the value $x_{fe}^v + x_{fe}^w$ for each edge e = (v, w) and each incident face f. First, we have to check whether the cyclic order of the edges around a vertex is compatible with the labels, i.e., in clockwise order we have outgoing edges labeled (top,·), incoming edges labeled (·,left), outgoing edges labeled (bottom,·), and incoming edges labeled (·,right). For a fixed port, edges bending to the right must precede edges bending to the left. We call an edge a middle edge of a port if it is the last edge bending to the left or the first edge bending to the right. Observe that each port has zero, one, or two middle edges.

If the compatibility check does not fail then the labels also determine the angles around the vertices, i.e., the variables x_{vf} for each vertex v and each incidence to a face f. Now, we check whether these values fulfill Conditions 1, 2, and 3'.

Finally, we have to check, whether Condition 4, i.e., the bend-or-end property can be fulfilled. To this end, we have to assign edges with concave bends to zero angles at an incident vertex in the same face. We must assign for each port of a vertex v, all but the middle edges to v. If at this stage an edge is assigned to two vertices, then G does not admit a planar L-drawing with the given port assignment. Otherwise, it remains to deal with the zero angles between two middle edges of a port. To this end, consider the following graph B. The nodes are on one hand the ports with two middle edges and on the other hand the edges that are middle edges of at least one port and that are not yet assigned to a vertex. A port of a vertex v and an edge e are adjacent in B if and only if e is a middle edge of v. Observe that B is a bipartite graph of maximum degree two and, thus, consists of paths, even length cycles, and isolated vertices. We have to test whether B has a matching in which every port node is matched. This is true if and only if no port is isolated and there is no maximal path starting and ending at a port node.

Appendix B.3 Omitted proofs of Sect. 4.1

Theorem 3. A planar st-graph admits an upward- (upward-rightward-) planar L-drawing if and only if it admits a bitonic (monotonically decreasing) pair.

Proof. Let G = (V, E) be a planar st-graph with n vertices.

" \Rightarrow ": The y-coordinates of an upward- (upward-rightward-) planar L-drawing of G yield a bitonic (monotonically decreasing) st-ordering π with respect to the embedding \mathcal{E} given by the L-drawing.

" \Leftarrow ": Given a bitonic (monotonically decreasing) st-ordering π of G, we construct an upward- (upward-rightward-) planar L-drawing of G using an idea of Gronemann [16]. For $i = 1, \ldots, n$, let $v_i \in V$ be the vertex with $\pi(v_i) = i$, set the y-coordinate of v_i to i, and let G_i be the subgraph of G induced by $V_i = \{v_1, \ldots, v_i\}$.

For the x-coordinates we construct a partial order \prec in such a way that, for $i=2,\ldots,n$, all vertices on the outer face of G_i are comparable and the L-drawing of G_i is planar, embedding preserving, and has the property that any edge from V_i to $V \setminus V_i$ can be added upward and in an embedding preserving way, no matter how we choose the x-coordinates of v_{i+1},\ldots,v_n .

During the construction, we augment G_i to G_i in such a way that the outer face $f_{\overline{G}_i}$ of \overline{G}_i is a simple cycle. We start by adding two artificial vertices v_{-1} and v_{-2} with y-coordinates -1 and -2, respectively, that are connected to v_1 and to each other. We set $v_{-2} \prec v_1 \prec v_{-1}$. Now let $i \in \{2, \ldots, n\}$ and assume that we have already fixed the relative coordinates of G_{i-1} . Let u_1, \ldots, u_k be the predecessors of v_i in ascending order with respect to \prec .

