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Enumeration of simple complete topological graphs ☆

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ABSTRACT

A *simple topological graph* $T = (V(T), E(T))$ is a drawing of a graph in the plane, where every two edges have at most one common point (an end-point or a crossing) and no three edges pass through a single crossing. Topological graphs G and H are *isomorphic* if H can be obtained from G by a homeomorphism of the sphere, and *weakly isomorphic* if G and H have the same set of pairs of crossing edges. We prove that the number of isomorphism classes of simple complete topological graphs on n vertices is $2^{\Theta(n^4)}$. We also show that the number of weak isomorphism classes of simple complete topological graphs with n vertices and $\binom{n}{4}$ crossings is at least $2^{n(\log n - O(1))}$, which improves the estimate of Harborth and Mengersen.

1. Introduction

A *topological graph* $T = (V(T), E(T))$ is a drawing of an (abstract) graph G in the plane with the following properties. The vertices of G are represented by a set $V(T)$ of distinct points in the plane and the edges of G are represented by a set $E(T)$ of simple curves connecting the corresponding pairs of points. We call the elements of $V(T)$ and $E(T)$ *vertices* and *edges* of T. The edges cannot pass through any vertices except their end-points. Any intersection point of two edges is either a common end-point or a *crossing*, a point where the two edges properly cross ("touching" of the edges is not allowed). We also require that any two edges have only finitely many intersection points and that no three edges pass through a single crossing. A topological graph is *simple* if every two edges have at most one common point (which is either a common end-point or a crossing). A topological graph is *complete* if it is a drawing of a complete graph.

☆ An extended abstract appeared in the proceedings of Eurocomb'07 [J. Kynčl, Enumeration of simple complete topological graphs, Electronic Notes in Discrete Mathematics 29 (2007) 295–299].

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We use two different notions of isomorphism to enumerate the topological graphs.

Topological graphs G and H are *weakly isomorphic* if there exists an incidence preserving one-to-one correspondence between $V(G)$, $E(G)$ and $V(H)$, $E(H)$ such that two edges of G cross if and only if the corresponding two edges of H do.

Topological graphs G and H are *isomorphic* if there exists a homeomorphism of the sphere which transforms G into H. (In the next section we state an equivalent combinatorial definition.)

Unlike the isomorphism, weak isomorphism can change the faces of the involved topological graphs, as well as the order in which one edge crosses other edges.

Pach and Tóth [9] proved the following lower and upper bounds on the number $T_{\mathrm{w}}(n)$ of weak isomorphism classes of simple complete topological graphs on n vertices. (I have proved the same bounds in [7] by a different and more complicated method, not having been aware of this result.)

Theorem 1 (*[7,9]*).

$$2^{\Omega(n^2)} \leq T_{\mathrm{w}}(n) \leq ((n-2)!)^n = 2^{O(n^2 \log n)}.$$

It is still an open problem which of these bounds is closer to the truth.

More precise estimate is known for the number $T_{\mathrm{w}}^{\mathrm{ext}}(n)$ of weak isomorphism classes of complete extendable topological graphs. A simple topological graph is called *extendable* if its edges can be extended to a pseudoline arrangement. A *pseudoline arrangement* is a collection of unbounded simple curves in the plane such that every two of the curves cross at most once and every common point of two curves is a proper crossing.

Theorem 2 (*[7]*).

$$T_{\mathrm{w}}^{\mathrm{ext}}(n) = 2^{\Theta(n^2)}.$$

The idea of the proof is the following. First, we find a close correspondence between complete extendable topological graphs on n vertices and pseudoarrangements of n points (i.e., arrangements of n points and $\binom{n}{2}$ pseudolines, where every pseudoline meets exactly two of the n points and every two pseudolines cross exactly once). Then, using the duality transform established by Goodman [4], we convert the problem into determining the number of non-isomorphic simple wiring diagrams of n pseudolines. This number is known to be of order $2^{\Theta(n^2)}$. (The best known upper bound, $2^{0.6988 \cdot n^2}$, has been established by Felsner [2].)

It is also known that the number of weak isomorphism classes of *geometric* graphs (i.e., (simple) topological graphs whose edges are drawn as straight-line segments) on n vertices is $2^{\Theta(n \log n)}$ [9].

