Simple realizability of complete abstract topological graphs simplified

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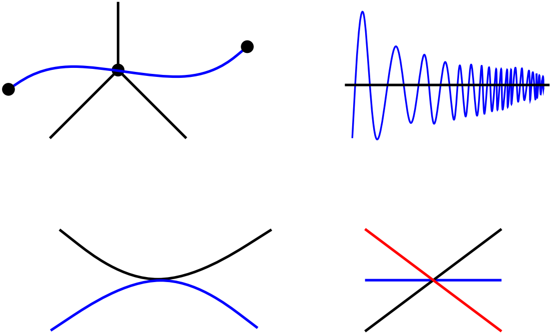
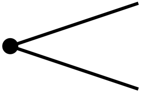


simple: any two edges have at most one common point



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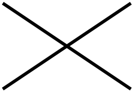

complete: $E = \binom{V}{2}$

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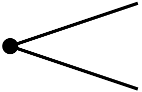

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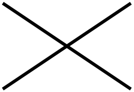

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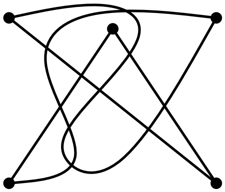
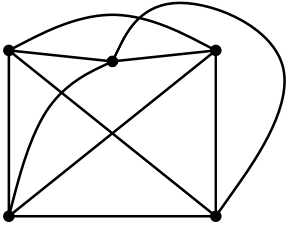


topological graph

drawing



simple complete topological graph

simple drawing of K_5

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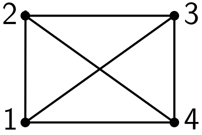
simple realization of A:



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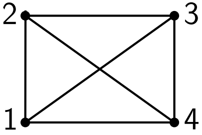

$A = (K_5, \emptyset)$ is not simply realizable

Simple realizability

instance: AT-graph A
question: is A simply realizable?

Previously known:

Theorem: (Kratochvíl and Matoušek, 1989)
Simple realizability of AT-graphs is NP-complete.

Theorem: (K., 2011)
Simple realizability of complete AT-graphs is in P.

“Unfortunately, the algorithm is of rather theoretical nature.”

— P. Mutzel, 2008

“The proof in [..] only gives a highly complex testing procedure, but no description in terms of forbidden minors or crossing configurations.”

— M. Chimani, 2011

Main result

def.: $(H, \mathcal{Y})$ is an **AT-subgraph** of $(G, \mathcal{X})$ if H is a subgraph of G and $\mathcal{Y} = \mathcal{X} \cap \binom{E(H)}{2}$

Theorem 1: Every complete AT-graph that is not simply realizable has an AT-subgraph on at most six vertices that is not simply realizable.

Theorem 2: There is a complete AT-graph A with six vertices such that all its induced AT-subgraphs with five vertices are simply realizable, but A itself is not.

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- Ábrego, Aichholzer, Fernández-Merchant, Hackl, Pammer, Pilz, Ramos, Salazar and Vogtenhuber (2015) generated a list of simple drawings of K_n for $n \leq 9$

Proof of Theorem 1 (sketch)

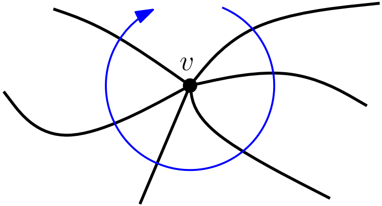
Let $A = (K_n, \mathcal{X})$ be a given complete AT-graph with vertex set $[n] = \{1, 2, \ldots, n\}$.

Main idea: take the previous "highly complex algorithm" and find a small obstruction every time it rejects the input.

three main steps:

1) computing the rotation system
2) computing the homotopy classes of edges with respect to a star
3) computing the minimum crossing numbers of pairs of edges

Step 1: computing the rotation system



AT-graph $\leftrightarrow$ rotation system

1a) rotation systems of 5-tuples (up to orientation)

1b) orienting 5-tuples (here 6-tuples needed)

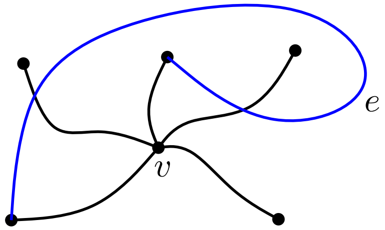
1c) rotations of vertices

1d) rotations of crossings

Ábrego et al. (pers. com.) verified that an abstract rotation system (ARS) of K_9 is realizable if and only if the ARS of every 5-tuple is realizable, and conjectured that this is true for any K_n.