If π is monotonically decreasing or if k=1, we first augment the graph. In the former case, we add to G an edge between v_i and the right neighbor of u_k on $f_{\overline{G}_{i-1}}$. In the latter case, let ℓ and r be the left and the right neighbor of u_1 on $f_{\overline{G}_{i-1}}$, respectively; see Fig. 4a. Following Gronemann [16], we add a dummy edge from either ℓ or r to v_i : Let s_{\max} be the successor of u_1 of maximum rank. We go in the circular order of the edges around u_1 from u_1v_i to the left. If we hit u_1s_{\max} before $u_1\ell$, we insert the edge rv_i into G, otherwise the edge ℓv_i . Note that inserting the dummy edge does not violate planarity since, on that side, u_k does not have any outgoing edge between u_kv_i and $f_{\overline{G}_{i-1}}$.

We now extend \prec . Let u_1, \ldots, u_k be the $k \geq 2$ predecessors of v_i in the possibly augmented graph; see Fig. 4b. Since G has a sink only on the outer face, we can place v_i anywhere between u_1 and u_k . Adding the two conditions $u_{k-1} \prec v_i \prec u_k$ also sure that all edges except (u_k, v_i) are rightward. But (u_k, v_i) was introduced only as a dummy edge for the case of a monotonically decreasing π .

Any linear order that is compatible with \prec yields unique x-coordinates in $\{1,\ldots,n\}$ for the vertices of G. Together with the y-coordinates that we fixed above, we now have positions for the vertices in an upward- (upward-rightward-) planar L-drawing of G. Finally, we remove the dummy edges that we inserted earlier.

Appendix B.4 Omitted Proofs of Section 4.2

Lemma 3. Let G = (V, E) be a planar st-graph with source s, sink t, and $(s, t) \notin E$. Then there exists a supergraph G' = (V', E') of G, where $V' = V \cup \{s'\}$ and

 $E' = E \cup \{(s', s), (s', t)\}$, such that (i) G' is an st-graph with source s' and sink t, and (ii) G' admits a bitonic (resp., monotonically increasing) st-ordering if and only if G does.

Proof. We prove the *if* direction. Let π' be a bitonic (resp., monotonically increasing) st-ordering of G' and let \mathcal{E}' be a planar embedding of G' compatible with π . We construct a ranking $\pi: V \to \{1, \ldots, |V|\}$ by setting $\pi(v) = \pi'(v) - 1$, for each $v \in V$. Also, we set \mathcal{E} to the restriction of \mathcal{E}' to G. Clearly, π is a bitonic (resp., monotonically increasing) st-ordering of G that is consistent with \mathcal{E} .

We now prove the *only if* direction. Let π be a bitonic (resp., monotonically increasing) st-ordering of G and let \mathcal{E} be a planar embedding of G compatible with π . We construct a ranking $\pi' = V' \to \{1, \ldots, |V'|\}$ as follows: We set (i) $\pi'(s') = 1$ and (ii) $\pi'(v) = \pi(v) + 1$, for each $v \in V$. We construct a planar embedding \mathcal{E}' of G' starting from \mathcal{E} by drawing s' in the outer face of \mathcal{E} and by routing edge (s',t) so that vertex t is the right-most successor of s' in the left-to-right order of the successors of s' around s'. We show that π' is a bitonic (resp., monotonically increasing) st-ordering of G' and that \mathcal{E}' is consistent with π' . Since, for each vertex $v \in V$, the ranks of the successors of v in v have all been decreased by 1 and since the left-to-right order of the successors of v is the same in v as in v, it follows that such ranks form a bitonic (resp., monotonically increasing) sequence in v if and only if they do so in v. Also, v and v are the successors of v and v and v are the ranks of the successors of v form a monotonically increasing sequence. This concludes the proof of the lemma. v

Lemma 4. Let G be a biconnected st-graph and let G^* be an augmentation of G such that $\langle G, G^* \rangle$ is v-bitonic, for each vertex v of G. Consider a node μ of the SPQR-tree of G and let $\operatorname{pert}^*(\mu)$ be the subgraph of G^* induced by the vertices of $\operatorname{pert}(\mu)$. Suppose that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is of $Type\ B$ and that $\operatorname{pert}(\mu)$ also admits an augmentation $\operatorname{pert}^+(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^+(\mu) \rangle$ is of $Type\ B$ and it is v-bitonic, for each vertex v of $\operatorname{pert}(\mu)$. There exists an augmentation G^+ of G such that $\langle G, G^+ \rangle$ is v-bitonic, for each vertex v of G, and such that the subgraph of G^+ induced by the vertices of $\operatorname{pert}(\mu)$ is $\operatorname{pert}^+(\mu)$.

