Analysis of the Q Factor in an RLC Series Circuit

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***Abstract*—The Quality Factor of an RLC series circuit has been experimentally determined at low resistances. By comparing calculations that assume an ideal circuit with the ones that take into account the additional resistance of the circuit components, the aim of this report is to show that the internal resistance of circuit elements is not negligible for low resistor values.**

I. Introduction

A series RLC circuit is a simple setup consisting of a resistor, capacitor and inductor placed in series with a Signal Generator. Current in the circuit can be modelled as a harmonic oscillator, meaning the resonance phenomenon can occur within the system. This happens in the series circuit when the potential differences on the capacitive and inductive reactances are equal in magnitude, but have opposite signs, hence cancelling their effect and causing a peak in the rate of flow of current.

II. Theory

The voltage across an inductor L is given by:

$$V_L = L\frac{d}{dt}I(t) \tag{1}$$

where $I(t)$ is the current depending on time t.

The voltage across a capacitor C is:

$$V_C = \frac{Q(t)}{C} \tag{2}$$

where $Q(t)$ is the charge depending on time t.

The voltage across a resistor R is:

$$V_R = I(t)R \tag{3}$$

where $I(t)$ is the current depending on time t.

The voltage produced by the Signal Generator is also a function of time, written as $V_m \sin \omega t$, where V_m and ω are the amplitude and the frequency of the generated voltage, respectively.

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Combining the three equations above, together with the time varying voltage from the Signal Generator, we obtain Kirchoff's Law for a Series RLC Circuit:

$$L\frac{d}{dt}I(t) + \frac{Q(t)}{C} + I(t)R = V_m \sin \omega t \tag{4}$$

Taking the time derivative of Equation (4) and using the definition of current $I(t) = \frac{d}{dt}Q(t)$, we obtain a linear, second order, inhomogenous differential equation for the current $I(t)$:

$$L\frac{d^2}{dt^2}I(t) + \frac{I(t)}{C} + R\frac{dI(t)}{dt} = V_m\omega \cos \omega t \tag{5}$$

An RLC circuit is a system where the voltage is an oscillating quantity. This circuit behaves like a damped harmonic oscillator, or a pendulum with friction, hence we derive a similar solution from the differential equation above. In this case, it is easier to solve for the circuit using complex numbers.

Suppose the Signal Generator voltage is a sinusoidal function of the form $V(t) = V_m \cos \omega t$, where the amplitude V_m is a real number. Using Euler's formula, we have that $V(t) = \Re(V_m e^{i\omega t})$.

We want to solve for the current I(t) in the series circuit, which is the same everywhere due to conservation of charge. We expect the current to be of the same form as the voltage, hence we guess: $I(t) = I_m e^{i\omega t}$, where I_m is the amplitude of the current (possibly a complex number).

Substituting the above into Equation (1), we get:

$$V_L = L\frac{d}{dt}I(t) = i\omega L I_m e^{i\omega t} \tag{6}$$

Now, we know that:

$$Q(t) = \int I(t)dt = I_m \int e^{i\omega t}dt = \frac{I_m}{i\omega}e^{i\omega t}$$

Hence, substitution into Equation (2) yields:

$$V_C = \frac{I_m e^{i\omega t}}{i\omega C} \tag{7}$$

Finally, we rewrite Equation (3) as:

$$V_R = RI(t) = RI_m e^{i\omega t} \tag{8}$$

Putting together Equations (6), (7) and (8) in Kirchoff's Law, we have:

$$RI_me^{i\omega t} + i\omega LI_me^{i\omega t} + \frac{I_me^{i\omega t}}{i\omega C} = V_me^{i\omega t} \tag{9}$$

Since $e^{i\omega t} \neq 0$, by cancelling the exponential term, we have:

$$\left(R + i\omega L + \frac{1}{i\omega C}\right)I_m = V_m \tag{10}$$

This is equivalent to Ohm's Law, where R is the resistance, $X_L = i\omega L$ is the inductive reactance and $X_C = \frac{1}{i\omega C}$ is the capacitive reactance. The total impedance Z of the circuit is therefore:

