Analysis of particle motion in the gravitational field of black holes

Evelyn-Andreea Ester
Supervised by Dr Stephen West
Department of Physics, Royal Holloway, University of London
(Dated: March 11, 2020)

Collision of two general geodesic particles around the Schwarzschild and Kerr black holes is studied and the center of mass energy for the collision is obtained. By comparison with Newtonian gravity, further analysis of the test particle orbits is made using numerical integration. We extend the discussion to particle motion around charged black holes.

I. INTRODUCTION

According to Birkhoff's theorem, the Schwarzschild metric is the most general spherically symmetric vacuum solution of the Einstein field equations. It describes a static black hole, i.e. a black hole that has neither electric charge nor angular momentum. It is important beacuse it illustrates the highly non-Euclidean character of spacetime geometry when gravity becomes strong and when appropriately truncated, it is the spacetime geometry of a black hole and of a collapsing star-as well as of a wormhole. The external field of any electrically neutral, spherical star satisfies the conditions of Birkhoff's theorem, whether the star is static, vibrating, or collapsing. Therefore its external field must be described by the Schwarzschild geometry [1].
In 1967, Israel published a theorem stating that the only static vacuum black holes are the Schwarzschild black holes (and hence are spherically symmetric). Israel extended his result to a uniqueness theorem for static black holes with charge: only Reissner-Nordstrøm black holes are allowed.
However, black holes in nature are likely to be highly rotating [2], and must therefore be described by the Kerr [3] metric, rather than the Schwarzschild metric. The Kerr metric is a generalization to a rotating body of the former. The exact solution for an uncharged, rotating black-hole, the Kerr metric, remained unsolved until 1963, when it was discovered by Roy Kerr. Israel's paper inspired researchers to see if the uniqueness result holds also for the Kerr case: does the Kerr metric describe all possible rotating black holes? This question is related to the conjecture that a black hole formed by gravitational collapse will asymptotically settle down to a member of the Kerr family. Carter showed that axisymmetric black holes could depend on only two parameters, the mass and angular momentum. He also showed that the Kerr metric was the only one that included a zero angular momentum black hole. These results strongly suggested that Kerr black holes were the only ones. Hawking proved that all stationary black holes must be either static or axisymmetric. He also showed that the topology of the event horizon had to be spherical, another assumption that was used in the Israel/Carter work. Finally, Robinson gave a definitive proof of Carter's results without comparing Kerr only to nearby solutions [4]. The charged Kerr solution, i.e. the Kerr-Newman metric describes a rotating, charged mass and is the most general of the asymptotically flat stationary black hole solutions to the Einstein-Maxwell equations. The theorems stated above show that the external gravitational and electromagnetic fields of a stationary black hole (a black hole that has settled down into its "final" state) are determined uniquely by the hole's mass M, charge e, and intrinsic angular momentum J -i.e., the black hole can have no "hair" (no other independent characteristics). We usually assume that the charge of a real black hole is negligible because macroscopic astrophysical charged objects will rapidly be neutralized by surrounding plasma. However, a small charge can be of astrophysical importance, e.g. the black hole endowed with electromagnetic structure has been used by Ruffini as a model to explain the energetics of GRB and the process of energy extraction from black holes [5].
Bañados, Silk and West [6] showed that Schwarzschild and Kerr black holes can act as particle accelerators and the center of mass energy of two test particles can be arbitrarily high if the collision occurs near the horizon of a Kerr black hole. Therefore, the extremal Kerr black hole surrounded by relic dark matter density spikes could be regarded as a Planck-energy scale collider, which might allow us to explore ultra high energy collisions and astrophysical phenomena. The discussion is extended to include the Reissner-Nordstrøm and Kerr-Newman backgrounds, which are the charged vacuum solutions of the Einstein-Maxwell equations.

A. Hamiltonian formalism in curved spacetime

The usual approach to treating general relativity as a field theory is based on the Lagrangian formulation. For some purposes (e.g. numerical relativity and canonical quantization), a Hamiltonian formulation is preferred. The Hamiltonian formulation of a field theory, like the Hamiltonian formulation of particle mechanics, requires choosing a preferred time variable. For a single particle, proper time may be used, and the Hamiltonian formulation remains manifestly covariant. For a continuous medium, the Hamiltonian formulation requires that a time variable be defined everywhere, not just along the path of one particle. Thus, the Hamiltonian formulation of general relativity requires a separation of time and space coordinates, known as a 3+1 decomposition. We know that the infinitesimal displacement vector in curved spacetime is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

We define the Lagrangian that describes the motion of a particle as

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{2}$$

and the four-momentum as

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu. \tag{3}$$

The Hamiltonian is therefore given by

$$\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \tag{4}$$

which satisfies the Hamilton equations

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu}, \quad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}. \tag{5}$$

The Hamilton-Jacobi equation is given by

$$\mathcal{H} = -\frac{\partial S}{\partial \lambda} = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{6}$$

where S is the Jacobi action and

$$\frac{\partial S}{\partial x^\mu} = p_\mu. \tag{7}$$

Also, the four-velocity $\mathbf{u}$ of a particle is given by

$$u_\mu = \frac{\partial x_\mu}{\partial \lambda}, \tag{8}$$

or, equivalently,

$$u_\mu = \frac{p_\mu}{\mu}, \tag{9}$$

where μ is the mass of the particle and λ is an affine parameter related to the proper time by $\tau = \mu\lambda$, which is equivalent to the normalizing condition

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\mu^2. \tag{10}$$

II. MOTION OF PARTICLES FOR SCHWARZSCHILD CASE

A. Schwarzschild geodesics

Writing equation (1) as a line element in coordinates $\{t, r, \theta, \phi\}$ known as Schwarzschild coordinates, Schwarzschild's solution takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{11}$$

where we have used natural units $c = G = \hbar = 1$.This metric has the following important properties:

- Time independent - the metric is independent of t. There is therefore a Killing vector $\boldsymbol{\xi}$ associated with this symmetry under displacements in the coordinate time t.

- Spherically Symmetric - The geometry of the 2-sphere of constant (t, r) is summarized by the line element

$$d\Omega^2 = r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{12}$$

  which describes the geometry of a sphere of radius r in flat three-dimensional space. Thus, the Schwarzschild geometry has the symmetries of a sphere with respect to changes in the angles θ and ϕ. Since the metric is independent of ϕ, we automatically have invariance under rotations about the z-axis. The Killing vector associated with this symmetry is

$$\eta^\alpha = (0, 0, 0, 1). \tag{13}$$

- If $M \ll r$, the line element can be expanded to give

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + d\Omega^2, \tag{14}$$

  which is the form of the static, weak-field metric from Newtonian gravity. Thus, we can consider that the geometry outside a spherically symmetric source is characterized by the total mass M and not on how the mass is radially distributed inside the source.

- Horizons - metric becomes singular at the radii $r = 0$ and $r = R_S = 2M$. The latter is a coordinate singularity that can be removed by transforming to Kruskal-Szekeres coordinates [1].

Since the metric is independent of time and spherically symmetric, it is easy to identify the conservation laws for energy and angular momentum. We can express these conservation laws in terms of the Killing vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$ as

$$E = -\boldsymbol{\xi} \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}, \tag{15}$$

$$l = \boldsymbol{\eta} \cdot \mathbf{u} = r^2 \sin^2\theta \frac{d\phi}{d\tau}, \tag{16}$$

where $\mathbf{u}$ is the particle's four-velocity. At large r, the constant E becomes energy per unit rest mass and at low velocities l becomes angular momentum per unit rest mass. Thus, we consider the above to be the conserved energy and angular momentum for particle orbits.

As we have discussed in appendix A, the conservation of angular momentum implies that the orbits lie in a plane, analogous to the Newtonian theory. This allows us to restrict the discussion for geodesic motion to the equatorial plane $\theta = \pi/2$. The normalization of the four-velocity , i.e.

$$\mathbf{u} \cdot \mathbf{u} = g_{\mu\nu} u^\mu u^\nu = -1 \tag{17}$$

yields another integral for the geodesic equation in addition to equations (15) and (16), therefore we can use Hamilton-Jacobi theory in order to derive the geodesic equations.

