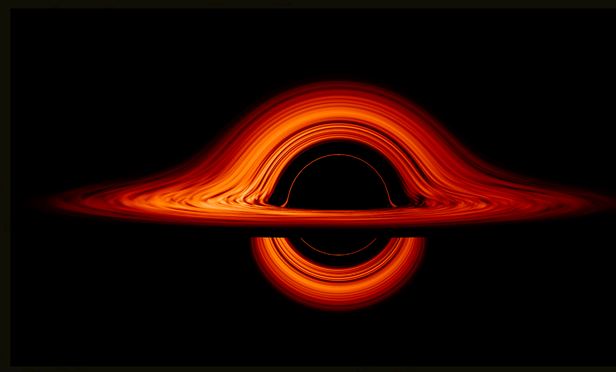
Analysis of particle motion in the gravitational field of black holes



Evelyn-Andreea Ester, Dr Stephen West

Royal Holloway, University of London

Project Abstract

Analyse how we can obtain arbitrarily high center of mass energies in the vicinity of the 4 classical types of black holes, as parametrised by their mass, charge and angular momentum. Additionally, investigate types of orbits for test particles in the Schwarzschild and Kerr backgrounds, using the effective potential and numerically integrating the obtained geodesics.

Introduction

The "no-hair theorem" states that the external gravitational and electromagnetic fields of a stationary black hole (a black hole that has settled down into its "final" state) are determined uniquely by the hole's mass M, charge Q, and intrinsic angular momentum a -i.e., the black hole can have no "hair" (no other independent characteristics). Bañados, Silk and West showed that Schwarzschild and Kerr black holes can act as particle accelerators and the center of mass energy of two test particles can be arbitrarily high if the collision occurs near the horizon of a Kerr black hole. Therefore, the extremal Kerr black hole could be regarded as a Planck-energy scale collider, which might allow us to explore ultra high energy collisions and astrophysical phenomena. The discussion is extended to include the Reissner-Nordstrøm and Kerr-Newman backgrounds, which are the charged vacuum solutions of the Einstein-Maxwell equations.

Analysis

Schwarzschild black hole

Consider the Schwarzschild background, which describes a static uncharged black hole. Due to spherical symmetry of the metric, energy and angular momentum of particles are conserved. Conservation of angular momentum implies the orbits of test particles lie in a plane, analogous to Newtonian theory. Therefore, we consider geodesic motion in the equatorial plane of the black hole, given by $\theta = \pi/2$. We use geodesics to derive the center of mass energy of two particles mass m_0 colliding in the vicinity of the black hole, given by:

$$\frac{1}{2m_0^2}\left(E^{\text{Schw}}_{\text{cm}}\right)^2 = \frac{1}{r^2(r-2)}\Big(2r^2(r-1) - l_1 l_2(r-2) - \sqrt{2r^2 - l_1^2(r-2)}\sqrt{2r^2 - l_2^2(r-2)}\Big). \quad (1)$$

In the limit as the particles approach the horizon, we have:

$$E^{\text{Schw}}_{\text{cm}}(r \to r_H) = \frac{m_0}{2}\sqrt{(l_2 - l_1)^2 + 16}. \quad (2)$$

Therefore the maximum is $E^{\text{Schw}}_{\text{cm}} = 2\sqrt{5}m_0$, which is a finite limit.

Kerr black hole

Consider the Kerr background, which describes a rotating uncharged black hole.The Kerr background is only axisymmetric, therefore the only three integrals of motion are energy, angular momentum about axis of symmetry and the norm of the four-velocity. These three conservation laws are not enough to reduce the problem to one involving quadratures only. However, a fourth constant of motion has been obtained by Carter by explicitly demonstrating the separability of the Hamilton-Jacobi equation.

