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THE FIBONACCI SEQUENCE

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Introduction

The Fibonacci sequence arose from a mathematical puzzle in the *Liber Abaci* (1202), a book written by Leonardo Pisano - also known as Fibonacci. Apart from introducing the Arabic numbers to Europe, the book posed a situation on the breeding of rabbits, giving rise to the sequence's recursive formula. We will look at this in the following section along with finding a formula that gives the nth term. In section 2, we will focus on patterns within the Fibonacci sequence and in section 3 we will look at the sequences applications in geometry. Finally, in section 4, we see how the sequence arises in nature.

1. FINDING THE N^{th} TERM

1.1. **Recursive formula.**

Definition 1.1. $[PCCM,\ section\ 2,\ def.1]$ The Fibonacci sequence, (F_n), is a sequence of numbers satisfying the recursion relation:

$$F_n = F_{n-1} + F_{n-2}$$

Example 1.2. In the introduction, a situation proposed by Fibonacci about the breeding of rabbits was mentioned, and that we can use this to formulate a recursive formula for the sequence. The question asked was, starting in January with a pair of mature rabbits in a field (one male, one female), given the following conditions:

- Newborns take 1 month to mature and start mating
- Offspring consist of 1 male and 1 female each time
- No rabbits die

How many rabbits would there be at the end of December?

To begin, we formulate a table comprising of the number of mature rabbit pairs at the beginning of the given month (M_n), the number of immature rabbit pairs at the beginning of the given month (I_n) and the total number of rabbit pairs at the end of each given month (F_n):

Month	M_n	I_n	F_n
Jan	1	0	1
Feb	1	1	2
Mar	2	1	3
Apr	3	2	5
May	5	3	8
...	...	...	...
Dec	144	89	233

The first thing to notice from the table is the total number of pairs of rabbits at the end of each month (F_n) is the sum of the pairs of mature rabbits at the start of the month (M_n) and the pairs of immature rabbits at the start of the month (I_n). We can write this as $F_n = M_n + I_n$, with n denoting the month (n=1,2,3,...12). We can first see from the table that M_n equals F_n in the previous month, we can denote this F_{n-1}. We can also see that I_n equals M_n in the previous month, so we can denote this M_{n-1} or F_{n-2}. Using this new notation, our equation now becomes $F_n = F_{n-1} + F_{n-2}$, this is the equation from Definition 1.1. Based on a similar in [VHAM].

1.2. **Alternative forms.** The formula derived in the previous subsection (1.1) is not the only recursive formula we can use to calculate the Fibonacci sequence. For this report, we will be using four versions:

$$
\begin{aligned}
&[1] : F_n = F_{n-1} + F_{n-2}, for\ n \geq 3 \\
&[2] : F_{n+1} = F_n + F_{n-1}, for\ n \geq 2 \\
&[3] : F_{n+2} = F_{n+1} + F_n, for\ n \geq 1 \\
&[4] : F_{n+3} = F_{n+2} + F_{n+1}, for\ n \geq 0
\end{aligned}
$$

We can easily see that replacing n with n + 1 in [1] we would get [2], or replacing n with n + 2 in [1] we would get [3], etc. Based on results in [VHAM].

1.3. **Q-Matrix.** So far, we have looked at recursive formulas. We will now look to find a way to compute F_n without knowing the two previous terms.

Definition 1.3. $[Mat1]$

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

From the definition, we can see that

$$Q^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Note, however, that $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2$ So, we can write:

$$Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$$

Theorem 1.4. $[VHAM, Theorem\ II]$ *For $n \geq 1$, the n^{th} power of Q is given by:*

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

Proof.

$$P(1) : Q^1 = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Assume P(k) is true for some fixed $k \in \mathbb{N}$:

$$P(k) : Q^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$$

We can now show that P(k+1) holds true for all $k \in \mathbb{N}$:

$$P(k+1) : Q^{k+1} = Q^k Q = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} F_{k+1} + F_k & F_{k+1} \\ F_k + F_{k-1} & F_k \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}$$

Hence proof is complete by mathematical induction. [VHAM]

□

1.4. **Deriving Binet's Formula.**

Definition 1.5. [$MT182$, $def.28$] An eigenvector of A is a non-zero vector $\underline{\text{w}}$ such that A$\underline{\text{w}}$ = $\lambda\underline{\text{w}}$ for some field element λ. The value of λ corresponding to an eigenvector is called an eigenvalue.

Definition 1.6. [$MT182$, $def.29$] Let A be a 2x2 matrix with entries in field $\mathbb{F}$. The characteristic equation of A is the quadratic equation det(Q-λI) = 0. The characteristic polynomial of A is the quadratic polynomial in λ given by det(Q-λI).