The Hamilton-Jacobi equation for Schwarzschild metric is

$$-\frac{\partial S}{\partial \lambda} = \frac{1}{2}\Bigg[-\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\partial S}{\partial t}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\Bigg]. \tag{18}$$

This allows us to derive the geodesic equations for the equatorial plane $\theta = \pi/2$ as

$$\frac{dt}{d\tau} = \left(1 - \frac{2}{r}\right)^{-1}, \tag{19}$$

$$\frac{dr}{d\tau} = -\frac{1}{r^2}\sqrt{r(2r^2 + 2l^2 - rl^2)}, \tag{20}$$

$$\frac{d\phi}{d\tau} = \frac{l}{r^2}, \tag{21}$$

where l is the angular momentum of the particle per unit rest mass. Consider now two particles with angular momenta l_1 and l_2 colliding at a radius r. The particles are assumed to be at rest at infinity. We want to compute the energy in the center of mass frame for the collision. Therefore, we define

$$E_{\rm cm}^2 = (p_1^\mu + p_2^\nu)^2, \tag{22}$$

therefore

$$E_{\rm cm} = m_0\sqrt{2}\sqrt{1 - g_{\mu\nu} u^\mu_{(1)} u^\nu_{(2)}}, \tag{23}$$

where $u^\mu_{1,2}$ are the four velocities of the two colliding particles. In the Schwarzschild background, this becomes

$$\frac{1}{2m_0^2}\left(E_{\rm cm}^{\rm Schw}\right)^2 = \frac{2r^2(r-1) - l_1 l_2 (r-2) - \sqrt{2r^2 - l_1^2(r-2)}\sqrt{2r^2 - l_2^2(r-2)}}{r^2(r-2)} \tag{24}$$

We observe that the horizon is located at $r = 2$. Calculating the limit at $r = 2$ with L'Hôpital's rule, we obtain that

$$\frac{1}{2m_0^2}\left(E_{cm}^{\rm RN}\right)^2 (r \to r_H) = 1 + \frac{1}{2}\left[\frac{q_{2(H)} - q_2}{q_{1(H)} - q_1} + \frac{q_{1(H)} - q_1}{q_{2(H)} - q_2}\right]. \tag{25}$$

The maximum center of mass energy occurs for l_2 and l_1 opposite with their maximum allowed values ± 4. Using these values in equation (24), one obtains the maximum $E_{cm} = 2\sqrt{5} m_0$, which is a finite limit. Note also that if $l_1 = l_2$, then $E_{cm} = 2m_0$.[6]

Using equations (15) and (16), we obtain from the metric that

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1. \tag{26}$$

Rewriting the above to show the correspondence with Newtonian mechanics, we have

$$\frac{E^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right]. \tag{27}$$

By defining the constant

$$\mathcal{E} \equiv \frac{E^2 - 1}{2}, \tag{28}$$

and the effective potential

$$V_{\rm eff}(r) \equiv \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}, \tag{29}$$

the correspondence becomes exact, i.e.

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\rm eff}(r). \tag{30}$$

This potential has extremum points at

$$r_{\rm min/max} = \frac{l^2}{2m} \mp \frac{l}{2}\sqrt{\frac{l^2}{m^2} - 12}. \tag{31}$$

B. Orbits in Schwarzschild geometry

The qualitative behaviour of an orbit depends on the relationship between $\mathcal{E}$ and the effective potential in equation (30), analogous to the Newtonian case. Thus, we have four kinds of orbits in the Schwarzschild geometry. If $l/M < \sqrt{12} = 3.46$, there are no real extrema and $V_{\rm eff} < 0$ for all r. If $l/M > \sqrt{12}$, then $V_{\rm eff}$ has one maximum and one minimum. The maximum lies above

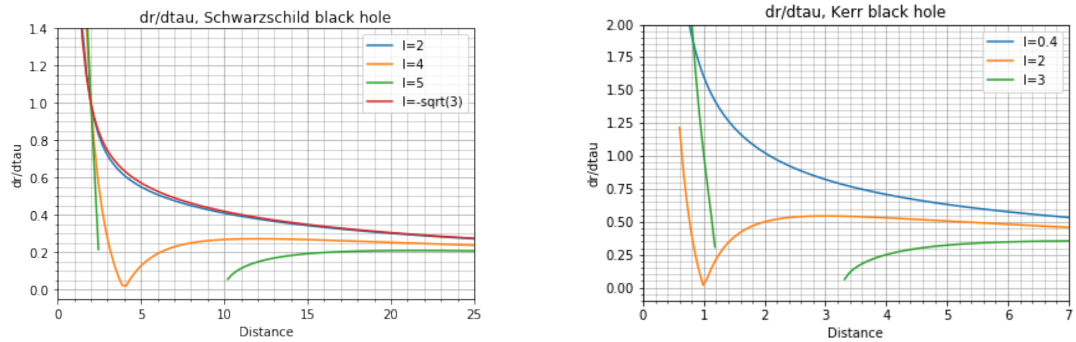

FIG. 1. For a Schwarzschild black hole, the plot shows the variation of $\dot{r}$ with radius for four different values of angular momentum $l = 2, 4, 5, -\sqrt{3}$. The case $l = 4$ is a critical case where geodesics start falling in. For a Kerr black hole with $a = 1$, the plot shows the variation of $\dot{r}$ with radius for three different values of angular momentum $l = 0.4, 2, 3$.

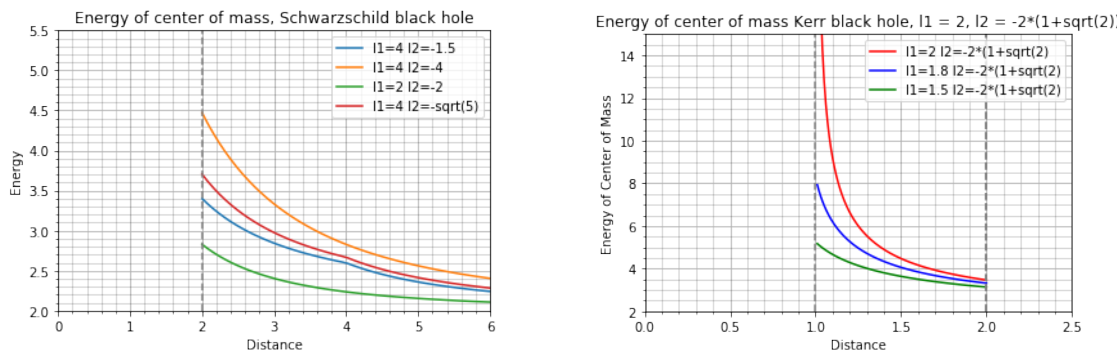

FIG. 2. For a Schwarzschild black hole, the plot shows the variation of $E_{\rm cm}$ with radius for the given combinations of l_1 and l_2. The kink corresponding to the critical geodesics with $l = 4$ is due to the fact that the particle orbits are precessing ellipses for which the energy has turning points as shown in fig... For a Kerr black hole, the plot shows the variation of $E_{\rm cm}$ with radius for the given combinations of l_1 and l_2. For $l = 2$, the center of mass energy blows up at the horizon.

$V_{\rm eff} = 0$ if $l/M > 4$ and otherwise lies below it. There is a centrifugal barrier with a maximum height.
Turning points occur at the radii r_*, where $\mathcal{E} = V_{\rm eff}(r_*)$, because that is where the radial velocity vanishes. If $l/M < \sqrt{12}$, there are no turning points for positive values of $\mathcal{E}$. An inwardly directed particle falls all the way to the origin. This is in contrast with Newtonian theory, where as long as $l \neq 0$, there is a positive centrifugal barrier that will reflect the particle. Circular orbits are possible at the radii given by equation (31). The orbit at the maximum is unstable, because a small increase in $\mathcal{E}$ will lead the particle to escape to infinity or collapse to $r = 0$. The orbit at the minimum is stable. There are bound orbits for $\mathcal{E} < 0$ that oscillate between two turning points. Orbits with $0 < \mathcal{E} < V_{\rm eff,max}$ are scattering orbits that come from infinity, orbit the center of attraction and then return. The orbits with $\mathcal{E} > V_{\rm eff,max}$ plunge into the center of attraction.

1. Radial plunge orbits

We consider the radial free fall of a particle from infinity, i.e. $l = 0$. This means we have $dt/d\tau = 1$, or, equivalently $\mathcal{E} = 0$. From equation (30), we thus have

$$0 = \frac{1}{2}\left(\frac{\partial r}{\partial \tau}\right)^2 - \frac{M}{r}, \tag{32}$$

which gives the radial component of the four-velocity $dr/d\tau$. Together with the time component $dt/d\tau$, the four-velocity is

$$u^\mu = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right). \tag{33}$$

By writing equation (32) in the form

$$r^{1/2} dr = -(2M)^{1/2} d\tau, \tag{34}$$

we can integrate both sides (we have taken the negative square root to correspond to a geodesic going inward), thus

$$r(\tau)=\left(\frac{3}{2}\right)^{2/3}(2M)^{1/3}(\tau_*-\tau)^{2/3}, \tag{35}$$

where τ_* is an arbitrary integration constant that fixes the proper time then $r=0$. Using equations (15) and (32), we find that

$$\frac{dt}{dr}=-\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1}, \tag{36}$$

which on integration gives

$$t=t_*+2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2}-2\left(\frac{r}{2M}\right)^{1/2}+\ln\left|\frac{(r/2M)^{1/2}+1}{(r/2M)^{1/2}-1}\right|\right]. \tag{37}$$

Several important features of radial plunge orbits can be seen from equations (34) and (37). As $t\to-\infty$, $r\to\infty$, so the particle is falling inward from infinity. We also see from equation (34) that from any fixed value of r on the trajectory, it takes a finite time to reach $r=2M$, but equation (37) shows it takes an infinite amount of coordinate time t. This is yet another indication that Schwarzschild coordinates display a coordinate pathology at $R_S=2M$.

2. *Bound orbits*

To find the shape of an orbit, we need to find the relationship between r and ϕ parameters. Thus, for $dr/d\tau$, we need to solve equation (30) and for $d\phi/d\tau$ we need to solve equation (16) with $\theta=\pi/2$, then divide the first into the second. We thus have

$$\frac{d\phi}{dr}=\pm\frac{l}{r^2}\left[E^2-\left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)\right]^{-1/2}. \tag{38}$$

The sign corresponds to the ϕ direction in which the particle moves with increasing r. By integrating, we find $\phi(r)$ and the result can be expressed in terms of elliptic functions.