We focus on enumeration with the stronger notion of isomorphism. This is motivated by earlier enumeration results for pseudoline arrangements [1,6], under a similar notion of isomorphism.

Let $T(n)$ be the the number of isomorphism classes of simple complete topological graphs. We prove the following asymptotic estimate on $T(n)$.

Theorem 3.

$$T(n) = 2^{\Theta(n^4)}.$$

We also improve the estimate of Harborth and Mengersen [5] on the number $T_{\mathrm{w}}^{\max}(n)$ of weak isomorphism classes of simple drawings of K_n with the maximum number of crossings. They proved that $T_{\mathrm{w}}^{\max}(n) \geq \mathrm{e}^{c\sqrt{n}}$ by considering the subgraphs formed by *empty* edges (i.e., the edges without crossing) and relating them with the partitions of an n-point set. Using the same construction but different enumeration method, we prove the following lower bound.

Theorem 4.

$$T_{\mathrm{w}}^{\max}(n) \geq 2^{n-5} \frac{(n-3)!}{n} \geq 2^{n(\log n - O(1))}.$$

An extended abstract of this paper appeared in proceedings of Eurocomb'07 [8].

2. Proof of Theorem 3

In this section we consider labeled graphs, i.e., the graphs whose vertices are distinguished by labels $1, 2, \ldots, n$. We require that an isomorphism of labeled graphs preserves the labels. But this makes almost no difference in the result, since the growth of the function $T(n)$ is much faster than $n!$, the number of distinct labelings of n vertices. In the case of Theorem 4, however, we will have to distinguish between labeled and unlabeled graphs.

First we state some additional definitions. A *rotation* of a vertex $v \in V(T)$ is the clockwise cyclic order in which the edges incident with v leave the vertex v. A *rotation system* of the topological graph T is the set of rotations of all its vertices. Similarly we define a *rotation* of a crossing c as the clockwise order in which the four portions of the two edges crossing at c leave the point c (note that each crossing has exactly two possible rotations). An *extended rotation system* of a topological graph is the set of rotations of all its vertices and crossings. Assuming that T and T' are drawings of the same abstract graph, we say that their (extended) rotation systems are *inverse* if for each vertex $v \in V(T)$ (and each crossing c in T) the rotation of v and the rotation of the corresponding vertex $v' \in V(T')$ are inverse cyclic permutations (and so are the rotations of c and the corresponding crossing c' in T'). For example, if T' is a mirror image of T, then T and T' have inverse (extended) rotation systems.

Now we can state the combinatorial definition of the isomorphism. Topological graphs G and H are *isomorphic* if (1) G and H are weakly isomorphic, (2) for each edge e of G the order of crossings with the other edges of G is the same as the order of crossings on the corresponding edge e' in H, and (3) the extended rotation systems of G and H are the same or inverse. This induces a one-to-one correspondence between the faces of G and H such that the crossings and the vertices incident with a face f of G appear along the boundary of f in the same (or inverse) cyclic order as the corresponding crossings and vertices in H appear along the boundary of the face f' corresponding to f. It follows from Jordan–Schönflies theorem that this definition is equivalent to the previous one (assuming that the graphs are drawn on the sphere).

Now we establish the lower bound on $T(n)$.

We denote by C_n a complete convex geometric graph with n vertices, i.e., a complete geometric graph whose vertices are in convex position (note that all these graphs are weakly isomorphic to each other).

Theorem 5. *The number of isomorphism classes of extendable graphs weakly isomorphic to C_n is at least $2^{\Omega(n^4)}$.*

Proof. For convenience, suppose that n is a multiple of 6. We partition the vertex set of the constructed graph(s) into six disjoint subsets $V_0, V_1, \ldots, V_5$, each of cardinality $\frac{n}{6}$. For each $k = 0, 1, \ldots, 5$, we place the vertices of the set V_k on the unit circle, inside a small neighborhood of the point $(\cos(\frac{k\pi}{3}), \sin(\frac{k\pi}{3}))$; see Fig. 1, left. For every two vertices $u \in V_k$ and $v \in V_l$ such that $|k - l| \neq 3$, we draw the edge uv as a straight-line segment. In Fig. 1, these edges are schematically represented by the dotted corridors. For $k \in \{0, 1, 2\}$, the edges between the sets V_k and V_{k+3} are drawn inside a narrow straight corridor C_k connecting these two sets such that all the crossings among this group of edges occur outside the region $C = C_0 \cap C_1 \cap C_2$, and for $k, l \in \{0, 1, 2\}$, $k \neq l$, all the crossings between the edges connecting V_k with V_{k+3} and the edges connecting V_l with V_{l+3} lie inside C. In the region C, the edges connecting V_2 with V_5 form $\frac{n^2}{36}$ parallel curves. Together with the edges connecting V_1 with V_4, they form an $\frac{n^2}{36} \times \frac{n^2}{36}$ grid inside C.