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

This can also be written as:

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{11}$$

From Equation (11), it is easy to deduce that $Z = |Z|e^{i\phi}$, where $|Z|$ is the magnitude and ϕ is the phase of the impedance Z. Hence:

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{12}$$

$$\phi = \arctan\frac{(\omega L - \frac{1}{\omega C})}{R} \tag{13}$$

Using Ohm's Law, we find the current I_m to be:

$$I_m = \frac{V_m}{|Z|e^{i\phi}} = |I_m|e^{-i\phi} \tag{14}$$

where the magnitude of the current $|I_m|$ is given by:

$$|I_m| = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{15}$$

and the phase ϕ is given by Equation (13).

Equations (13) and (14) tell us that the current through the circuit is out of phase with the applied voltage by a value of ϕ. The current through an inductor "lags" the voltage by $\pi/2$ and the current through a capacitor "leads" the voltage by $\pi/2$. [1]

Resonance and Q Factor of a Circuit:

The series resonance is obtained when $I(t)$ has a maximum value. Equation (13) indicates that the amplitude of the current $I_0 = \frac{V_0}{Z}$ reaches a maximum when Z is at a minimum, i.e. when $X_L = X_C$, or $\omega L = \frac{1}{\omega C}$, leading to:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{16}$$

where ω_0 is called the resonant frequency.
At resonance, the impedance becomes $Z = R$, the amplitude of the current is $I_0 = \frac{V_0}{R}$ and the phase is $\phi = 0$.

The Quality Factor Q describes how underdamped an oscillator or resonator is and characterizes a resonator's bandwidth relative to its centre frequency. Another definition of Q is the ratio of the energy stored in the oscillating resonator to the energy dissipated per cycle by damping processes. By definition:

$$Q = \frac{\omega_0}{\Delta\omega} \tag{17}$$

where ω_0 is the resonant frequency obtained in Equation (16) and $\Delta\omega = \omega_+ - \omega_-$, the values of the angular driving frequency for which the power is equal to half the maximum power at resonance (equivalently, values for which the current amplitude is $\frac{1}{\sqrt{2}}I_0$). $\Delta\omega$ is called full width at half maximum (FWHM).

Knowing that $I_0 = \frac{V_0}{R}$ at resonance, the width can be calculated from Equation (15) as follows:

$$\frac{V_0}{\sqrt{2}R} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{18}$$

Rearranging terms, the equation becomes:

$$X_L = X_C + \alpha R \tag{19}$$

where $\alpha = \pm 1$.

Define $\tau_L = \frac{L}{R}$. Substituting the formulae for capacitive and inductive reactances and Equation (16) into Equation (19), we obtain a quadratic expression for the angular driving frequency:

$$\omega^2 - \omega\frac{\alpha}{\tau_L} - \omega_0^2 = 0 \tag{20}$$

The roots of this equation are:

$$\omega_{1,2} = \frac{\alpha}{2\tau_L} \pm \sqrt{\omega_0^2 + \frac{1}{(2\tau_L)^2}} \tag{21}$$

Considering only the positive root, we have:

$$\omega_\pm = \frac{\pm 1}{2\tau_L} + \sqrt{\omega_0^2 + \frac{1}{(2\tau_L)^2}} \tag{22}$$

Hence,

$$\Delta\omega = \omega_+ - \omega_- = \frac{1}{\tau_L} = \frac{R}{L} \tag{23}$$

$$Q = \frac{\omega_0}{\Delta\omega} = \omega_0\tau_L \tag{24}$$

Combining Equations (16) and (24), we derive the following equivalent formulae for Q: [2]