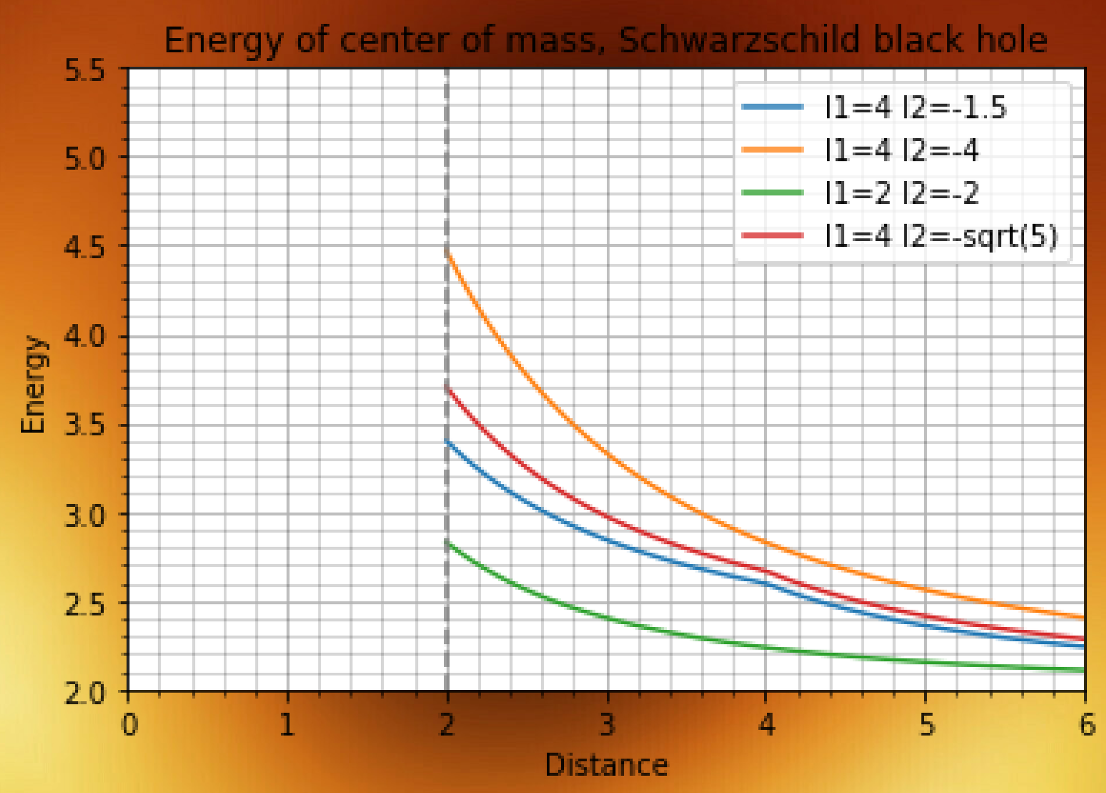
Analogous to the Schwarzschild case, we consider collision of two particles in the equatorial plane and derive the center of mass energy as