We partition the crossings of this grid into $\frac{n^2}{18} - 1$ horizontal rows R_i (parallel diagonals of the grid). Each (horizontal) edge e connecting V_0 with V_3 is drawn along one of the horizontal rows R_i (each edge is assigned to a different row). In the neighborhood of each crossing c in R_i we can decide whether the edge e passes above or below c; see Fig. 1, right. These two possibilities give us two non-isomorphic graphs, and the choices can be made independently at each crossing of the grid. If we lead the horizontal edges along the middle $\frac{n^2}{36}$ rows, we can make the choice at at least $\frac{n^4}{1728}$ different crossings, which gives us at least $2^{n^4/1728}$ non-isomorphic extendable drawings of C_n. $\square$

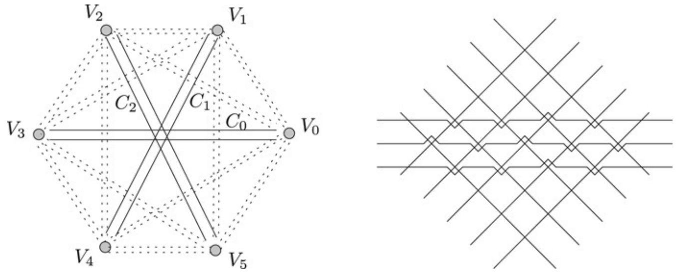

Fig. 1. An illustration of the proof of Theorem 5.

An *abstract topological graph* (briefly an *AT-graph*) is a pair (G, R) where G is a graph and $R \subseteq \binom{E(G)}{2}$ is a set of pairs of its edges. For a topological graph T which is a drawing of G we define R_T as a set of pairs of edges having at least one common crossing and we say that (G, R_T) is an AT-graph of T. A topological graph T is called a *realization* of (G, R) if $R_T = R$. If (G, R) has a realization, we say that (G, R) is *realizable*. We say that the AT-graph (G, R) is *simply realizable* if it has a *simple realization*, i.e., a drawing which is a simple topological graph. Note that two topological graphs are weakly isomorphic if and only if they are realizations of the same abstract topological graph.

A trivial upper bound for $T(n)$ is $2^{O(n^4 \log n)}$, since there are at most $(O(n^2))!$ orders of crossings on every edge and at most $2^{O(n^4)}$ different extended rotation systems. In the rest of this section we improve this trivial upper bound by eliminating the '$\log n$' from the exponent. For extendable graphs, we can get the result directly by using the upper bound on the number of non-isomorphic arrangements of $\binom{n}{2}$ pseudolines. However, we must choose a more tricky approach for the non-extendable graphs.

We begin with a key observation which establishes a connection between simply realizable complete AT-graphs and the rotation systems of their drawings.

Proposition 6. (1) *The rotation system of a simple complete topological graph G uniquely determines which pairs of edges of G cross, i.e., two simple complete topological graphs with the same rotation system are weakly isomorphic.*

(2) *If two simple complete topological graphs are weakly isomorphic, then their rotation systems are either the same or inverse.*

Note that these properties are a specialty of complete graphs: one rotation system may correspond to many weakly non-isomorphic non-complete simple topological graphs, and non-complete graphs can be drawn with many different rotation systems. Trivial examples of such graphs are paths and stars.

We will denote the rotation system of a topological graph G as $\mathcal{R}(G)$ and we will represent it as a sequence of rotations of its vertices. The rotation $\mathcal{R}(v)$ of a vertex v will be represented by a cyclic sequence of the labels of the remaining vertices.

The first part of this proposition, which actually establishes the upper bound on $T_{\mathrm{w}}(n)$, was also proved by Pach and Tóth [9], and the second part was independently proved by Gioan [3]. We include the proof here for completeness.

