Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems
Institute for Informatics

Stick Graphs with Length Constraints

Steven Chaplick, Philipp Kindermann, Andre Löffler, Florian Thiele, Alexander Wolff, Alexander Zaft, and **Johannes Zink**

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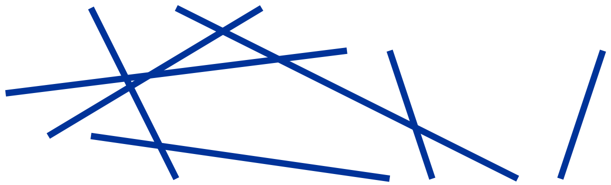

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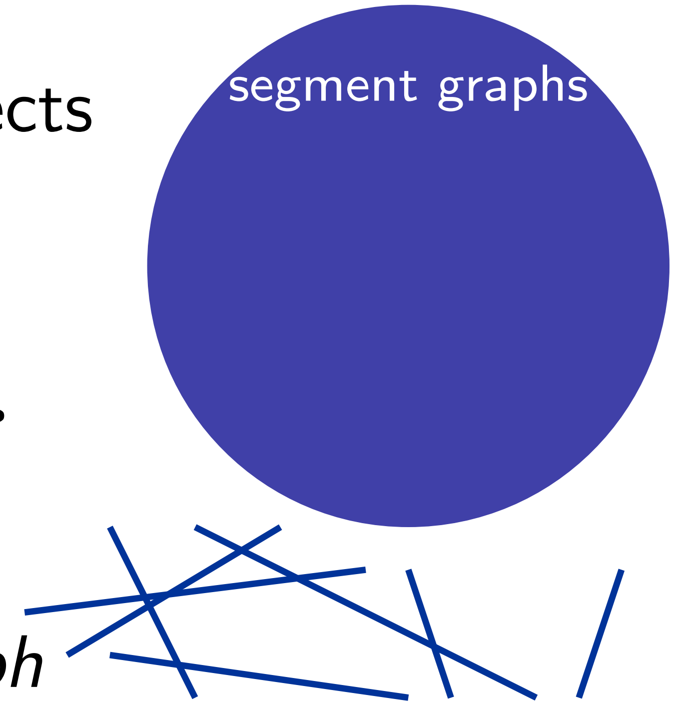

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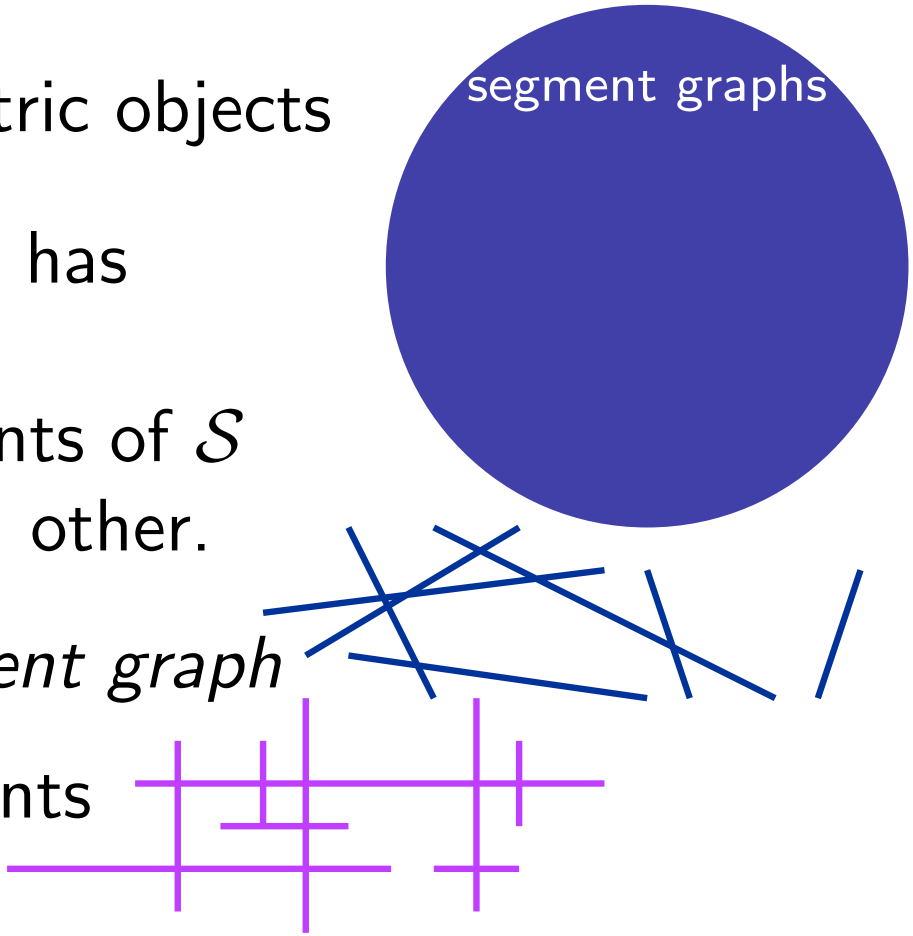

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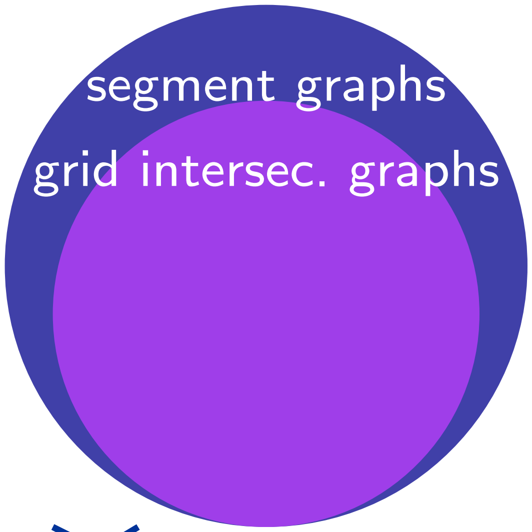

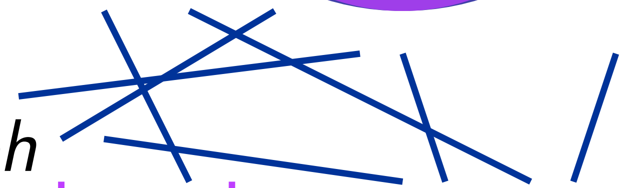

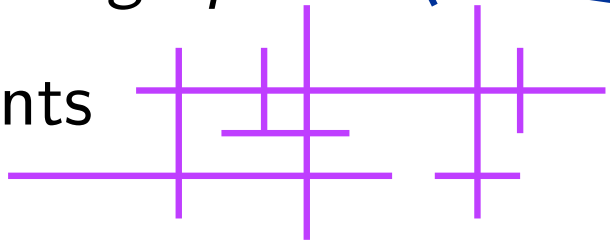

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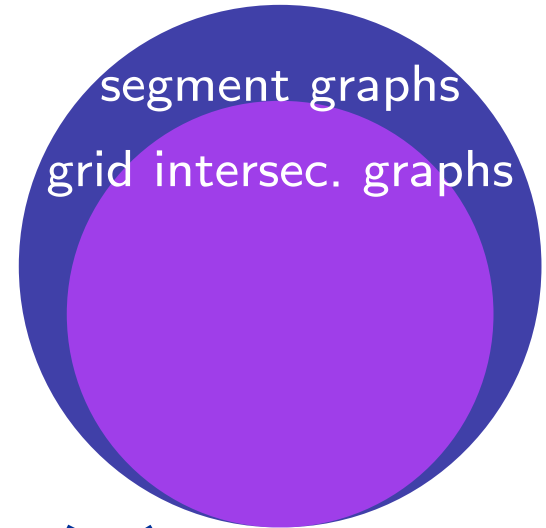

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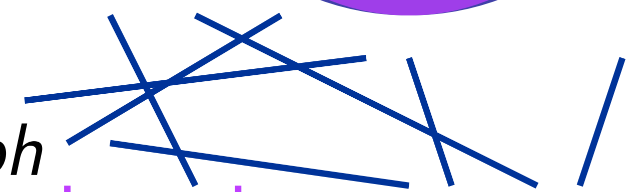

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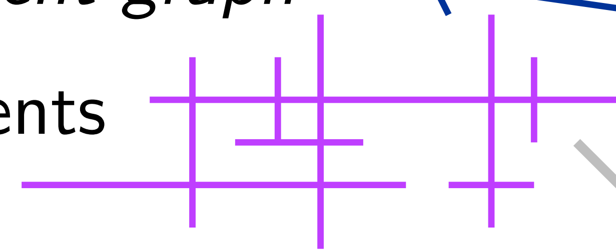

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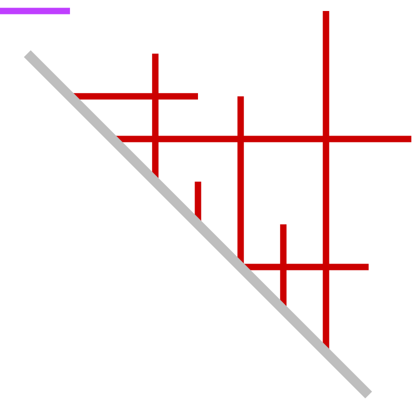

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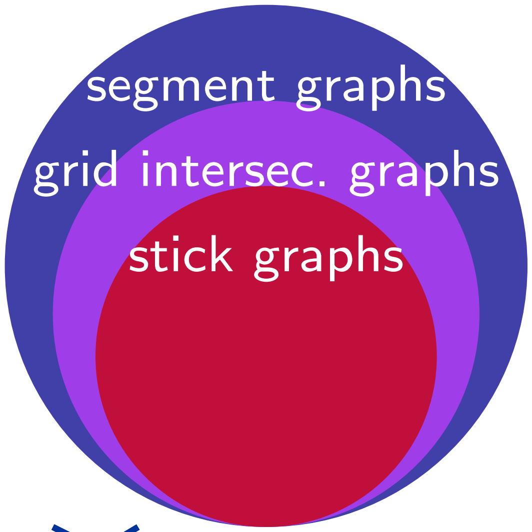

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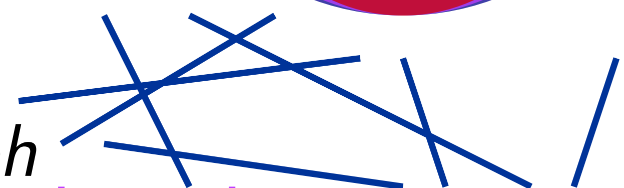

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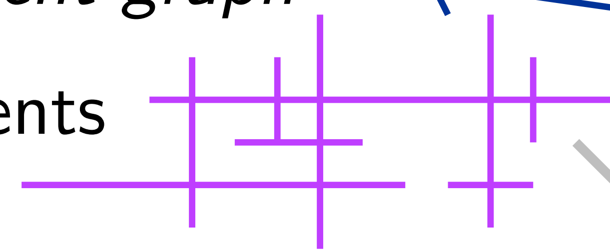

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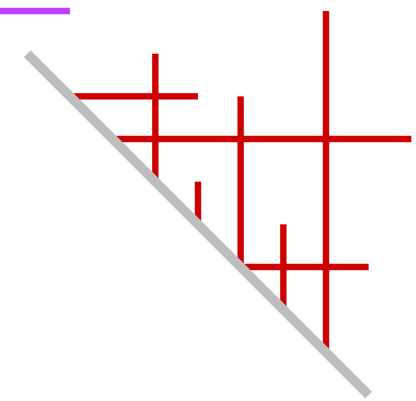

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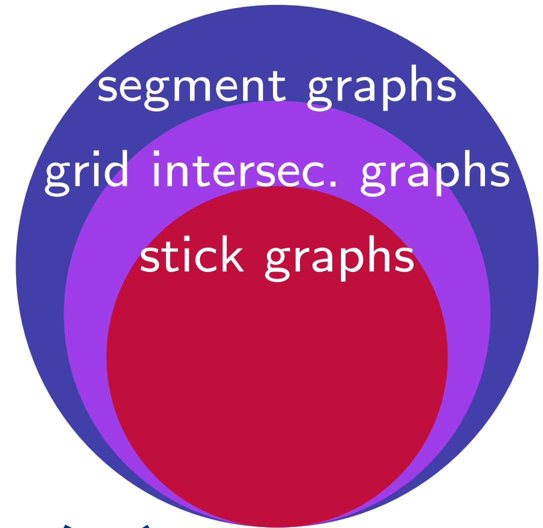

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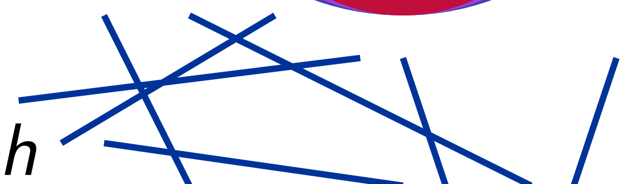

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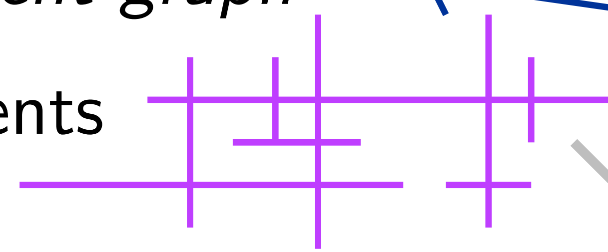

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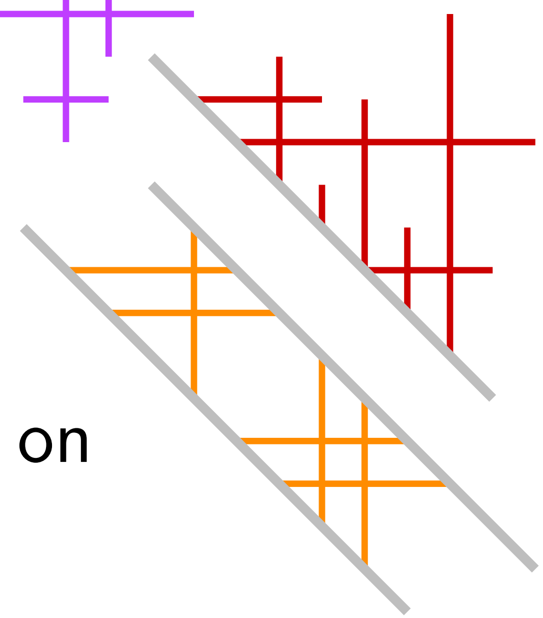

- $\mathcal{S}$: horizontal & vertical segments grounded on two parallel lines

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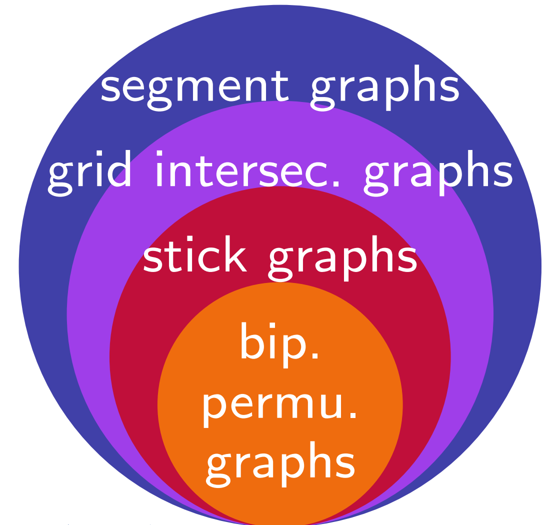

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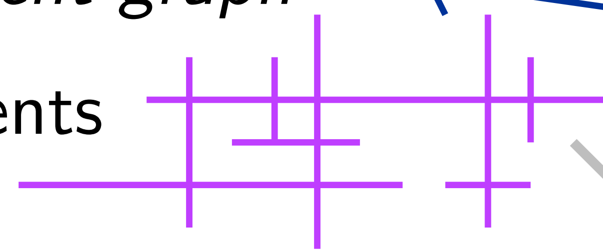
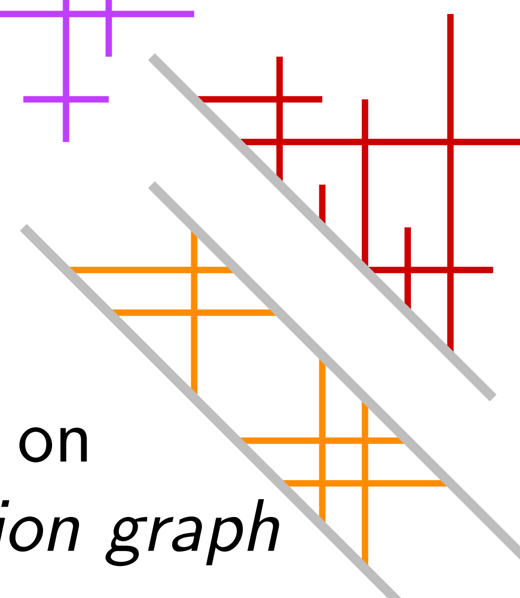


- $\mathcal{S}$: horizontal & vertical segments grounded on a line of slope -1 $\Rightarrow$ G: *stick graph*
- $\mathcal{S}$: horizontal & vertical segments grounded on two parallel lines $\Rightarrow$ G: *bipartite permutation graph*



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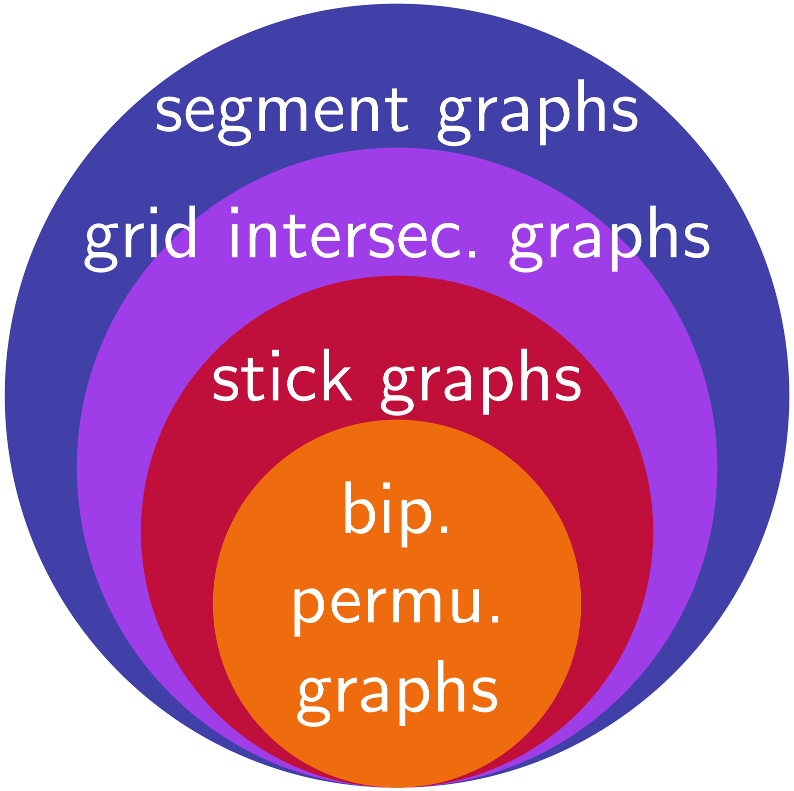
How hard for each class?



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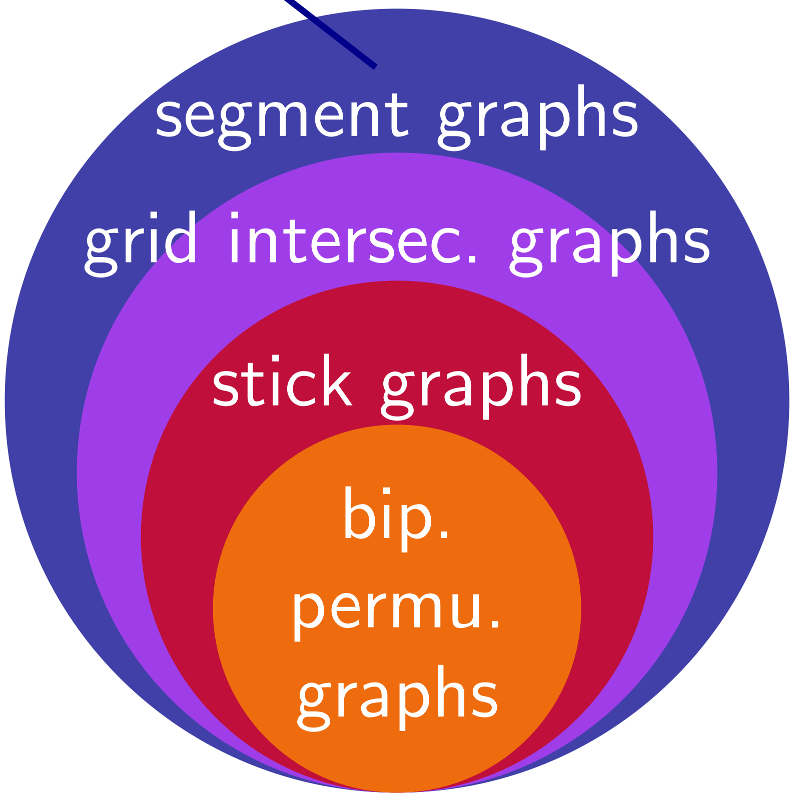

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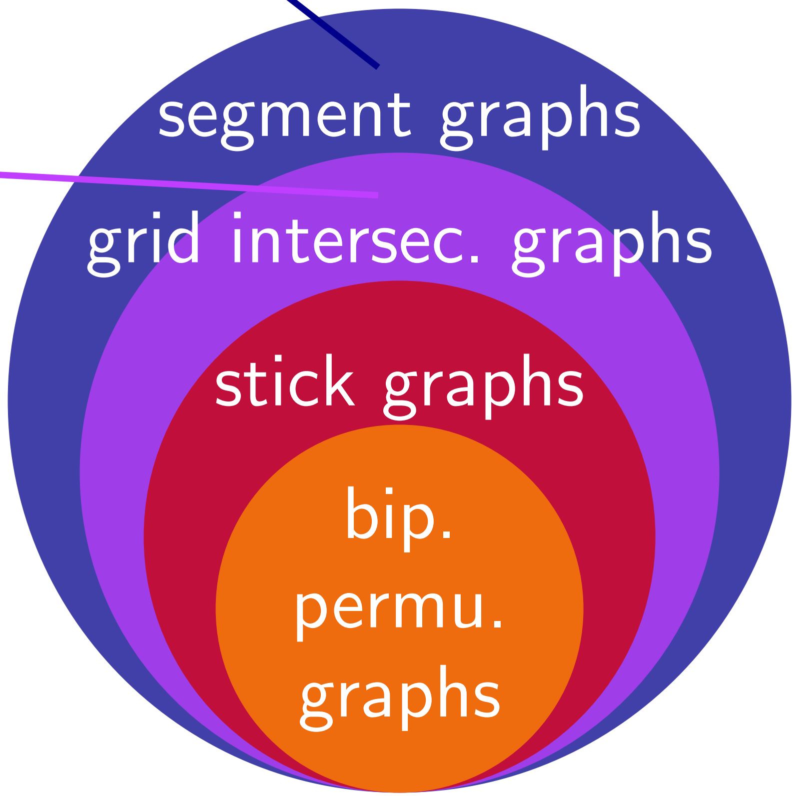

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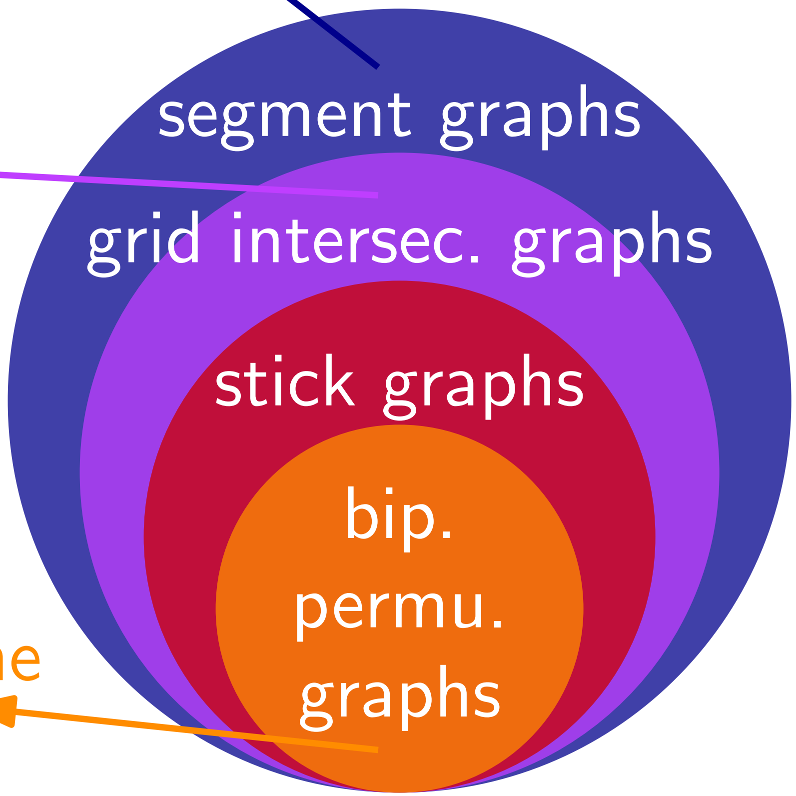

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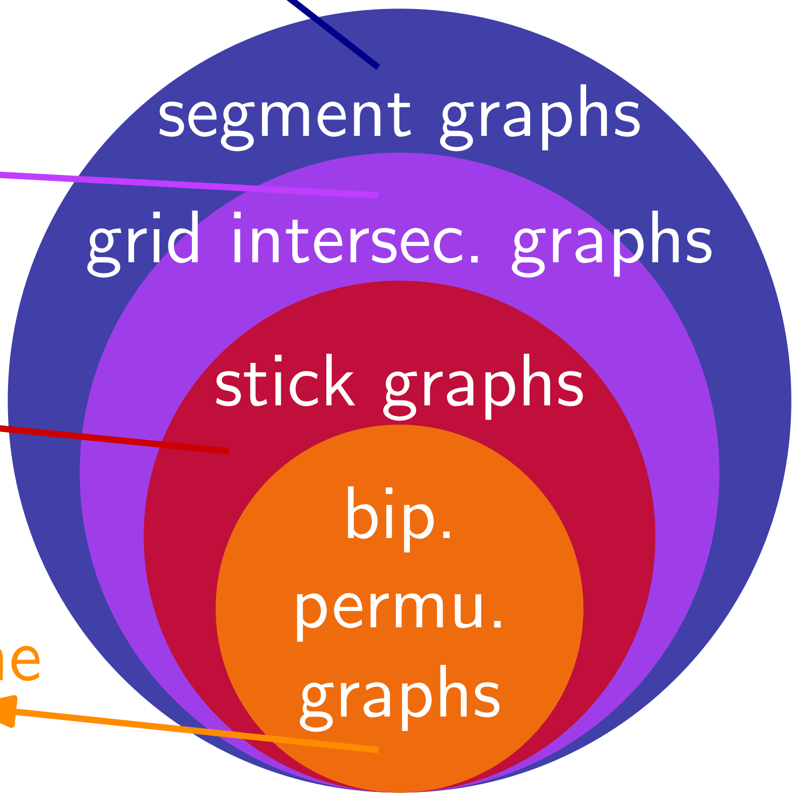