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} \tag{25}$$

Substituting these formulae into Equation (12), we also derive another expression for the magnitude of the impedance Z:

$$|Z| = R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \tag{26}$$

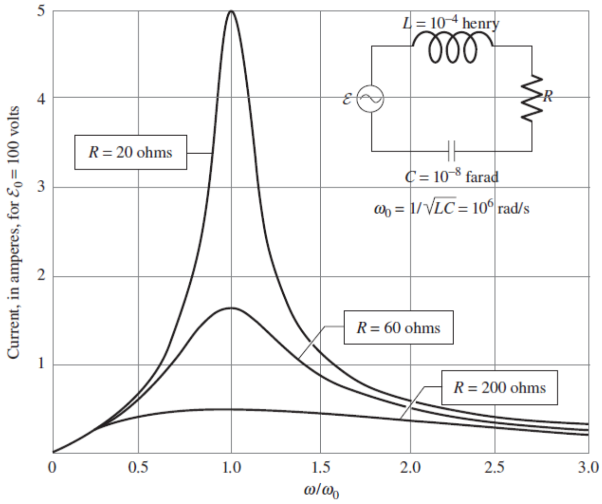


Fig. 1. Example of Resonance Curves for an RLC Circuit; the sharpness of the curve depends on the value of the resistor [3]

III. Experimental Procedure

In order to explore resonance, this experiment used a variable Signal Generator to supply an alternating current to a series RLC circuit. A digital oscilloscope was used to measure the Peak-to-Peak input voltage V_{out} (i.e. the voltage across the Signal Generator) and the Peak-to-Peak voltage V_R across the resistor. Additionally, the digital oscilloscope measured the phase difference between these two signals.

Using two different resistors with values $R_1 = 100\Omega$ and $R_2 = 200\Omega$, two different capacitors of unknown capacitances C_1 and C_2 and two different inductors of unknown inductances L_1 and L_2, the experimental data has been sampled on the following combinations: $R_1C_1L_1, R_2C_1L_1, R_1C_2L_1$ and $R_1C_1L_2$.

To construct the resonance curves needed to determine Q, we used an equivalent formula to Equation (17):

$$Q = \frac{f_0}{\Delta f}$$

where $\omega = 2\pi f$ and f is the input frequency that was varied on the function generator. The digital oscilloscope measured the corresponding phase difference between the two signals for each of the combinations above. The resonant frequency f_0 was the frequency at which the two signals were completely in phase.

IV. Analysis of the Data

For resonance, Equation (13) also implies that:

$$\phi = \arctan\frac{(V_L - V_C)}{R} = 0$$

Hence, for resonance we should have:

$$V_L = -V_C \textit{ and } V_{out} = V_R$$

However, in this experiment, we observe a difference between V_{out} and V_R at resonance. This is due to an additional resistance in the circuit that acts as a voltage divider, caused by the output impedance of the Signal Generator and the resistance of the inductor. [4] The voltage divider equation is:

$$V_R = V_{out}\frac{R}{Z} \tag{27}$$

At resonance, $X_L = X_C$ so $Z = R + R'$, where we consider R' to be the additional resistance. We also need to rewrite Equations (13) and (15) for the phase and current as:

$$\arctan\phi = \frac{X_L - X_C}{R + R'} \tag{28}$$

and

$$I = \frac{V_{out}}{\sqrt{(R + R')^2 + (X_L - X_C)^2}} \tag{29}$$

We now present two methods for calculating the Q Factor and the unknown capacitances and inductances. The first method is based on direct calculations involving the measured data and takes into account rms values of current and voltage, whereas the second is based on measuring the ratio of voltage transmitted to the incident voltage. The latter is more accurate, because it deals with the problem of internal impedance of the Signal Generator and simply scales the impedance of the circuit to the impedance of the resistor alone.