Proof. First, observe that, by removing from G^* all the edges (gray edges in Fig. 5a) connecting a vertex in $\operatorname{pert}(\mu)$ that is not a successor of s_{μ} and a vertex not in $\operatorname{pert}(\mu)$ that is not a successor of s_{μ} , we obtain an augmentation G^{\diamond} of G such that (i) the subgraph of G^{\diamond} induced by the vertices of $\operatorname{pert}(\mu)$ is $\operatorname{pert}^*(\mu)$ and (ii) $\operatorname{pair} \langle G, G^{\diamond} \rangle$ is of v-bitonic, for any vertex v of G^9 . Therefore, in the following we assume that $G^* = G^{\diamond}$.

Let \mathcal{E} be a planar embedding of G^* . Consider the subgraph G_{μ}^- obtained by removing from G^* all the vertices of $V(\operatorname{pert}(\mu)) \setminus \{s_{\mu}, v_{\mu}\}$ an their incident edges. Let \mathcal{E}_{μ}^- be the planar embedding of G_{μ}^- induced by \mathcal{E} . Let f be the face of \mathcal{E}_{μ}^- whose boundary used to enclose the removed vertices. Observe that, the poles s_{μ} and t_{μ} of μ belong to f. Let v_l and v_r be successors of s_{μ} belonging to G_{μ}^- such

⁹ We remark that these edges are never introduced by our algorithm, however, for the sake of generality we make no assumption on their absence in this proof.

that v_l and v_r are predecessors in G^* of first successors of pert*(μ). Observe that, since we assumed $G^* = G^{\diamond}$, there exists exactly two vertices satisfying these properties.

Let \mathcal{E}_{μ}^{+} be a planar embedding of $\operatorname{pert}^{+}(\mu)$ in which s_{μ} and t_{μ} are incident to the outer face. We now obtain plane graph $G^{+} = G_{\mu}^{-} \cup \operatorname{pert}^{+}(\mu)$ as follows. First, we embed $\operatorname{pert}^{+}(\mu)$ in the interior of f, identifying s_{μ} in $\operatorname{pert}^{+}(\mu)$ with s_{μ} in f and t_{μ} in $\operatorname{pert}^{+}(\mu)$ with t_{μ} in f. Then, we insert two directed edges between a vertex in G_{μ}^{-} and a vertex of $\operatorname{pert}^{*}(\mu)$ as follows. We add a directed edge from v_{l} to a first successor of s_{μ} in $\operatorname{pert}^{+}(\mu)$. Also, we add a directed edge from v_{r} to the other first successor of s_{μ} in $\operatorname{pert}^{+}(\mu)$, if μ is not a Q-node, or to the same first successor of s_{μ} in $\operatorname{pert}^{+}(\mu)$ to which v_{l} is now adjacent, otherwise.

To see that the directed graph G^+ is an st-graph, observe that the added edges do not introduce any directed cycle as there exists no directed path from a vertex in pert⁺(μ) to a vertex in G^-_{μ} . Also, by construction, the subgraph of G^+ induced by the vertices of pert(μ) is pert⁺(μ).