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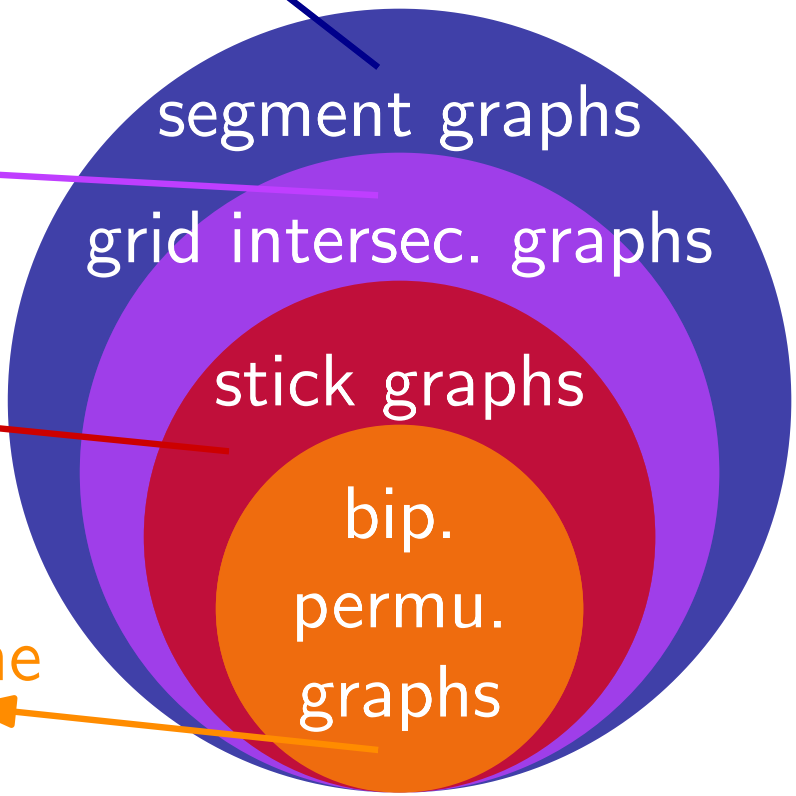

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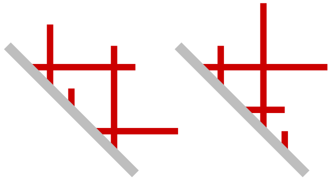

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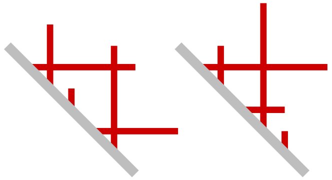

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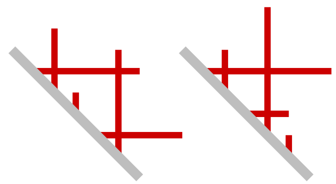

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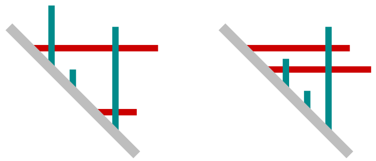

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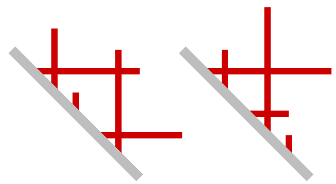

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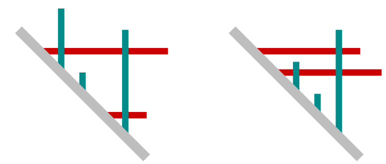

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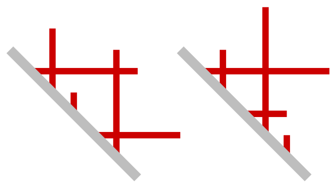

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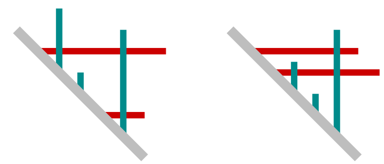

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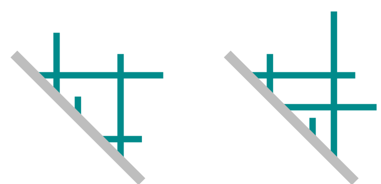

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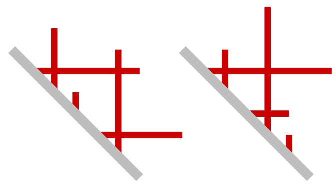

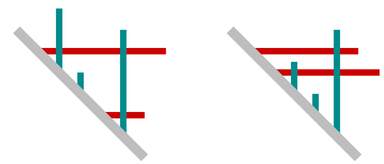

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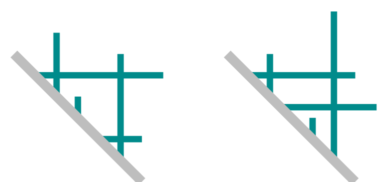

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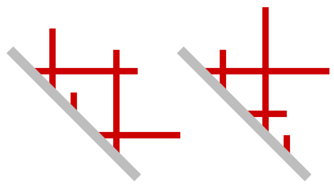

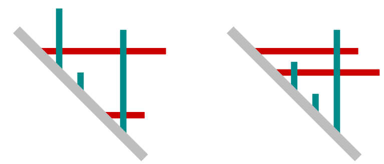

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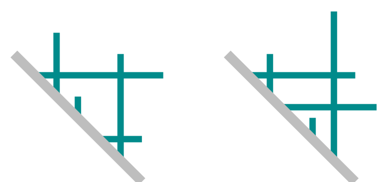

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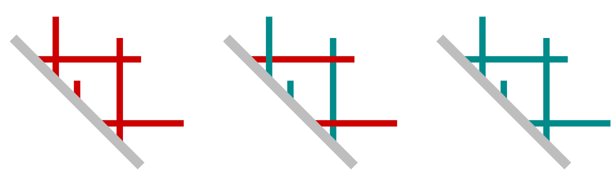

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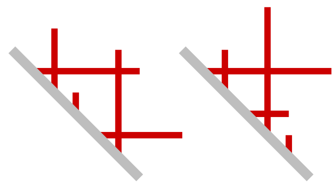

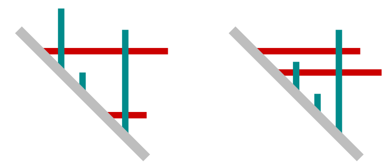

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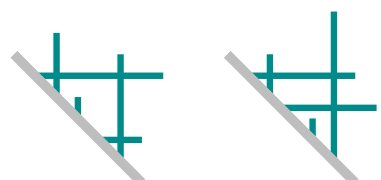

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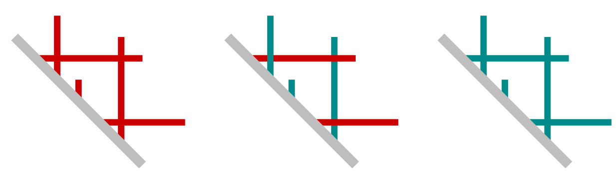

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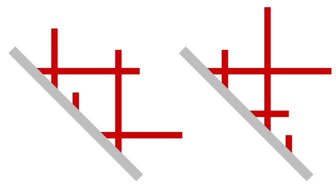

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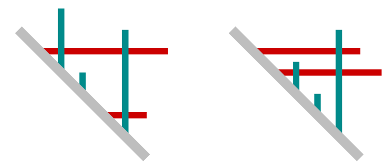

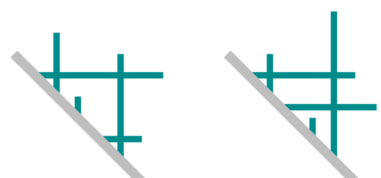

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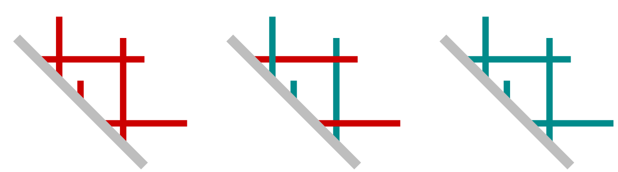
  ... if a permutation of the vertices in A and a permutation of the vertices in B is given?

First investigated by De Luca et al. [GD'18]



- **$\mathbf{STICK_{\star}^{fix}}$:**
  ... if a stick length for each vertex (and possibly permutations of A, or A and B) is given?

new

Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

$\star$	**STICK**$_\star$	**STICK**$^{\textbf{fix}}_\star$
A		
AB		

Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

$\star$	**STICK**$_\star$	**STICK**$_\star^{\mathbf{fix}}$
	?	**?**
A	**?**[1]	**?**
AB	$O(\|A\|\|B\|)$ [De Luca et al. GD'18]	**?**

[1] an $O(|A|^3|B|^3)$ time algorithm proposed by De Luca et al. turned out to be wrong

Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

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for a bipartite graph $G = (A \cup B, E)$

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$\star$	**STICK**$_\star$	**STICK**$_\star^{\text{fix}}$
	?	**?**
A	**?**[1] $O(\lvert A\rvert \lvert B\rvert)$	**?**
AB	$O(\lvert A\rvert \lvert B\rvert)$ [De Luca et al. GD'18]	**?**

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Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

$\star$	$\textbf{STICK}_\star$	$\textbf{STICK}_\star^{\textbf{fix}}$
	?	**?** NP-complete
A	**?**[1] $O(\lvert A\rvert\lvert B\rvert)$	**?**
AB	$O(\lvert A\rvert\lvert B\rvert)$ [De Luca et al. GD'18] $O(\lvert E\rvert)$	**?**

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Complexity of Recognition

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Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

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	?		**?**	NP-complete
A	**?**[1]	$O(\|A\|\|B\|)$	**?**	NP-complete
AB	$O(\|A\|\|B\|)$ [De Luca et al. GD'18]	$O(\|E\|)$	**?**	in general: NP-complete

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Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

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AB	$O(\lvert A\rvert\lvert B\rvert)$ [De Luca et al. GD'18]	$O(\lvert E\rvert)$	**?**	in general: NP-complete w/o isolated vtc.: $O((\lvert A\rvert + \lvert B\rvert)^2)$

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Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

$\star$	**STICK**$_\star$	**STICK**$_\star^{\text{fix}}$
	next **?**	**?** NP-complete
A	**?**[1] $O(\|A\|\|B\|)$	**?** NP-complete
AB	$O(\|A\|\|B\|)$ [De Luca et al. GD'18] $O(\|E\|)$	**?** in general: NP-complete w/o isolated vtc.: $O((\|A\| + \|B\|)^2)$

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Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

our results

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A	**?**[1]	$O(\|A\|\|B\|)$	**?**	NP-complete
AB	$O(\|A\|\|B\|)$ [De Luca et al. GD'18]	$O(\|E\|)$	**?**	in general: NP-complete; w/o isolated vtc.: $O((\|A\| + \|B\|)^2)$

next

afterwards

[1] an $O(|A|^3|B|^3)$ time algorithm proposed by De Luca et al. turned out to be wrong

Algorithm for STICK_A

$\star$	**$\mathrm{STICK}_\star$**	**$\mathrm{STICK}_\star^{\mathrm{fix}}$**
	?	NP-complete
A	$O(\lvert A\rvert\lvert B\rvert)$	NP-complete
AB	$O(\lvert E\rvert)$	in general: NP-complete w/o isolated vtc.: $O((\lvert A\rvert + \lvert B\rvert)^2)$

Algorithm for STICK_A

- Sweep-line along the ordered vertical sticks in A:
  enter event (i) and *exit event* $(i\!\rightarrow)$ for each $a_i \in A$

Algorithm for STICK$_{\text{A}}$

- Sweep-line along the ordered vertical sticks in A:
  enter event (i) and *exit event* $(i\!\rightarrow)$ for each $a_i \in A$
- Let $p \in \{\ i,\ i\!\rightarrow\ \}$,
  $G^i :=$ subgraph induced by $a_1, \ldots, a_i$ and their neighbors,
  $B^p :=$ vertices in B that intersect the sweep-line at event p

Algorithm for $\mathrm{STICK_A}$

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  - contains two types of nodes: leaves and non-leaves

Algorithm for STICK$_A$

- Sweep-line along the ordered vertical sticks in A:
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- Let $p \in \{\ i,\ i{\rightarrow}\ \}$,
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- (Rooted) tree data structure $\mathcal{T}^p$:
  - contains two types of nodes: leaves and non-leaves
  - each leaf corresponds to a vertex in B^p

Algorithm for STICK_A

- Sweep-line along the ordered vertical sticks in A:
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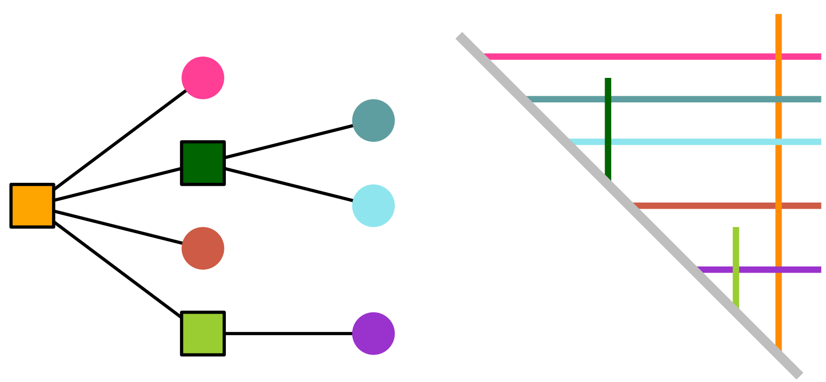

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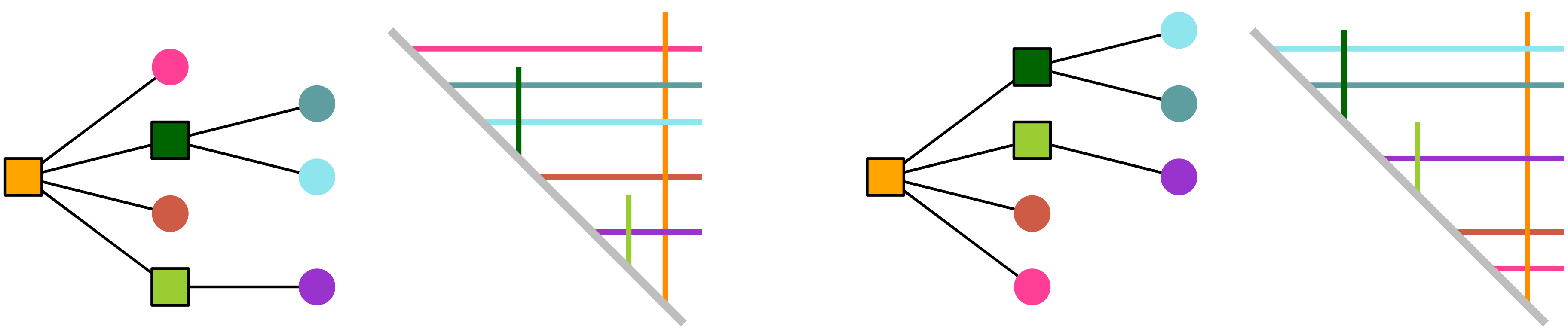
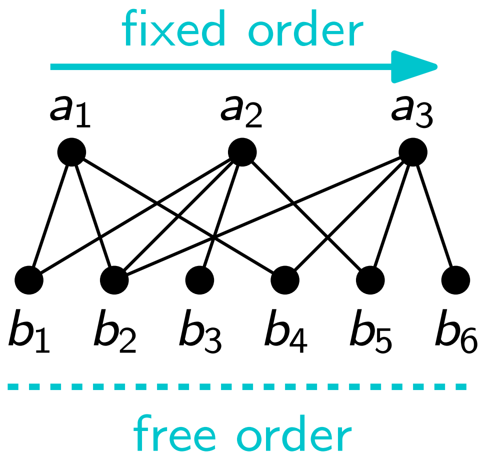


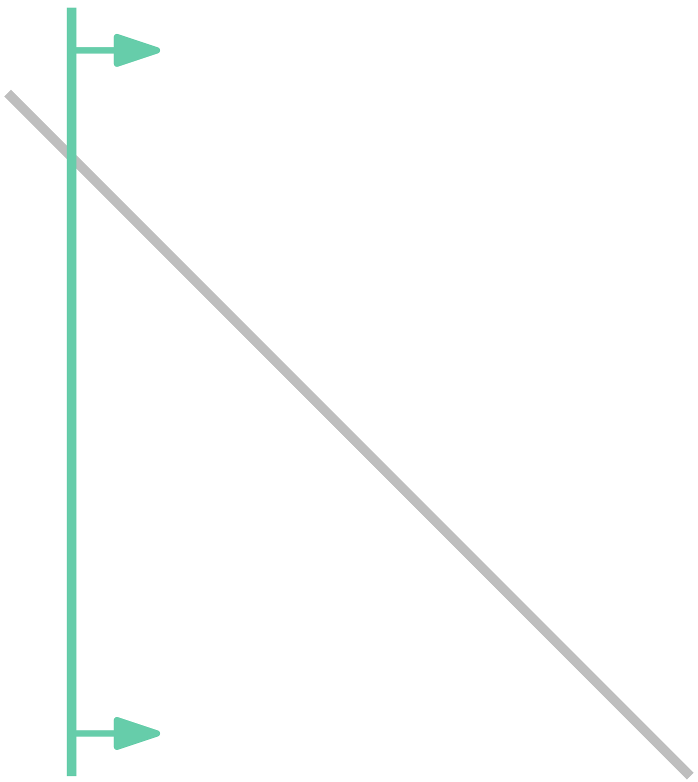
Example for STICK_A

$i = 0$ Event: *Start* $B^0 = \emptyset$

G:



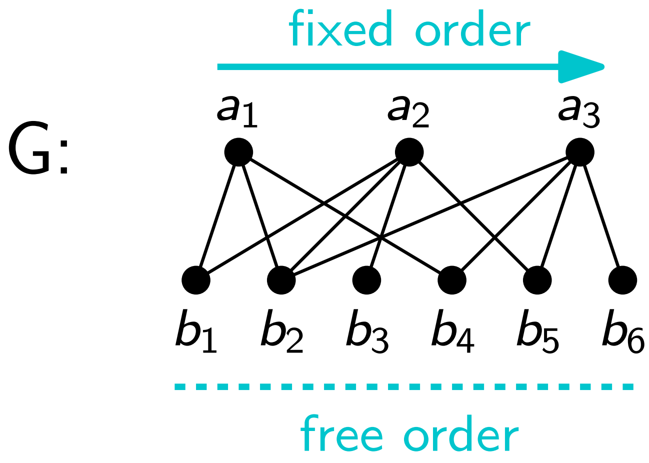
G^0 :

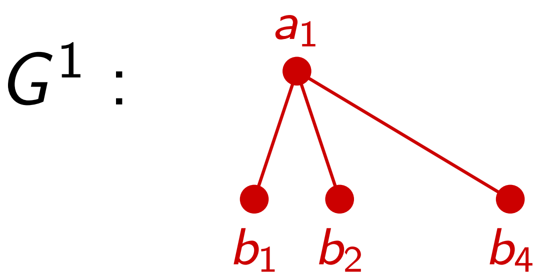


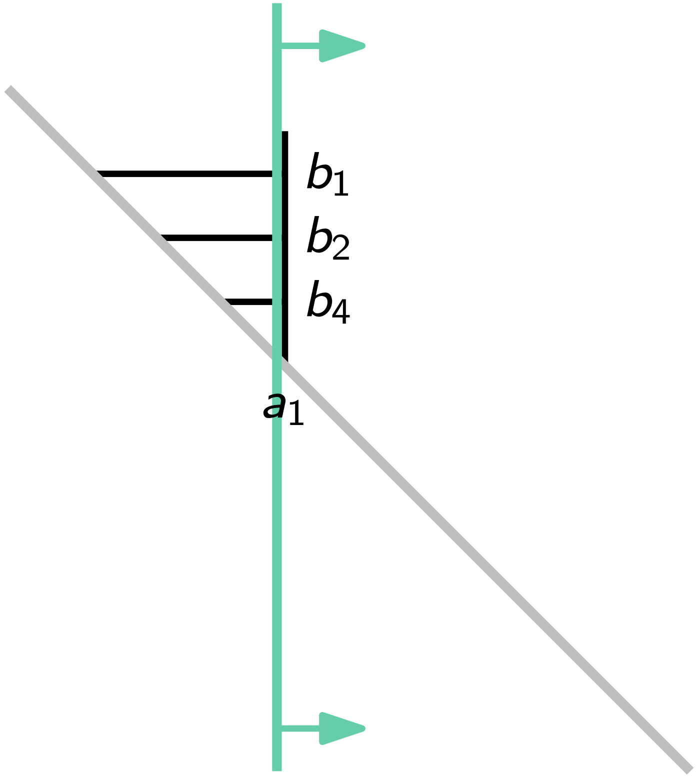
$\mathcal{T}^0$:

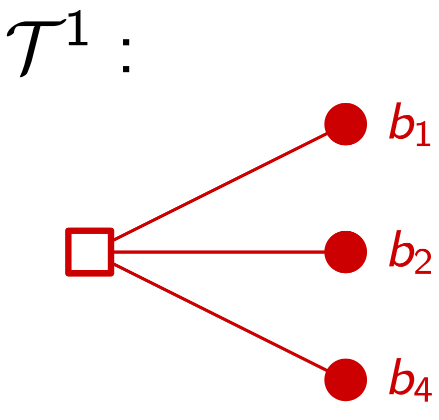
Example for STICK_A

$i = 1$ Event: 1 $B^1 = \{b_1, b_2, b_4\}$









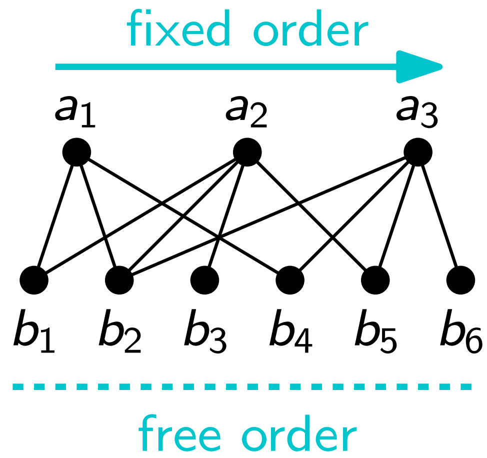
Example for STICK$_A$

$i = 1$

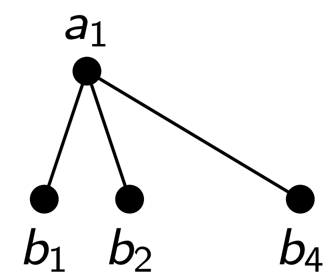
Event: $1_\rightarrow$

$B^{1\rightarrow} = \{b_1, b_2, b_4\}$

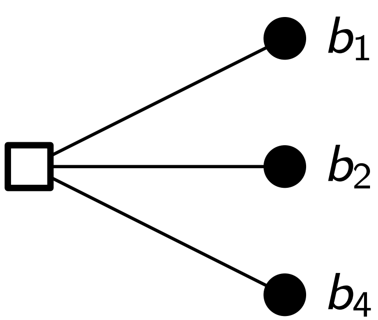
G:

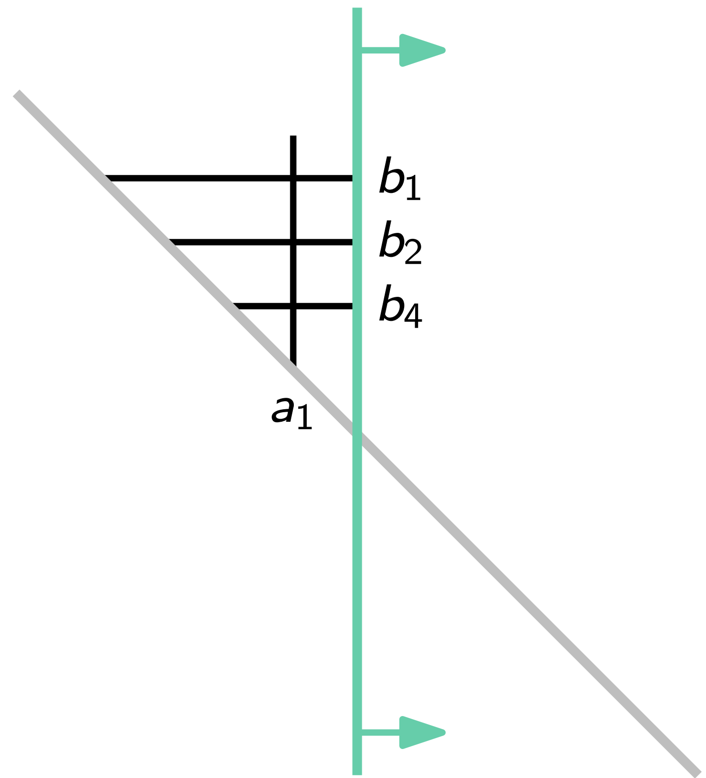


G^1 :



$\mathcal{T}^{1\rightarrow}$:



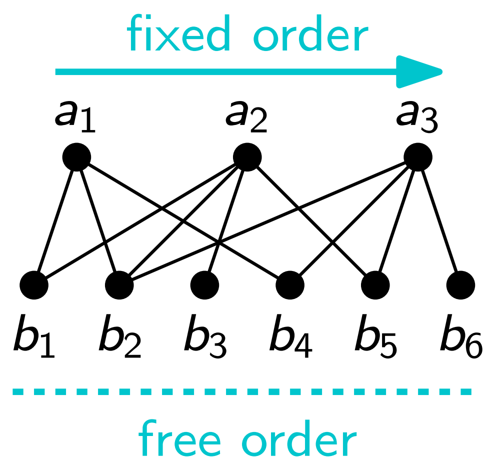


Example for STICK_A

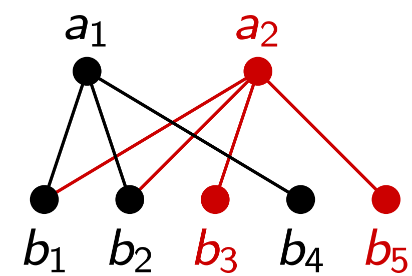
$i = 2$ Event: 2

$B^2 = \{b_1, b_2, b_3, b_4, b_5\}$

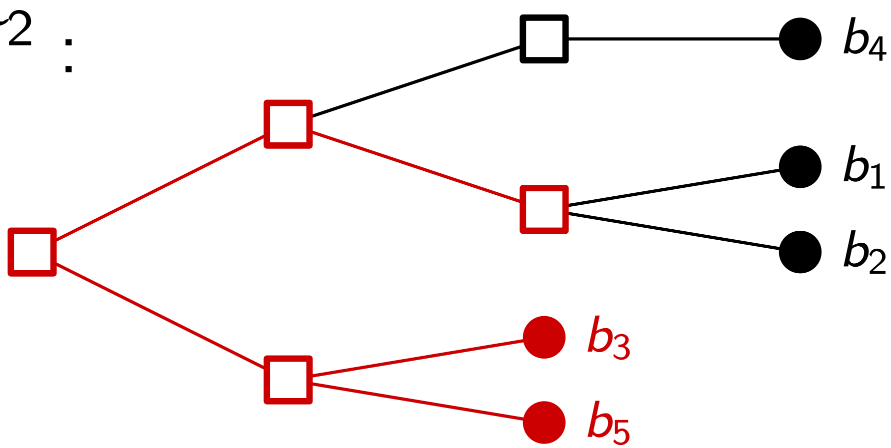
G:

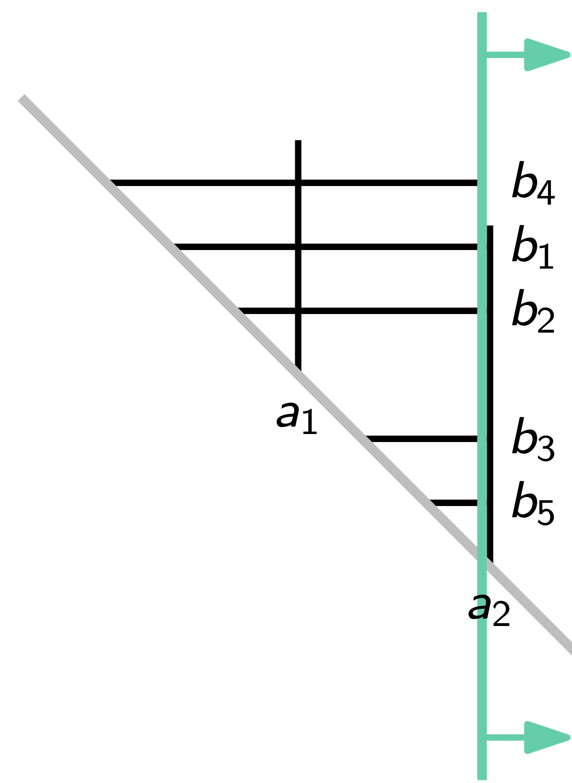


G^2 :



$\mathcal{T}^2$:

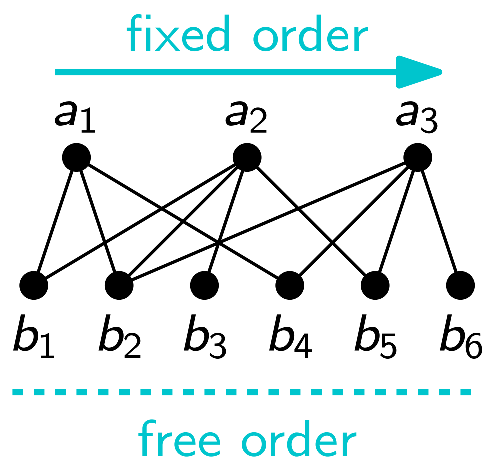




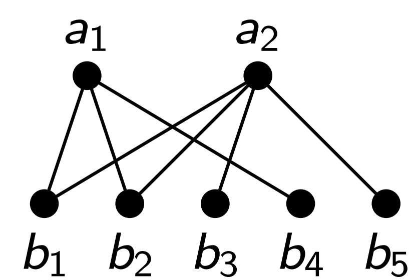
Example for STICK_A

$i = 2$ Event: $2\!\rightarrow$ $B^{2\rightarrow} = \{b_1, b_2, b_3, b_4, b_5\}$

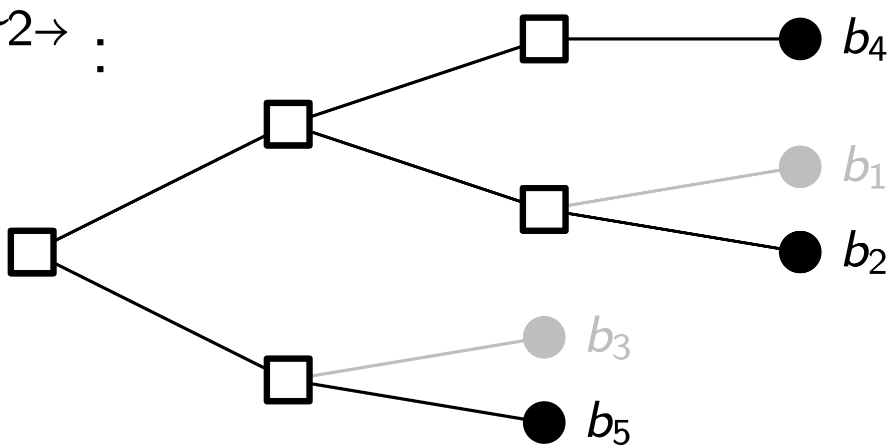
G:

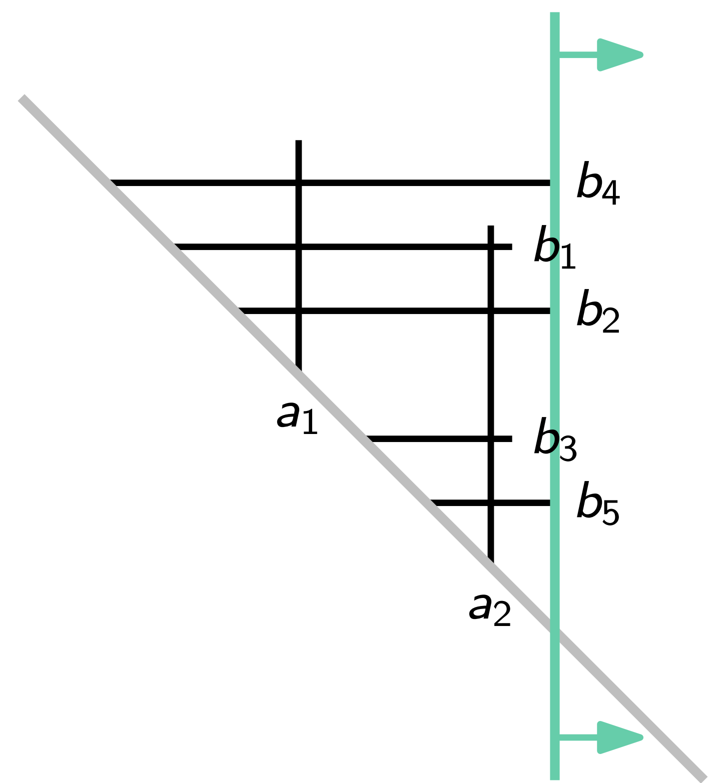


G^2 :



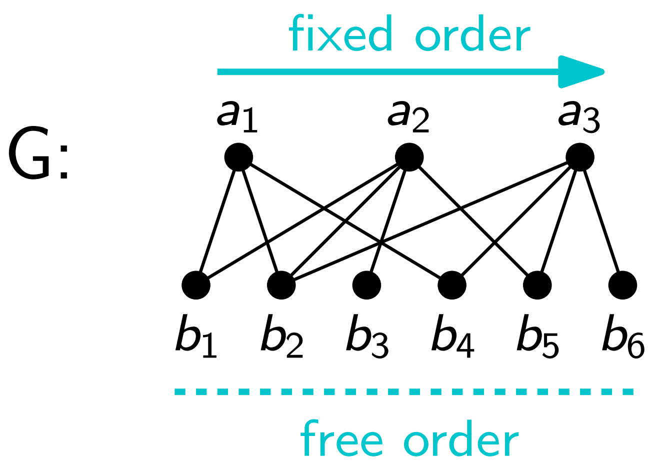
$\mathcal{T}^{2\rightarrow}$:

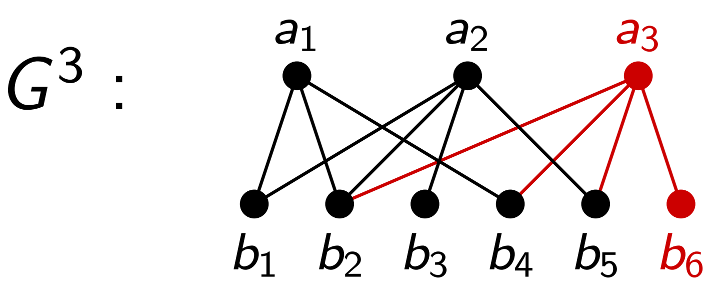


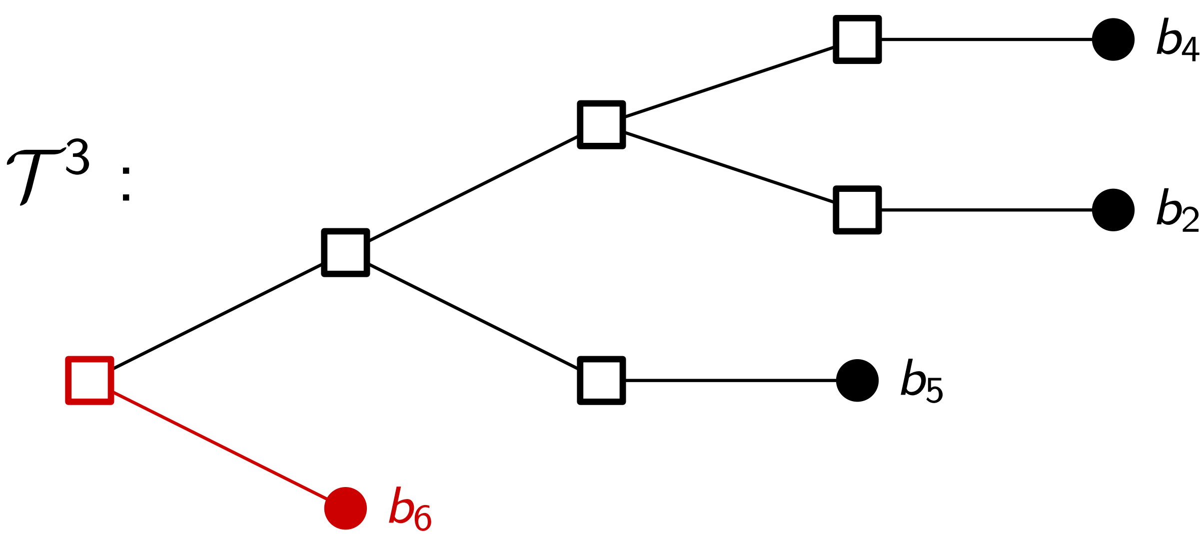


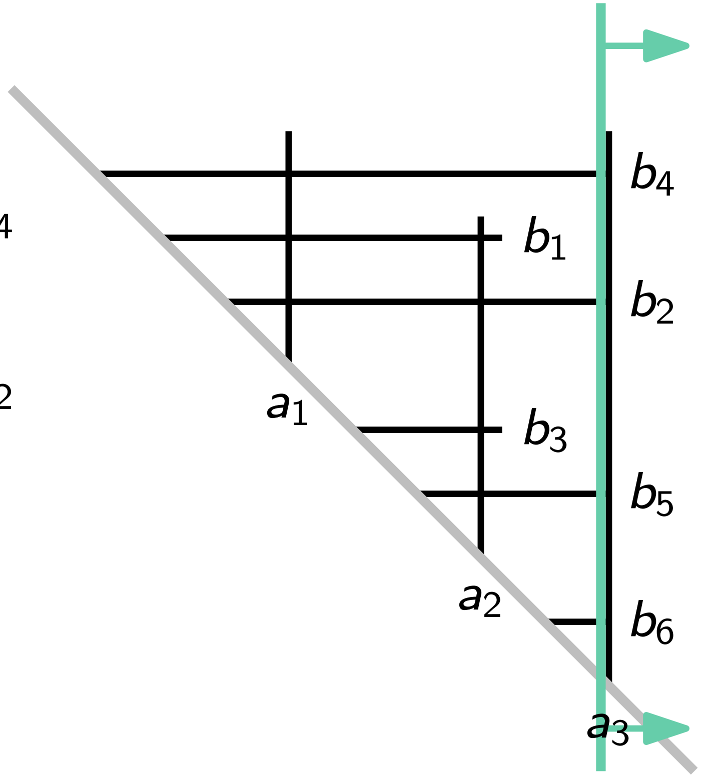
Example for STICK_A

$i = 3$ Event: 3 $B^3 = \{b_2, b_4, b_5, b_6\}$



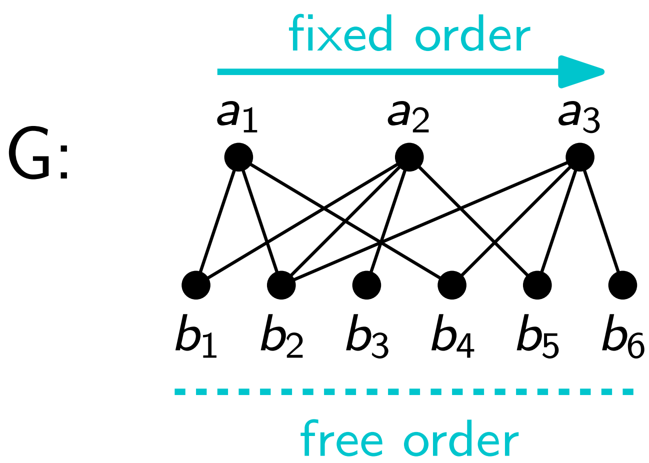


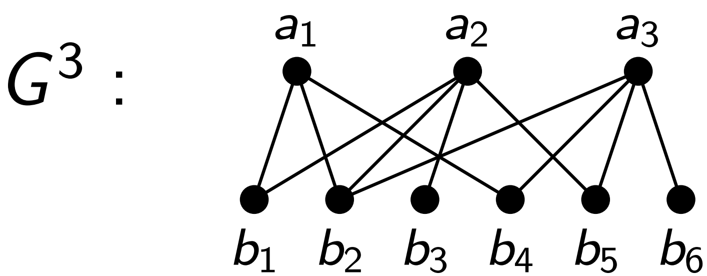


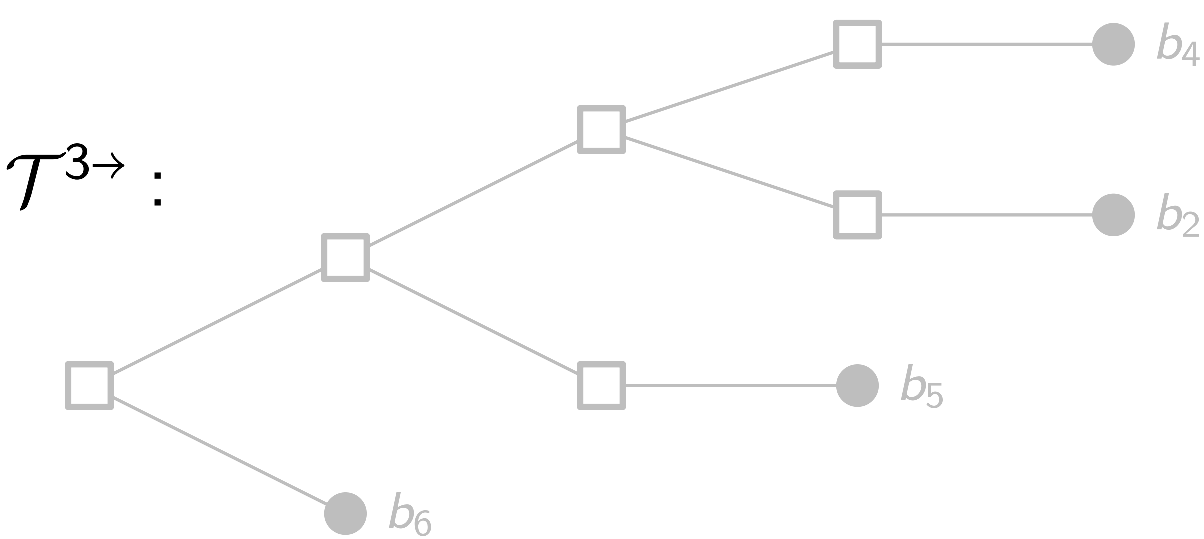


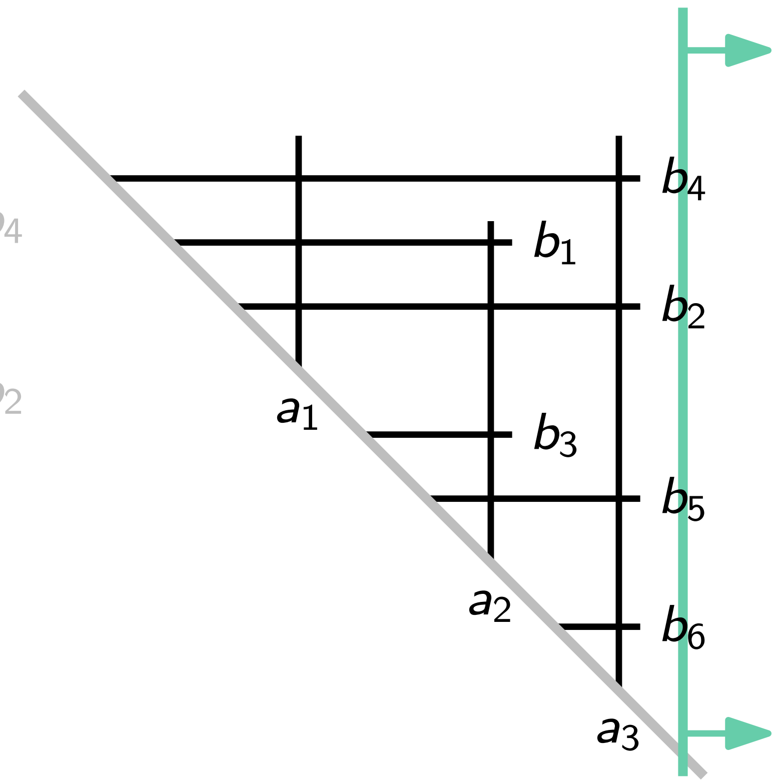
Example for STICK_A

$i = 3$ Event: $3\rightarrow$ $B^{3\rightarrow} = \{b_2, b_4, b_5, b_6\}$



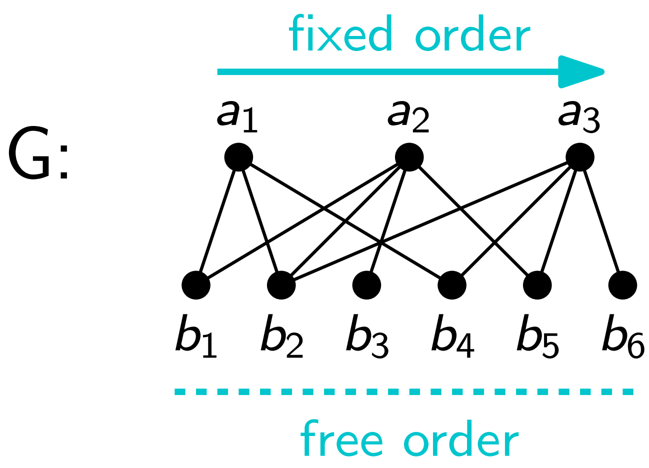


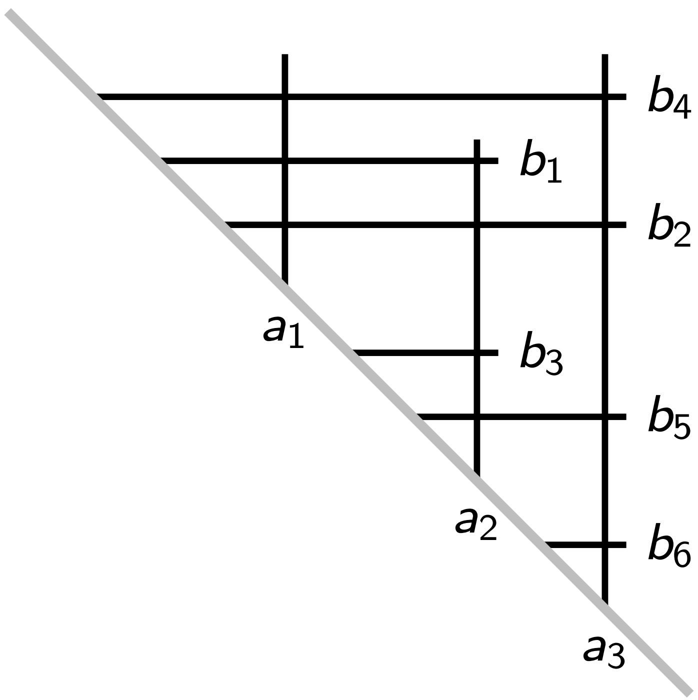




Example for STICK_A

$i = 3$ Event: *End*

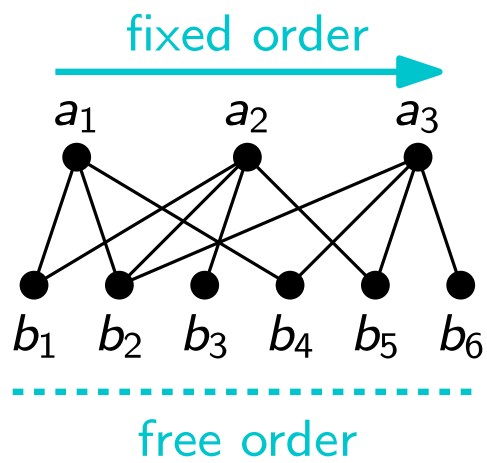




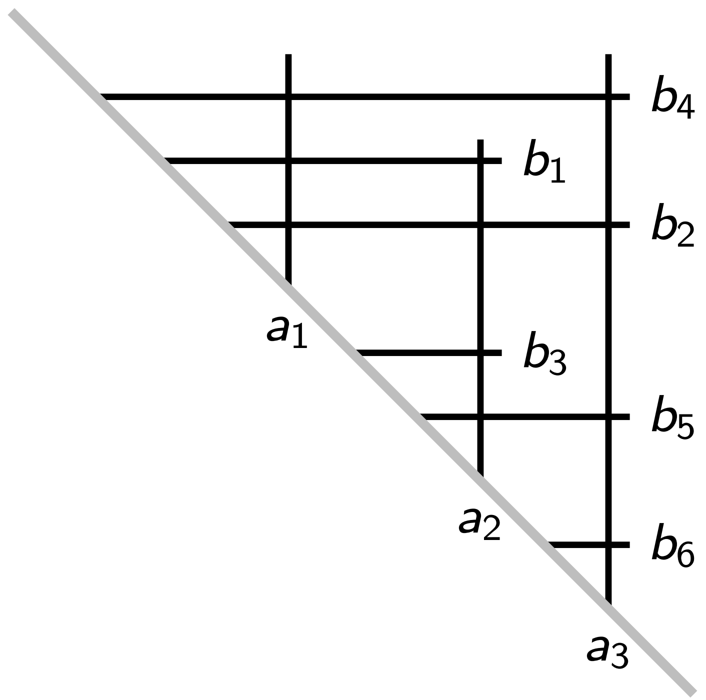
Example for STICK_A

$i = 3$ Event: *End*

G:



Runtime in $O(|A| \cdot |B|)$



$\mathsf{STICK}^{\mathsf{fix}}_{\mathsf{AB}}$ with isolated vertices

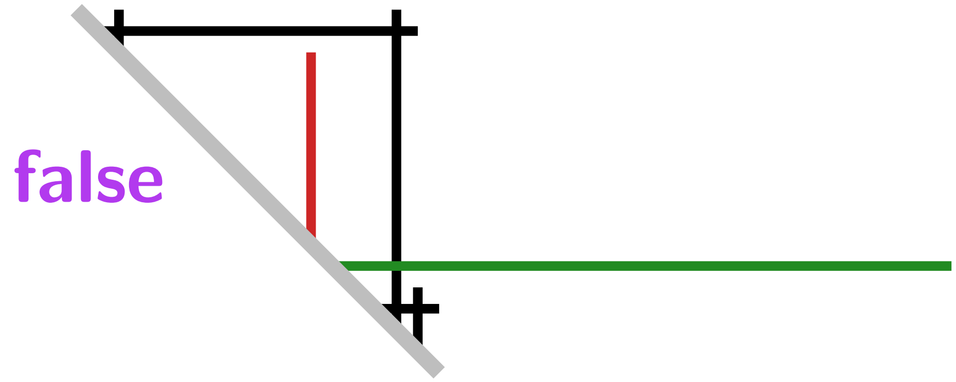
$\star$	$\mathsf{STICK}_\star$	$\mathsf{STICK}^{\mathsf{fix}}_\star$
	?	NP-complete
A	$O(\|A\|\|B\|)$	NP-complete
AB	$O(\|E\|)$	in general: NP-complete w/o isolated vtc.: $O((\|A\|+\|B\|)^2)$

Hardness of STICK$^{\text{fix}}_{\text{AB}}$

- NP-hardness by reduction from MONOTONE-3-SAT

Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

- NP-hardness by reduction from MONOTONE-3-SAT
- Variable gadget:



Hardness of $\mathrm{STICK}^{\mathrm{fix}}_{\mathrm{AB}}$

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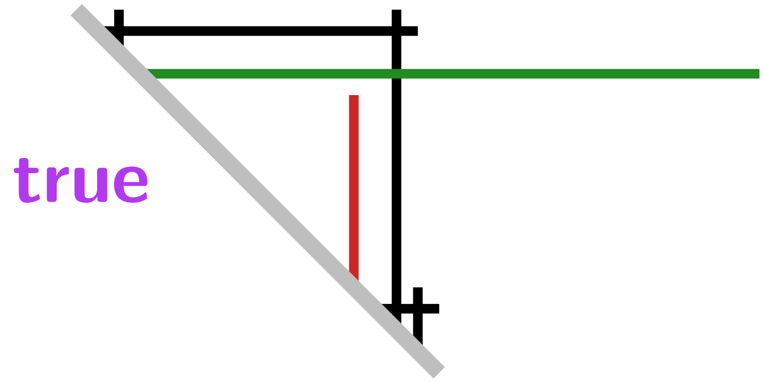

Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

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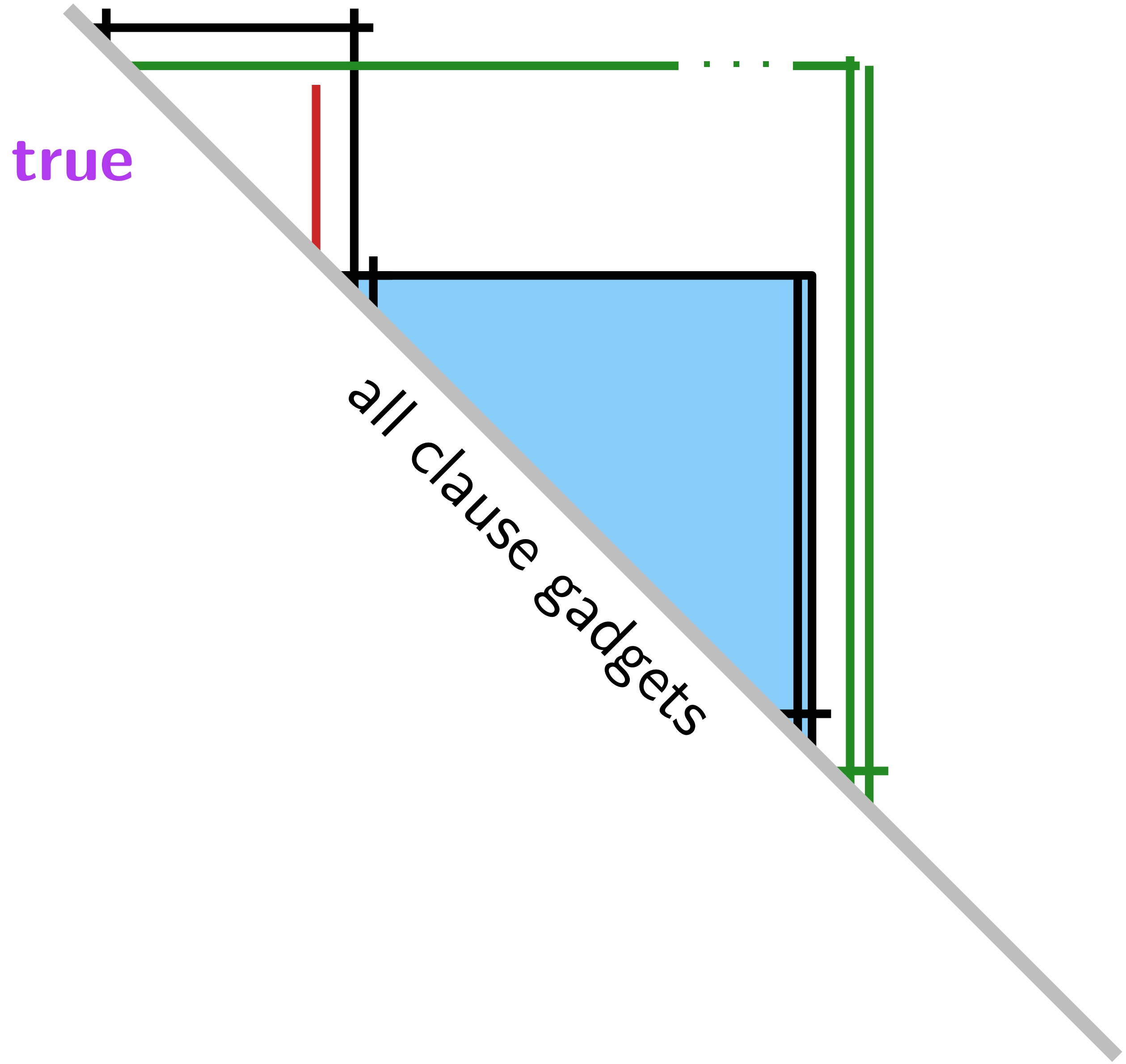

Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

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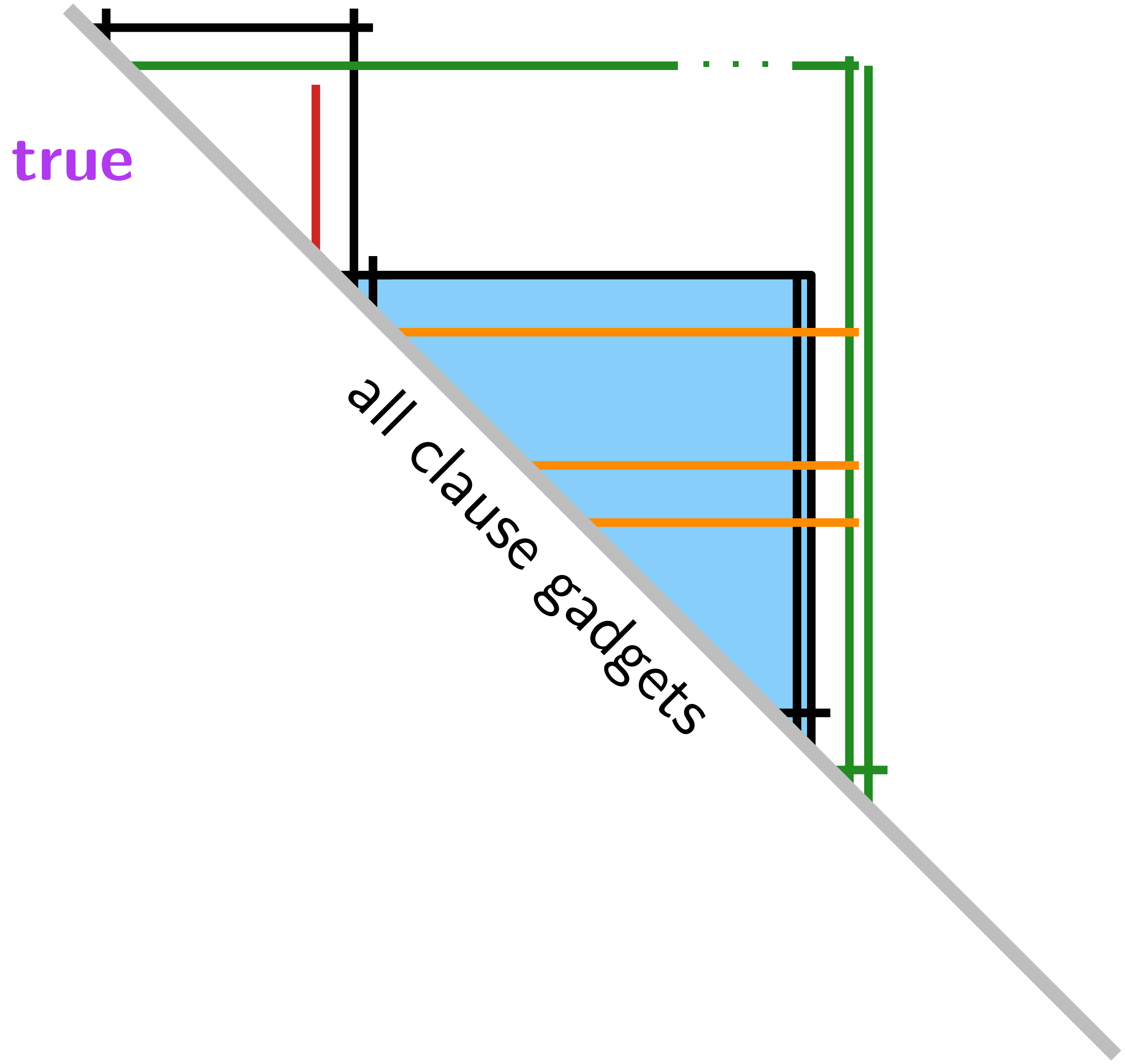
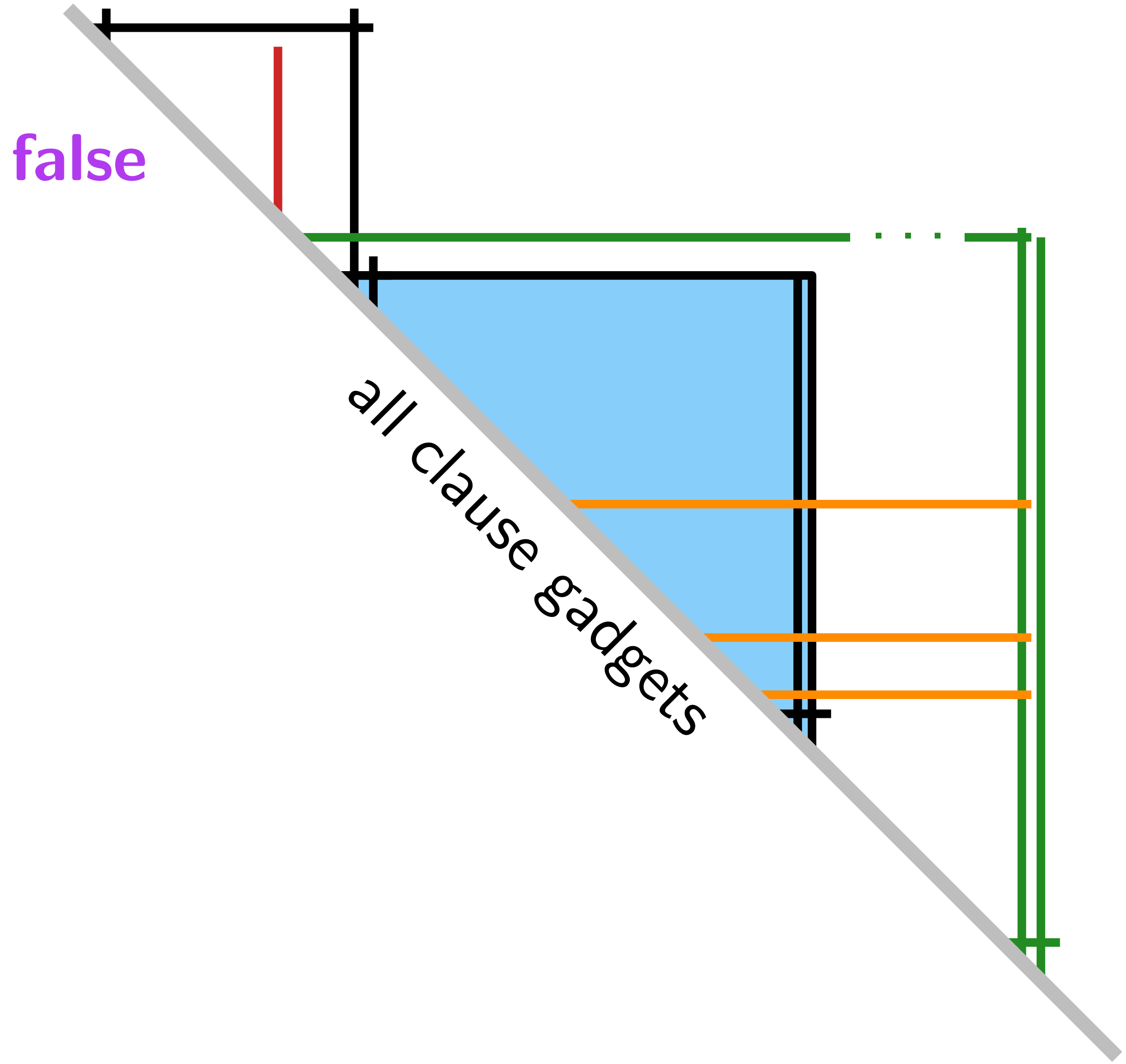
- Variable gadget:



Hardness of $\text{STICK}_{\text{AB}}^{\text{fix}}$

- NP-hardness by reduction from MONOTONE-3-SAT
- Variable gadget:



Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

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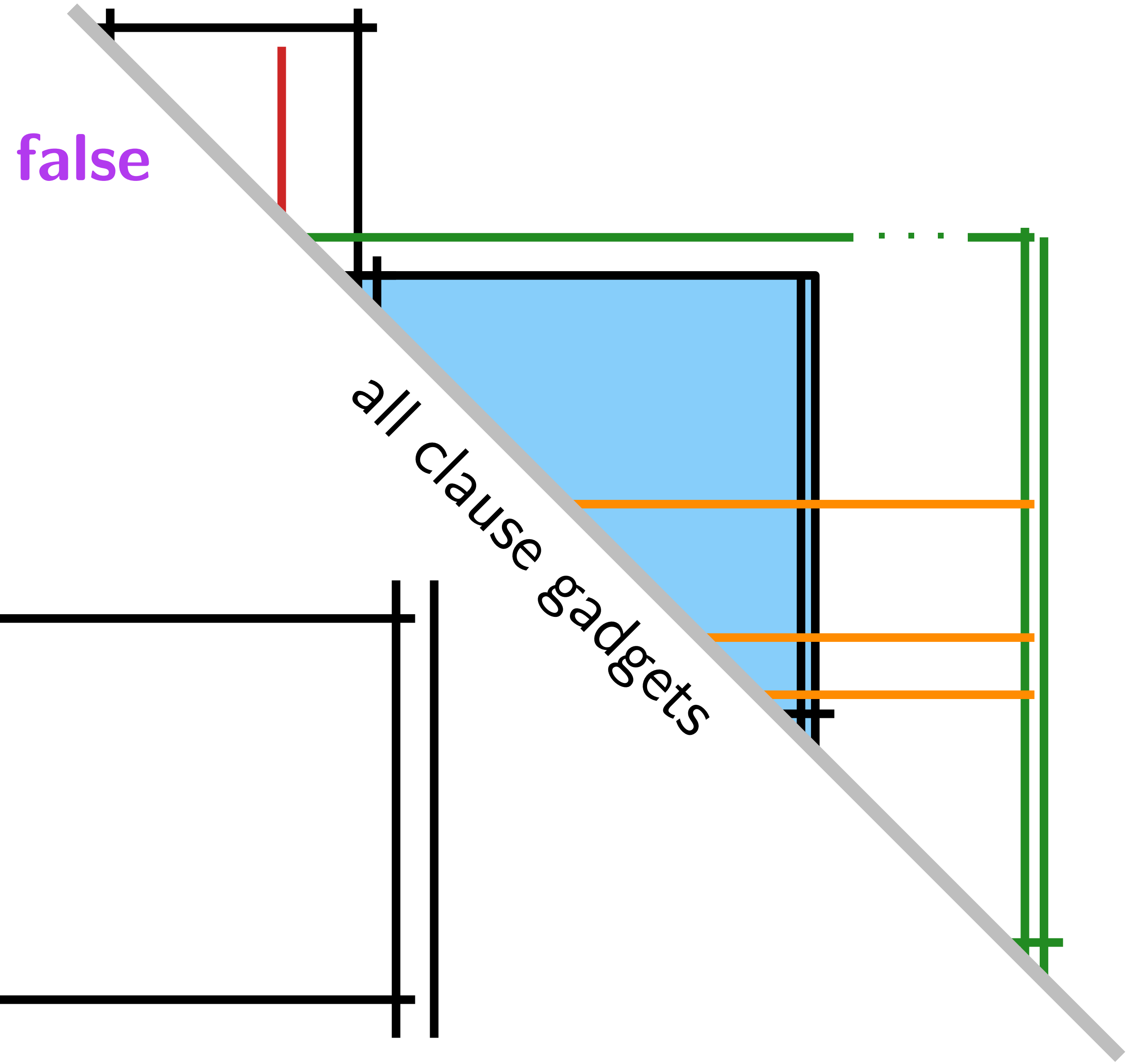
- Variable gadget:



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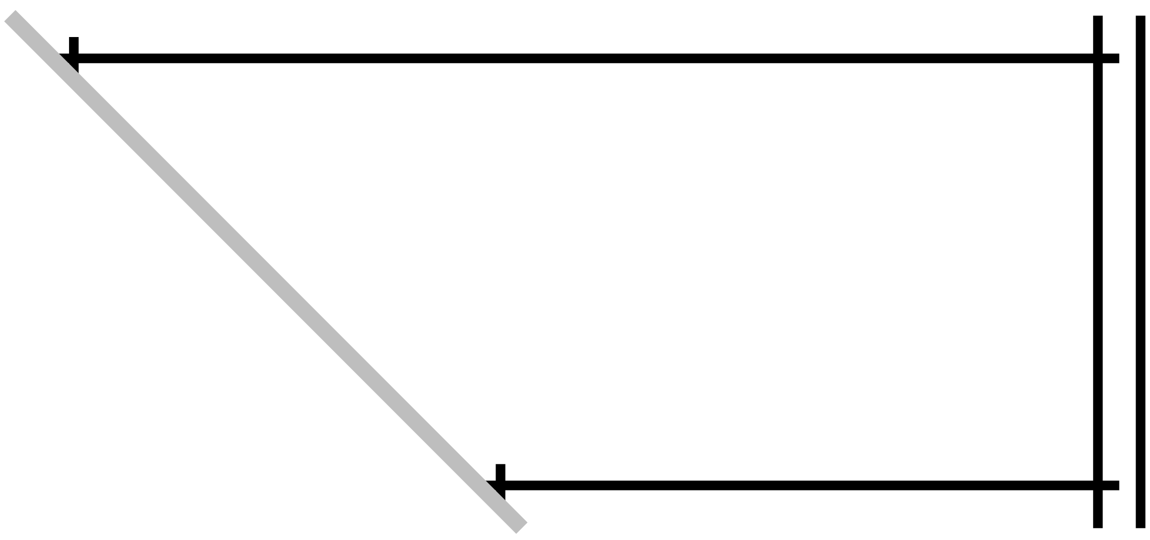

Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

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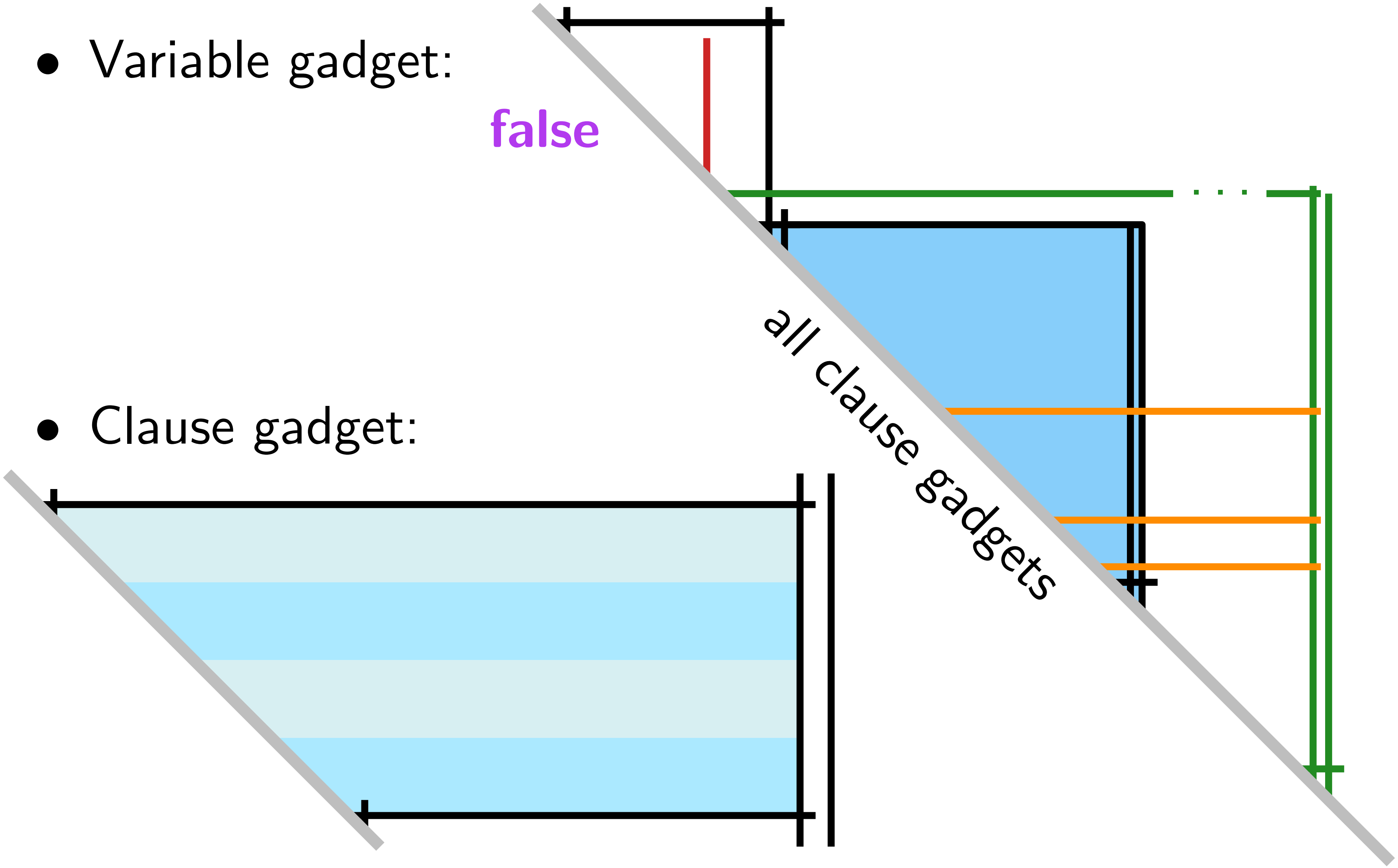

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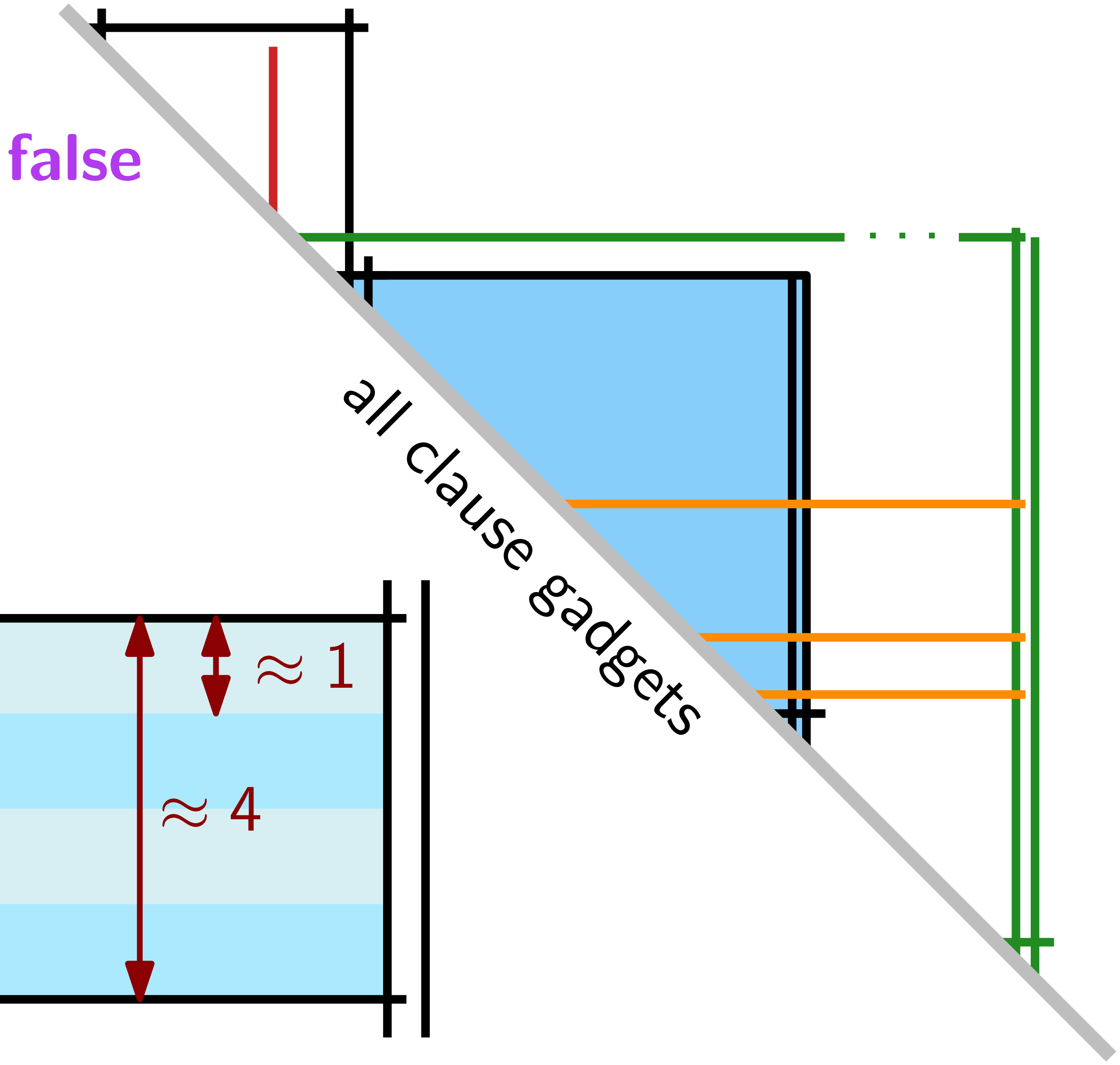

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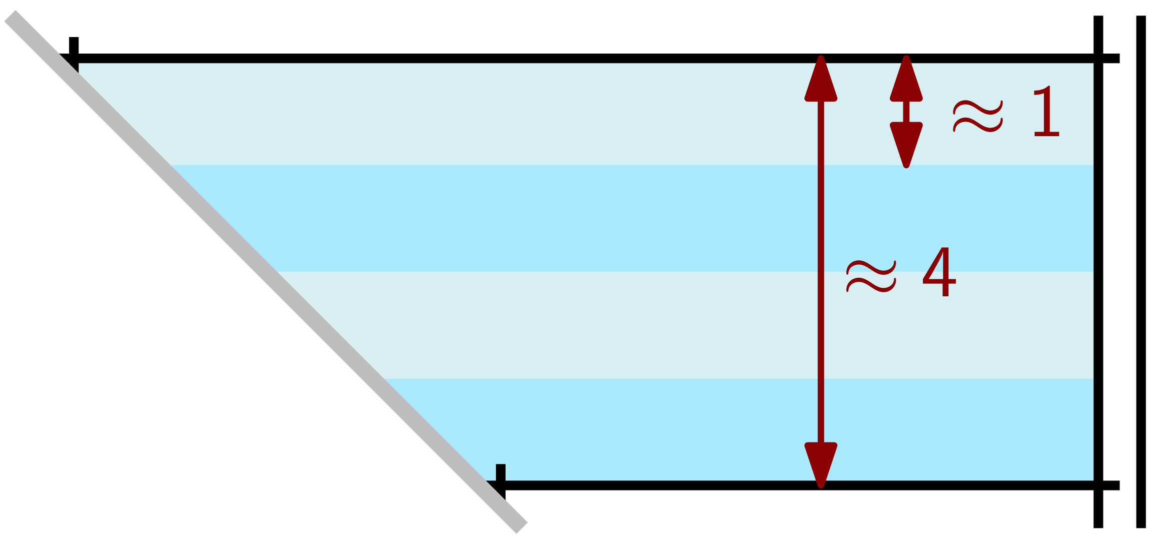