Proof. Both statements trivially hold for graphs with at most 3 vertices. Thus, we further consider only graphs with $n \geq 4$ vertices.

(1) Let G be a simple complete topological graph with n vertices and the rotation system $\mathcal{R}(G)$. We first observe that the AT-graph of G is uniquely determined by the AT-graphs of all 4-vertex complete subgraphs of G and that the rotation system of every subgraph of G is uniquely determined by $\mathcal{R}(G)$ and can be easily derived from $\mathcal{R}(G)$ by deleting appropriate rotations and labels.

So we only need to verify that the rotation system of a simple complete topological graph H on 4 vertices uniquely determines the AT-graph of H.

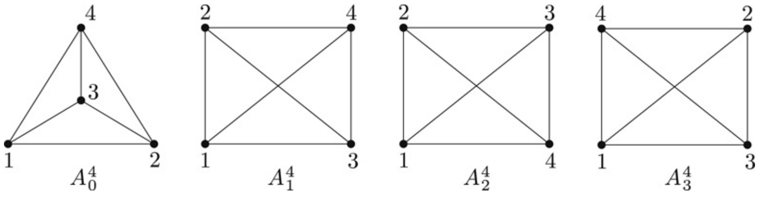

Fig. 2. Drawings of all realizable AT-graphs on 4 vertices.

Let $V(H) = \{1, 2, 3, 4\}$. There are exactly four distinct complete AT-graphs with the vertex set $V(H)$: $A_0^4 = (K_4, \emptyset)$, $A_1^4 = (K_4, \{\{\{1, 4\}, \{2, 3\}\}\})$, $A_2^4 = (K_4, \{\{\{1, 3\}, \{2, 4\}\}\})$ and $A_3^4 = (K_4, \{\{\{1, 2\}, \{3, 4\}\}\})$; see Fig. 2 for the drawings of these four graphs.

Each vertex of H has two possible rotations, so there are $2^4 = 16$ possible combinations of rotations. However, we show that only 8 of them are rotation systems of a simple drawing of K_4.

There are four non-isomorphic simple drawings of K_4 with labeled vertices and they correspond exactly to the four drawings in Fig. 2. Since an isomorphism of topological graphs preserves or inverts the rotation system, each of the graphs A_i^4, $i = 0, 1, 2, 3$, can be drawn with exactly two mutually inverse rotation systems (either the same way as in Fig. 2 or as a mirror image of that drawing). It remains to verify that no rotation system corresponds to more than one of the graphs A_i^4. For this purpose we list the four rotation systems of the drawings of the graphs A_i^4 with the rotation $(1, 2, 3)$ at the vertex 4:

Graph	Rotation system
A_0^4	$((2, 4, 3), (1, 3, 4), (1, 4, 2), (1, 2, 3))$
A_1^4	$((2, 4, 3), (1, 4, 3), (1, 2, 4), (1, 2, 3))$
A_2^4	$((2, 3, 4), (1, 3, 4), (1, 2, 4), (1, 2, 3))$
A_3^4	$((2, 3, 4), (1, 4, 3), (1, 4, 2), (1, 2, 3))$

This finishes the proof of the first part of the proposition. We have also proved the second part for $n = 4$.

(2) It remains to prove the second statement for graphs with $n \geq 5$ vertices. First we separately consider the case $n = 5$ and then we use it to extend the statement to graphs with more than five vertices.

There are 5 non-isomorphic simple drawings of K_5 (see [5] or Fig. 3) and each of them is a realization of a different AT-graph (i.e., the weak isomorphism of simple drawings of K_5 implies the isomorphism). As for 4-vertex graphs, it follows that each AT-graph on 5 vertices can be drawn with only two mutually inverse rotation systems.

Let A be a simply realizable complete AT-graph with the vertex set $\{1, 2, \ldots, n\}$, $n \geq 6$. We know that each complete 5-vertex subgraph of A has only two possible rotation systems. Suppose that the rotation system of $A[\{1, 2, 3, 4, 5\}]$, the induced subgraph of A with the vertices 1, 2, 3, 4, 5, is fixed (in some simple realization of A). We show that then the rotation system of every other 5-vertex complete subgraph of A is uniquely determined.