We now show that the pair $\langle G, G^+ \rangle$ is v-bitonic, for any v in G. Clearly, any vertex $v \notin \{s_{\mu}, v_l, v_r\}$ has the same successors in G^+ as in G^* , therefore $\langle G, G^+ \rangle$ is v-bitonic. Further, by construction, $\langle G, G^+ \rangle$ is s_{μ} -bitonic, that is, $\langle G, G^+ \rangle$ is of Type B; refer to Fig. 5b. Finally, since $v_l(v_r)$ is not adjacent in G to any vertex in pert (μ) , the subgraph of G^+ induced by the successors of $v_l(v_r)$ in G is the same as the subgraph of G^* induced by the successors of $v_l(v_r)$ in G. This concludes the proof.

Details for the R-node Case

R-node. Recall that, by Observation 1, the skeleton of a node μ of T is an st-graph between its poles s_{μ} and t_{μ} .

For each vertex $v \neq t_{\mu}$, let e_1, \ldots, e_k be the virtual edges exiting v in the order in which they appear clockwise around v in $skel(\mu)$, and let μ_i be the node of T corresponding to e_i . First, observe that if there exists more than one virtual edge e_i exiting from v whose corresponding child μ_i is of Type B, then node μ does not admit an augmentation pert* (μ) such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic. In fact, as shown for the P-node case, this implies that s_{μ} would have more than one apex. We aim at (i) selecting a flip for each pert*(μ_i) and (ii) adding an edge between a vertex in pert* (μ_i) and a vertex in pert* (μ_{i+1}) , with $i=1,\ldots,k-1$, in order to obtain an augmentation pert*(μ) of pert(μ) such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic. In particular, such edges will either be directed from the last successor of v in pert* (μ_i) to a first successor of v in pert* (μ_{i+1}) (right edges) or from the last successor of v in pert*(μ_{i+1}) to a first successor of v in pert* (μ_i) (left edges). Observe that, in any augmentation pert* (μ) of pert (μ) such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic, for each pair of consecutive virtual edges e_i and e_{i+1} exiting v, either a left edge or the alternative right edge is introduced connecting a vertex in $pert(\mu_i)$ with a vertex in $pert(\mu_{i+1})$.

We assign a label in $\{L, R\}$ to some of the faces of $skel(\mu)$ as follows. For each face f of $skel(\mu)$ incident to two consecutive virtual edges exiting v, we say that v is the *source vertex* of f if it is the source of the st-graph induced by the

edges incident to f, and t_f is the sink vertex of f if it is the sink of the st-graph induced by the edges incident to f. Consider the two virtual edges $e_l = (v, v_l)$ and $e_r = (v, v_r)$ exiting v and incident to f, where e_l precedes e_r in the clockwise order of the edges exiting v. If $v_l = t_f$ and $(s_f, v_l) \in \operatorname{pert}^*(\mu_l)$, we assign label L to f. If $v_r = t_f$ and $(s_f, v_r) \in \operatorname{pert}^*(\mu_r)$, we assign label R to f. Observe that, in any augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is s_{μ} -bitonic, faces with label L (with label R) must be traversed by a left edge (resp. right edge). In fact, vertex t_f is also the sink of the st-graph induced by the edges incident to the face of pert(μ) corresponding to f; also, the alternative edges with respect to those inserted would exit t_f and hence would introduce a directed cycle in pert*(μ). We remark that, augmenting an unlabeled face with any of the two alternative edges does not introduce any directed cycles. This is due to the fact that there exists no directed path connecting an internal vertex in pert* (μ_l) with an internal vertex in pert* (μ_r) . Hence, in the following we can assume that the obtained augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ is an acyclic st-graph.

Based on the type of the children μ_1, \ldots, μ_k of μ and on the labeling of the faces of which v is the source vertex, one of the following three claims applies.

Claim 1 If no child of μ corresponding to a virtual edge exiting v is of Type B and if v is not the source of two faces of $skel(\mu)$ labeled L and R, respectively, then $pert(\mu)$ can be augmented in such a way that μ is of Type M.

