Enumeration of Simple Complete Topological Graphs

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Graph: $G = (V, E), |V| < \infty, E \subseteq \binom{V}{2}$

Topological graph: a drawing of a graph in the plane

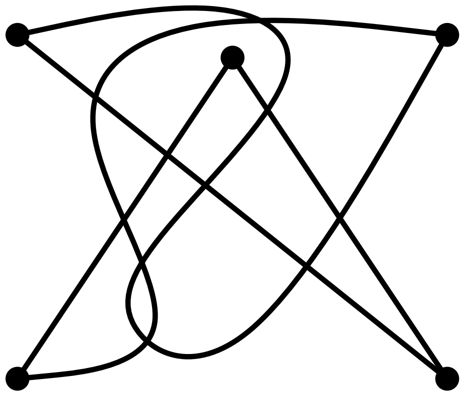
vertices $=$ points

edges $=$ simple curves

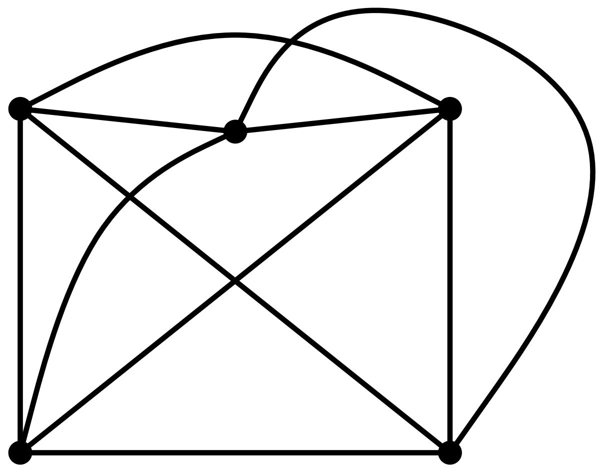
- edges do not pass through any vertices other than their end-points
- any two edges have only finitely many common points
- any intersection point of two edges is either a common end-point or a **crossing** (no touching allowed)
- at most two edges can intersect in one crossing

simple: any two edges have at most one common point

complete: $E = \binom{V}{2}$



topological graph

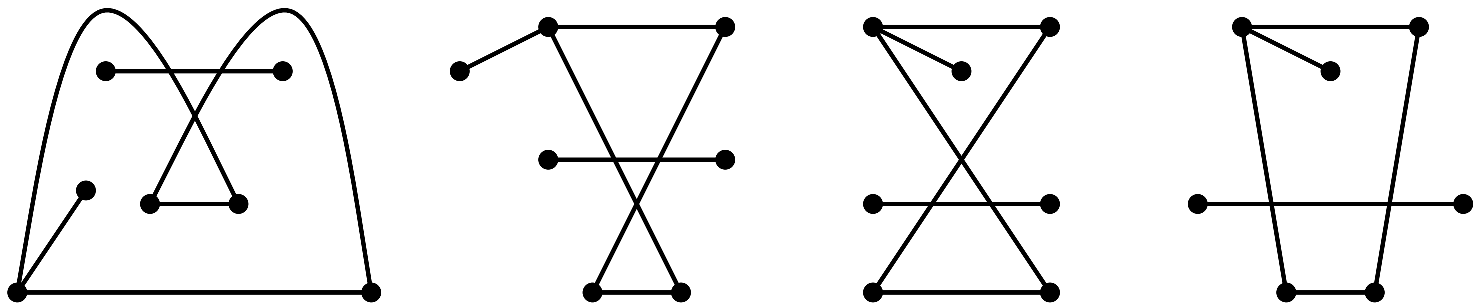


simple complete topological graph

Topological graphs G, H are

isomorphic if there exists a homeomorphism (of the sphere) which maps G onto H

weakly isomorphic if the same pairs of edges cross in G and in H



$T(n)$ = number of isomorphism classes

$T_{\mathrm{w}}(n)$ = number of weak isomorphism classes

of simple complete topological graphs on n vertices

Theorem [J. Pach, G. Tóth, 2004]:

$$2^{\Omega(n^2)} \leq T_{\mathrm{w}}(n) \leq 2^{O(n^2 \log n)}$$

Theorem 1:

$$T(n) = 2^{\Theta(n^4)}$$

Lower bounds are attained even for **extendable** graphs

Remark: The number of weak isomorphism classes of complete **geometric** graphs on n vertices is $2^{O(n \log n)}$