C. Numerical Implementation

It can be shown that the force on a particle of mass $m<<M$ outside a spherically symmetric distribution of mass M is given by

$$\mathbf{F}=-\frac{Mm}{r^3}\mathbf{r}\left(1+\frac{3l^2}{r^2}\right). \tag{39}$$

In the following, we will use the Schwarzschild radius $R_S=2M$ to introduce the dimensionless variables

$$\rho\equiv\frac{r}{R_S}, \tag{40}$$

$$T\equiv\frac{\tau}{t_0}, \tag{41}$$

where t_0 is some characteristic time of the system and τ is the proper time. Note that t_0 has dimension 1 in natural units, thus we can choose $t_0=R_S$.

In the appendix we have shown that a system can be described via an equation of the form

$$\mathcal{E}=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+V_{\text{eff}}(r), \tag{42}$$

where $\mathcal{E}$ is a constant and

$$V_{\text{eff}}(r)\equiv\frac{M}{r}+\frac{l^2}{2r^2}-\frac{Ml^2}{r^3} \tag{43}$$

is the Newtonian effective potential discussed in Appendix A, with an additional relativistic correction. In classical mechanics, there exist bound orbits (circles and ellipses) and scattering orbits (hyperbolae). There is only one circular orbit and it is stable. By contrast, in general relativity there are three types of orbits: radial plunging, scattering orbits and bound orbits - precessing ellipses (see figure (3)). By using Newton's 2nd Law and the newly introduced variables, we can rewrite equation (39) as

$$\frac{d^2\rho}{dT^2}=-\frac{1}{2\rho^2}\left(1+\frac{3l^2}{\rho^2}\right), \tag{44}$$

where we have redefined l as l/R_S. Classical orbits lie in the plane due to conservation of angular momentum (see Appendix) The same is true in a spherically symmetric background, e.g. Schwarzschild. We will choose the particle to move in the (x,y) plane.
Let $\rho\equiv(x,y), u\equiv\partial x/\partial\tau, v\equiv\partial y/\partial\tau$. We can write (in natural units) equation (39) as four coupled first-order differential equations,

$$\frac{dx}{dT}=u, \tag{45}$$

$$\frac{du}{dT}=-\frac{x}{2\rho^3}\left(1+\frac{3l^2}{\rho^2}\right), \tag{46}$$

$$\frac{dy}{dT}=v, \tag{47}$$

$$\frac{dv}{dT}=-\frac{y}{2\rho^3}\left(1+\frac{3l^2}{\rho^2}\right). \tag{48}$$

We now implement the fourth-order Runge-Kutta method discussed in Appendix B in order to get the figure 3 below.

III. MOTION OF PARTICLES FOR KERR CASE

The treatment of gravitational collapse assumes exact spherical symmetry, with the Schwarzschild geometry being the only spherical solution of the Einstein field equations. However, realistic gravitational collapse is not spherically symmetric. From the perspective of a distant observer, the endstate is indistinguishable

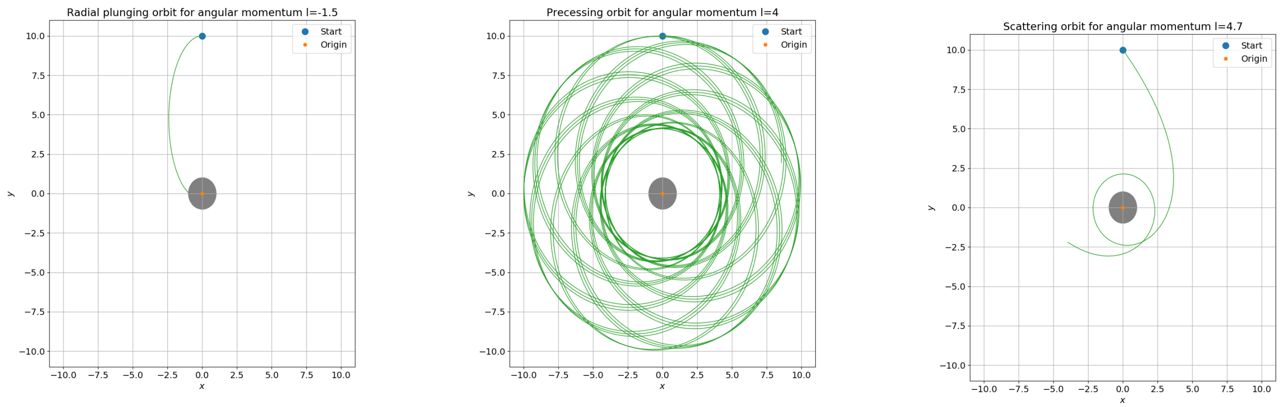

FIG. 3. Types of orbits in Schwarzschild geometry, plotted for different values of the angular momentum l.

from a time-independent Kerr black hole, characterised by just a mass M and angular momentum J, with a horizon that conceals the singularity within it. At the time of writing, there is no rigorous proof from the Einstein equation that a generic gravitational collapse that proceeds far enough inevitably forms a black hole, concealing singularities from observers outside. This is a conjecture called the cosmic censorship conjecture, assumed to hold for realistic astrophysical situations.[7]

The Kerr family of solutions of the Einstein-Maxwell equations is the most general class of solutions known which could represent the field of a rotating neutral or electrically charged body in asymptotically flat space. Although the symmetries provide only 3 constants of the motion, a fourth one turns out to be obtainable from the unexpected separability of the Hamilton-Jacobi equation, with the result that the equations, not only of geodesics but also of charged-particle orbits can be integrated completely in terms of explicit quadratures.

It can be conjectured that the low-angular-momentum Kerr fields may be the only examples of stationary axisymmetric asymptotically flat vacuum solutions of Einstein-Maxwell equations. The original form of the Kerr family of solutions is given in terms of coordinates u, r, θ, ϕ, which can be interpreted on a manifold formed by taking the topological product of a 2-plane on which u and r are Cartesian coordinates running from $-\infty$ to ∞ and a 2-sphere on which θ and ϕ are ordinary spherical coordinates (ϕ is periodic with 2π and θ runs from 0 to π). These solutions are stationary and axisymmetric with Killing vectors $\partial/\partial u$ and $\partial/\partial\phi$ and they are invariant under the discrete transformation of inversion about the equatorial hyperplane $\theta = \pi/2$. Both the metric and the electromagnetic field forms are analytic, except on the stationary ring $r = 0$, $\theta = \pi/2$, where Σ^2 vanishes. In fact, the curvature itself becomes singular as $\Sigma^2 \to 0$, except in the special case when both e and m vanish. In this special case there must still be a singularity of the geometry at $\Sigma^2 = 0$, although the metric is then flat everywhere else. In all cases the metric and the electromagnetic field are well-behaved throughout the rest of the manifold, except for the usual trivial degeneracy of spherical coordinates at $\theta = 0, \theta = \pi$.

The generalisation of the solutions to include an electromagnetic field was achieved by transformation of the Schwarzschild solution (to which they reduce in the case when a vanishes). the charged generalisation of the empty-space Kerr metrics has been obtained by applying an analogous transformation to the charged spherical solution of Reissner and Nordstrøm (which is likewise the limiting case to which the charged solutions reduce when a vanishes). [8]

The spacetime around a rotating black hole with mass M and angular momentum J is described by the Kerr metric,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \left(\frac{4aMr\sin^2\theta}{\Sigma}\right)dtd\phi + \left(\frac{\Sigma}{\Delta}\right)dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \tag{49}$$

where a is the angular momentum of the black hole per unit mass (also called the Kerr parameter) and the functions Δ and Σ have the forms

$$\Delta = r^2 + a^2 - 2Mr \tag{50}$$

$$\Sigma^2 = r^2 + a^2\cos^2\theta. \tag{51}$$

(52)

The (t, r, θ, ϕ) are called Boyer-Lindquist coordinates, analogous to the Schwarzschild coordinates for a non-rotating black hole.

A number of important properties of the Kerr geometry follow from this line element:

- Asymptotically flat - for $r \ll M$ and $r \ll a$, the line element becomes

$$ds^2 \approx -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4Ma}{r^2}\sin^2\theta(rd\phi)dt + ..., \quad (53)$$

where the leading terms in each metric coefficient as r becomes large and the $1/r$ corrections to that behaviour (if any) have been retained. This asymptotic form establishes that the Kerr geometry approaches the geometry of flat spacetime far from the black hole.

- Stationary, Axisymmetric - the metric is independent of t (stationary) and ϕ (axisymmetric). The two Killing vectors that correspond to these symmetries are $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$:

$$\xi^\alpha = (1,0,0,0), \quad (54)$$
$$\eta^\alpha = (0,0,0,1), \quad (55)$$

where the components are given in the usual order t, r, θ, ϕ. In addition, the Kerr metric is unchanged under reflection in the equatorial plane $\theta = \pi/2$. These are symmetries specific for rotating bodies, however as we see from the dependence of g_{rr} and g_{tt} on θ, the geometry is not spherically symmetric.