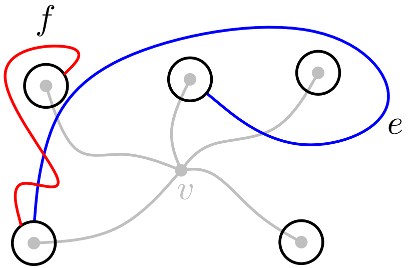
Step 2: computing the homotopy classes of edges

- Fix a vertex v and a topological spanning star $S(v)$, drawn with the rotation computed in Step 1
- for every edge e not in $S(v)$, compute the order of crossings of e with the edges of $S(v)$.



Step 2: computing the homotopy classes of edges

- Fix a vertex v and a topological spanning star $S(v)$, drawn with the rotation computed in Step 1
- for every edge e not in $S(v)$, compute the order of crossings of e with the edges of $S(v)$.
- drill small holes around the vertices, fix the endpoints of the edges on the boundaries of the holes



Step 3: computing the minimum crossing numbers

$\mathrm{cr}(e,f)$ = minimum possible number of crossings of two curves from the homotopy classes of e and f

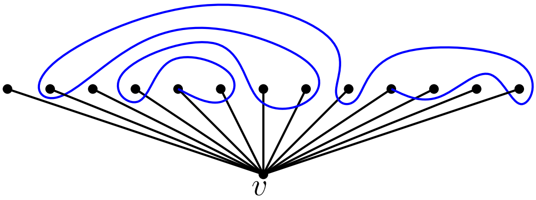
$\mathrm{cr}(e)$ = minimum possible number of self-crossings of a curve from the homotopy class of e

Fact: (follows e.g. from Hass–Scott, 1985) It is possible to pick a representative from the homotopy class of every edge so that in the resulting drawing, all the crossing numbers $\mathrm{cr}(e,f)$ and $\mathrm{cr}(e)$ are realized simultaneously.

We need to verify that

- $\mathrm{cr}(e) = 0$,
- $\mathrm{cr}(e,f) \leq 1$, and
- $\mathrm{cr}(e,f) = 1 \Leftrightarrow \{e,f\} \in \mathcal{X}$.

3a) characterization of the homotopy classes



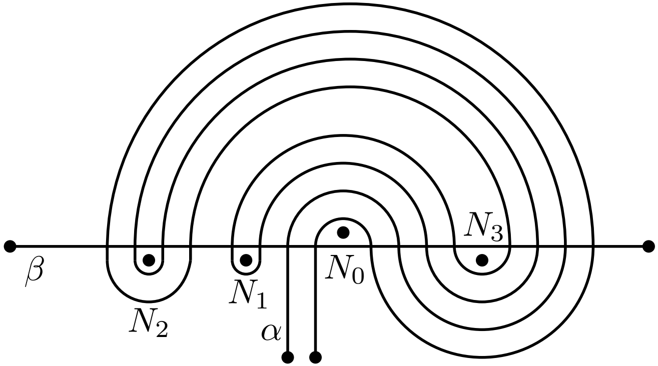
3b) parity of the crossing numbers (4- and 5-tuples)

3c) crossings of adjacent edges (5-tuples)

3d) multiple crossings of independent edges (5-tuples)

Picture hanging without crossings

remove one nail:



similar concept with crossings:
Demaine et al., Picture-hanging puzzles, 2014.

Independent $\mathbb{Z}_2$-realizability

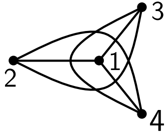
- T is an **independent $\mathbb{Z}_2$-realization** of $(G, \mathcal{X})$ if
  - T is a drawing of G and
  - $\mathcal{X}$ is the set of pairs of independent edges that cross an odd number of times in T
- AT-graph A is **independently $\mathbb{Z}_2$-realizable** if it has an independent $\mathbb{Z}_2$-realization

Obs.: simple realization $\Rightarrow$ independent $\mathbb{Z}_2$-realization

Example:

$A = (K_4, \{\{\{1,3\},\{2,4\}\},\{\{1,2\},\{3,4\}\},\{\{1,4\},\{2,3\}\}\})$

independent $\mathbb{Z}_2$-realization of A:



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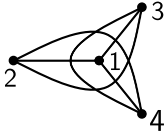

$A = (K_5, \emptyset)$ is not independently $\mathbb{Z}_2$-realizable (Hanani–Tutte)

def.: Call an AT-graph $(G, \mathcal{X})$ even (or an even *G*) if $|\mathcal{X}|$ is even, and odd (or an odd *G*) if $|\mathcal{X}|$ is odd.

Theorem 3: Every complete AT-graph that is not independently $\mathbb{Z}_2$-realizable has an AT-subgraph on at most six vertices that is not independently $\mathbb{Z}_2$-realizable.

More precisely, a complete AT-graph is independently $\mathbb{Z}_2$-realizable if and only if it contains no even K_5 and no odd $2K_3$ as an AT-subgraph.