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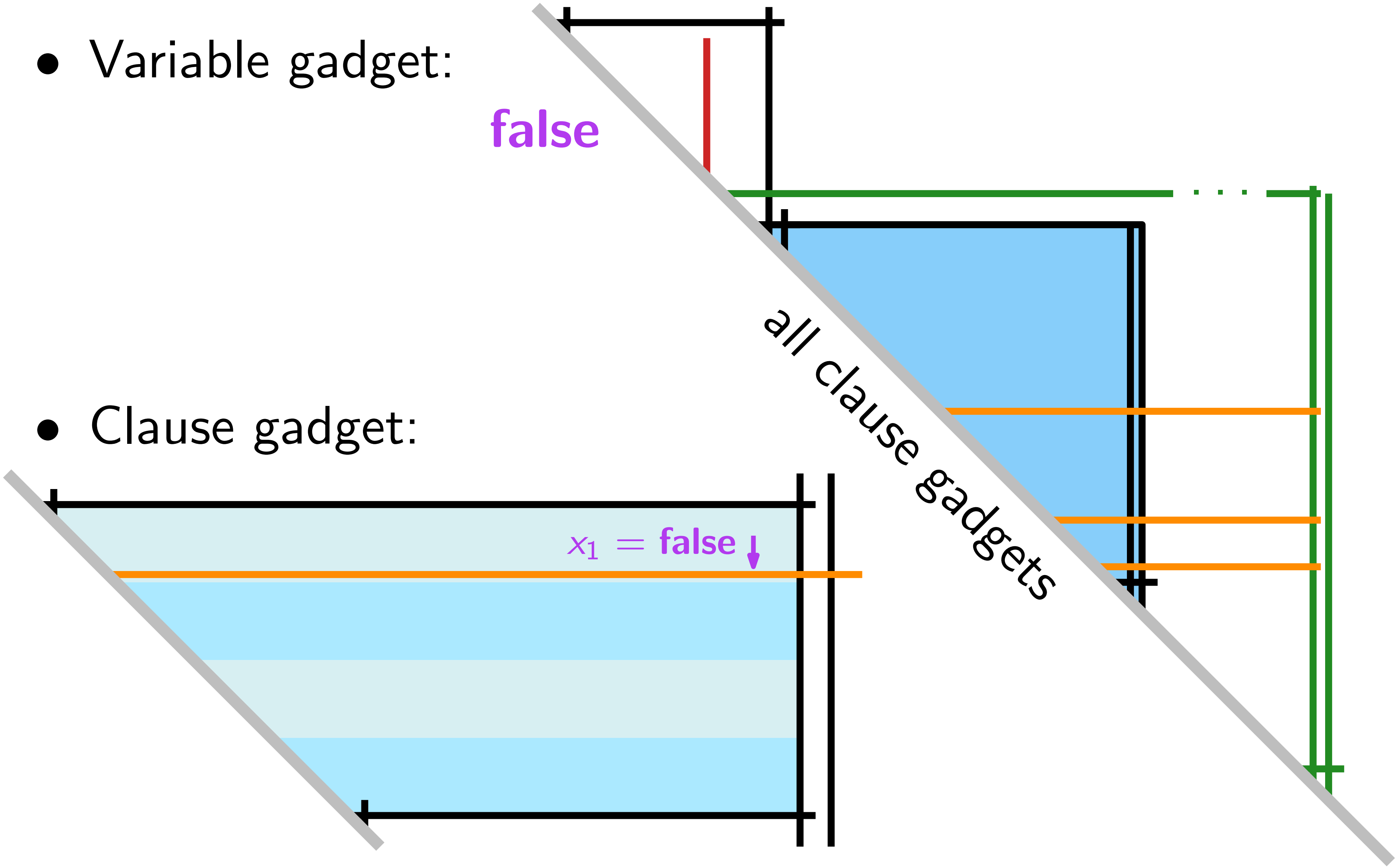

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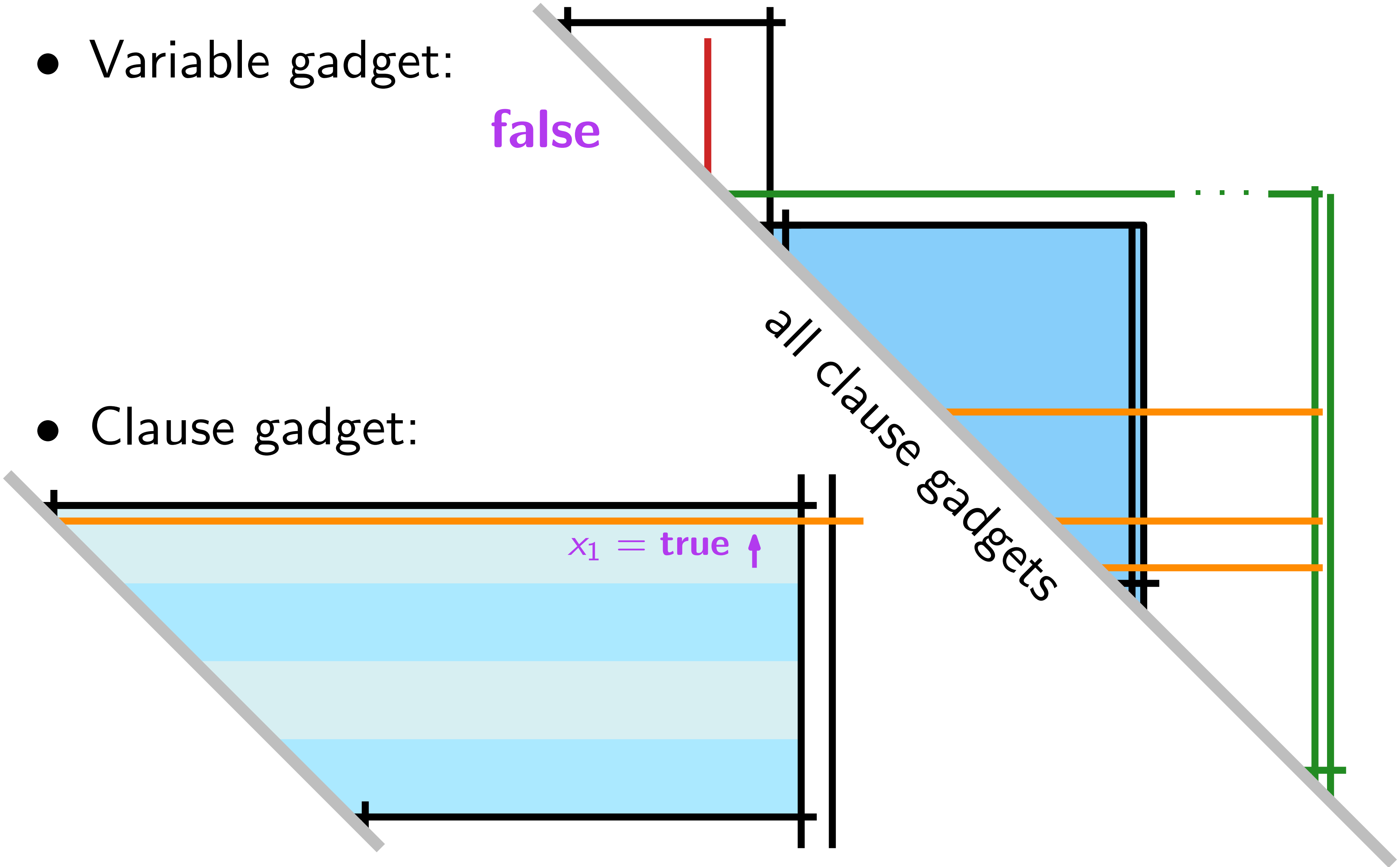

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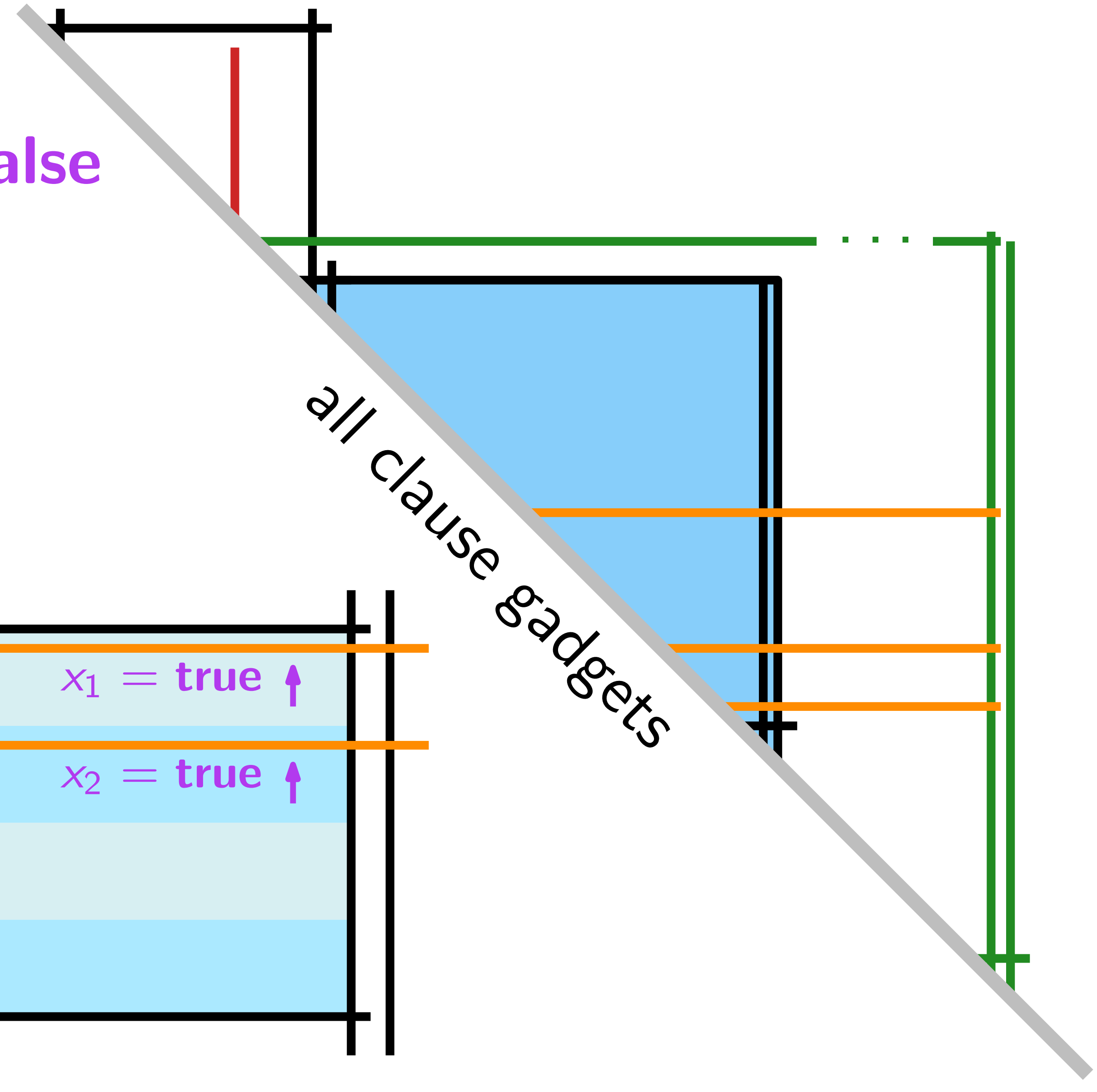

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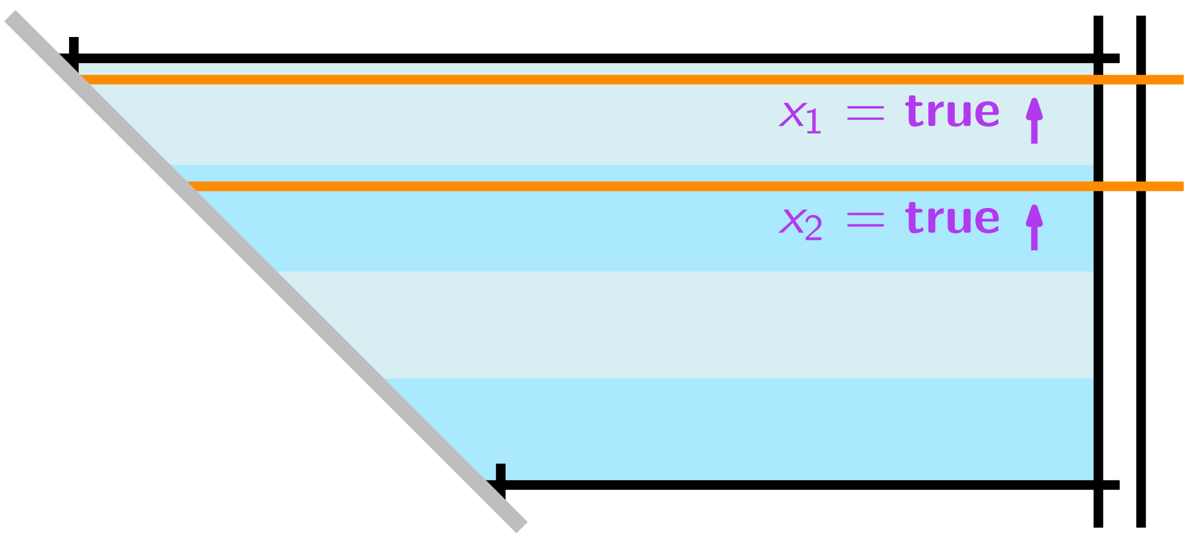

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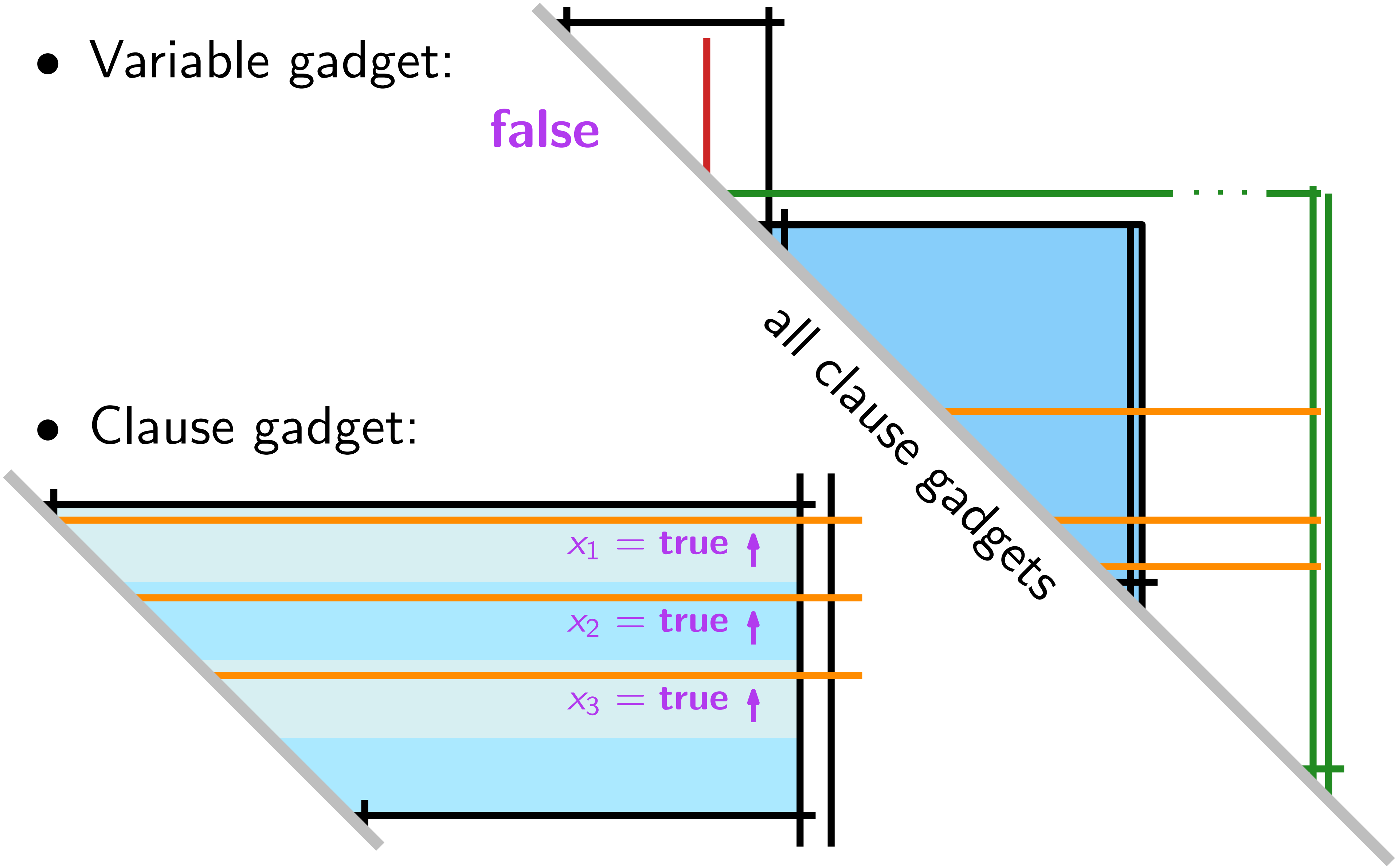

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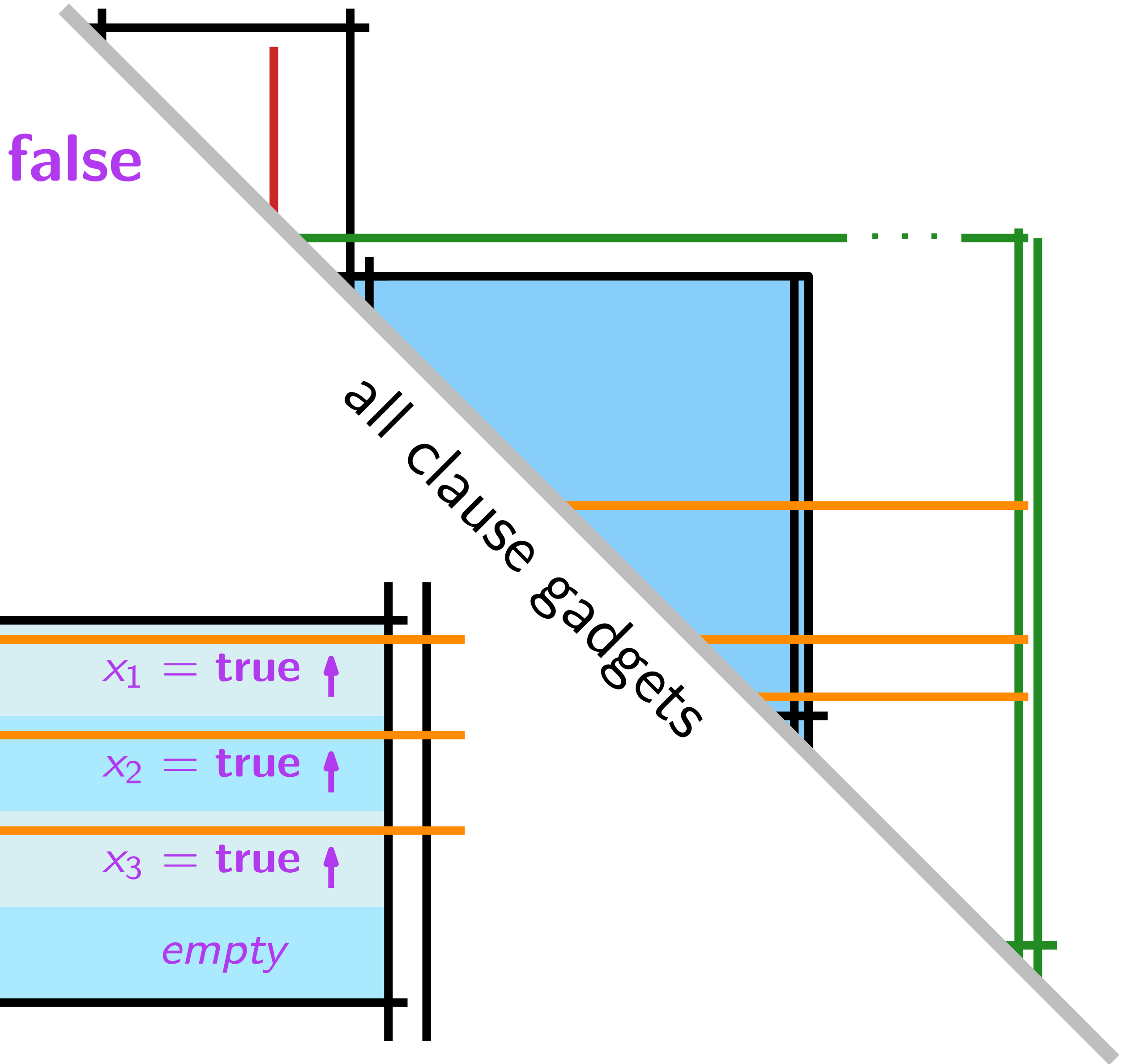

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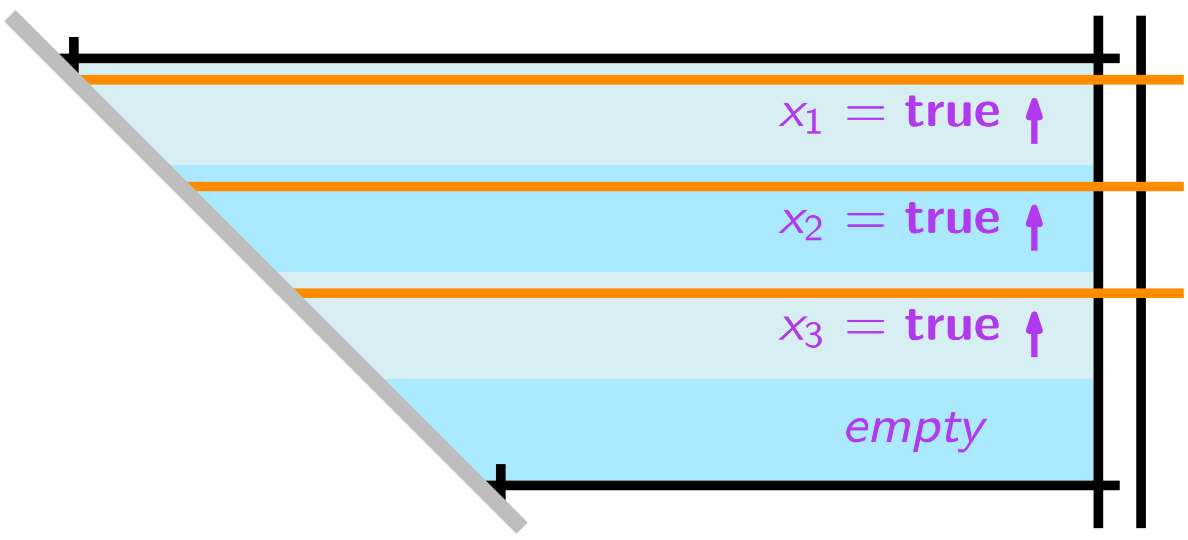

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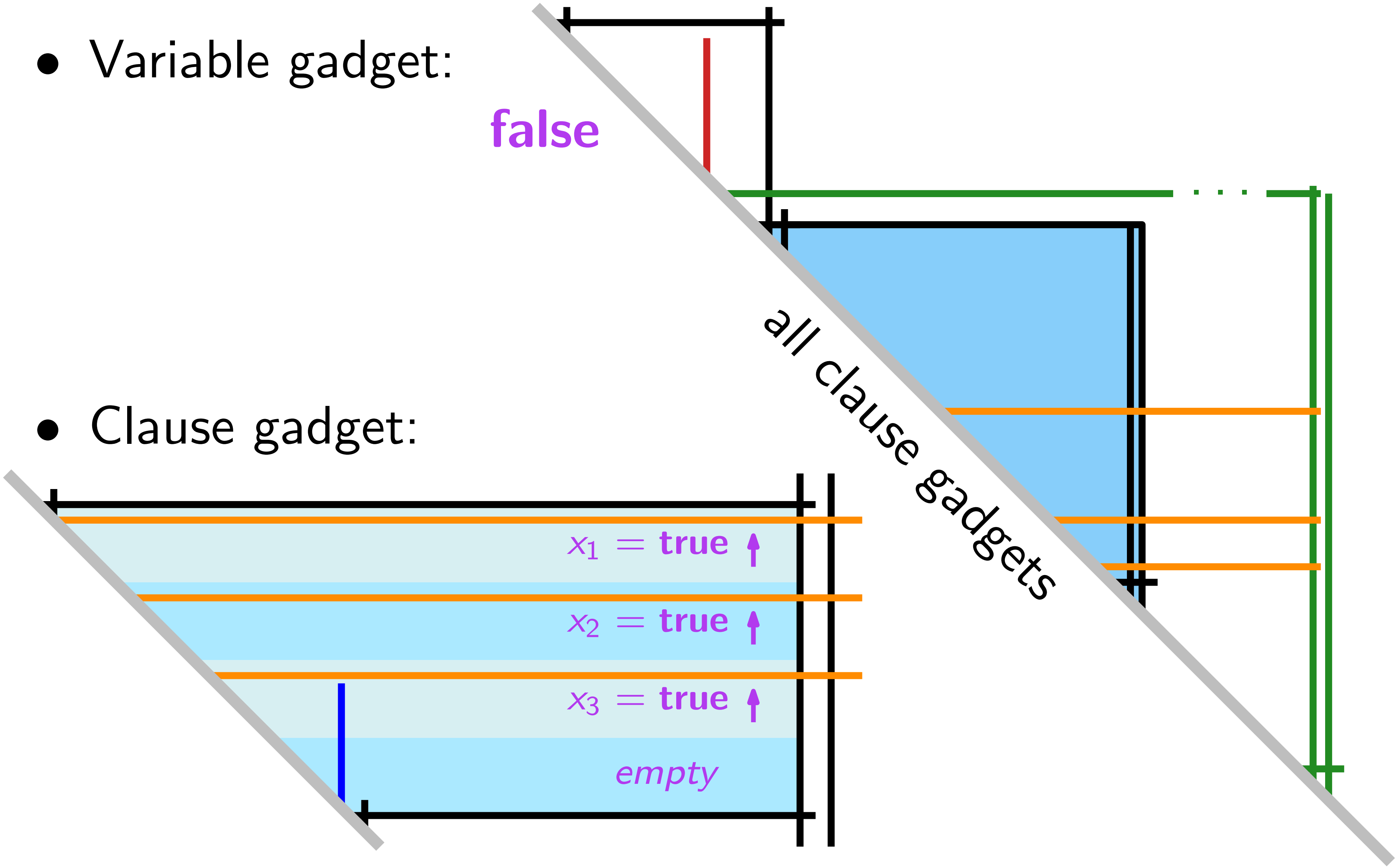