Lemma. *Let B and C be two 5-vertex complete subgraphs of A with exactly 4 common vertices. Then the rotation system $\mathcal{R}(B)$ uniquely determines the rotation system $\mathcal{R}(C)$.*

Proof of Lemma. Without loss of generality, let $V(B) = \{1, 2, 3, 4, 5\}$, $V(C) = \{1, 2, 3, 4, 6\}$ and let the rotation of the vertex 1 in $\mathcal{R}(B)$ be $(2, 3, 4, 5)$. Then the rotation of 1 in $A[\{1, 2, 3, 4\}]$ is $(2, 3, 4)$ and it must be a subsequence of a rotation of 1 in $\mathcal{R}(C)$. But this always happens for exactly one of the pairs of inverse cyclic permutations of the set $\{2, 3, 4, 6\}$, thus the rotation of 1 in C is uniquely determined and so is the whole rotation system of C. $\square$

By repeated use of this lemma we obtain that the rotation system of every complete subgraph of A on 5 (and also 4) vertices is uniquely determined by $\mathcal{R}(A[\{1, 2, 3, 4, 5\}])$. It remains to show

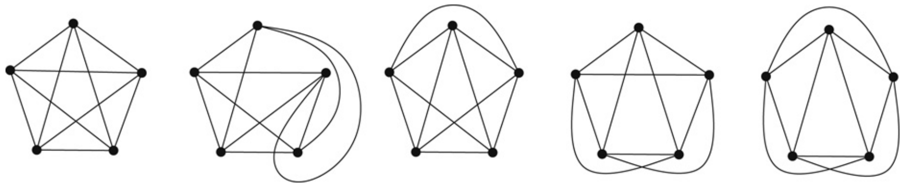

Fig. 3. All five non-isomorphic simple drawings of K_5 [5].

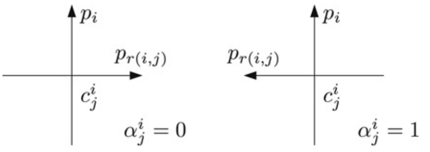

Fig. 4. An encoding of the crossings on the pseudochord p_i.

that this also uniquely determines the rotation of each vertex in A. But this easily follows from the fact that a cyclic order of a finite set X is uniquely determined by the cyclic order of all 3-element subsets of X (actually, it suffices to know the orders of the triples containing one fixed vertex). It follows that a simple realization of A can have only two possible rotation systems. □

An *arrangement of pseudochords* is a finite set M of simple curves in the plane with end-points on a common simple closed curve C_M, such that all the curves from M lie in the region bounded by C_M and every two curves in M have at most one common point, which is then a proper crossing. The elements of M are called *pseudochords*. The arrangement M is *simple* if no three pseudochords from M share a common crossing. A *perimetric order* of M is the counter-clockwise cyclic order of the end-points of the pseudochords of M on C_M. Note that the perimetric order of M determines which pairs of pseudochords cross and which do not. Two (labeled) arrangements of pseudochords are *isomorphic* if they have the same perimetric order and the same orders of crossings on the corresponding pseudochords.

The following proposition is a generalization of Felsner's [2] enumeration of simple wiring diagrams.

Proposition 7. *The number of non-isomorphic simple arrangements of n pseudochords with fixed perimetric order inducing k crossings is at most 2^k.*

Proof. Let $M = \{p_1, p_2, \dots, p_n\}$ be an arrangement of pseudochords with a given perimetric order. Fix an arbitrary orientation for each pseudochord $p_i \in M$ and denote by a_i and b_i its initial and terminal end-points. Further, denote by $c_1^i, c_2^i, \dots, c_{k_i}^i$ the crossings of p_i with other pseudochords from M, in the order from a_i to b_i. To every p_i we assign a vector $\alpha^i = (\alpha_1^i, \alpha_2^i, \dots, \alpha_{k_i}^i) \in \{0, 1\}^{k_i}$, where $\alpha_j^i = 0$ if the pseudochord $p_{r(i,j)}$ crossing p_i at c_j^i is oriented from the left to the right (i.e., if the rotation of c_j^i is $(a_i, a_{r(i,j)}, b_i, b_{r(i,j)})$), in the other case $\alpha_j^i = 1$; see Fig. 4.