Direct Calculation from Set of Measurements

Because the oscilloscope measured the phase difference between signals, we substitute Equation (29) into Equation (28) and deduce the following formula:

$$I = \frac{V_{out}}{R + R'}\cos\phi$$

using the trigonometric identity $\frac{1}{\sqrt{1+\tan^2\phi}} = \cos\phi$.
At resonance:

$$I_{0,rms} = \frac{V_{out,rms}}{R+R'}$$

because the two signals are in phase and $\cos\phi = 1$ because $\phi = 0$. By calculating:

$$V_{R',rms} = \frac{1}{\sqrt{2}}(V_{out} - V_R); \quad V_{R,rms} = \frac{1}{\sqrt{2}}V_R$$

We deduce the current and the additional resistance:

$$I_{R,rms} = \frac{V_{R,rms}}{R}; \quad R' = \frac{V_{R',rms}}{I_{R,rms}}$$

In order to determine the full width at half maximum, we calculate $\frac{1}{\sqrt{2}}I_{R,rms}$ and look at the corresponding frequency f from the set of measurements. It follows that $\Delta f = 2(f_0 - f)$. The Q Factor is given by Equation (17).
Using Equation (25), we also determine the capacitances C_1, C_2 and the inductances L_1, L_2.
We summarise the findings in the following tables:

TABLE I
DIFFERENCES IN Q CAUSED BY THE ADDITIONAL R'

	$R_1C_1L_1$	$R_2C_1L_1$	$R_1C_2L_1$	$R_1C_1L_2$
$f_0(kHz)$	158.8	160	50.1	339.2
$R'(\Omega)$	18.5	17.33	18.5	7
$Q_{forR+R'}$	6.015	3.5	2.717	2.92
$Q_{forR'=0}$	8	4	3.125	3.4

TABLE II
CALCULATION OF CAPACITANCES AND INDUCTANCES

$R_1C_1L_1$	C_1	L_1
With R+R'	1.4 nF	0.7 mH
With R	1.25 nF	0.8 mH
$R_2C_1L_1$	C_1	L_1
With R+R'	1.3 nF	0.7 mH
With R	1.24 nF	0.8 mH
$R_1C_2L_1$	C_2	L_1
With R+R'	9.8 nF	1.00 mH
With R	10.0 nF	1.00 mH
$R_1C_1L_2$	C_1	L_2
With R+R'	4.3 nF	0.14 mH
With R	1.38 nF	0.16 mH

Statistical Determination using Lorentzian Curve

Because in this experiment we measured voltages, in order to analyse the resonance curves, it is easier to look at the plot of voltage versus frequency and expect to see the same shape with a magnitude difference of R. Since Ohm's Law holds:

$$\frac{V_R}{V_{out}} \cdot \frac{V_{out}}{R} = \frac{V_R}{R} = I_R = I_C = I_L$$

So the plot of voltage versus frequency is the same curve as the current through the circuit, but with the magnitude change of $\frac{V_{out}}{R}$.

From the voltage divider equation, we have that the voltage transfer function (defined as the ratio of voltage transmitted to the incident voltage) is:

$$\frac{V_R}{V_{out}} = \frac{R_{tot}}{Z} = \frac{1}{\frac{Z}{R_{tot}}}$$

where $R_{tot} = R + R'$ is the total resistance of the circuit.

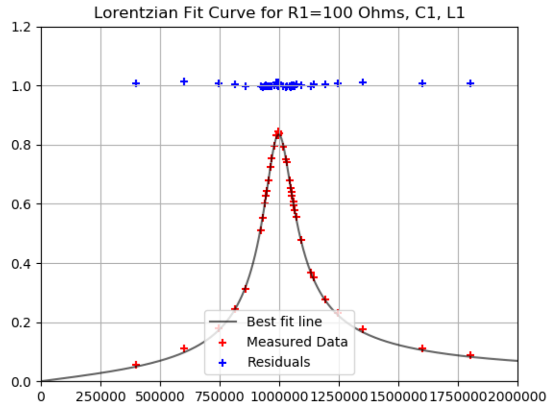
Because V_R is normalised by the source voltage V_{out}, it follows that the graph of $\frac{V_R}{V_{out}}$ is equivalent to the graph of the admittance Y (i.e. inverse of impedance), normalised by $R + R'$.
A widely used method for estimating the behaviour of second-order systems (such as the series RLC circuit) is the Lorentzian distribution. [5] Therefore, using Equation (26) as the expression for Z, the function to be optimised with a non-linear least squares fit is:

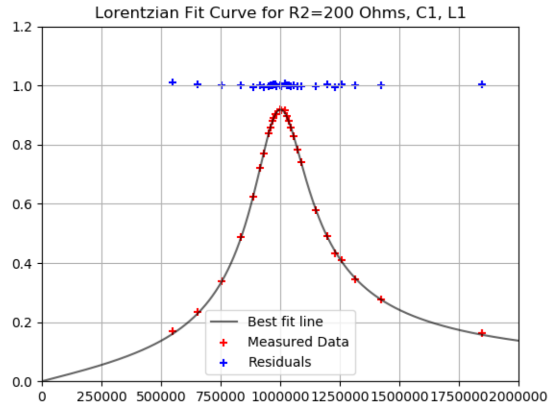
$$Y(\omega) = \frac{1}{\frac{R}{R_{tot}}\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

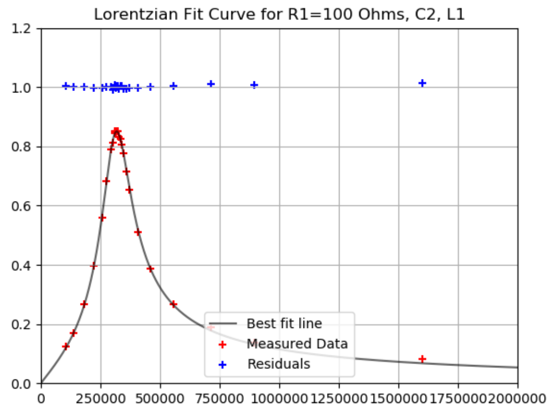
By fitting this artificial curve to the measurements, we obtain the values of the resistance in the circuit. Using Equation (25), we also deduce values for Q, L, and C. The results are summarised in the following table:

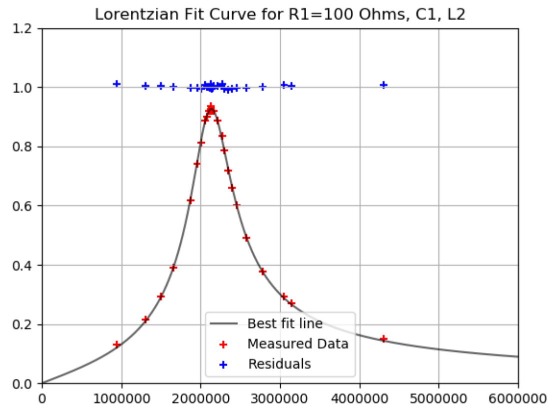
TABLE III
LORENTZIAN FIT RESULTS

	$R_1C_1L_1$	$R_2C_1L_1$	$R_1C_2L_1$	$R_1C_1L_2$
$R+R'$	119.91Ω	217.82Ω	117.59Ω	107.98Ω
L	1 mH	1 mH	1 mH	0.2 mH
C	1.04 nF	1.038 nF	10.27 nF	1.033 nF
Q_{fit}	8.012	4.416	2.6	4.19









V. Conclusion

Quality Factor determination from experimental data is not accurate for a low value of the resistor if it assumes that the internal resistances of the circuit components are negligible. When the value of the resistor is large, these internal resistances can indeed be neglected, because the ratio between R and $R + R'$ is very close to 1. However, if the resistor has a small value, we need to consider the effects of an additional resistance for an accurate determination of the Q Factor. Although values can be approximated directly from the data sets, for more precise results, the Lorentzian estimation of parameters is preferred.

References

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