Lemma 1.7. [$MT182$, *Lemma* 14] *If we have found non-proportional eigenvectors v and w, then the 2x2 matrix P which has these eigenvectors as its columns will be invertible. Moreover, if the two eigenvalues are λ_1 and λ_2, then*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Definition 1.8. [$MT182$]

Let there be an mxm matrix, D, which has values along the NW-SE diagonal, and zeros elsewhere:

$$[D]_{i,j} = \begin{cases} values, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Then D is said to be a diagonal matrix.

Proposition 1.9: $A^n = PD^nP^{-1}$ (proved by induction in [MT1820, chapter 2].)

Lemma 1.9. [Gue] *Let L be a linear operator on $\mathbb{R}^2$ given by matrix Q.*

For any vector $\begin{bmatrix} x \\ y \end{bmatrix}$*, we can see that:*

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x \end{bmatrix}$$

This can be extended to the Fibonacci sequence:

$$L\left(\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

Thus, we can calculate any nth Fibonacci number by applying the L operator n times on the vector:

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

resulting in:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Q^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To derive Binet's formula,we begin by finding the eigenvalues of the Q matrix. Using the 2x2 identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$det(Q - \lambda I_2) = det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$= det\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - 1 \cdot 1 = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0, [5] \text{ is the characteristic equation of Q.}$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

We say $\lambda_1 = \phi = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \varphi = \frac{1-\sqrt{5}}{2}$ [6]

For λ_1:

$$Q - \lambda_1 \cdot I_2 = \begin{bmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{bmatrix}$$

Note: $\lambda_1\lambda_2 = 1$ and $\lambda_1 + \lambda_2 = 1$

$$\begin{bmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0$$

Using row reduction:

$$\begin{bmatrix} \lambda_2 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \to \begin{bmatrix} 1 & -\lambda_1 \\ \lambda_2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\lambda_1 \\ 0 & 1 \end{bmatrix}$$

Solving simultaneous equations, the eigenvector for λ_1 is:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

Similarly, for λ_2, we find the eigenvector:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

As in Lemma 1.7:

$$P = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \text{ and } P^{-1} = \tfrac{1}{\sqrt{5}} \cdot \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

Multiplying Q by P and P^{-1}:

$$P^{-1}QP = \tfrac{1}{\sqrt{5}} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

By proposition 1.9: $Q^n = P^{-1} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} P$

Applying Q^n to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Q^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tfrac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \tfrac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_1^n - \lambda_2^n \end{bmatrix}$$

Yielding Binet's formula:

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

[*Gue*]

2. SEQUENTIAL PATTERNS

2.1. **Summation.**

In this section we will be finding and proving some identities which stem from the recursive formula. To start, here is a table of the first 10 Fibonacci numbers summed cumulatively, with n being the position in the sequence and F_n being the n^{th} term in the sequence:

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
Sum	1	2	4	7	12	20	33	54	88	143

First, notice the sequence of summations looks a lot like the Fibonacci sequence, but every term in this sequence is one less than a Fibonacci number. What happens when we sum the first 5 Fibonacci numbers?

$$1 + 1 + 2 + 3 + 5 = 12 = 13 - 1$$

Writing the numbers in terms of F_n we arrive at:

$$F_1 + F_2 + F_3 + F_4 + F_5 = F_7 - 1$$

This can be written more concisely:

$$\sum_{k=1}^{5} F_k = F_7 - 1$$

If we now look at the sum of the first 6 Fibonacci numbers, we find:

$$1 + 1 + ... + 8 = 20 = 21 - 1$$

which in terms of F_n gives:

$$F_1 + F_2 + ... + F_6 = F_8 - 1$$

or

$$\sum_{k=1}^{6} F_k = F_8 - 1$$

From these two cases, we see what looks like a general pattern emerge:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

Let us add F_{n+1} to both sides:

$$\sum_{k=1}^{n+1} F_k = F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

We arrive at recursive formula [3]:

$$F_{n+3} = F_{n+2} + F_{n+1}$$

We will prove this result by mathematical induction.

Proposition 2.1: $\sum_{k=1}^{n} F_k = F_{n+2} - 1$

Proof. Recall, $F_1 = 1, F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$, this is [3].

$$P(n) = F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$$

P(1) is true as $F_1 = F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$
Now assume P(k) is true for some fixed $k \in \mathbb{N}$:

$$P(k) : F_1 + F_2 + F_3 + ... + F_k = F_{k+2} - 1$$

We can now show that P(k+1) holds for all $k \in \mathbb{N}$:

$$P(k+1) : F_1+F_2+F_3+...+F_{k+1} = F_{(k+1)+2}-1 \Rightarrow (F_1+F_2+F_3+...+F_k)+F_{k+1} = F_{k+3}-1$$

Since P(k) is assumed true, we can replace $(F_1 + F_2 + F_3 + ... + F_k)$ with $(F_{k+2} - 1)$. We now have:

$(F_{k+2}-1)+F_{k+1} = F_{k+3}-1 \Rightarrow (F_{k+2}+F_{k+1})-1 = F_{k+3}-1 \Rightarrow F_{k+2}+F_{k+1} = F_{k+3}$

This is clearly true as if we let n = k + 1 in [3], we have $F_{k+3} = F_{k+2}+F_{k+1}$.
Hence, proof complete by mathematical induction.
Based on proof in [MBVH].