The sum of the lengths of the vectors α^i is equal to $\sum_{i=1}^n k_i = 2k$, but every crossing is encoded by two bits of different values. Hence, there are at most 2^k different sequences $(\alpha^1, \alpha^2, \dots, \alpha^n)$ encoding an arrangement with a given perimetric order and a fixed orientation of pseudochords.

It remains to show that this encoding is injective, i.e., that we can uniquely reconstruct the isomorphism class of M from the vectors $\alpha^1, \alpha^2, \dots, \alpha^n$ by identifying the pseudochords $p_{r(i,j)}$.

We proceed by induction on k. For the arrangements without crossings there is only one isomorphism class with a fixed perimetric order. Now, suppose that we can reconstruct the isomorphism class for the arrangements with at most $k - 1$ crossings and take a sequence $\alpha = (\alpha^1, \alpha^2, \dots, \alpha^n)$ encoding an arrangement M of oriented pseudochords $\{p_1, p_2, \dots, p_n\}$ with a given cyclic order of the end-points $a_i, b_i, i = 1, 2, \dots, n$.

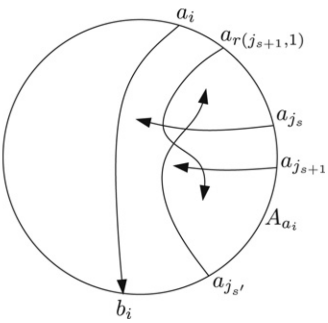

Fig. 5. $p_{j_{s'}}$ cannot be the first pseudochord crossing p_{j_s}.

We can suppose that M does not contain empty pseudochords, i.e., the pseudochords without crossings (we can remove the empty pseudochords and investigate the order of crossings in the resulting arrangement M'). Each pseudochord p_i divides the (topological) circle C_M into two open arcs, A_{a_i} and A_{b_i}, where A_{a_i} is the arc starting at a_i (and ending at b_i) in the clockwise direction. Clearly, there exists a pseudochord p_i such that one of the arcs A_{a_i}, A_{b_i} contains at most one end-point of each pseudochord $p_j, j \neq i$. Suppose that it is the arc A_{a_i} (the other case is symmetric). We can also suppose that all the pseudochords crossing p_i start in the arc A_{a_i} (if not, we sequentially revert the orientation of each pseudochord p_j ending in A_{a_i} and change the value of every bit $\alpha_{j'}^{i'}$ corresponding to a crossing on p_j). Let $a_{j_1}, a_{j_2}, \ldots, a_{j_t}$ be the clockwise order of the end-points of all the pseudochords crossing p_i. Put $j_0 = i$. We observe that $\alpha^{j_0} = (0, 0, \ldots, 0)$ and that $\alpha_1^{j_t} = 1$ (the pseudochord crossing p_{j_t} closest to its initial end-point a_{j_t} is one of the $p_{j_0}, p_{j_1}, \ldots, p_{j_{t-1}}$). It follows that there exists $s \in \{0, 1, \ldots, t-1\}$ such that $\alpha_1^{j_s} = 0$ and $\alpha_1^{j_{s+1}} = 1$.

Claim. *The first crossing on the pseudochords p_{j_s} and $p_{j_{s+1}}$ is their common crossing, i.e., $r(j_s, 1) = j_{s+1}$ and $r(j_{s+1}, 1) = j_s$.*

Proof of Claim. The first pseudochord which crosses p_{j_s} must initiate in the arc $A_{a_{j_s}} \cap A_{a_i}$. Hence, it is one of the $p_{j_{s+1}}, p_{j_{s+2}}, \ldots, p_{j_t}$. For contradiction, suppose that $r(j_s, 1) = j_{s'}$, where $s' \in \{s+2, s+3, \ldots, t\}$. But then the pseudochord $p_{r(j_{s+1},1)}$ is forced to cross $p_{j_{s'}}$ before $p_{j_{s+1}}$ (see Fig. 5) and then terminate in the arc A_{a_i}, which contradicts the choice of p_i. $\square$

As we have identified the first crossing $c = c_1^{j_s} = c_1^{j_{s+1}}$ on two pseudochords with adjacent initial end-points on C_M, we can make an induction step and swap the end-points a_{j_s} and $a_{j_{s+1}}$ in the perimetric order of M. In this way we obtain a perimetric order inducing an arrangement with $k-1$ crossings. We also delete the first value from the vectors α^{j_s} and $\alpha^{j_{s+1}}$ and obtain an encoding α'. From the induction hypothesis, α' uniquely determines the isomorphism class of an arrangement M' which is a "sub-arrangement" of M obtained from M by deleting the initial parts of p_{j_s} and $p_{j_{s+1}}$, including their common crossing c, and redrawing C_M appropriately. It follows that also the isomorphism class of M is uniquely determined by the given perimetric order and the encoding sequence α. $\square$

The following theorem together with the upper bound on $T_w(n)$ gives the upper bound on $T(n)$ in Theorem 3.