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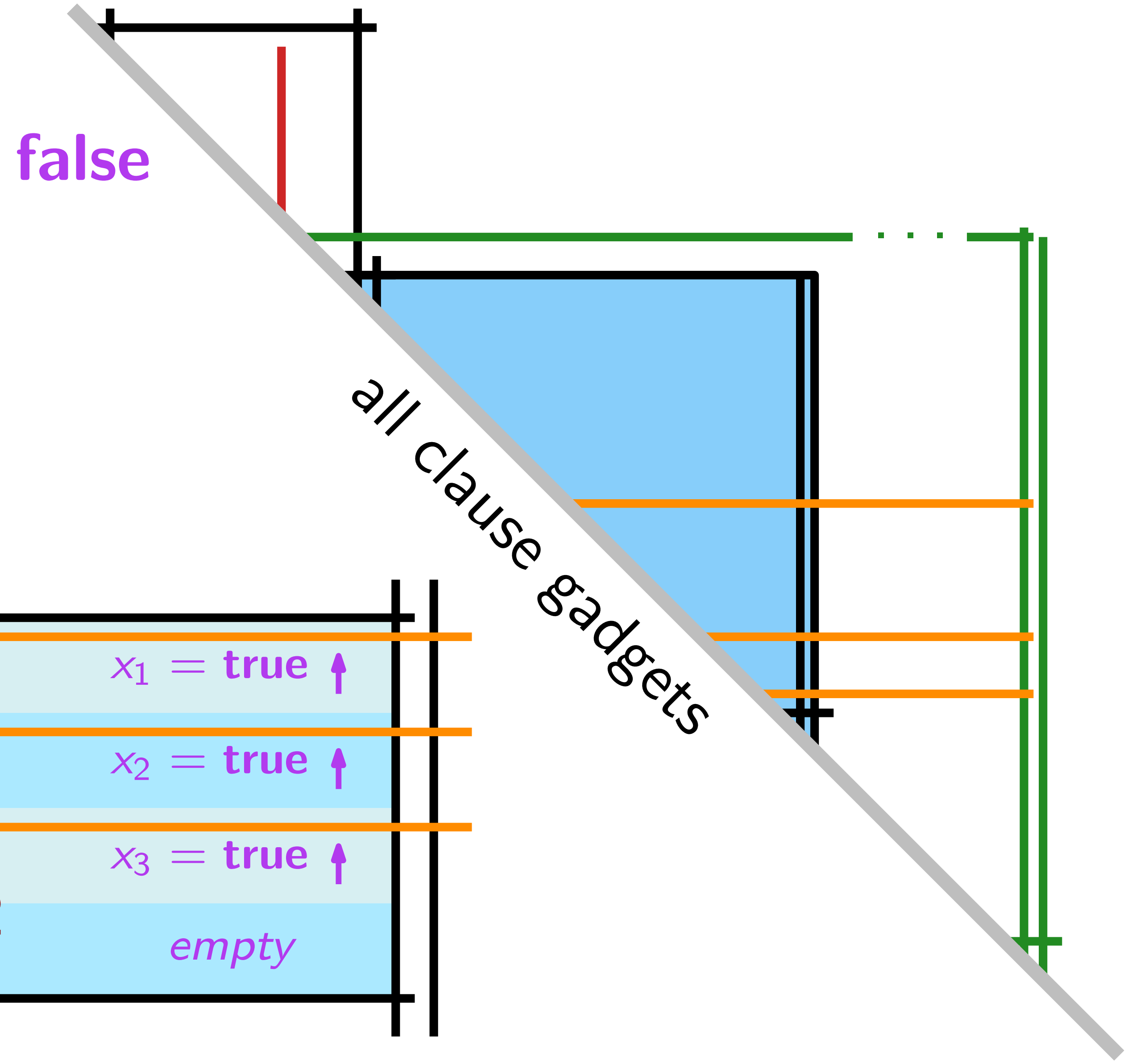

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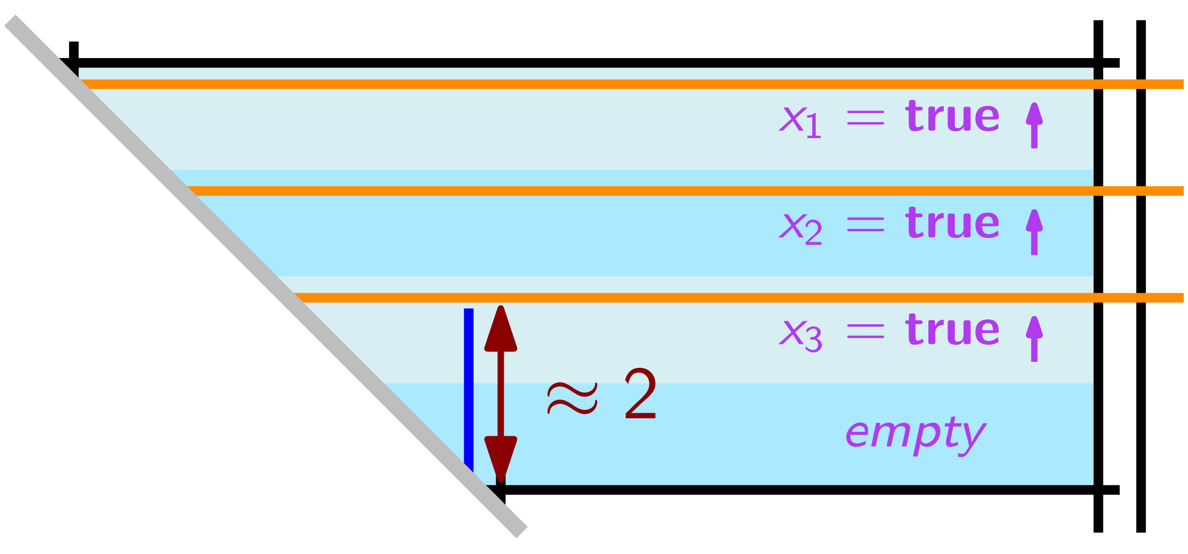

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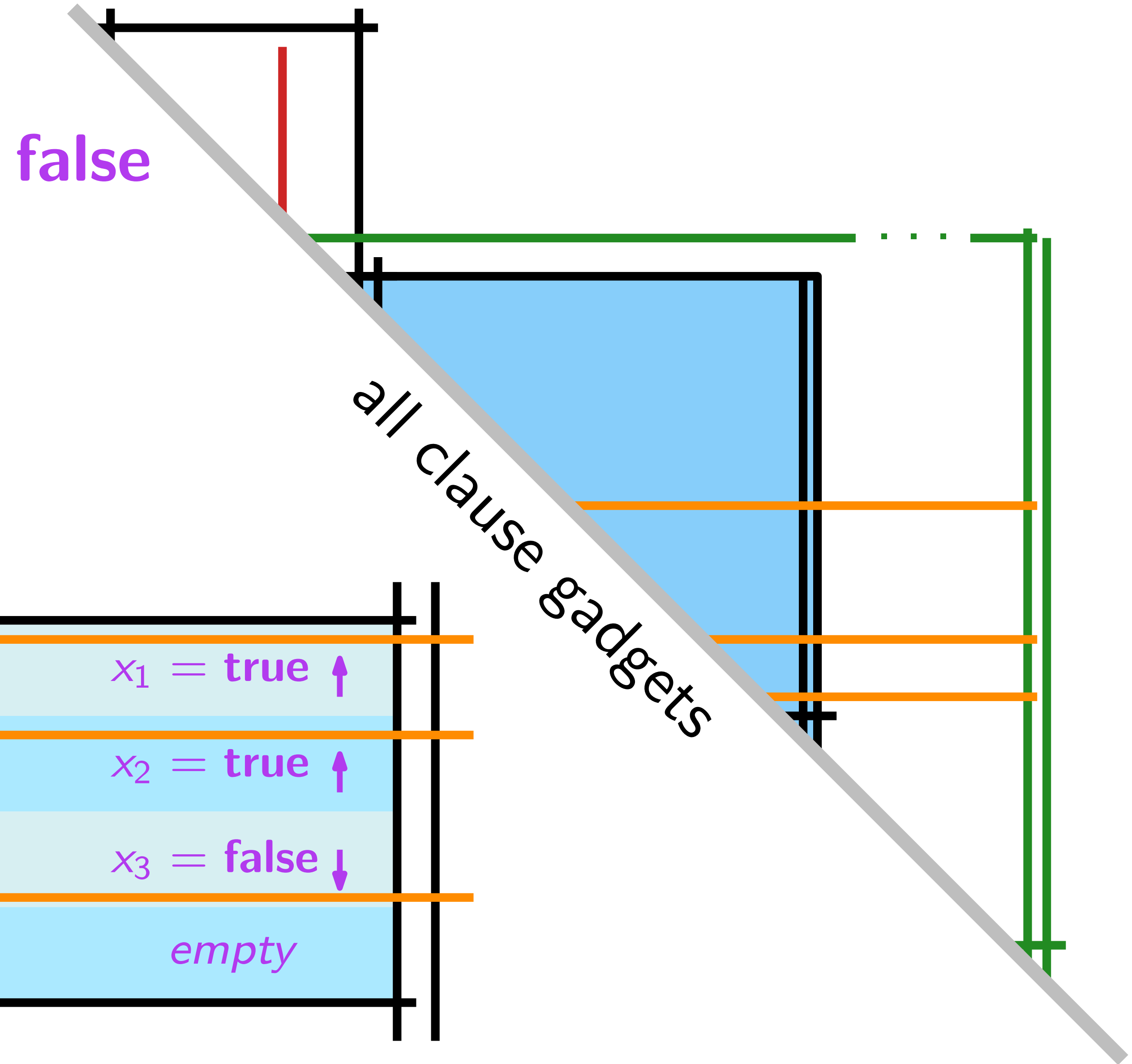

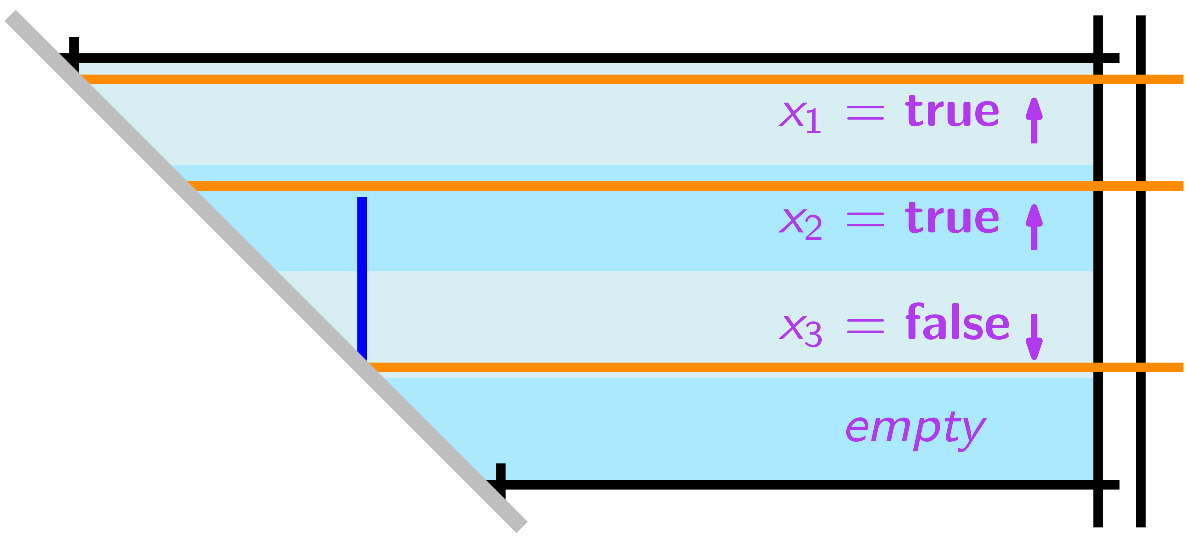

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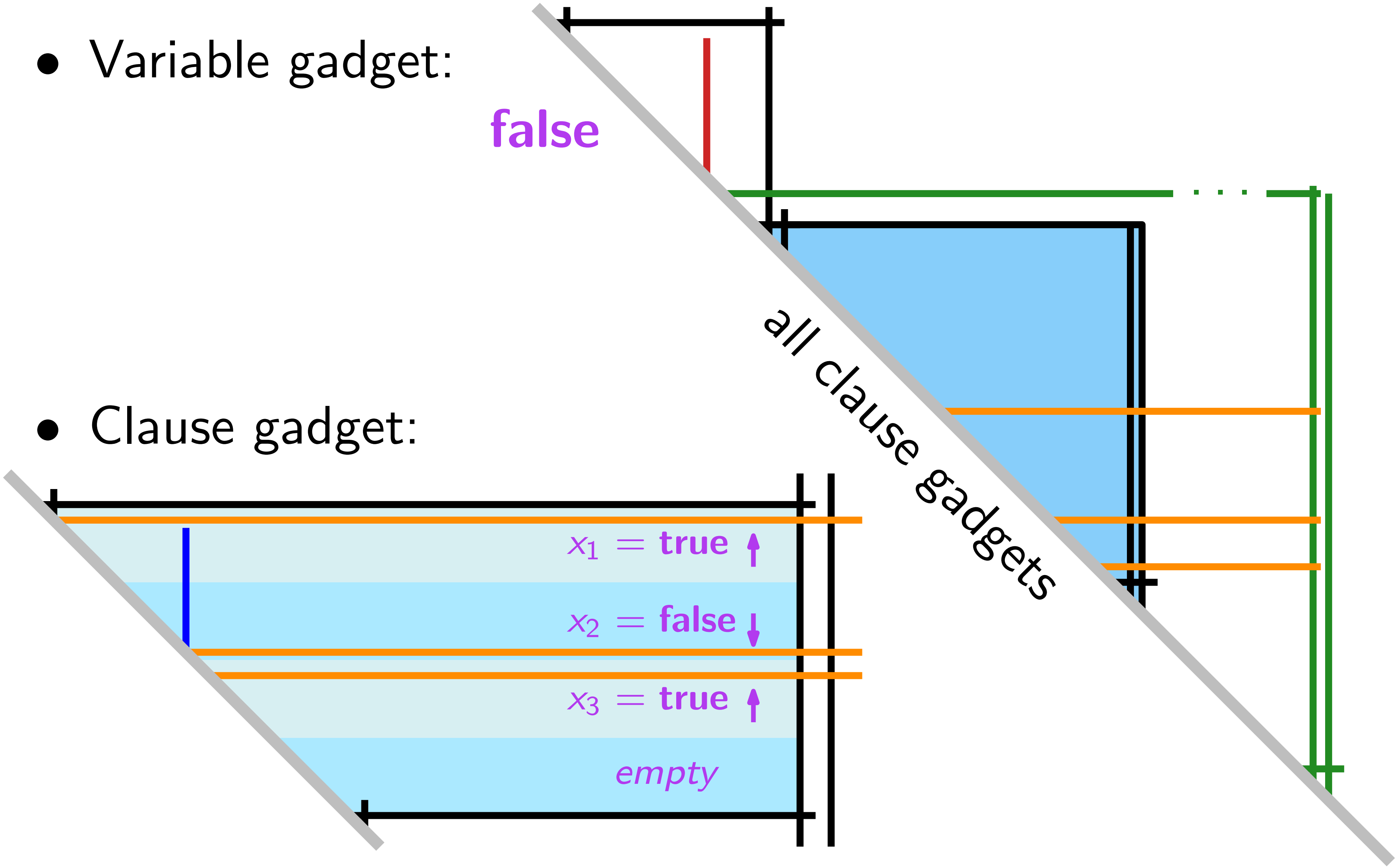

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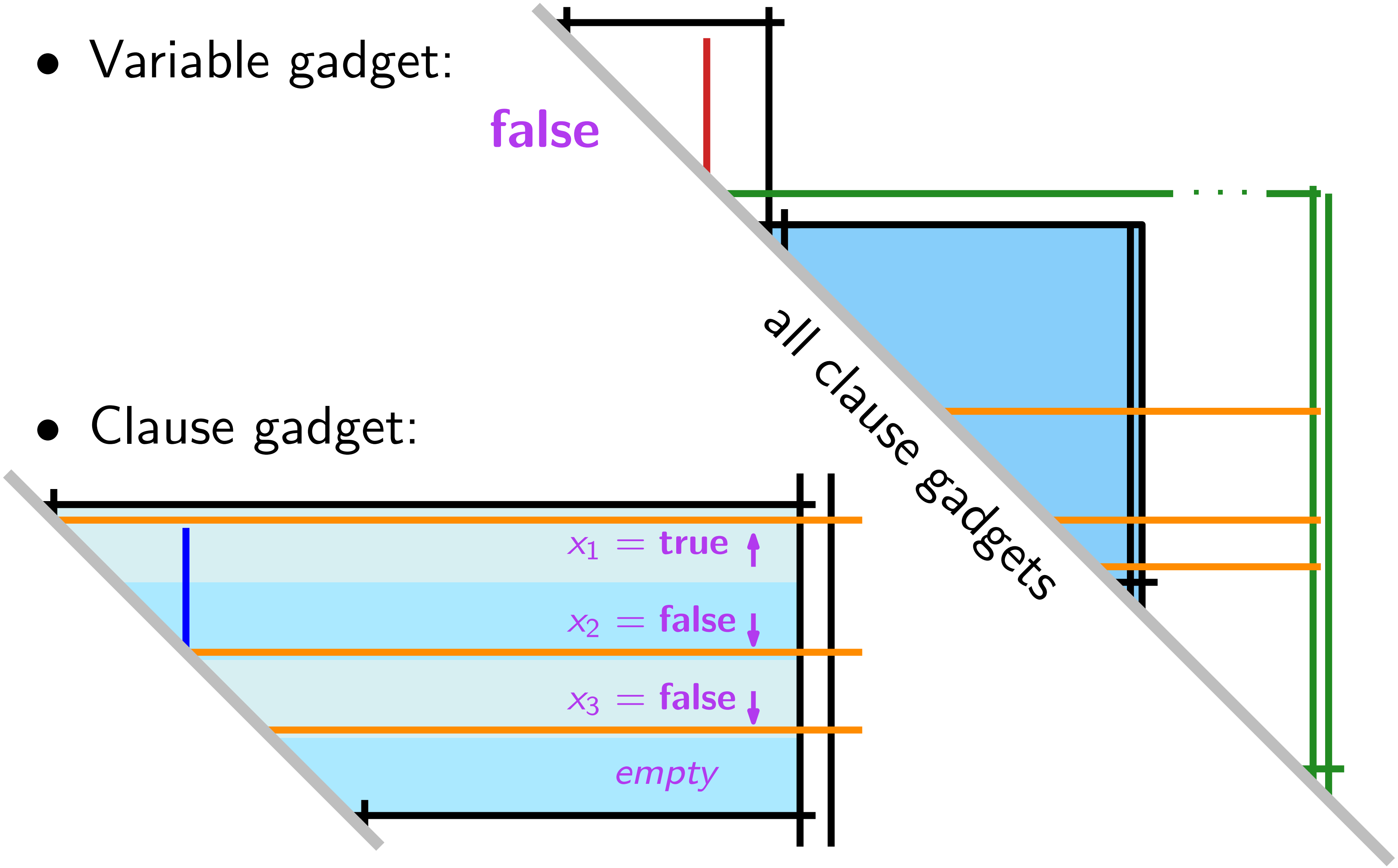

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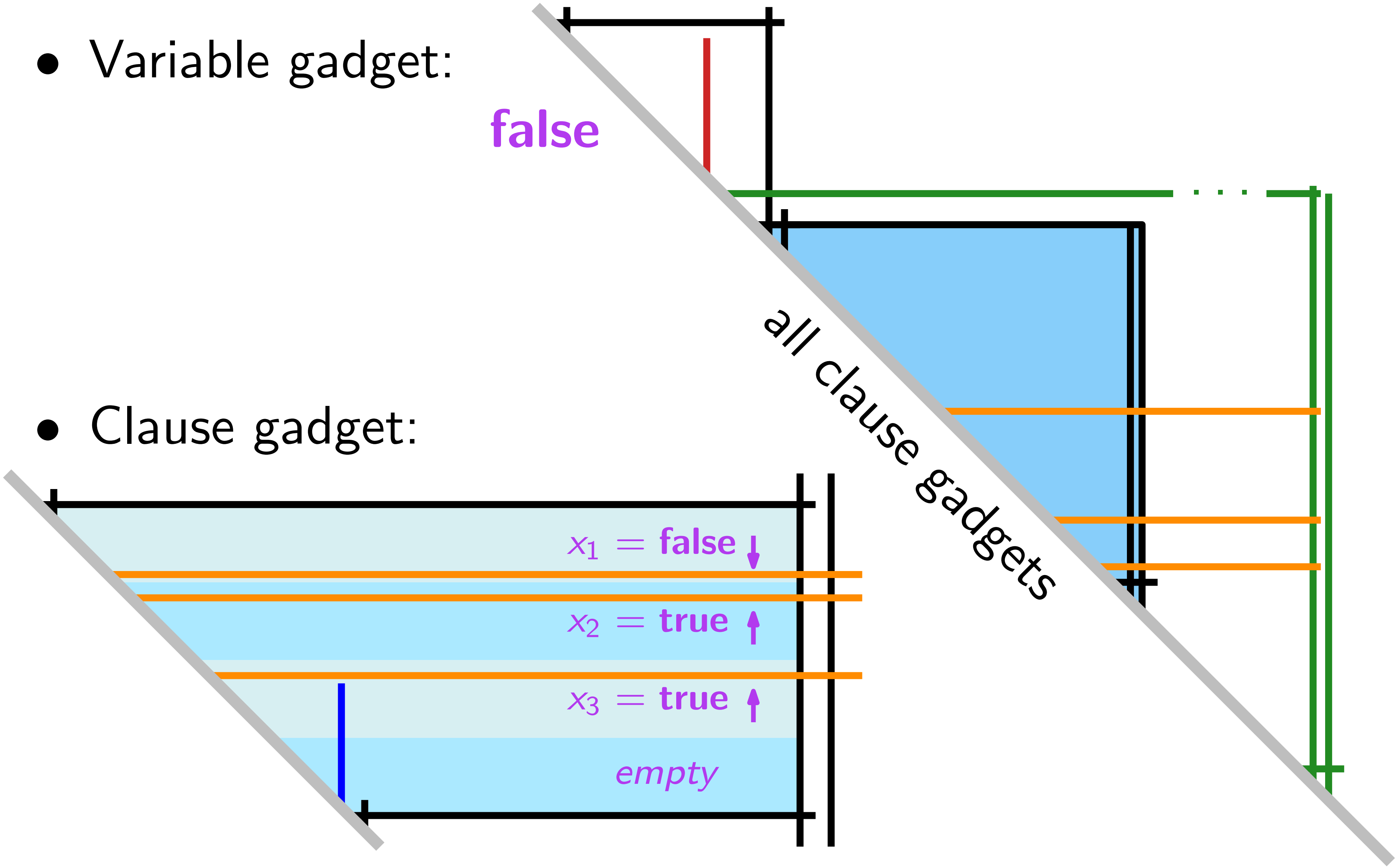

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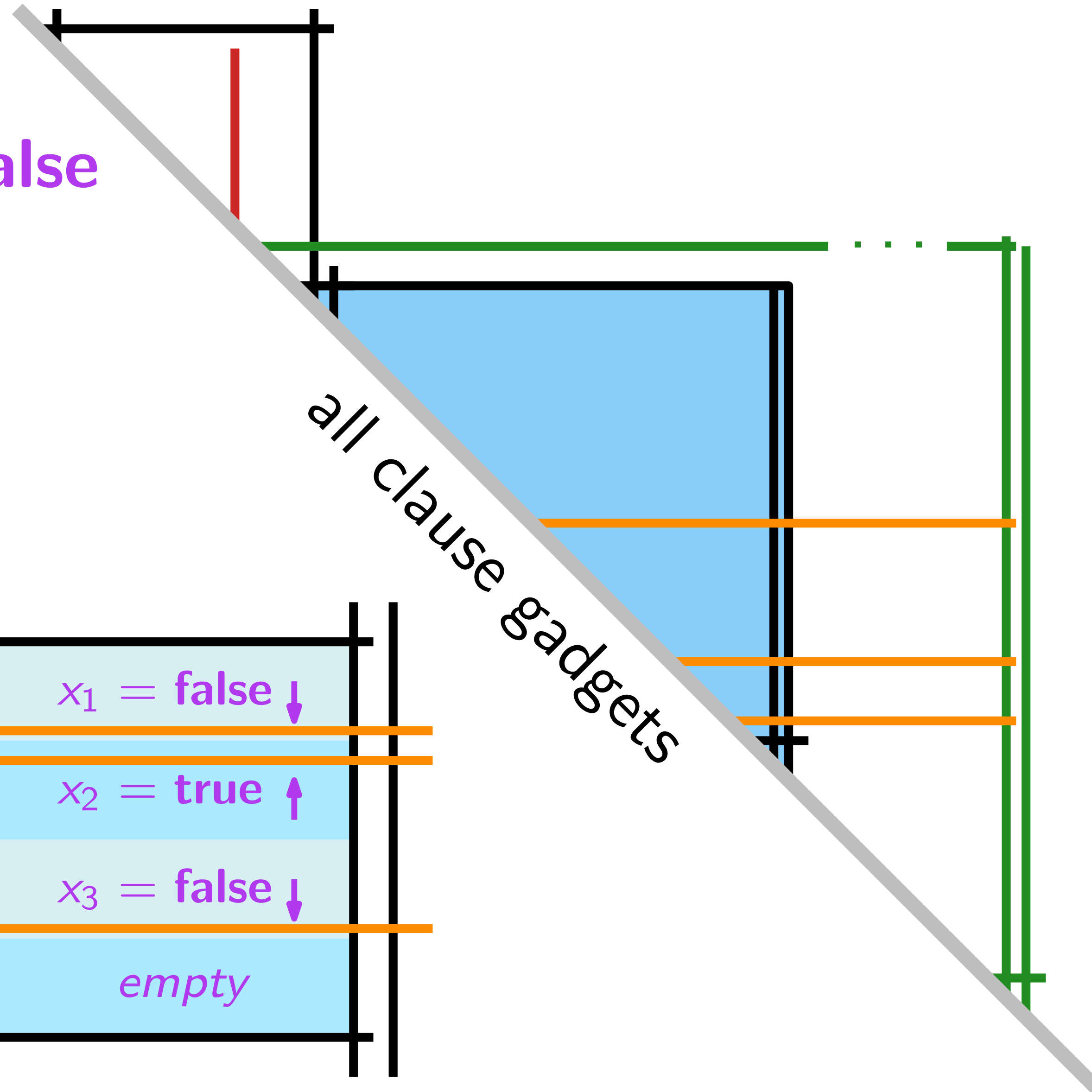