□

2.2. **Powers.** Now we will look at the sequence of squares. This is a table showing the summation of F_n^2, with n being the position in the sequence and F_n being the n^{th} term in the sequence:

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55
F_n^2	1	1	4	9	25	64	169	441	1156	3025
Sum	1	2	6	15	40	104	273	714	1870	4895

Summing the first few F_n^2 terms cumulatively gives:
$1 + 1 = 2 \Rightarrow F_1^2 + F_2^2 = 2 = 1 \times 2 = F_2F_3$
$1 + 1 + 4 = 6 \Rightarrow F_1^2 + F_2^2 + F_3^2 = 6 = 2 \times 3 = F_3F_4$
$1 + 1 + 4 + 9 = 15 \Rightarrow F_1^2 + F_2^2 + F_3^2 + F_4^2 = 15 = 3 \times 5 = F_4F_5$
$1 + 1 + 4 + 9 + 25 = 40 \Rightarrow F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 = 40 = 5 \times 8 = F_5F_6$
The general pattern that seems to emerge is $F_1^2+F_2^2+F_3^2+..+F_n^2 = F_nF_{n+1}$, or more concisely: $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$. We will prove this result by mathematical induction.

Proposition 2.2: $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$

Proof.

$$P(n) : F_1^2 + F_2^2 + F_3^2 + ... + F_n^2 = F_n F_{n+1}$$

P(1) true as $F_1 = 1$, $F_2 = 1$, $F_1^2 = 1$ thus $F_1 F_2 = 1 = F_1^2$
Now assume that P(k) is true for some fixed $k \in \mathbb{N}$:

$$P(k) : F_1^2 + F_2^2 + F_3^2 + ... + F_k^2 = F_k F_{k+1}$$

We can now show that P(k+1) holds for all $k \in \mathbb{N}$:

$$P(k+1) : F_1^2 + F_2^2 + F_3^2 + ... + F_{k+1}^2 = F_{k+1} F_{(k+1)+1} = F_{k+1} F_{k+2}$$

LHS $= F_1^2 + F_2^2 + F_3^2 + ... + F_{k+1}^2 = (F_1^2 + F_2^2 + F_3^2 + ... + F_k^2) + F_{k+1}^2 = (F_k F_{k+1}) + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1}(F_{k+2}) =$ RHS

Hence, proof complete by mathematical induction. Based on proof in [VHAM].

□

3. GEOMETRY

The Fibonacci sequence is characterised by the recursion relation:

$$F_{n+1} = F_n + F_{n-1} \text{ [2]}$$

This is a linear second order recursion relation that has the auxiliary equation: $\lambda^2 = \lambda + 1$ [7] or $\lambda^2 - \lambda - 1 = 0$ [5], known as the Fibonacci quadratic equation. From section 1.4, the roots of this equation are:

$$\lambda_1 = \frac{1+\sqrt{5}}{2} (\Phi, \text{ the golden ratio}) \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2} (1\text{-}\Phi) \text{ [6]}$$

We can gain more insight into the properties of the golden ratio by choosing a geometric approach. In fact, the concept of golden ratio division appeared more than 2400 years ago as evidenced in art and architecture. The Greek sculptor Phidias made the Parthenon statues in a way that seems to embody the golden ratio; Plato describes the five possible regular solids (the Platonic solids), some of which are related to the golden ratio.
Although its properties were mentioned in the works of the ancient Greeks, Pythagoras and Euclid, the connection between the golden mean and Fibonacci numbers was made by J. Kepler in Book 1 of *Harmonices Mundi*, where he discusses Euclidean geometry in general and the *proportio divina* in particular. In Chapter 15 of Book 3, he develops it from the Fibonacci series, tracing the gradual approach of Fibonacci's ratios to the mean.

Definition 3.1. [*Mat*2, *Euclid's "Elements"*] A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.