Proof. Suppose that v is not the source of any R-labeled face (resp., of any L-labeled face). For each $i=1,\ldots,k$, we select the flip of \mathcal{E}_i such that the last successor of v in $\operatorname{pert}^*(\mu_i)$ lies on the left path of $\operatorname{pert}^*(\mu_i)$ (resp., on the right path of $\operatorname{pert}^*(\mu_i)$). For each $i=1,\ldots,k-1$, we add a left edge (resp., right edge) directed from the last successor of v in $\operatorname{pert}^*(\mu_{i+1})$ (resp., in $\operatorname{pert}^*(\mu_i)$) to the first successor of v in $\operatorname{pert}^*(\mu_i)$ (resp., in $\operatorname{pert}^*(\mu_{i+1})$). By Case (i) of Observation 2, the introduced edges do not affect planarity. Finally, by the fact that all the nodes corresponding to the virtual edges exiting v are of Type M and by the choice of the left and right edges, the obtained augmentation $\operatorname{pert}^*(\mu)$ of $\operatorname{pert}(\mu)$ is such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is v-monotonic.

Claim 2 If exactly one child μ_b of μ corresponding to a virtual edge exiting v is of Type B, then $pert(\mu)$ can not be augmented in such a way that μ is of Type M while it can be augmented in such a way that μ is of Type B if and only if all the faces of $skel(\mu)$ of which v is the source vertex labeled R (resp., labeled L) precede e_b (resp., follow e_b) clockwise around v.

Proof. Clearly, in this case node μ cannot be of Type M. First, observe that if vertex v is the source of an L-labeled face f_L of skel(μ) that precedes the virtual edge e_b clockwise around v, then node μ cannot be of Type B either. In fact, in any augmentation pert*(μ) of pert(μ) the subgraph of pert*(μ) induced by the successors of v in pert(μ) would contain the left edge traversing f_L that points away from the apex of v in pert*(v). Analogously, observe that if vertex v is the source of an v-labeled face v-labeled fac

clockwise around v, then node μ cannot be of Type B. Therefore, it remains to consider the case in which all the faces of skel(μ) of which v is the source vertex that are labeled R (resp., labeled L) precede e_B (resp., follow e_b) clockwise around v. We can then augment $\operatorname{pert}(\mu)$ in such a way that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is strictly v-bitonic as follows. For $i=1,\ldots,b-1$, we select the flip of \mathcal{E}_i such that the last successor of v in $\operatorname{pert}^*(\mu_i)$ lies on the right path of $\operatorname{pert}^*(\mu_i)$; also, for $i=b+1,\ldots,k$, we select the flip of \mathcal{E}_i such that the last successor of v in $\operatorname{pert}^*(\mu_i)$ lies on the left path of $\operatorname{pert}^*(\mu_i)$. For $i=1,\ldots,b-1$, we add a right edge directed from the last successor of v in $\operatorname{pert}^*(\mu_{i+1})$; also, for $i=b,\ldots,k-1$, we add a left edge directed from the last successor of v in $\operatorname{pert}^*(\mu_{i+1})$; also, for $i=b,\ldots,k-1$, we add a left edge directed from the last successor of v in $\operatorname{pert}^*(\mu_{i+1})$; to a first successor of v in $\operatorname{pert}^*(\mu_i)$. Clearly, the obtained augmentation of $\operatorname{pert}(\mu)$ is such that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is strictly v-bitonic and, by Observation 1, $\operatorname{pert}^*(\mu)$ is also planar.

Claim 3 If no child of μ corresponding to a virtual edge exiting v is of Type B and v is the source vertex of at least one R-labeled face and of one L-labeled face, then $pert(\mu)$ can not be augmented in such a way that μ is of Type M while it can be augmented in such a way that μ is of Type B if and only if all the faces of $skel(\mu)$ of which v is the source vertex labeled R precede the faces labeled L clockwise around v.