- Metric reduces to Schwarzschild for $a = 0$.

- Horizons - metric becomes singular when Σ or Δ vanishes. The singularity at $\Sigma = 0$ is a physical singularity - it has infinite spacetime curvature. The quantity Δ vanishes at the radii

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \quad (56)$$

assuming that $a \leq M$. Since the radius r_+ exists only for this condition, the angular momentum J is therefore limited by its mass squared. Black holes with the limiting value $a = M$ are called extreme Kerr black holes and are of astrophysical importance. When a black hole swallows matter and radiation from an accretion disk, its mass M and angular momentum $J = Ma$ change, i.e. the hole evolves. The details of the evolution have been calculated by Thorne (1974), where it was shown that the accreting matter would spin the hole up to $a = M$, but the radiation emitted by the disk and swallowed by the hole produces a counteracting torque, which prevents spin-up beyond a limiting state of $a \approx 0.998$.[9]

The orbits of test particles will generally not be confined to a plane. Orbits in Schwarzschild geometry stay in a plane because of the conservation of the test particle's angular momentum, itself a consequence of the geometry's spherical symmetry. But the Kerr geometry is not spherically symmetric, only axisymmetric, therefore only the component of the angular momentum along the symmetry axis is conserved. In general, geodesic motion in a stationary axisymmetric spacetime will allow two integrals of motion: the energy and the angular momentum about the axis of symmetry. The norm of the four-velocity will also be conserved by virtue of its parallel propagation. These three conservation laws will not, in general, suffice to reduce the problem of solving the equations of geodesic motion to one involving quadratures only. However, the existence of a fourth quantity that is conserved along a geodesic was first discovered by Carter by explicitly demonstrating the separability of the Hamilton-Jacobi equation. [10]

The Hamilton-Jacobi equation for the Kerr geometry is given by

$$-2\frac{\partial S}{\partial \lambda} = -\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Delta\Sigma^2}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{4aMr}{\Sigma^2\Delta}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi} + \frac{\Delta}{\Sigma^2}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{\Sigma^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2. \quad (57)$$

It is convenient to write this as

$$2\frac{\partial S}{\partial \lambda} = -\frac{1}{\Sigma^2\Delta}\left[(r^2+a^2)\frac{\partial S}{\partial t} + a\frac{\partial S}{\partial \phi}\right]^2 + \frac{1}{\Sigma^2\sin^2\theta} \cdot \left[a\sin^2\theta\frac{\partial S}{\partial t} + \frac{\partial S}{\partial \phi}\right]^2 + \frac{\Sigma^2}{\Delta}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{\Sigma^2}\left(\frac{\partial S}{\partial \theta}\right)^2. \quad (58)$$

Assuming that the variables can be separated, we seek a solution of the form

$$S(t,r,\theta,\phi) = -\frac{1}{2}\mu^2\lambda - Et + l\phi + S_r(r) + S_\theta(\theta). \quad (59)$$

For this chosen form of S, equation (58) becomes

$$-\mu^2\Sigma^2 = \frac{1}{\Delta}\left[(r^2+a^2)E - al\right]^2 - \frac{1}{\sin^2\theta}(aE\sin^2\theta - l)^2 - \Delta\left(\frac{\partial S_r}{\partial r}\right)^2 - \left(\frac{\partial S_\theta}{\partial \theta}\right)^2. \quad (60)$$

With the aid of the identity

$$\frac{(aE\sin^2\theta - l)^2}{\sin^2\theta} = \left(\frac{l^2}{\sin^2\theta} - a^2E^2\right)\cos^2\theta + (l - aE)^2, \quad (61)$$

we can rewrite equation (60) as

$$\left\{\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 - \frac{1}{\Delta}\left[(r^2+a^2)E - al\right]^2 + (l-aE)^2 - \mu^2r^2\right\} + \left\{\frac{\partial S_\theta}{\partial \theta} + \left(\frac{l^2}{\sin^2\theta} - a^2E^2\right)\cos^2\theta - \mu^2a^2\cos^2\theta\right\} = 0. \quad (62)$$

The separability of the equation is now manifest and we can infer that

$$\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 = \frac{1}{\Delta}[(r^2+a^2)E-al]^2 - [\mathcal{C}+(l-aE)^2+\mu^2 r^2], \tag{63}$$

$$\left(\frac{\partial S_\theta}{\partial \theta}\right)^2 = \mathcal{C} - \left(\frac{l^2}{\sin^2\theta} - a^2E^2 + \mu^2 a^2\right)\cos^2\theta, \tag{64}$$

where $\mathcal{C}$ is Carter's separation constant. With the abbreviations

$$R = [(r^2+a^2)E-al]^2 - \Delta + (l-aE)^2 + \mu^2 r^2] \tag{65}$$

$$\Theta = \mathcal{C} - [a^2(\mu^2-E^2) + l^2\sin^{-2}\theta]\cos^2\theta, \tag{66}$$

the solution for S is

$$S = -\frac{1}{2}\mu^2\lambda - Et + l\phi + \int^r \frac{\sqrt{R(r)}}{\Delta}\,dr + \int^\theta \sqrt{\Theta(\theta)}. \tag{67}$$

The basic equations governing the motion can be deduced from the solution above by setting to zero the partial derivatives of S with respect to the different constants of the motion - $\mathcal{C}, \mu^2, E, l$. Therefore, we have

$$\frac{\partial S}{\partial \mathcal{C}} = \frac{1}{2}\int \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial \mathcal{C}}dr + \frac{1}{2}\int \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial \theta}d\theta = 0, \tag{68}$$

that leads to the equation

$$\int^r \frac{dr}{\sqrt{R}} = \int^\theta \frac{d\theta}{\sqrt{\Theta}}. \tag{69}$$

Similarly, we find

$$\tau = \int^r \frac{r^2}{\sqrt{R}}dr + a^2\int^\theta \frac{\cos^2\theta}{\sqrt{\Theta}}d\theta, \tag{70}$$

$$t = \frac{1}{2}\int^r \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial E}dr + \frac{1}{2}\int^\theta \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial E}d\theta \tag{71}$$

$$= \tau E + 2M\int^r r[r^2E - a(l-aE)]\frac{dr}{\Delta\sqrt{R}}, \tag{72}$$

$$\phi = -\frac{1}{2}\int^r \frac{1}{\Delta\sqrt{R}}\frac{\partial R}{\partial l}dr - \frac{1}{2}\int^\theta \frac{1}{\sqrt{\Theta}}\frac{\partial \Theta}{\partial l}d\theta \tag{73}$$

$$= a\int^r [(r^2+a^2)E-al]\frac{dr}{\Delta\sqrt{R}} + \int^\theta (l\sin^{-2}\theta - aE)\frac{d\theta}{\sqrt{\Theta}}. \tag{74}$$

If we identify $\mathcal{C} = \mathcal{K} - (l-aE)^2$, we find the equations of motion for a test particle to be

$$\Sigma^4\dot{r}^2 = R, \tag{75}$$

$$\Sigma^4\dot{\theta}^2 = \Theta, \tag{76}$$

$$\Sigma^2\dot{\phi} = \frac{1}{\Delta}[2aMrE + (\Sigma^2 - 2Mr)l\sin^{-2}\theta, \tag{77}$$

$$\Sigma^2\dot{t} = \frac{1}{\Delta}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta)E - 2aMrl]. \tag{78}$$

For calculating the $E_{\rm cm}$ energy, we will restrict our discussion to orbits confined in the equatorial plane $\theta = \pi/2$. In this case, the equations of motion become

$$\frac{dr}{d\tau} = \pm\frac{1}{r^2}\sqrt{T^2 - \Delta(m_0^2r^2 + (l-aE)^2)}, \tag{79}$$

$$\frac{d\phi}{d\tau} = -\frac{1}{r^2}[(aE-l) + aT/\Delta], \tag{80}$$

$$\frac{dt}{d\tau} = -\frac{1}{r^2}[a(aE-l) + (r^2+a^2)T/\Delta], \tag{81}$$

where $T \equiv E(r^2+a^2) - la$. Here E is the total energy of the particle and $l = p_\phi$ is the component of angular momentum per unit mass parallel to the symmetry axis. By applying equation (23), we find the equivalent of equation (**??**) for a rotating background to be

$$\left(E_{\rm cm}^{\rm Kerr}\right)^2 = \frac{2m_0^2}{r(r^2-2r+a^2)}\left(2a^2(1+r) - 2a(l_2+l_1) - l_2l_1(-2+r) + 2(-1+r)r^2\right) - \sqrt{2(a-l_2)^2 - l_2^2r + 2r^2}\sqrt{2(a-l_1)^2 - l_1^2r + 2r^2}. \tag{82}$$

The horizon is now located at the larger root of $r^2 - 2r + a^2 = 0$, that is $r_+ = 1 + \sqrt{1-a^2}$. Analogous to the Schwarzschild case, we can apply L'Hôpital's rule and find the limiting value of $E_{cm}^{\rm Kerr}$ at the horizon r_+ for the case $a = 1$ as

$$E_{cm}^{\rm Kerr}(r \to r_+) = \sqrt{2}m_0\sqrt{\frac{l_2-2}{l_1-2} + \frac{l_1-2}{l_2-2}}, \tag{83}$$

which is finite for arbitrary values of l_1 and l_2. This is true except when $l = 2$. A new phenomena appears if one of the particles participating in the collision has the critical angular momentum $l = 2$. In this case the limit ceases to exist, and the center of mass energy blows up at the horizon. These equations can be used to express the radial motion of a test particle

$$\frac{E^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r, E, l), \tag{84}$$

governed by the effective potential

$$V_{\text{eff}}(r, E, l) = -\frac{M}{r} + \frac{l^2 - a^2(E^2 - 1)}{2r^2} - \frac{M(l - aE)^2}{r^3}. \tag{85}$$

Using the same method as for the Schwarzschild case, we obtain the plots of the Kerr orbits in figure (4) below. See also Appendix C for the effective potential and energy plots.