Graphs with maximum number of crossings

$T_{\mathrm{w}}^{\max}(n)$ = number of weak isomorphism classes of simple complete topological graphs on n vertices with $\binom{n}{4}$ crossings

Theorem [H. Harborth, I. Mengersen, 1992]:

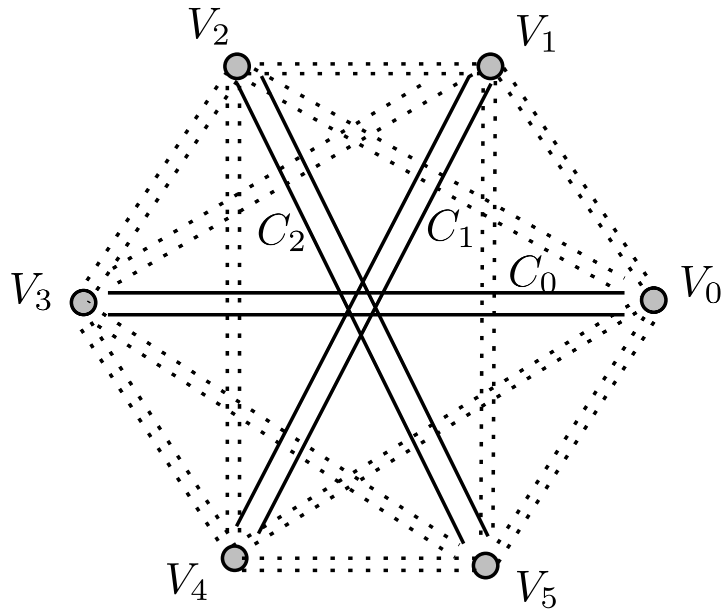
$$T_{\mathrm{w}}^{\max}(n) \geq e^{c\sqrt{n}}$$

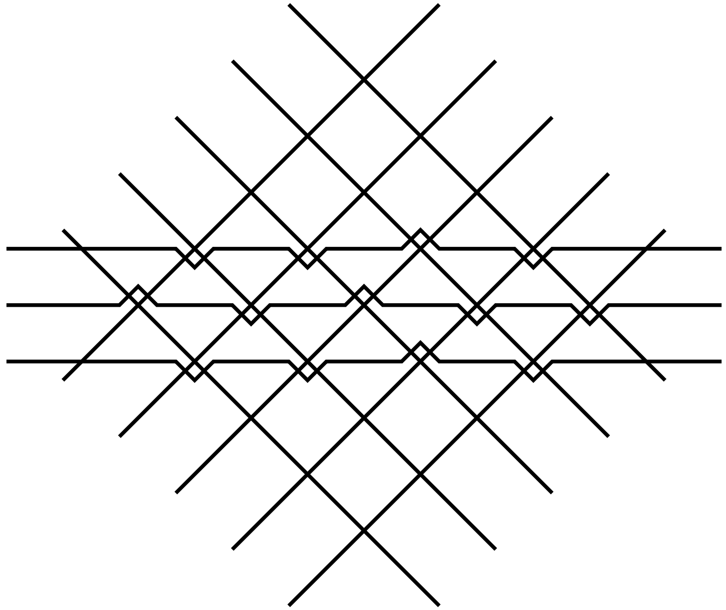
Theorem 2:

$$T_{\mathrm{w}}^{\max}(n) \geq 2^{n(\log n - O(1))}$$

Proof of Theorem 1

Lower bound: $T(n) \geq 2^{\Omega(n^4)}$





Upper bound: $T(n) \leq 2^{O(n^4)}$

Proposition: There is a two-to-one correspondence between rotation systems and weak isomorphism classes of simple complete topological graphs.

star-cut representation:

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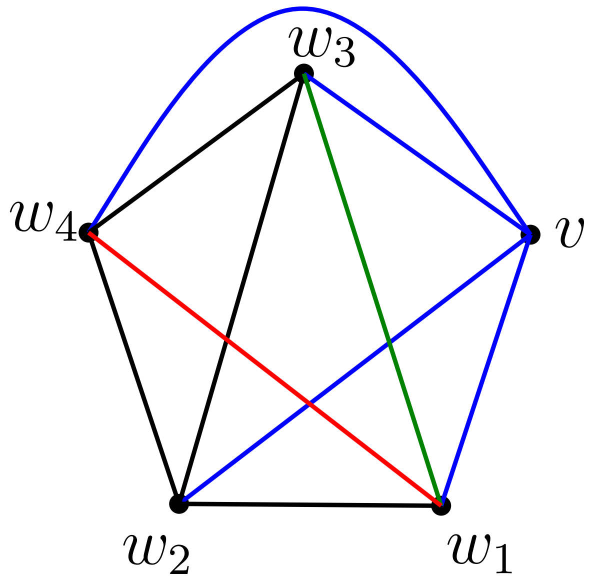

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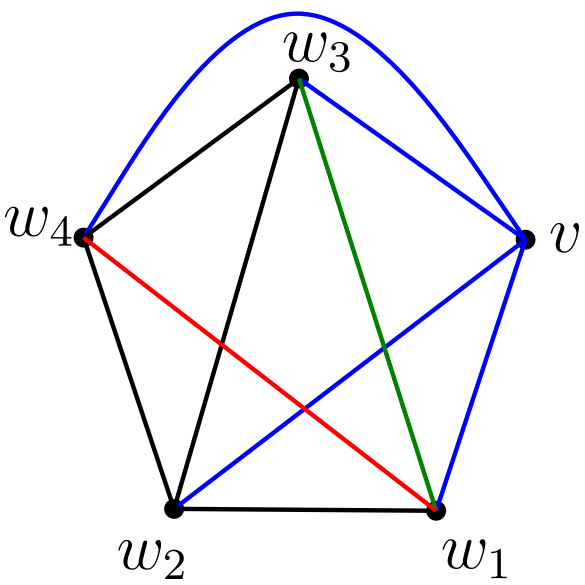

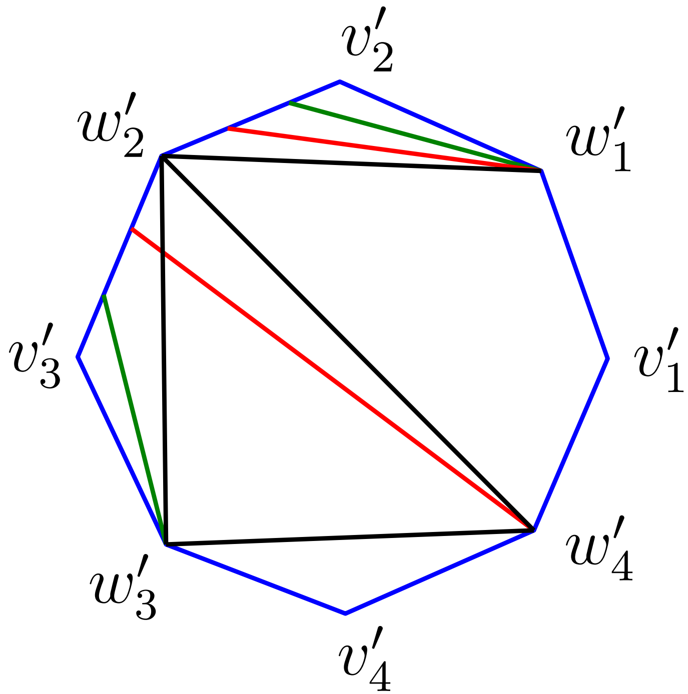

Proposition: The number of non-isomorphic simple arrangements of n pseudochords with fixed perimetric order inducing k crossings is at most 2^k.

$2^{O(n^2 \log n)}$ weak isomorphism classes

$O(n^3)$ pseudochords

$\Rightarrow 2^{O(n^3 \log n)}$ perimetric orders

$O(n^4)$ crossings

$\Rightarrow 2^{O(n^4)}$ arrangements

$\Rightarrow 2^{O(n^4)}$ isomorphism classes