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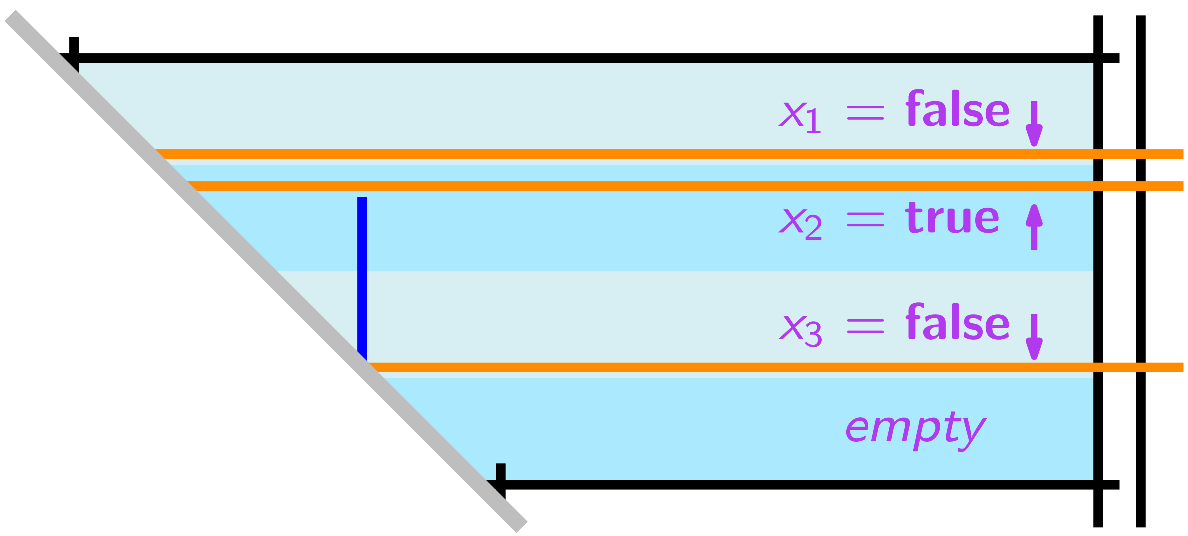

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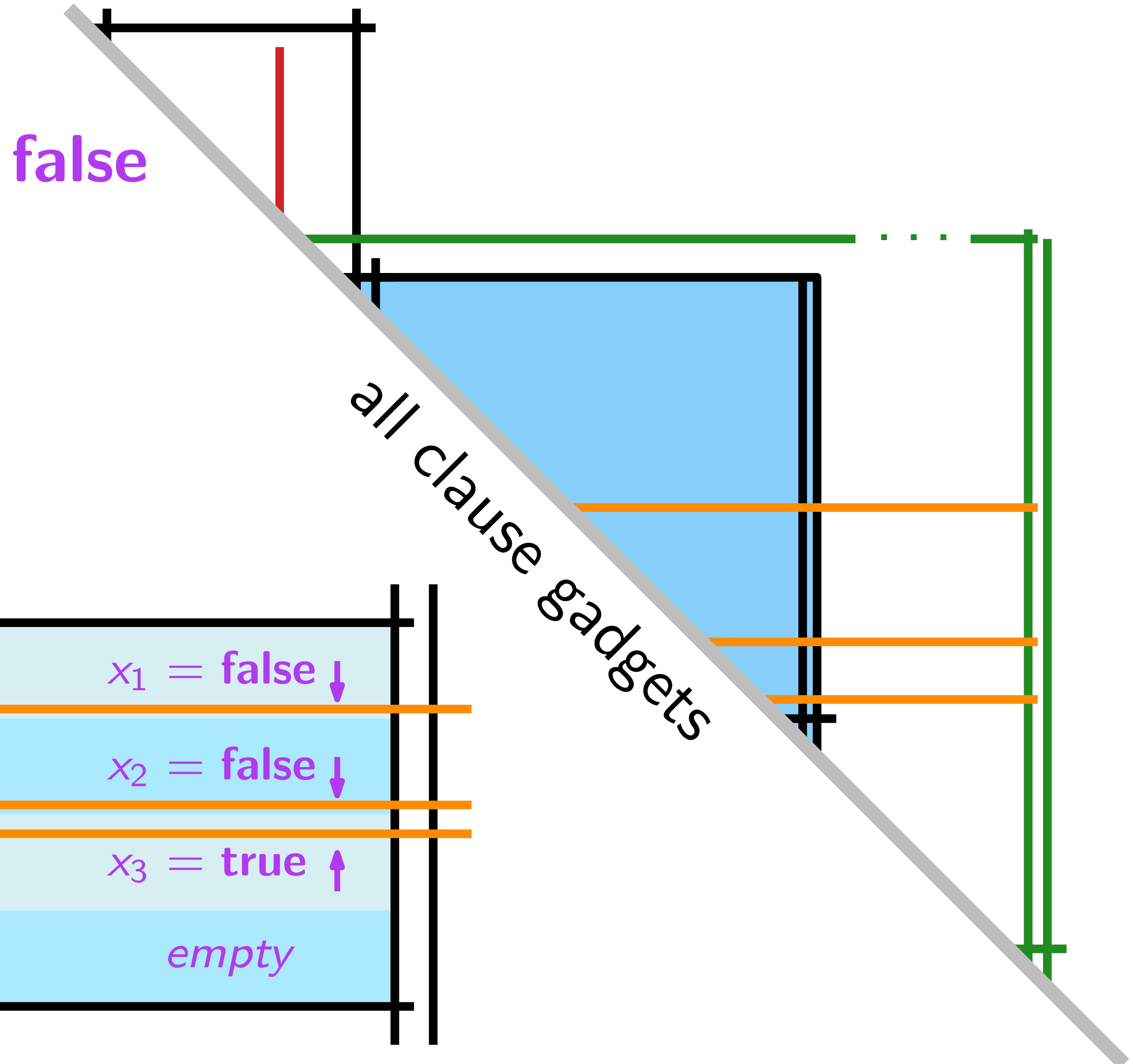

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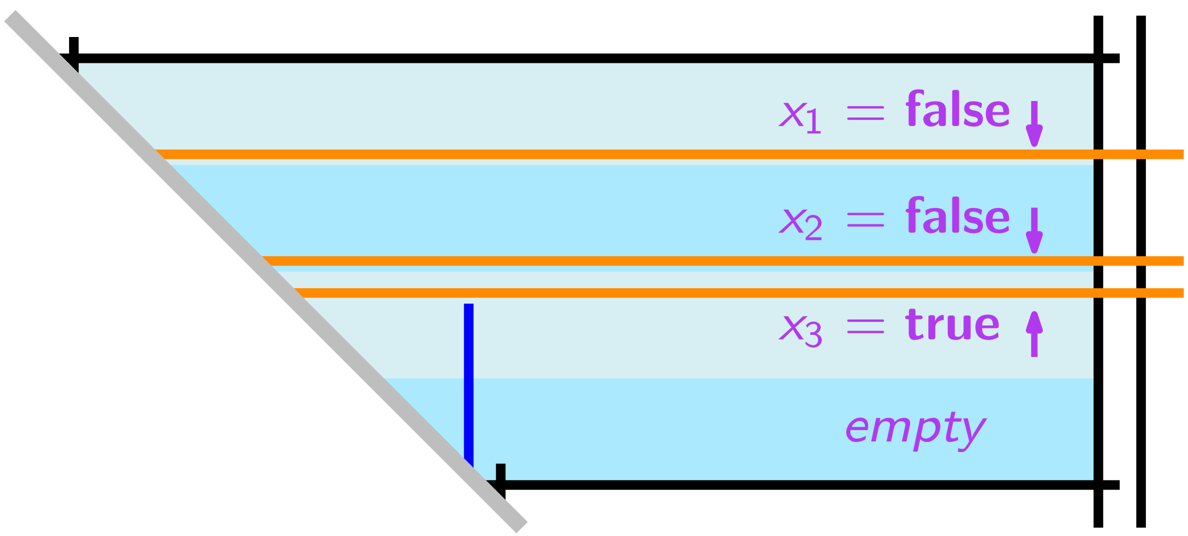

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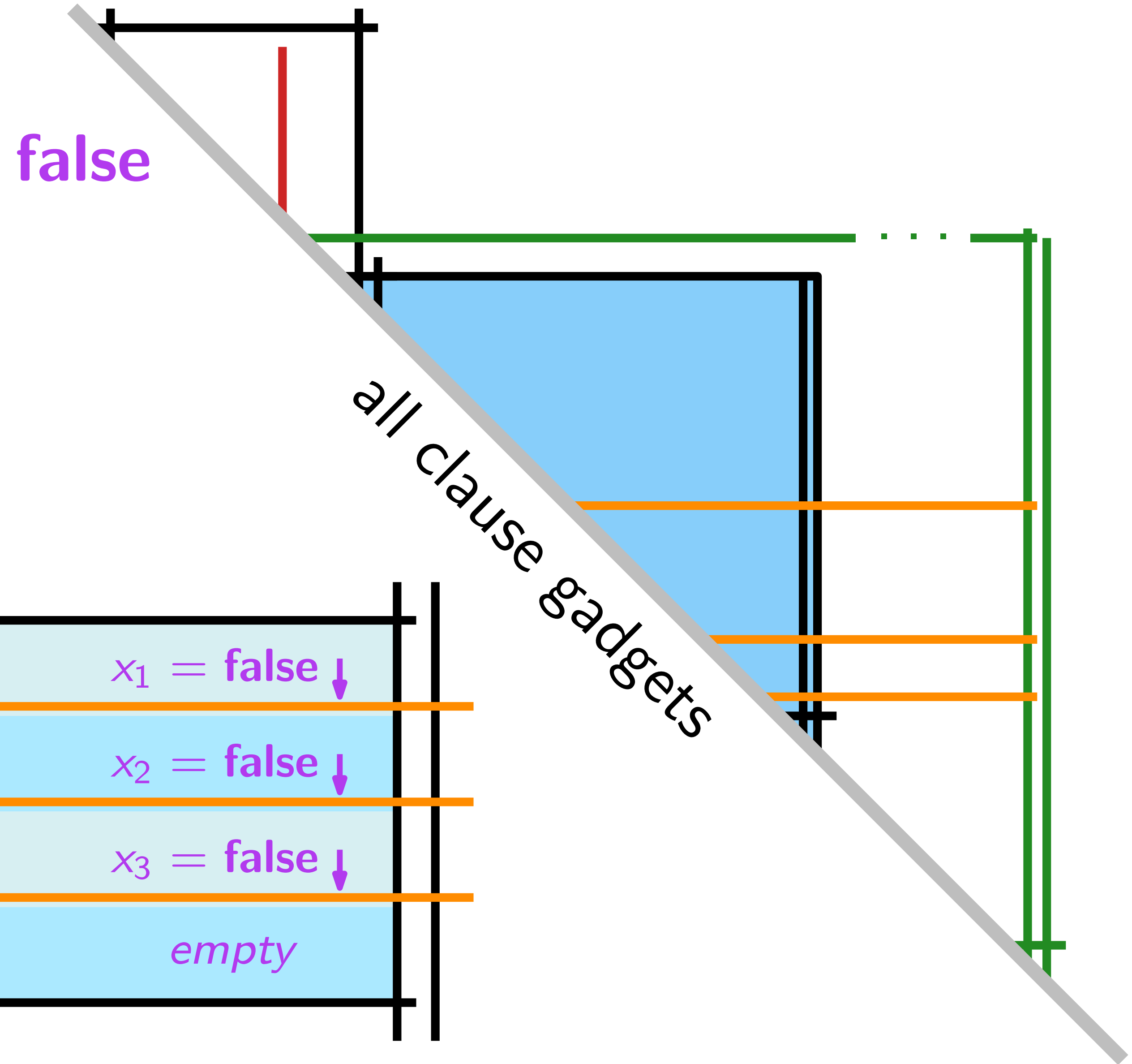

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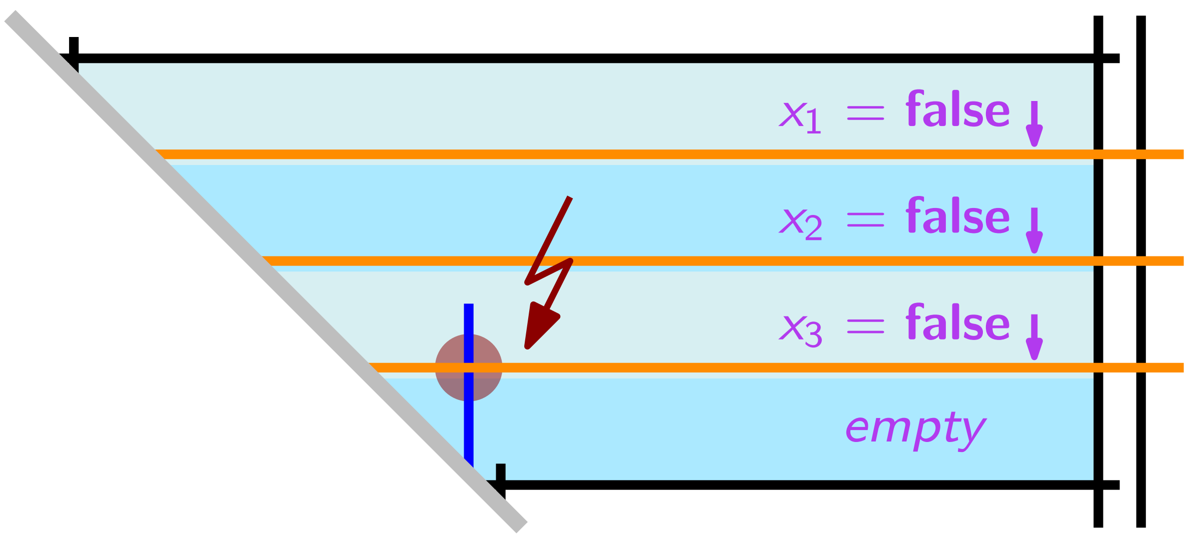
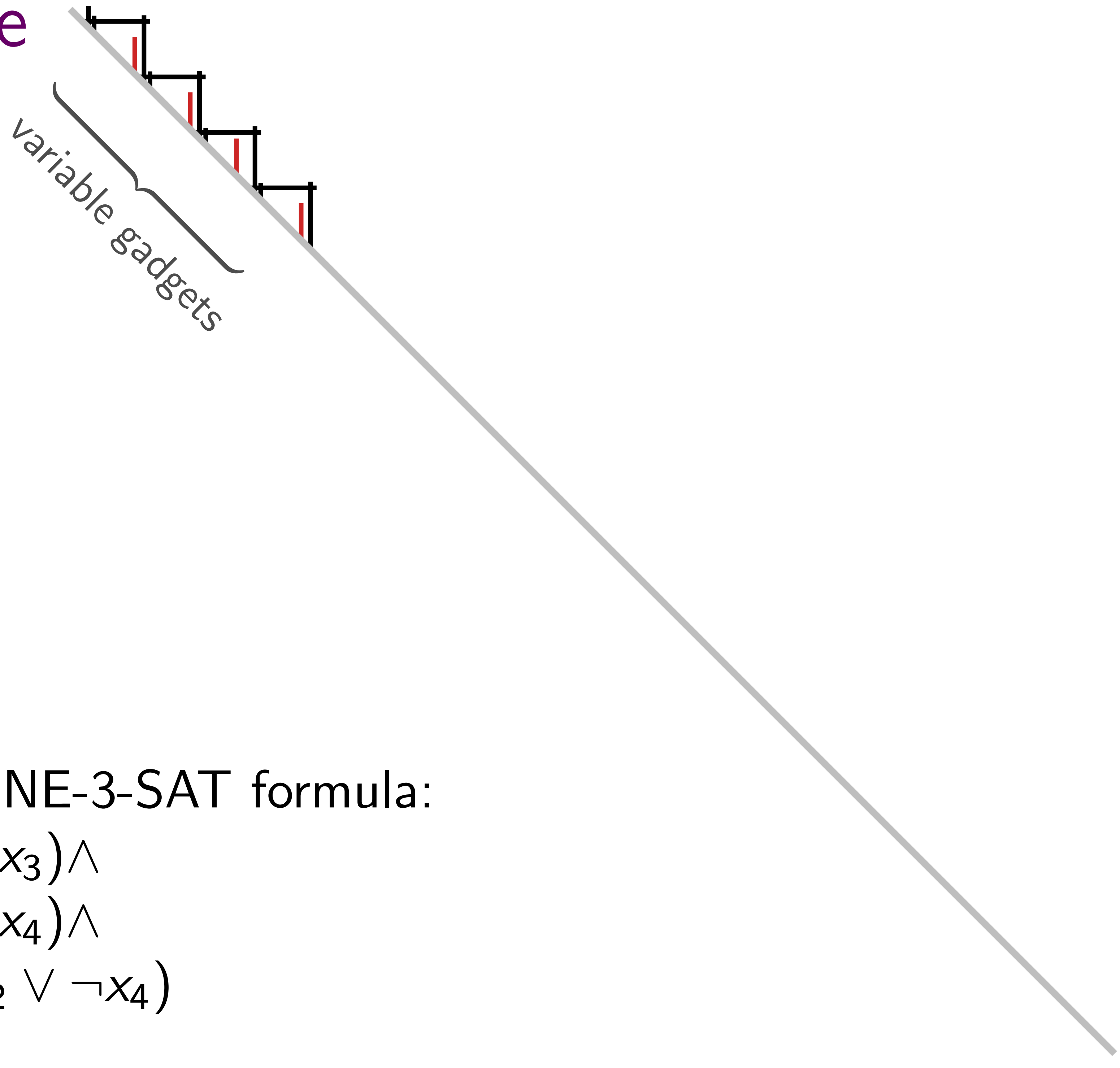


Example

MONOTONE-3-SAT formula:

$(x_1 \vee x_2 \vee x_3) \wedge$
$(x_2 \vee x_3 \vee x_4) \wedge$
$(\neg x_1 \vee \neg x_2 \vee \neg x_4)$

Example



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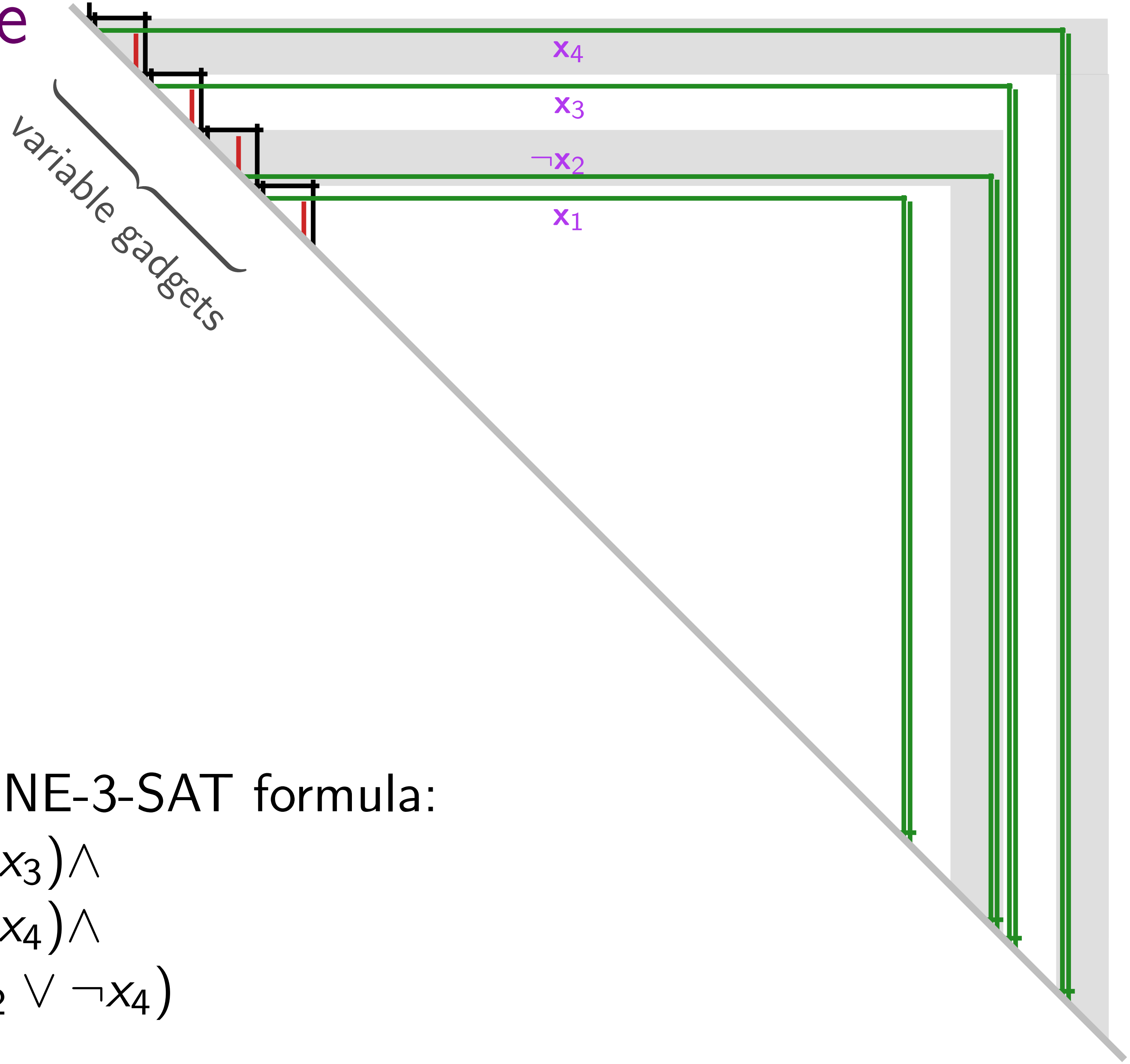

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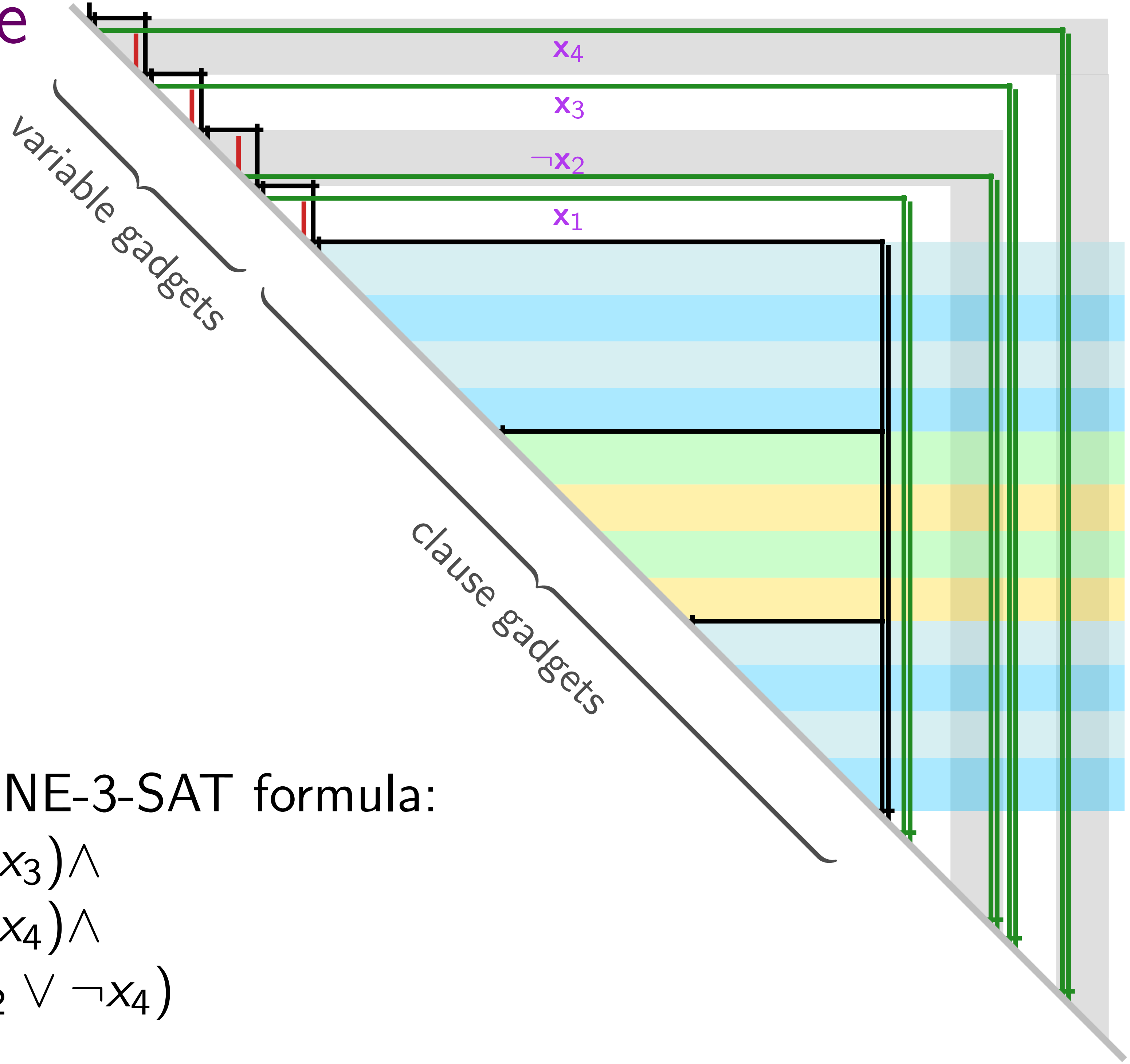