IV. MOTION OF PARTICLES IN REISSNER-NORDSTRØM CASE

We will now discuss the form of the metric outside a static spherically symmetric charged body. The exterior of such an object is not a vacuum, since it is filled with a static electric field. We must therefore solve the Einstein field equations for a static spherically symmetric space-time in the presence of a non-zero energy–momentum tensor, this time representing the electromagnetic field of the object. The result is

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{86}$$

The Reissner-Nordstrøm metric has an intrinsic singularity at $r = 0$ analogous to the (uncharged) Schwarzschild case, but it also posesses a coordinate singularity for

$$\Delta = 1 - \frac{2M}{r} + \frac{e^2}{r^2} = 0, \tag{87}$$

i.e. the coordinate singularities occur on the surfaces $r = r_\pm$, where

$$r_\pm = M \pm (M^2 - e^2)^{1/2}. \tag{88}$$

This means we have the following cases:

- Case 1: $M^2 < e^2$. In this case $r_\pm$ are both imaginary, and so no coordinate singularities exist. The metric is therefore regular for all positive values of r. Since the function Δ always remains positive, the coordinate t is always timelike and r is always spacelike. Thus, the intrinsic singularity at $r = 0$ is a timelike line, as opposed to a spacelike line in the Schwarzschild case. This means that the singularity does not necessarily lie in the future of timelike trajectories and can, in principle, be avoided. In the absence of any event horizons $r = 0$ is a naked singularity, i.e. it is visible to the outside world. However, due to the cosmic censorship hypothesis, this case is not considered physically realistic.

- Case 2: $M^2 > e^2$. In this case, $r_\pm$ are both real so there exist two coordinate singularities on the surfaces $r = r_\pm$. The situation at $r = r_+$ is similar to the Schwarzschild case at $r = 2M$. For $r > r_+$, we have $\Delta > 0$, so the coordinates t and r are timelike and spacelike respectively. In the region $r_- < r < r_+$, Δ becomes negative and so the physical nature of the coordinates t, r is interchanged. Thus, a massive particle that enters the surface $r = r_+$ from outside must necessarily move in the direction of decreasing r, making $r = r_+$ an event horizon. The major difference from the Schwarzschild geometry is that the irreversible infall of the particle need only continue to the surface $r = r_-$, since for $r < r_-$, Δ is again positive and so t, r recover their usual properties.

- Case 3: $M^2 = e^2$. This case is called extreme Reissner-Nordstrøm black hole; $\Delta > 0$ everywhere, except for $r = M$, where it becomes null.

We now consider again geodesic motion in the equatorial plane of the black hole, given by $\theta = \pi/2$ and use these geodesics to derive the center of mass energy of two particles colliding in the vicinity of the black hole, given by:

$$\frac{1}{2m_0^2}\left(E_{\text{cm}}^{\text{RN}}\right)^2 = \frac{1 + (E_1 - \frac{q_1 Q}{r})(E_2 - \frac{q_2 Q}{r}) - \sqrt{(E_1 - \frac{q_1 Q}{r})^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2}\sqrt{(E_2 - \frac{q_2 Q}{r})^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2}}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)m^2} \tag{89}$$

In the limit as the particles approach the horizon, we have:

$$\frac{1}{2m_0^2}\left(E_{\text{cm}}^{\text{RN}}\right)^2 (r \to r_H) = 1 + \frac{1}{2}\left[\frac{q_{2(H)} - q_2}{q_{1(H)} - q_1} + \frac{q_{1(H)} - q_1}{q_{2(H)} - q_2}\right], \tag{90}$$

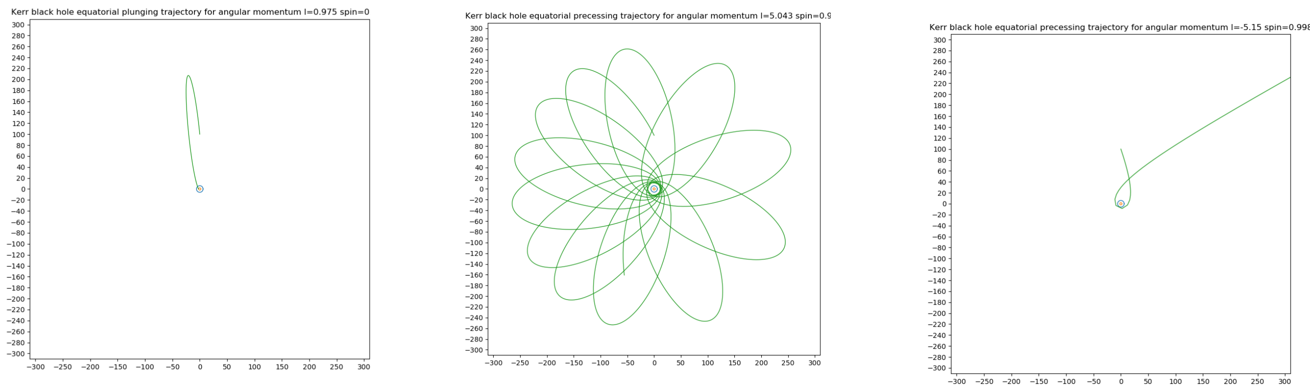

FIG. 4. Types of orbits in Kerr geometry, plotted for different values of the angular momentum l.

which is arbitrarily high for $q_1 = q_{1(H)}$, where $q_i = \frac{E_i r_H}{Q}$. Therefore we conclude that although the black hole is static, we can still obtain arbitrarily high center of mass energies due to the black hole's charge [11].

V. MOTION OF PARTICLES IN KERR-NEWMAN CASE

The Kerr-Newman metric can be expressed as

$$\begin{aligned} ds^2 &= \Sigma^2 d\theta^2 - 2a\sin^2\theta dr d\phi + 2dr du + \Sigma^{-2}[(r^2+a^2)^2 \\ &- \Delta a^2 \sin^2\theta] \sin^2\theta d\phi^2 - 2a\Sigma^{-2}(2Mr - e^2)\sin^2\theta d\phi du \\ &- [1 - \Sigma^{-2}(2Mr - e^2)]du^2. \end{aligned} \tag{91}$$

The corresponding covariant form of the electromagnetic field tensor is

$$\begin{aligned} F &= 2e\Sigma^{-4}[(r^2 - a^2\cos^2\theta)dr\wedge du - 2a^2 r\cos\theta\sin\theta d\theta\wedge du \\ &- a\sin^2\theta(r^2 - a^2\cos^2\theta)dr\wedge d\phi \\ &+ 2ar(r^2+a^2)\cos\theta\sin\theta d\theta\wedge d\phi], \end{aligned} \tag{92}$$

where

$$\Sigma^2 = r^2 + a^2\cos^2\theta, \tag{93}$$

$$\Delta = r^2 - 2Mr + a^2 + e^2, \tag{94}$$

and $\wedge$ denotes the antisymmetric tensor product.

Despite its many advantages, the coordinate system has the drawback that it does not display the full symmetry of the space. We thus introduce the new time and angle coordinates t and ϕ, defined by:

$$\begin{aligned} dt &= du - (r^2+a^2)\Delta^{-1}dr \\ d\phi &= d\varphi - a\Delta^{-1}dr \end{aligned} \tag{95}$$

and obtain the metric tensor form as

$$\begin{aligned} ds^2 &= \Sigma^2\Delta^{-1}dr^2 + \Sigma^2 d\theta^2 + \Sigma^{-2}\sin^2\theta[a dt - (r^2+a^2)d\phi]^2 \\ &- \Sigma^{-2}\Delta[dt - a\sin^2\theta d\phi]^2. \end{aligned} \tag{96}$$

In this new system, the electromagnetic field tensor from equation (92) becomes

$$\begin{aligned} F &= 2e\Sigma^{-4}(r^2 - a^2\cos^2\theta)dr\wedge[dt - a\sin^2\theta d\phi] \\ &- 4e\Sigma^{-4}ar\cos\theta\sin\theta d\theta\wedge[a dt - (r^2+a^2)d\phi]. \end{aligned} \tag{97}$$

In this system (Boyer-Lindquist form), it is clear that the metrics reduce to the familiar Schwarzschild and Reissner-Nordstrøm forms when a vanishes.

Just as the parameter a couples with the mass to give the angular momentum, it couples with the charge to give an asymptotic magnetic dipole moment ea. There is no freedom of variation of the gyromagnetic ratio, which is simply e/m. This is exactly the same as the gyromagnetic ratio predicted for a spinning particle by the Dirac equation, which is obeyed to quite a high accuracy by the electron.