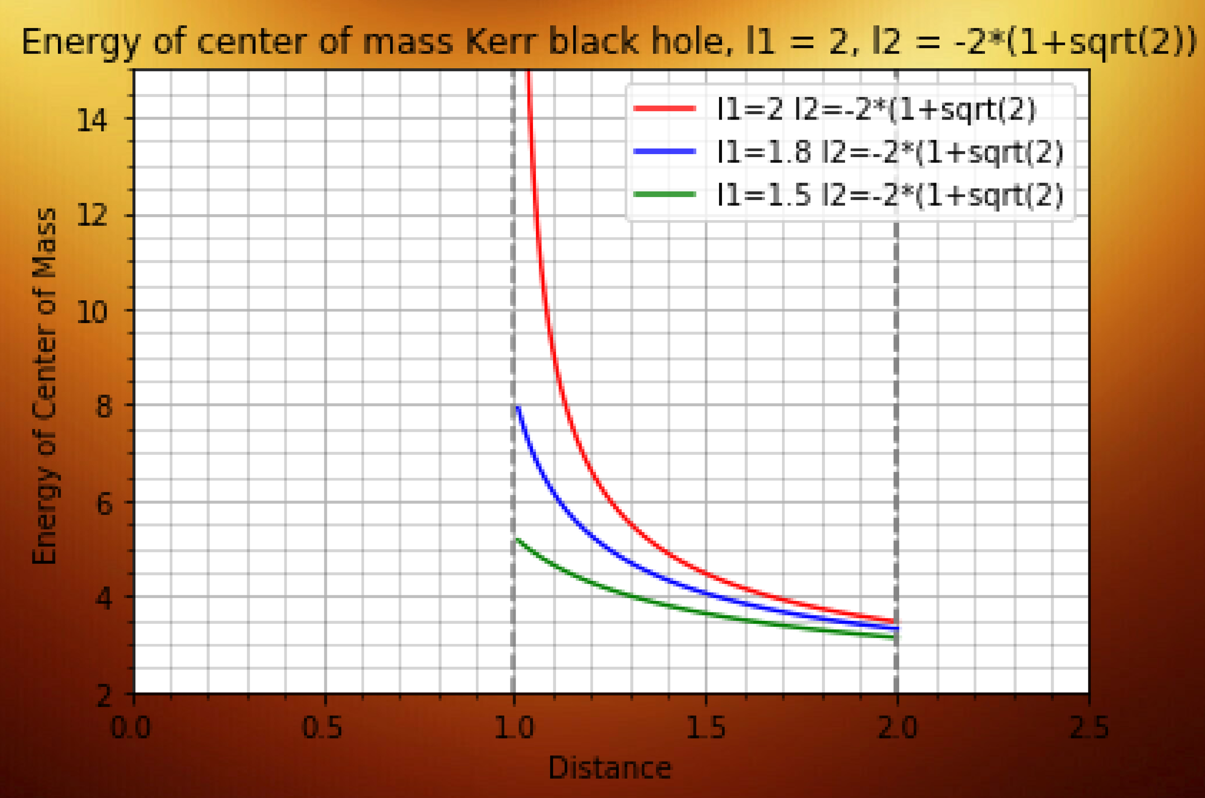
$$\left(E^{\text{Kerr}}_{\text{cm}}\right)^2 = \frac{2m_0^2}{r(r^2-2r+a^2)}\Big(2a^2(1+r) - 2a(l_2+l_1) - l_1 l_2(r-2) + 2(r-1)r^2 - \sqrt{2(a-l_2)^2 - l_2^2 r + 2r^2} \times \sqrt{2(a-l_1)^2 - l_1^2 r + 2r^2}\Big). \quad (3)$$

In the limit as the particles approach the horizon, we have

$$E^{\text{Kerr}}_{\text{cm}}(r \to r_+) = \sqrt{2}m_0\sqrt{\frac{l_2-2}{l_1-2} + \frac{l_1-2}{l_2-2}}. \quad (4)$$

Therefore if one of the particles has angular momentum $l = 2$, we obtain $E^{\text{Kerr}}_{\text{cm}} = \infty$ [1].





Reissner-Nordstrøm black hole

The charged analogue of a Schwarzschild black hole is the Reissner-Nordstrøm black hole. In order for coordinate singularities to exist, we impose the condition $M^2 \geq Q^2$. Again, we consider geodesic motion in the equatorial plane of the black hole, given by $\theta = \pi/2$. Use geodesics to derive the center of mass energy of two particles with charges q_1, q_2 colliding in the vicinity of the black hole, given by:

$$\frac{1}{2m_0^2}\left(E^{\text{RN}}_{\text{cm}}\right)^2 = \frac{1}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2}\Bigg[1 + \left(E_1 - \frac{q_1 Q}{r}\right)\left(E_2 - \frac{q_2 Q}{r}\right)\Bigg] - \sqrt{(E_1 - \frac{q_1 Q}{r})^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2} \times \sqrt{(E_2 - \frac{q_2 Q}{r})^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2}\Bigg) \quad (5)$$

In the limit as the particles approach the horizon, we have:

$$\frac{1}{2m_0^2}\left(E^{\text{RN}}_{\text{cm}}\right)^2(r \to r_H) = 1 + \frac{1}{2}\left[\frac{q_{2(H)} - q_2}{q_{1(H)} - q_1} + \frac{q_{1(H)} - q_1}{q_{2(H)} - q_2}\right] \quad (6)$$

which is arbitrarily high for $q_1 = q_{1(H)}$, where $q_i = \frac{E_i r_H}{Q}$. Therefore we conclude that although the black hole is static, we can still obtain arbitrarily high center of mass energies due to the black hole's charge [2].