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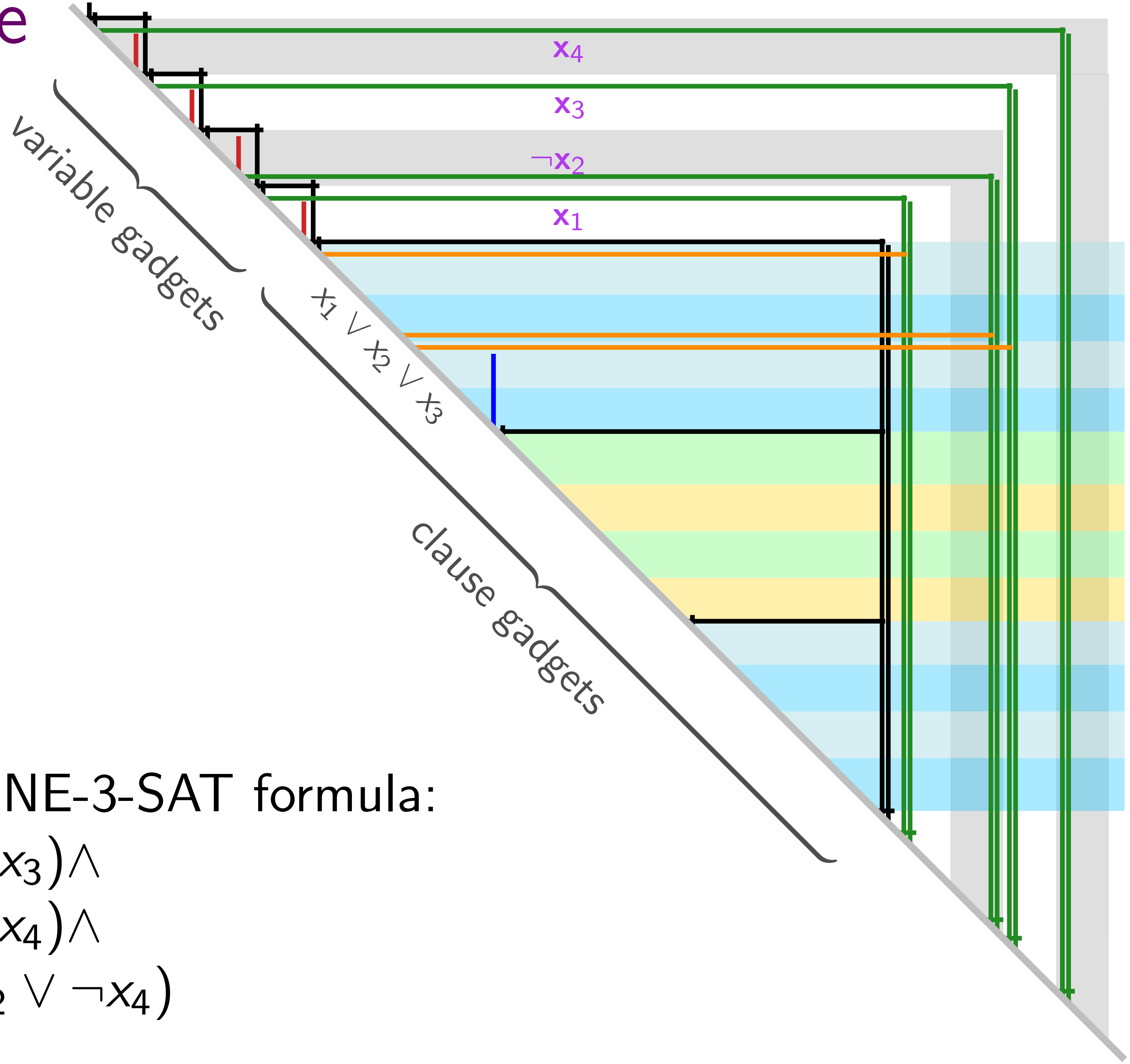

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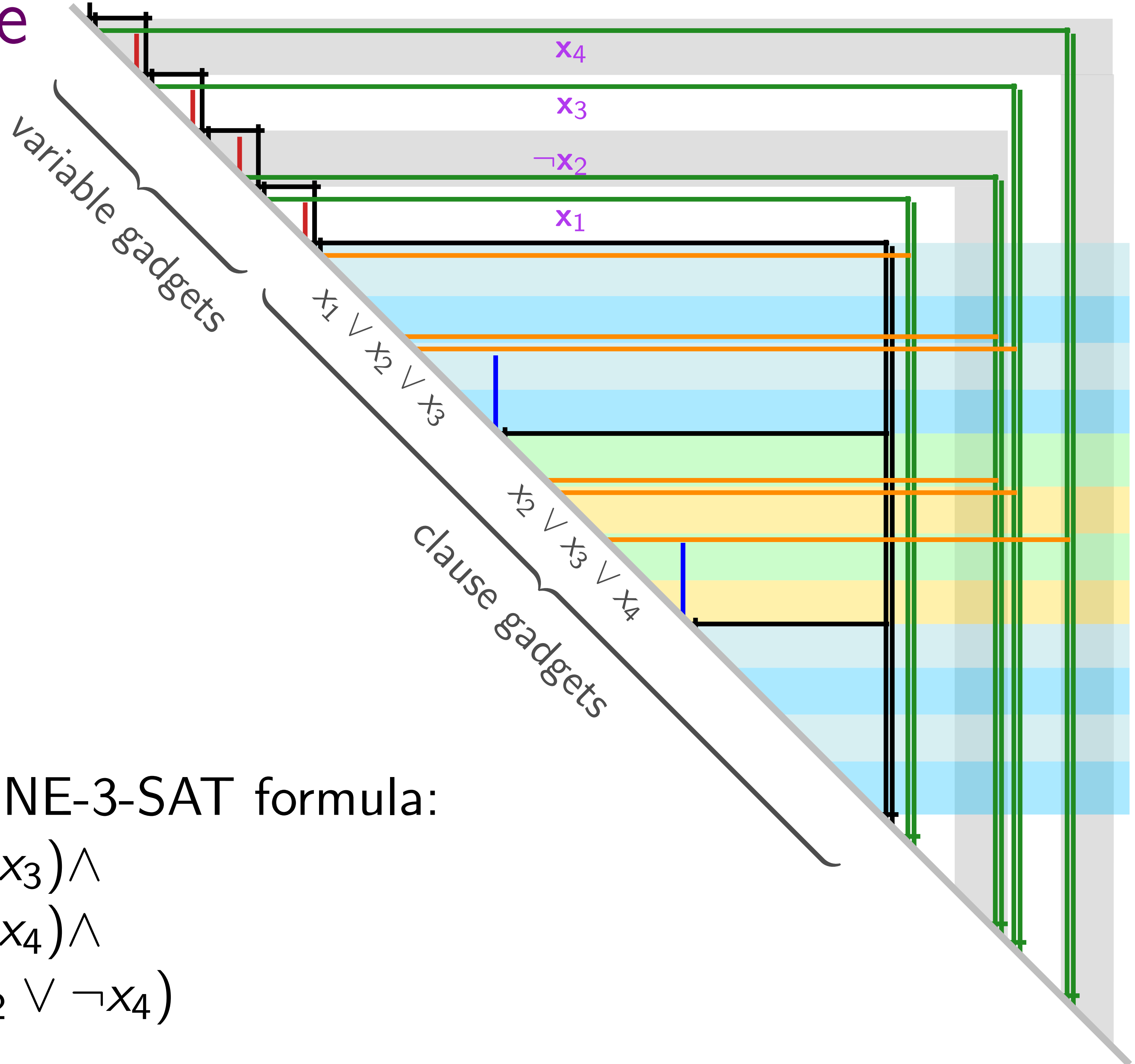

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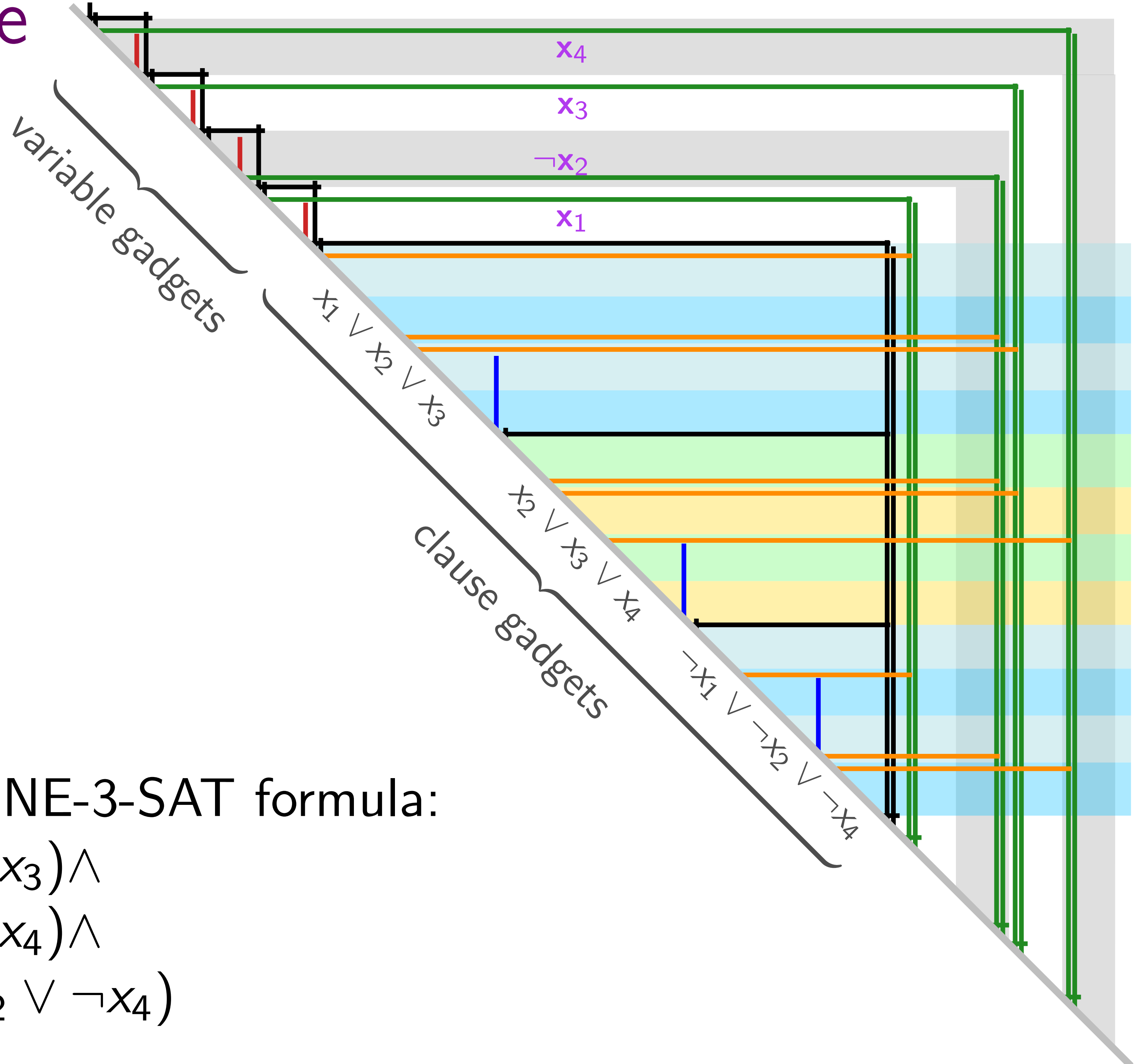

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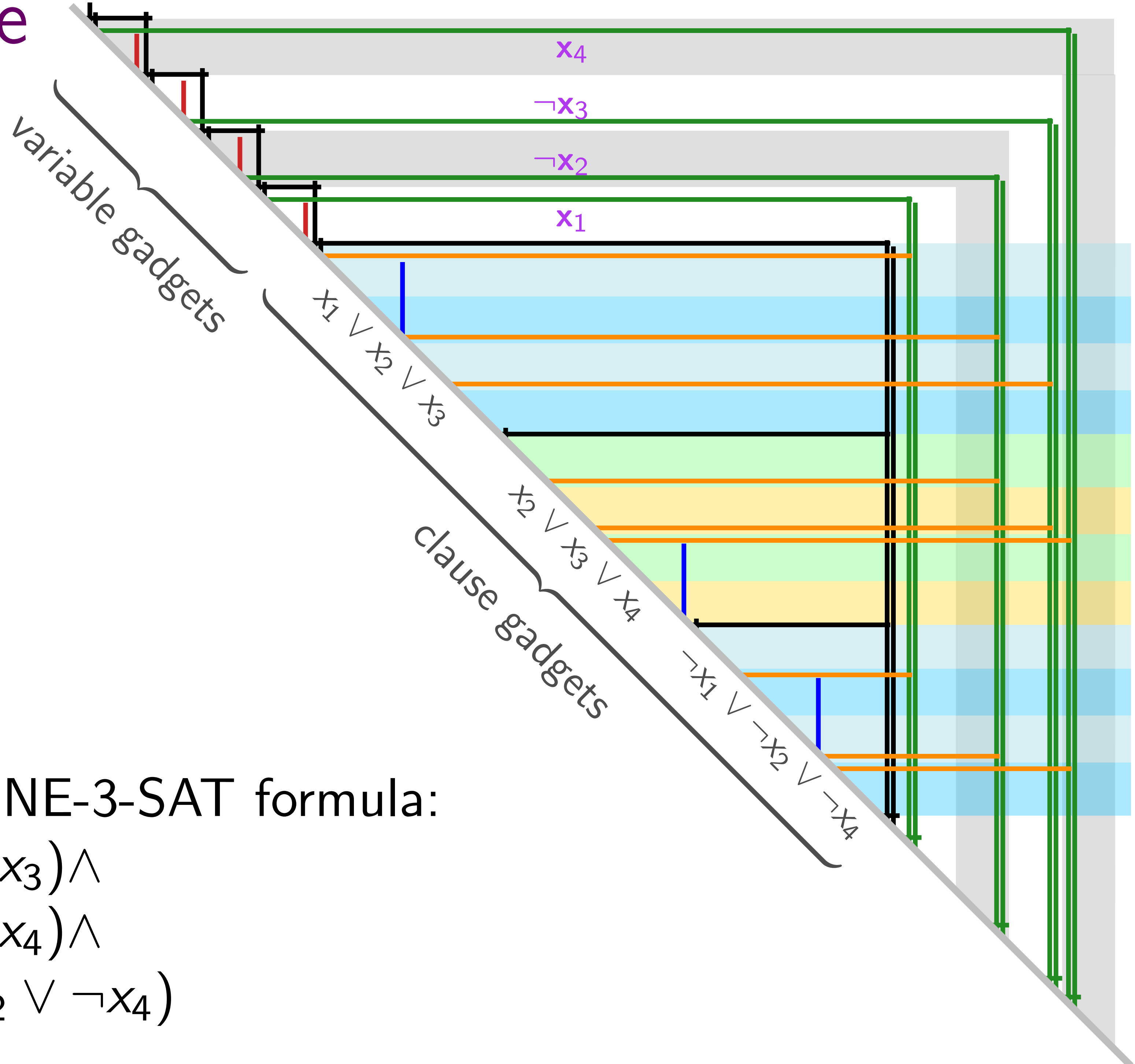

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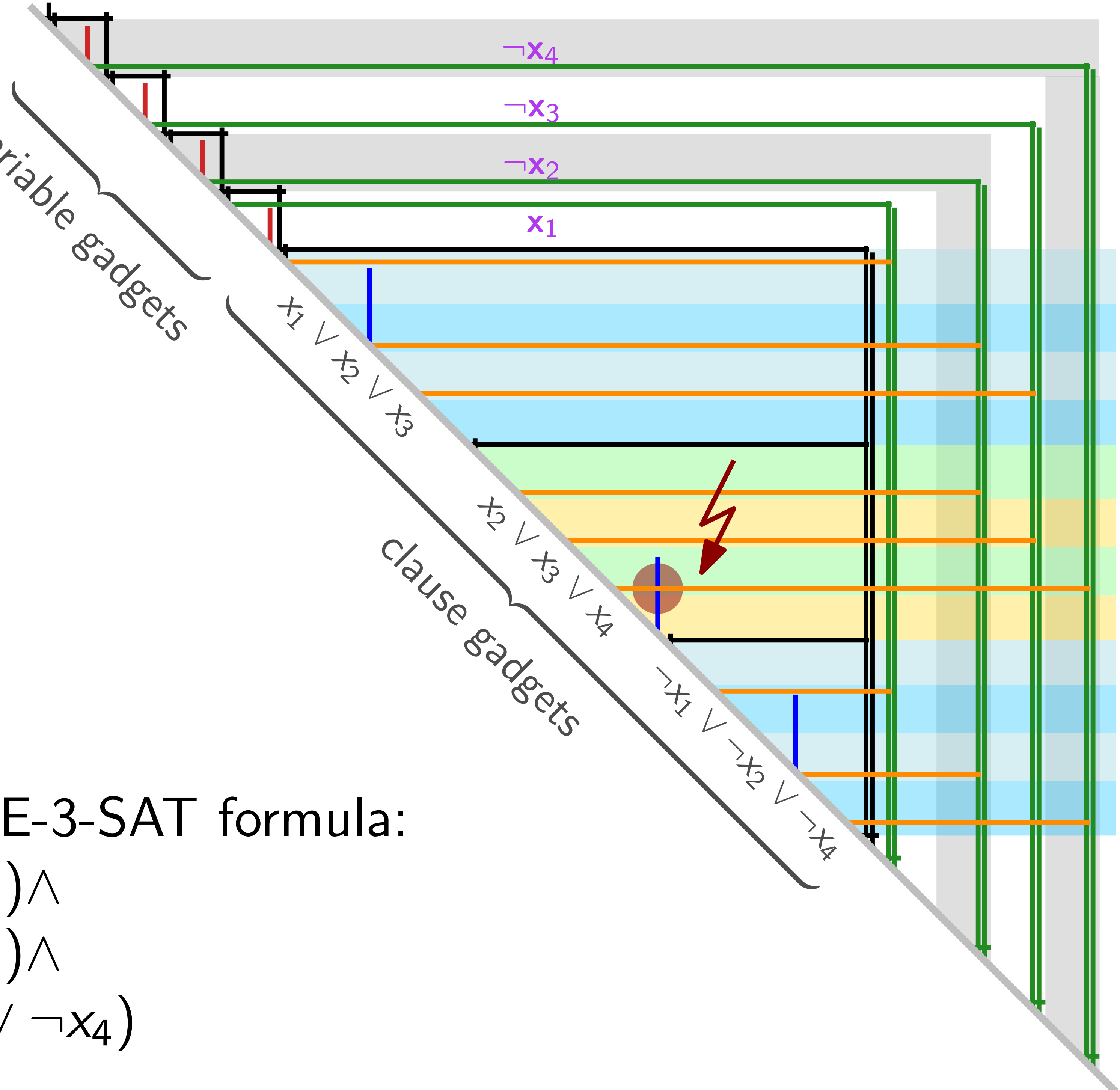

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$\mathsf{STICK}^{\mathsf{fix}}_{\mathsf{AB}}$ without isolated vertices

$\star$	$\textbf{STICK}_\star$	$\textbf{STICK}^{\textbf{fix}}_\star$
	?	NP-complete
A	$O(\|A\|\|B\|)$	NP-complete
AB	$O(\|E\|)$	in general: NP-complete w/o isolated vtc.: $O((\|A\|+\|B\|)^2)$

Uniqueness Lemma

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Lemma: In all stick representations of an instance of $\mathsf{STICK}_{\mathsf{AB}}$, the order of vertices on the ground line is the same after removing all isolated vertices. This order can be found in time $O(|E|)$.

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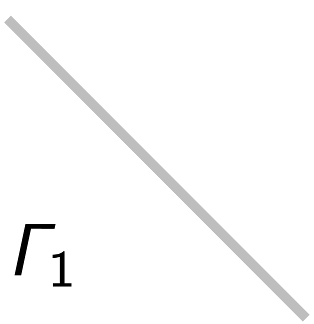

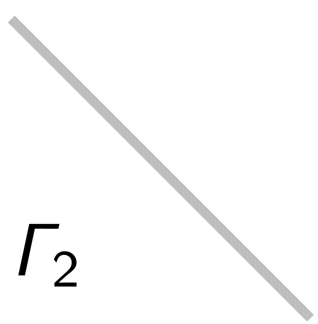

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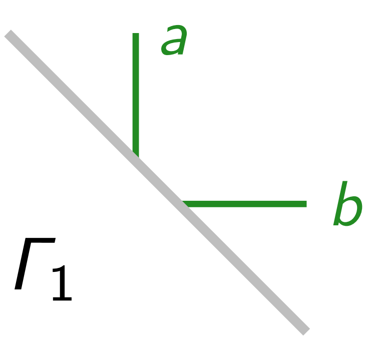

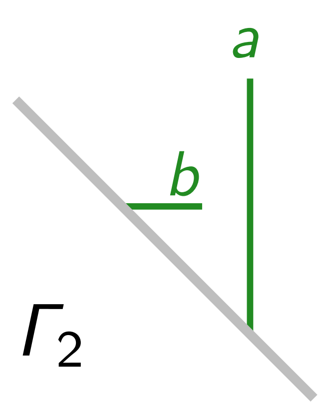

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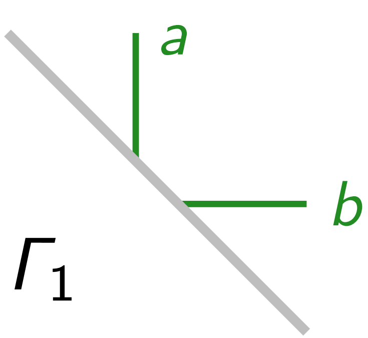

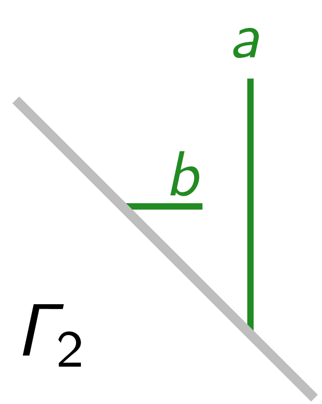

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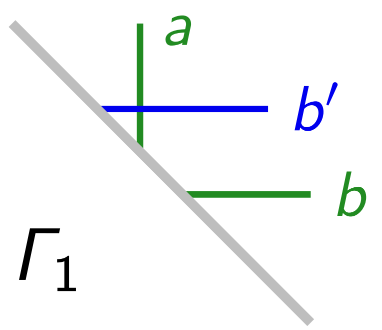

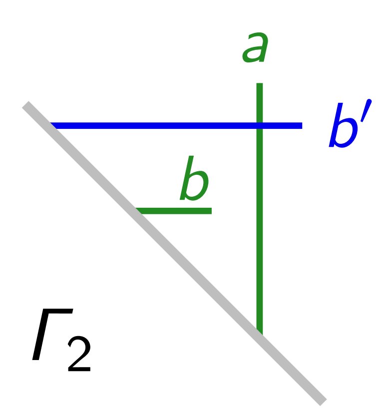

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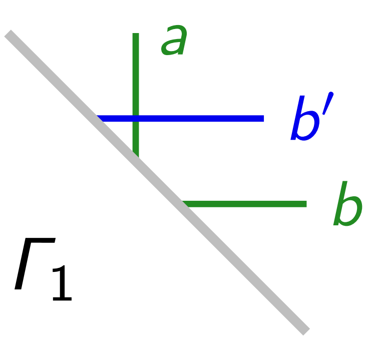

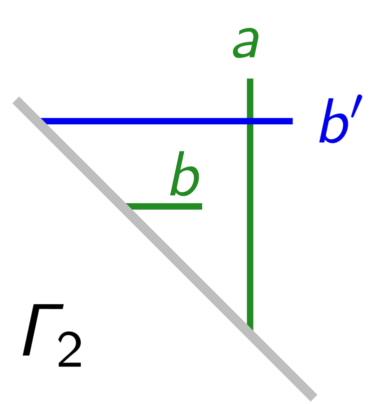

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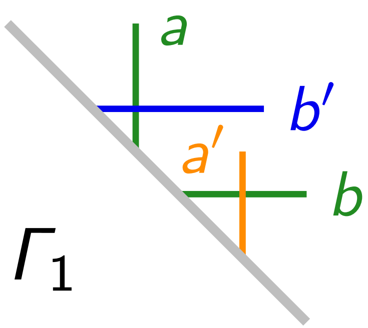

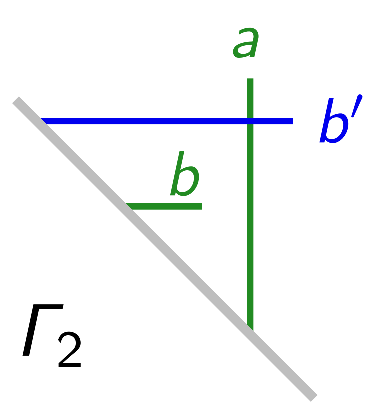

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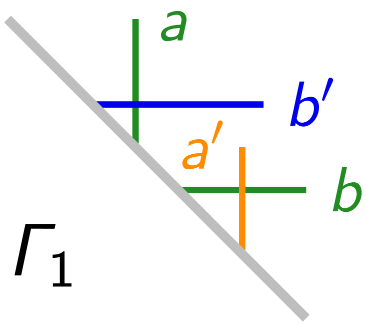
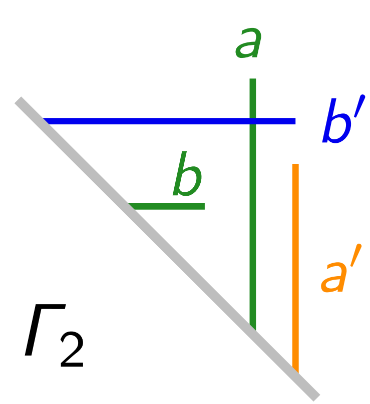


Γ_1



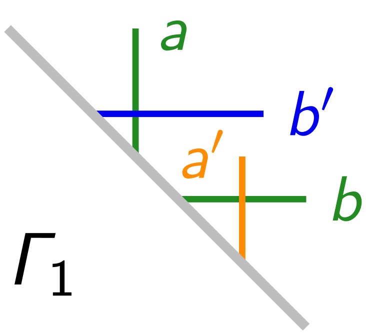
Γ_2

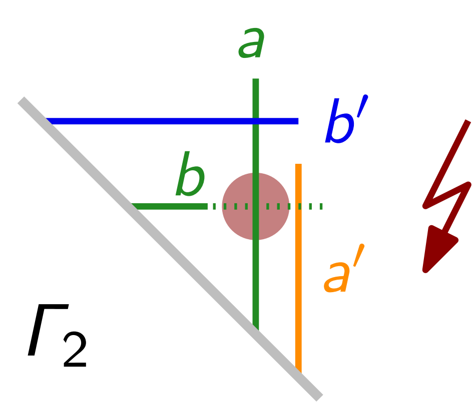
Uniqueness Lemma

Lemma: In all stick representations of an instance of STICK_{AB}, the order of vertices on the ground line is the same after removing all isolated vertices. This order can be found in time $O(|E|)$.

Proof (Sketch):

- Assume: stick representations Γ_1, Γ_2 have different order
- W.l.o.g. $a < b$ in Γ_1 and $b < a$ in Γ_2 $\Rightarrow ab \notin E$
- b' is adjacent to $a \Rightarrow b' < b$
- a' is adjacent to $b \Rightarrow a < a'$





Linear Program for $\text{STICK}_{\text{AB}}^{\text{fix}}$

- No isolated vertices $\Rightarrow$ Compute ordering $v_1, \ldots, v_n$

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$\Rightarrow$ **Isolated vertices make $\text{STICK}_{\text{AB}}^{\text{fix}}$ NP-hard**

Summary

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	still open	NP-complete
A	$O(\|A\|\|B\|)$	NP-complete
AB	$O(\|A\|\|B\|)$ $O(\|E\|)$ [De Luca et al. GD'18]	in general: NP-complete w/o isolated vtc.: $O((\|A\| + \|B\|)^2)$

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A	$O(\lvert A\rvert \lvert B\rvert)$	NP-complete (by reduction from MONO-3-SAT)
AB	$O(\lvert A\rvert \lvert B\rvert)$ [De Luca et al. GD'18] $O(\lvert E\rvert)$	in general: NP-complete w/o isolated vtc.: $O((\lvert A\rvert + \lvert B\rvert)^2)$