Proof. First, observe that if vertex v is the source of two faces f_L and f_R of $\mathrm{skel}(\mu)$ labeled L and R, respectively, such that f_L precedes f_R clockwise around v, then there exists no augmentation $\mathrm{pert}^*(\mu)$ of $\mathrm{pert}(\mu)$ such that $\langle \mathrm{pert}(\mu), \mathrm{pert}^*(\mu) \rangle$ is v-bitonic. In fact, in any augmentation $\mathrm{pert}^*(\mu)$ of $\mathrm{pert}(\mu)$ the subgraph of $\mathrm{pert}^*(\mu)$ induced by the successors of v in $\mathrm{pert}(\mu)$ would contain the left edge traversing f_L followed by the right edge traversing f_R ; clearly, this precludes a bitonic path. Therefore, it only remains to consider the case in which all the faces labeled R precede all the faces labeled L in the clockwise order around v. Node μ cannot be of Type M as the existence of an apex vertex of v is implied by the presence of both a left and a right edge.

We can then augment $\operatorname{pert}(\mu)$ in such a way that $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is strictly v-bitonic as follows. Let e_c be any virtual edge exiting v such that all the R-labeled faces precede e_c clockwise around v and such that all the L-labeled faces follow e_c clockwise around v. We apply the same strategy as in the proof of Claim 2 to select a flip for each embedding \mathcal{E}_i and to introduce left and right edges to obtain $\operatorname{pert}^*(\mu)$, where e_c has the role of e_b . Therefore, $\operatorname{pert}^*(\mu)$ is planar and $\langle \operatorname{pert}(\mu), \operatorname{pert}^*(\mu) \rangle$ is strictly v-bitonic. Also, the apex of v is the last successor of v in $\operatorname{pert}^*(\mu_c)$.

Appendix B.5 Omitted Proofs of the Open Problems Section

Lemma 5. A graph with n vertices that admits a planar, upward-planar, or upward-rightward-planar L-drawing has at most 4n-6, 3n-6, or 2n-3 edges and these bounds are tight.

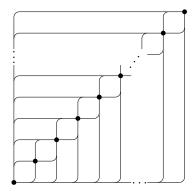


Fig. 7: An L-planar graph with n vertices and 4n-6 edges.

Proof. In the following let n denote the number of vertices of the considered graph.

planar: Consider for each port of a vertex the furthest bend. Recall that the bend on any edge is the furthest bend of at least one of its end vertices. On the other hand each vertex has at most four furthest bends. Thus there can be at most 4n edges. Consider now the outer face. The topmost (bottommost, rightmost, leftmost) vertex doesn't have a furthest bend at its top (bottom, right, left) port. Moreover in a maximal L-planar drawing there are at least two edges e_1 and e_2 on the outer face such that its bend is a furthest bend of both end vertices: Consider the bottommost vertex v. If v is neither the leftmost nor the rightmost vertex, let u_1 and u_2 be the leftmost and rightmost vertex such that there is an edge $e_1 = (u_1, v)$ and $e_2 = (u_2, v)$, respectively. If v is the leftmost (rightmost) vertex, let u be the rightmost (leftmost) vertex such that there is an edge $e_1 = (u, v)$ and let w be the topmost vertex such that there is an edge $e_2 = (v, w)$. This yields the 4n-6 bound. Finally, Fig. 7 indicates a graph with 4n-6 edges.

upward-planar: By Corollary 1, every maximal undirected graph oriented according to a bitonic st-ordering is a directed graph with 3n-6 edges admitting an upward-planar L-drawing. Since upward-planar graphs must be acyclic, they cannot contain 2-cycles. Thus, there are at most 3n-6 edges. upward-rightward-planar: Each vertex has at most two furthest bends. The bottommost vertex has no furthest bend to the left, the rightmost vertex has no furthest bend to the top and in a maximal upward-rightward planar L-drawing there is at least one bend that is furthest for both end vertices. Hence, there are at most 2n-3 edges. Omitting all but the upward-rightward edges in Fig. 7 yields a graph with 2n-3 edges.