Theorem 8. *A complete AT-graph with n vertices has at most $2^{O(n^4)}$ non-isomorphic simple realizations.*

Proof. Let G be a simple realization of a given complete AT-graph A. By Proposition 6(2), G can have two different rotation systems. Let us fix one of them.

Now we introduce a *star-cut representation* of the graph G. Choose an arbitrary vertex v and denote by $w_1, w_2, \ldots, w_{n-1}$ the remaining vertices of G so that $\mathcal{R}(v) = (w_1, w_2, \ldots, w_{n-1})$. Let $S(v)$ denote the union of all the edges vw_i of G ($S(v)$ is a "topological star" with the central vertex v). If we consider

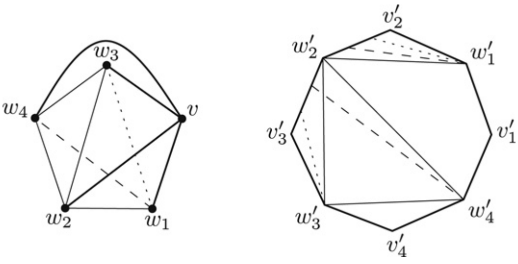

Fig. 6. A simple drawing of K_5 and its star-cut representation.

G drawn on the sphere S^2, the set $S^2 \setminus S(v)$ is mapped by a homeomorphism Φ onto an open regular $2(n-1)$-gon D in the plane. We can visualize this by cutting the sphere along the edges of the star $S(v)$ and then unpacking the resulting surface in the plane. The map Φ^{-1} can be continuously extended to the closure of D, giving a natural correspondence between the vertices and edges of D and the vertices and edges in $S(v)$: each vertex w_i corresponds to one vertex w_i' of D and the vertex v of G corresponds to $n-1$ vertices $v_1', v_2', \ldots, v_{n-1}'$ of D. If Φ preserves the orientation, the counter-clockwise order of the vertices of D is $v_1', w_1', v_2', w_2', \ldots, v_{n-1}', w_{n-1}'$. Each edge $vw_i \in E(G)$ splits into two adjacent edges $v_i'w_i'$ and $v_{i+1}'w_i'$; see Fig. 6. During the cutting operation every edge e_k of G not incident with v can be cut into several pieces. Since e_k crosses each edge of $S(v)$ at most once, it is cut into at most n pieces $e_{k,j}$. Every crossing of the edge e_k with an edge vw_i corresponds to two end-points of two different pieces $e_{k,j}, e_{k,j'}$ lying on the edges $v_i'w_i'$ and $v_{i+1}'w_i'$.

Let C be the boundary of D. In the neighborhood of every vertex v_i' we redraw a small portion of C through the interior of D such that it avoids the vertex v_i', and we shorten the edges incident with v_i' appropriately. We obtain a topological circle C' and an arrangement $\mathcal{A}_{G,v}$ of $O(n^3)$ pseudochords $e_{k,j}$ with end-points on C', which has $O(n^4)$ crossings.

Clearly, we can reconstruct the original graph G from $\mathcal{A}_{G,v}$ by reverting the star-cut operation, i.e., the isomorphism class of G is determined by the perimetric order and the isomorphism class of $\mathcal{A}_{G,v}$ and by the orientation of its edges $e_{k,j}$. The arrangement $\mathcal{A}_{G,v}$ has at most $O(n^3)! = 2^{O(n^3 \log n)}$ different perimetric orders (including the orientations of the pseudochords) and, by Proposition 7, at most $2^{O(n^4)}$ different isomorphism classes. It follows that the number of non-isomorphic simple drawings of a given complete AT-graph A is at most $2^{O(n^4)}$. □

3. Graphs with maximum number of crossings

The maximum number of crossings in a simple complete topological graph with n vertices is $\binom{n}{4}$, since every induced subgraph on four vertices has at most one crossing. An example of such a graph is the convex graph C_n.