In the Boyer-Lindquist coordinates, there are two horizons occuring where Δ vanishes. When M is greater than the critical value

$$M^2 = a^2 + e^2, \tag{98}$$

$\Delta = 0$ for the values of r (both positive), defined by

$$r_\pm = M \pm (M^2 - a^2 - e^2)^{1/2}, \tag{99}$$

and is negative in between them. In the intermediate region the solution changes character. We can see from equation (96) that this region cannot be regarded as stationary in the strict sense since there are no longer any

timelike vectors in the planes (r=const, θ=const) of the Killing vectors, but instead r has taken over the role of a timelike variable. In the limiting case when a and e both vanish, the inner horizon collapses onto the central singularity and the outer horizon becomes the well-known Schwarzschild horizon at $r = 2M$. When equation (98) is satisfied, the two horizons coalesce at $r = M$. When $a^2+e^2 > M^2$, there are no Killing horizons and the manifold is geodesically complete except for those geodesics which reach the central singularity at $\Sigma^2 = 0$. However, when $a^2 + e^2 \leq M^2$, although the local failure of the metric can be bypassed by reverting to the Kerr-Newman form, the manifold above remains incomplete as $r \to r_\pm$, since there are geodesics for which the coordinate u becomes unbounded with a finite affine distance. However, Carter has obtained analytic extensions that are geodesically complete.
The equations of motion of a test particle with mass μ and charge ϵ are given by

$$\frac{D^2x^\mu}{D\tau^2} = \frac{\epsilon}{\mu}F^\mu_\sigma\left(\frac{Dx^\sigma}{D\tau}\right), \tag{100}$$

where $D/D\tau$ represents covariant differentiation with respect to proper time τ satisfying equation (10) and F is the electromagnetic field tensor. The Lagrangian in this case is

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \epsilon A_\mu\dot{x}^\nu, \tag{101}$$

where A is the vector potential

$$A = e\Sigma^{-2}r(dt - a\sin\theta d\phi), \tag{102}$$

satisfying $F = 2dA$. In order to derive the Hamiltonian, we introduce the genralized momenta as

$$\pi_\mu = p_\mu + \epsilon A_\mu = g_{\mu\nu}\dot{x}^\nu + \epsilon A_\mu, \tag{103}$$

and therefore we obtain

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}(\pi_\mu - \epsilon A_\mu)(\pi_\nu - \epsilon A_\nu). \tag{104}$$

The general form of the Hamilton-Jacobi equation is

$$-\frac{dS}{d\lambda} = \frac{1}{2}g^{\mu\nu}\left[\left(\frac{\partial S}{\partial x^\mu}\right) - \epsilon A_\mu\right]\left[\left(\frac{\partial S}{\partial x^\mu}\right) - \epsilon A_\nu\right]. \tag{105}$$

The Jacobi action S must then take the form

$$S = -\frac{1}{2}\mu^2\lambda - Et + l\phi + S_r(r) + S_\theta(\theta). \tag{106}$$

Inserting equation (106) in equation (105), we obtain

$$\begin{aligned} &p_\theta^2 + (aE\sin\theta - l\sin^{-1}\theta)^2 + a^2\mu^2\cos^2\theta = \mathcal{K}, \\ &\Delta p_r^2 - \Delta^{-1}[(r^2+a^2)E - al - \epsilon er]^2 + r^2 = -\mathcal{K}. \end{aligned} \tag{107}$$

Thus, we have that

$$\pi_t = p_t + \epsilon A_t = -E, \tag{108}$$

$$\pi_\phi = p_\phi + \epsilon A_\phi = l, \tag{109}$$

$$\pi_\theta = \frac{dS_\theta}{d\theta} = \sqrt{\Theta}, \tag{110}$$

$$\pi_r = \frac{dS_r}{dr} = \Delta^{-1}(\sqrt{R}). \tag{111}$$

The equations above provide a complete set of first integrals of the motion.The first two equations correspond to conservation of energy E and angular momentum about the symmetry axis ϕ_z. In addition, we have the rest mass μ as a constant of motion. The functions $\Theta(\theta)$, $P(r)$, and $R(r)$ are defined by

$$\Theta = \mathcal{C} - cos^2\theta[a^2(\mu^2 - E^2) + l^2\sin^{-2}\theta], \tag{112}$$

$$P = E(r^2 + a^2) - la + \epsilon er, \tag{113}$$

$$R = P^2 - \Delta(\mu^2r^2 + \mathcal{K}), \tag{114}$$

where $\mathcal{C} = \mathcal{K} - (l - aE)^2$. The final solution for the Jacobi action becomes

$$\begin{aligned} S = -\frac{1}{2}\mu^2\lambda - Eu + l\phi + \int^\theta(\sqrt{\Theta})\,d\theta \\ + \int^r \Delta^{-1}(\sqrt{R})\,dr, \end{aligned} \tag{115}$$

where the signs of the two square roots are independent of each other, and where the lower limits of integration are not specified, since only changes of the action are important. By successively setting to zero the partial derivatives with respect to the constants of motion, we obtain the following equations describing test-particle orbits:

$$\int^\theta\frac{d\theta}{\sqrt{\Theta}} = \int^r\frac{dr}{\sqrt{R}}, \tag{116}$$

$$\lambda = \int^\theta\frac{a^2\cos^2\theta}{\sqrt{\Theta}}d\theta + \int^r\frac{r^2}{\sqrt{R}}dr, \tag{117}$$

$$t = \int^\theta\frac{-a(aE\sin^2\theta - l)}{\sin^2\theta\sqrt{\Theta}}d\theta + \int^r\frac{(r^2+a^2)P}{\Delta\sqrt{R}}dr, \tag{118}$$

$$\phi = \int^\theta\frac{-(aE\sin^2\theta - l)}{\sin^2\theta\sqrt{\Theta}}d\theta + \int^r\frac{aP}{\Delta\sqrt{R}}dr. \tag{119}$$

Therefore we can now obtain the first order differential equations system

$$\Sigma^4\dot{\theta} = \sigma_\theta\sqrt{\Theta}, \tag{120}$$

$$\Sigma^4\dot{r} = \sigma_r\sqrt{R}, \tag{121}$$

$$\Sigma^4\dot{t} = -a(aE\sin^2\theta - l) + \frac{(r^2+a^2)P}{\Delta}, \tag{122}$$

$$\Sigma^4\dot{\phi} = -(aE - l\sin^{-2}\theta) + \frac{aP}{\Delta}. \tag{123}$$

The sign functions $\sigma_\theta = \pm 1$ and $\sigma_r = \pm 1$ correspond to outgoing/ingoing geodesics and are independent from each other but must be used consistently in all four equations. Then, the radial equation for the time-like particle moving along geodesics is given by setting to zero equation (30), where V_{eff} is

$$V_{\text{eff}} = -\frac{R^2}{2\Sigma^4} \tag{124}$$

The circular orbit of a particle can be found by imposing the conditions

$$V_{\text{eff}} = 0, \quad \frac{dV_{\text{eff}}}{dr} = 0. \tag{125}$$

Since we are interested in causal geodesics, we also need the condition $dt/d\lambda > 0$. As $r \to r_+$ for the timelike particle, this reduces to

$$E \geq \frac{al}{2(a^2 + r_+^2)} = l\Omega_+, \tag{126}$$

where we have used the angular velocity of the Kerr-Newman black hole

$$\Omega_+ = \frac{a}{(r_+^2 + a^2)} = \frac{a}{(2M^2 - e^2 + 2M\sqrt{M^2 - e^2 - a^2})}. \tag{127}$$

Equivalently, we need $l \leq l_c$, where $l_c \equiv \Omega_+^{-1} E$ is the critical angular momentum. Using equation (23) and taking $E = 1$, we have that the center of mass energy in the equatorial plane for two colliding charged particles with the same rest mass m_0 is

$$\begin{aligned}
\frac{1}{2m_0^2}\left(E_{\text{cm}}^{\text{KN}}\right)^2 = -\frac{1}{r^2\Delta}\Big\{ & -2r^4 + r^3[2 + e(\epsilon_1 + \epsilon_2)] - r^2(2a^2 + e^2 - l_1 l_2 + e^2\epsilon_1\epsilon_2) - 2a^2 r + 2r[a(l_1 + l_2) - l_1 l_2] \\
& + e^2(a - l_1)(a - l_2) + aer[a(\epsilon_1 + \epsilon_2) - (l_2\epsilon_1 + l_1\epsilon_2)] + \sqrt{(a^2 + r^2 - al_1 - er\epsilon_1)^2 - \Delta[r^2 + (a - l_1)]^2} \\
& \times \sqrt{(a^2 + r^2 - al_2 - er\epsilon_2)^2 - \Delta[r^2 + (a - l_2)^2]}\Big\}.
\end{aligned} \tag{128}$$

For two uncharged particles we then find that the limiting value of E_{cm} is

$$E_{\text{cm}}(r \to r_+) = 2m_0\sqrt{1 + \frac{(l_1 - l_2)^2}{(l_1 - l_c)(l_2 - l_c)}\frac{l_c}{4a}}. \tag{129}$$

We have that the effective potential for a particle with critical angular momentum l_c on the equatorial plane of an extremal black hole is

$$V_{\text{eff}} = -\frac{(r-1)^2(r - r_c)}{r^4}, \tag{130}$$

where $r_c = (1 - a^2)/2a^2$. For a particle falling freely from rest at infinity to reach the horizon, we need $V_{\text{eff}} \leq 0$ for any $r \geq 1$, or equivalently, $r_c \leq 1$. Therefore, the range for the black hole spin must be

$$\frac{1}{\sqrt{3}} \leq a \leq 1. \tag{131}$$

VI. CONCLUSIONS

In this paper, we have analysed the possibility that black holes can act as natural particle accelerators. By varying the parameters describing stationary black holes, we have derived the geodesics governing the motion of test particles and the center of mass energy for their collision. We have also examined particle orbits in order to have a deeper insight into the collision process. It is pointed out in [6] that the center of mass energy of the collision for two particles in the background of an extremal Kerr black hole can approach an arbitrarily high value if one of the particles has angular momentum $l = 2$. When extended to the Kerr-Newman case, an unlimited center of mass energy requires that the collison takes place at the horizon of an extremal black hole, one of the colliding particles has critical angular momentum and the spin of the black hole satisfies $1/\sqrt{3} \leq a \leq 1$.