Kerr-Newman black hole

The charged analogue of a Kerr black hole is the Kerr-Newman black hole. Since the spacetime is not spherically symmetric, we consider the general geodesics and similarly to the Kerr case we obtain a fourth constant of motion which allows us to fully integrate the equations of motion. We consider geodesics in the equatorial plane of the black hole, given by $\theta = \pi/2$, and use them to derive the center of mass energy of two particles with charges q_1, q_2 colliding in the vicinity of the black hole, given by:

$$\frac{1}{2m_0^2}\left(E^{\text{KN}}_{\text{cm}}\right)^2 = -\frac{1}{r^2\Delta}\Big\{-2r^4 + r^3[2 + Q(q_1+q_2)] - r^2(2a^2 + Q^2 - l_1 l_2 + Q^2 q_1 q_2) - 2a^2 r + 2r[a(l_1+l_2) - l_1 l_2] + Q^2(a-l_1)(a-l_2) + aQr[a(q_1+q_2) - (l_2 q_1 + l_1 q_2)] + \sqrt{(a^2 + r^2 - a l_1 - Q r q_1)^2 - \Delta[r^2 + (a-l_1)]^2} \times \sqrt{(a^2 + r^2 - a l_2 - Q r q_2)^2 - \Delta[r^2 + (a-l_2)^2]}\Big\}, \quad (7)$$

where $\Delta = r^2 + a^2 - 2Mr + Q^2$. In the limit as the particles approach the horizon, we have:

$$E^{\text{KN}}_{\text{cm}}(r \to r_+) = 2m_0\sqrt{1 + \frac{(l_1 - l_2)^2}{(l_1 - l_c)(l_2 - l_c)}\frac{l_c}{4a}}. \quad (8)$$

For a particle falling freely from rest at infinity to reach the horizon, we need $r_c \leq 1$, where $r_c = (1-a^2)/2a^2$. Therefore, we conclude that in order to still obtain arbitrarily high center of mass energies for the charged rotating case, the range for the black hole spin must be $\frac{1}{\sqrt{3}} \leq a \leq 1$ [3].

Further analysis: shape of particle trajectories

To further analyse shapes of orbits, consider effective potentials:

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}, \quad (9)$$

$$V_{\text{eff}}(r, E, l) = -\frac{M}{r} + \frac{l^2 - a^2(E^2-1)}{2r^2} - \frac{M(l-aE)^2}{r^3}. \quad (10)$$

for Schwarzschild and Kerr respectively. By analogy with the Newtonian case, use numerical integration methods to obtain radial plunging, precessing and scattering orbits. We conclude that the kink observed in the Schwarzschild center of mass energy graph is due to the fact that the trajectory of a particle with critical angular momentum 4 is a precessing ellipse, which produces turning points for the kinetic/potential energies.

Future work

Consider lowering a small charge into a Schwarzschild black hole. We will show that, as it is lowered, the electric field remains well-behaved and all the multipole moments except the monopole moment vanish. The result will be a Reissner-Nordstrøm black hole. By analogy, we will use the Kerr metric to derive an expression for the electromagnetic field of a point charge at rest on the symmetry axis near a rotating Kerr black hole. In the limit as the charge approaches the horizon, we find that the electromagnetic field becomes identical to that of the Kerr-Newman solution. We will conclude that lowering a charge into a Kerr black hole produces a Kerr-Newman black hole.

References

[1] https://arxiv.org/pdf/0909.0169.pdf

[2] https://arxiv.org/pdf/1007.4598.pdf

[3] https://arxiv.org/pdf/1006.1056.pdf

[4] https://www.nasa.gov/mission_pages/chandra/news/black-hole-image-makes-history

[5] https://www.nasa.gov/feature/goddard/2019/nasa-visualization-shows-a-black-hole-s-warped-world/