Harborth and Mengersen [5] introduced a construction of many weakly non-isomorphic simple complete topological graphs with n vertices and $\binom{n}{4}$ crossings. The construction begins with the graph $G_4 = C_4$ and then proceeds in $n-4$ steps. In the kth step, we first select a vertex $v_k \in V(G_{k+3})$ and two edges $e_k, e_k' \in E(G_{k+3})$ incident with v_k and adjacent in its rotation. Then, we add a new vertex w_k in a small neighborhood of v_k between the edges e_k and e_k'. Finally, we draw the edges between w_k and the remaining vertices of G_{k+3} around the vertex v_k and then along the edges incident with v_k so that they do not cross e_k'; see Fig. 7. We obtain a simple complete graph G_{k+4} with $\binom{k+4}{4}$ crossings.

Harborth and Mengersen [5] showed that this construction gives graphs where the subgraph formed by empty edges is almost any union of paths on n vertices, except the empty graph and the graphs with a single path having at most three edges. This yields a lower bound $e^{\Omega(\sqrt{n})}$ on the number $T_{\mathrm{w}}^{\max}(n)$.

We prove a better lower bound given by this construction by inductive enumeration of labeled graphs. The main idea is that almost every choice of the vertex v_k and the edges e_k, e_k' yields a different set of pairs of crossing edges.

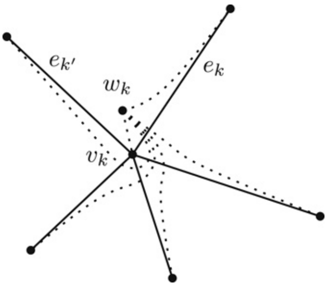

Fig. 7. Extending a complete graph by a vertex w_k.

Theorem 9. *The number of weak isomorphism classes of labeled simple complete topological graphs with n vertices and $\binom{n}{4}$ crossings is at least $2^{n-5}(n-1)!(n-3)!$.*

Proof. The theorem holds for $n = 4$, since there are exactly 3 weakly non-isomorphic simple drawings of K_4 with one crossing; see Fig. 2. It remains to prove that the addition of a vertex w_k to the graph G_{k+3} in the kth step can produce at least $2(k+3)(k+1)$ weakly non-isomorphic graphs G_{k+4}.

Let $(u_1, u_2, \ldots, u_{k+2})$ be the rotation of the vertex v_k in G_{k+3}. We claim that if v_k is fixed, then each choice of the edges e_k and e'_k yields a different weak isomorphism class of G_{k+4}. Suppose, without loss of generality, that $e_k = v_k u_1$ and $e'_k = v_k u_{k+2}$. Then for each $i \in \{1, 2, \ldots, k+2\}$, the number of crossings of the edge $v_k u_i$ with the edges incident with w_k is $k + 2 - i$. Hence, we can uniquely determine the edges e_k, e'_k from the AT-graph of G_{k+4}.

Among the edges incident with w_k, the edge $w_k v_k$ is always empty and the edges $w_k u_i$, where $i \geq 2$, have at least one crossing. The edge $w_k u_1$ leading close to the edge $e_k = v_k u_1$ is empty if and only if e_k is empty. Hence, if e_k is not empty, the AT-graph of G_{k+4} determines the vertex v_k as the only vertex of G_{k+4} which is joined to w_k by an empty edge. If e_k is empty, the AT-graph of G_{k+4} gives us exactly two possibilities for the vertex v_k.

There are $2(k+3)(k+2)$ different choices of the triple (v_k, e_k, e'_k), four for every fixed edge e_k. Every simple complete topological graph with n vertices and the maximum number of crossings has at most n empty edges [5], thus there are at most $2(k+3)$ pairs of triples (v_k, e_k, e'_k) yielding two weakly isomorphic graphs G_{k+4}. It follows that in the kth step we can obtain at least $2(k+3)(k+2) - 2(k+3) = 2(k+3)(k+1)$ weakly non-isomorphic graphs G_{k+4}. □

By dividing the lower bound from the last theorem by $n!$, the number of distinct labelings of the n vertices, we directly obtain the statement of Theorem 4.

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