Future work

We can consider dropping a small charge into a static black hole, (e.g. Schwarzschild), then its electromagnetic field may create a sufficiently large stress energy to destroy the event horizon and make it singular, or if the topology of the event horizon does not change (i.e. event horizon is not destroyed), then we find that the resultant metric is the Reissner-Nordstrøm metric. This is because a non-spherical charge distribution occuring as the body approaches the horizon must give rise to a spherical electric field, otherwise the horizon becomes either singular, destroyed, multiply connected or disconnected. If one of the latter posibilities were to occur, a static black hole would not be found in nature, since a charged particle would fall in and destroy it. We will show that, as a charge is lowered into a non-rotating black hole, the electric field remains well-behaved and all the multipole moments except the monopole moment vanish. The result will be a Reissner-Nordstrøm black hole [12].

By analogy, we will use the Kerr metric to derive an expression for the electromagnetic field of a point charge at rest on the symmetry axis near a rotating Kerr black hole. This is a generalisation of the previously obtained solution for the field of a point charge near a (non-rotating) Schwarzschild black hole. Unlike the Schwarzschild case, the charge is found to give rise to magnetic fields as seen by a stationary or locally nonrotating observer. The calculation of the field of a point charge at rest near a rotating Kerr black hole is made possible by deriving equations for components of the electromagnetic (and gravitational) perturbations of a Kerr black hole which can be solved by separation of variables (Teukolsky). Thus, the equations for the components of the electromagnetic field can be reduced to ordinary differential equations and handled by the same methods used for a point charge in the vicinity of Schwarzschild black holes. The total electromagnetic field can then be recovered from the full set of Maxwell's equations. Although the charge is at rest, we will find that here, magnetic fields are found to occur (unlike the Schwarzschild case). In particular, the electromagnetic field of the point charge has a nonvanishing magnetic dipole moment. In the limit as the charge approaches the horizon of a Kerr black hole, we find that the electromagnetic field becomes identical to that of the Kerr-Newman (i.e. charged Kerr) solution. Thus, we will conclude that lowering a charge into a Kerr black hole produces a Kerr-Newman black hole [13].

Appendix A: Central Forces in Newtonian Gravity

The three-dimensional motion of a particle in a central force potential is given by the equation

$$m\ddot{\mathbf{x}} = -\nabla V(r), \tag{A1}$$

where the potential depends only on $r = |x|$. We know that the angular momentum is

$$\mathbf{L} = m\mathbf{x} \times \dot{\mathbf{x}} \tag{A2}$$

and it is conserved in a central potential since

$$\frac{d\mathbf{L}}{dt} = m\mathbf{x} \times \ddot{\mathbf{x}} = -\mathbf{x} \times \nabla V = 0. \tag{A3}$$

From the conservation of angular momentum, we know that the position of a particle always lies in a plane perpendicular to $\mathbf{L}$, because $\mathbf{L} \cdot \mathbf{x} = 0$. Similarly, $\mathbf{L} \cdot \dot{\mathbf{x}} = 0$, hence the velocity of the particle also lies in the same plane. We use this to define a coordinate system in which the angular momentum points in the z direction and all motion takes place in the (x, y) plane. We then define the transformations

$$x = r\cos\theta, \tag{A4}$$

$$y = r\sin\theta, \tag{A5}$$

in order to express both the velocity and the acceleration of the particle in polar coordinates. The unit vectors pointing in the direction of increasing r and θ are, respectively,

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{A6}$$

For an angular displacement, these basis vectors will also change, therefore

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \hat{\mathbf{r}} \tag{A7}$$

If we consider the position of a particle to be given by $\mathbf{x} = r\hat{\mathbf{r}}$, then

$$\dot{\mathbf{x}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}. \tag{A8}$$

Differentiating again gives the expression for acceleration in polar coordinates,

$$\begin{aligned} \ddot{\mathbf{x}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{d\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\dot{\theta} && \text{(A9)} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}. && \text{(A10)} \end{aligned}$$

Now since $V = V(r)$, the equation of motion (A1) can be written in polar coordinates using

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} = -\frac{dV}{dr}\hat{\mathbf{r}}. \tag{A11}$$

Evaluating the $\hat{\boldsymbol{\theta}}$ component, we find that

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \tag{A12}$$

From this it follows that the angular momentum $l \equiv r^2\dot{\theta}$ is conserved. This can also be derived from equation (A2), by substituting the expressions for $\mathbf{x}$ and $\dot{\mathbf{x}}$, i.e. equations (A8) and (A9). It follows that

$$\mathbf{L} = mr^2\dot{\theta}\left(\hat{\mathbf{r}} \times \hat{\theta}\right). \tag{A13}$$

Since $\hat{\mathbf{r}}$ and $\hat{\theta}$ are orthogonal, the magnitude of the angular momentum is $|\mathbf{L}| = ml$, where l is the angular momentum per unit mass. The $\hat{\mathbf{r}}$ component of the angular momentum is

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{dV}{dr}. \tag{A14}$$

Using the fact that $l = r^2\dot{\theta}$ is conserved, we can write this as

$$m\ddot{r} = -\frac{dV}{dr} + \frac{ml^2}{r^3}. \tag{A15}$$

We now have that

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr}, \tag{A16}$$

where V_{eff} is called the effective potential and is given by

$$V_{\text{eff}}(r) = V(r) + \frac{ml^2}{2r^2}. \tag{A17}$$

The term $ml^2/2r^2$ is the angular momentum barrier/centrifugal barrier, because it stops the particle from getting too close to the origin. We now write the energy of the full three-dimensional problem as

$$E = \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + V(r) \tag{A18}$$

$$= \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r). \tag{A19}$$

It follows that the energy of the 3D system coincides with the energy of the effective 1D system we reduced the problem to. The efective potential is the real potential energy, together with a contribution from the angular kinetic energy.

1. Particle orbits

Starting with a central potential $V(r) = -k/r$, corresponding to an attractive inverse square law for $k > 0$, we have that the effective potential is

$$V_{\text{eff}} = -\frac{k}{r} + \frac{ml^2}{2r^2}. \tag{A20}$$

The minimum of the effective potential occurs at $r_0 = ml^2/k$ and takes the value $V_{\text{eff}}(r_0) = -k^2/2ml^2$. Note that the radial position of the minimum depends on the angular momentum l and also that the size of the orbit depends on the angular velocity $\dot{\theta}^2 = k/mr_0^3$. Then the possible forms of motion can be characterised by their energy E as follows:

- $E = E_{\text{min}} = -k^2/2ml^2$. Here the particle sits at the bottom of the well r_0 and stays there for all time. However, it also has angular velocity $\dot{\theta} = l/r_0^2$, therefore it has fixed radial position and it is moving in angular direction. This type of trajectory is a circular orbit about the origin (bound orbit).

- $E_{\text{min}} < E < 0$. Here the 1D system sits at the bottom of the well, oscillating between two points. Since $l \neq 0$, the particle also has angular velocity in the plane. This describes an orbit for which the radial distance r depends on time. For a central potential $V = -k/r$, this is an ellipse (bound orbit).

- $E > 0$. Now the particle can sit above the horizontal axis; it comes from infinity, reaches a minimum distance r and then rolls back out to infinity. For a central potential $V = -k/r$, the trajectory is a hyperbola (scattering orbit).

Appendix B: Numerical Integration

1. Basic methods

The simplest method for iterative integration of an ordinary differential equation is known as the Euler method, given by

$$y_{n+1} = y_n + hf(x_n, y_n), \tag{B1}$$

which advances a solution from x_n to $x_n + h$. Here, the derivative at the starting point of each interval is extrapolated to find the next function value. This formula is unsymmetrical, because it advances the solution through an interval h, but uses derivative information only at the beginning of the corresponding interval. This means that a step's error is only one power of h smaller than the correction term, i.e. $\mathcal{O}(h^2)$ added to equation (B1).
There are several reasons why Euler's method is not recommended for practical use: it is not very accurate compared to other methods run at equivalent stepsize and it is not generally stable.
We consider now the use of a step like equation (B1) to take a "trial" step to the midpoint of the interval. We then use the values of x and y at that midpoint to compute the "real" step across the whole interval. Mathematically, we have

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right), \\ y_{n+1} &= y_n + k_2 + \mathcal{O}(h^3). \end{aligned} \tag{B2}$$

The symmetrization cancels out the first-order error term, making this method second order (a method is called n^{th} order if its error term is $\mathcal{O}(h^{n+1})$).Second-order accuracy is obtained by using the initial derivative at each step to find a point halfway across the interval, then using the midpoint derivative across the full width of the interval. This is known as the second order Runge-Kutta/midpoint method. Its basic idea is that by adding a right combination of the RHS terms $f(x, y)$, we can eliminate the error terms order by order.

2. Fourth-order Runge-Kutta method

The (classical) fourth-order Runge-Kutta method is a numerical technique for solving an initial value problem of the form

$$\frac{dy}{dx} = f(x,y), \quad y(0) = y_0, \tag{B3}$$

which is based on

$$y_{n+1} = y_n + (a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4)/h, \tag{B4}$$

where knowing the value of $y = y_n$ at x_n can yield the value of y_{n+1} at x_{n+1}, and

$$h = x_{n+1} - x_n. \tag{B5}$$

Equating (B4) to the first 5 terms of the Taylor series, we obtain

$$\begin{aligned} y_{n+1} = y_n &+ \frac{dy}{dx}\bigg|(x_{n+1} - x_n) + \frac{1}{2!}\frac{d^2y}{dx^2}\bigg|(x_{n+1} - x_n)^2 \\ &+ \frac{1}{3!}\frac{d^3y}{dx^3}\bigg|(x_{n+1} - x_n)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\bigg|(x_{n+1} - x_n)^4 \end{aligned} \tag{B6}$$

Using equation (B5), we get

$$\begin{aligned} y_{n+1} = y_n &+ f(x_n, y_n)h + \frac{1}{2!}f'(x_n, y_n)h^2 \\ &+ \frac{1}{3!}f''(x_n, y_n)h^3 + \frac{1}{4!}f'''(x_n, y_n)h^4. \end{aligned} \tag{B7}$$

Based on the above, one of the usual solutions used is

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= hf(x_n + h, y_n + k_3), \\ y_{n+1} &= y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)/h + \mathcal{O}(h^5). \end{aligned} \tag{B8}$$

In each step, the derivative is evaluated four times: once at the initial point, twice at a trial midpoint and once at a trial endpoint in order to calculate the final value of the function. This method is superior to the midpoint method if at least twice as large a step is possible with equation (B8) for the same accuracy. The Runge-Kutta method treats every step in identical manner. This is mathematically well-defined, since any point along the trajectory of an ODE can serve as an initial point in the next iteration. Note that this formula is same as Simpson's 1/3 rule, if $f(x, y)$ were only a function of x. There are other versions of the 4^{th} order method such as

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right), \\ k_3 &= hf\left(x_n + \frac{2h}{3}, y_n - \frac{k_1}{3} + k_2\right), \\ k_4 &= hf(x_n + h, y_n + k_1 - k_2 + k_3), \\ y_{n+1} &= y_n + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4)/h + \mathcal{O}(h^5) \end{aligned} \tag{B9}$$

This formula is the same as Simpson's 3/8 rule, if $f(x, y)$ is only a function of x [14].

Appendix C: Effective potential and energy plots

[1] C. W. Misner, K. S. Thorne, J. A. Wheeler, *et al.*, *Gravitation* (Macmillan, 1973).
[2] J. M. Bardeen, Nature **226**, 64 (1970).
[3] R. P. Kerr, Physical review letters **11**, 237 (1963).
[4] S. A. Teukolsky, Classical and Quantum Gravity **32**, 124006 (2015).
[5] L. Chang-Qing, Chinese Physics Letters **30**, 100401 (2013).
[6] M. Banados, J. Silk, and S. M. West, Physical review letters **103**, 111102 (2009).
[7] J. B. Hartle, Gravity: An introduction to einstein's general relativity (2003).
[8] B. Carter, Physical Review **174**, 1559 (1968).
[9] K. S. Thorne, The Astrophysical Journal **191**, 507 (1974).
[10] S. Chandrasekhar, Clarendon Press, Oxford, UK **2**, 2 (1983).
[11] O. B. Zaslavskii, JETP Letters **92**, 571–574 (2010).
[12] J. M. Cohen, L. S. Kegeles, C. Vishveshwara, and R. M. Wald, Annals of Physics **82**, 597 (1974).
[13] J. M. Cohen and R. M. Wald, Journal of Mathematical Physics **12**, 1845 (1971).
[14] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical recipes in c (1988).

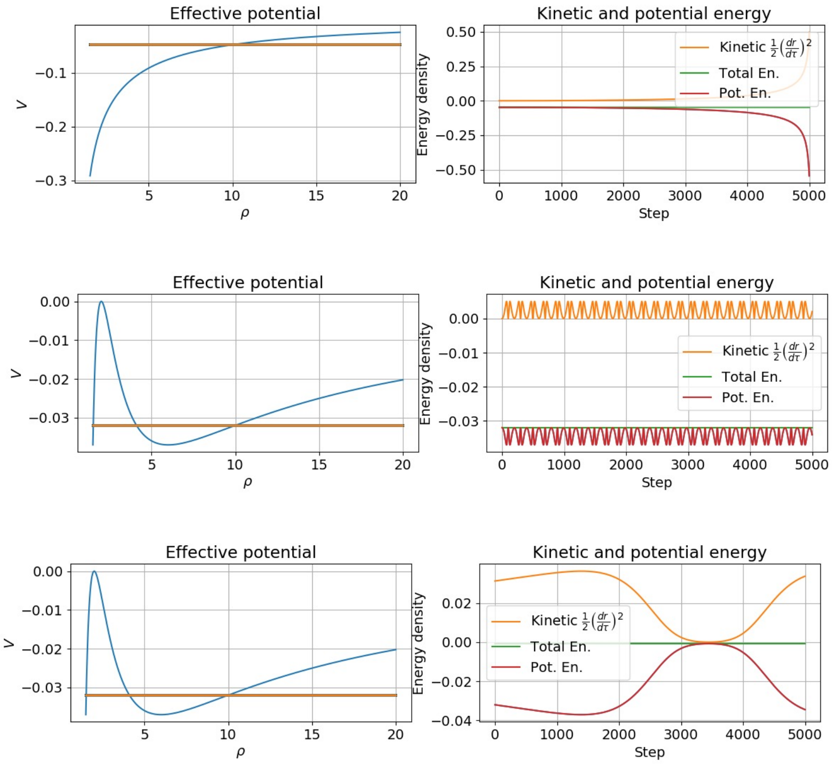

FIG. 5. Plots of effective potential and energies for Schwarzschild black hole

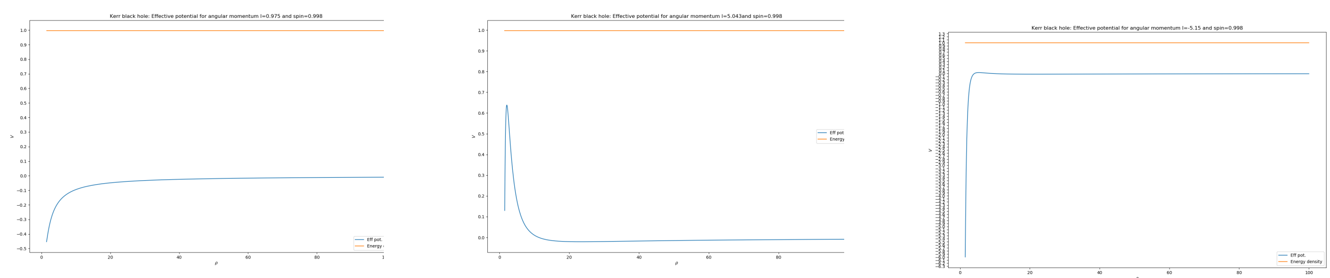

FIG. 6. Effective potentials for Kerr black hole

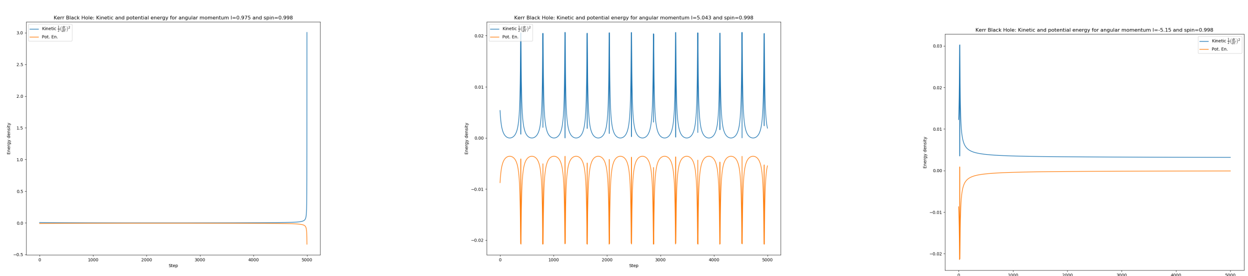

FIG. 7. Energies for Kerr black hole