Using this definition, consider a line AC divided by a point B in such a way that the ratio of the lengths of the two segments is equal to the ratio of the length of the longer segment to the entire line. If length AB is arbitrarily set equal to 1 and the length of the total line is λ, the segment BC= $\lambda - 1$, the ratios of lengths can be expressed as: $\frac{\lambda}{1} = \frac{1}{\lambda - 1}$ [8] which can be rearranged to give [5], with roots, [6]. [Dun]

Construction 1: Golden rectangle

- Start with square ABCD of side length 1.
- Connect midpoint E of side AB with vertex C.
  Since $EB = \frac{1}{2}$ and $BC = 1$, using Pythagoras' Theorem:
  $EC = \sqrt{EB^2 + BC^2} = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$
- Draw arc CF using point E as the center and segment EC as a radius.
  Since $EC = EF = \frac{\sqrt{5}}{2}$, we have $AF = AE + EF = \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \Phi$.

- Construct a parallel line to AD passing through F and a parallel line to AF passing through D. Label the intersection point of the two lines G.
  The rectangle AFGD is called a golden rectangle, because its sides are in the ratio $1 : \Phi$. See below.

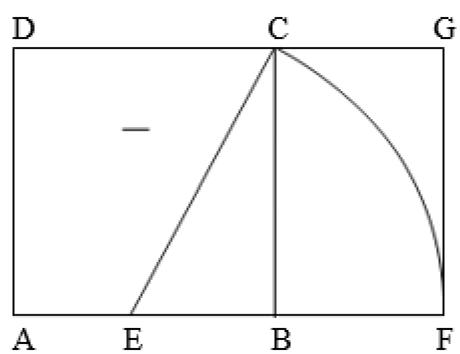

Figure 1

Construction 2: Golden isosceles triangle (i.e. an isosceles triangle which the side is in the ratio Φ to the base).

- Start with an isosceles triangle ABC, having the base angle twice the vertex angle (i.e.$\angle BAC = \angle BCA = 72°$ and $\angle ABC = 36°$).
- Bisect $\angle BCA$. It follows that $\angle BCD = \angle DCA = 36°$.

- Assume the base AC has length 1 and side BC has length x. Now we have $\angle BAC = \angle DAC = 72°$,$\angle DCA = \angle ABC = 36°$ and $\angle ADC = \angle BCA = 72°$, so triangles ACD and ABC are similar. Since they are similar, we have $\frac{AC}{AB} = \frac{CD}{BC} = \frac{AD}{AC} \Rightarrow \frac{1}{x} = \frac{x-1}{1}$,or $\frac{x}{1} = \frac{1}{x-1}$ [8]. So, ABC is a golden triangle, see below.

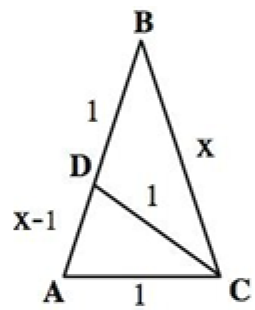

Figure 2

Construction 3: Pentagon

- Construct line AB and let C be a point on AB such that $\frac{AC}{CB} = \Phi$.
- Draw a circle center A and radius AB.
- Mark D on the circle such that AC=CD=BD.
  By **Construction 2**, triangle ABD is a golden isosceles triangle.
- Starting with this triangle, draw a circle passing through vertices A, B and D.

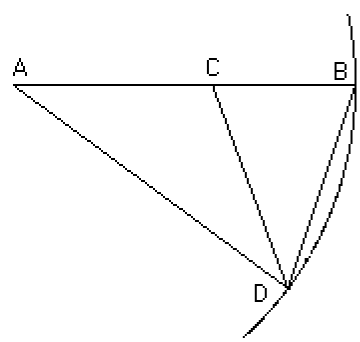

Figure 3.1

- Bisect $\angle ADB$ and let E be the intersection point of this line and the circle.
- Similarly, bisect $\angle ADB$ and let F be the intersection point of this line and the circle.
- Draw the pentagon AEBDF. (below)

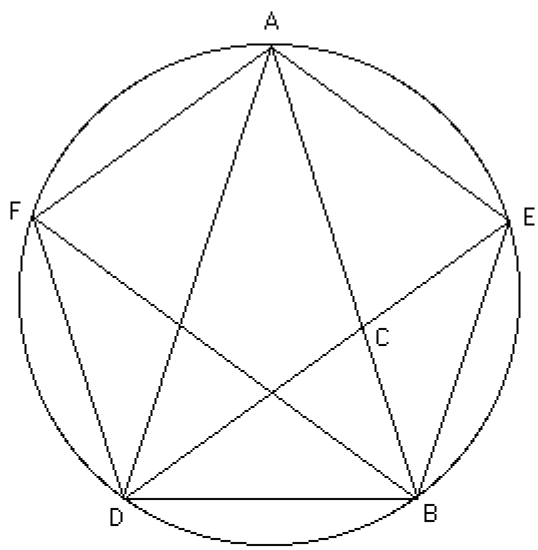

Figure 3.2

Construction 4: The logarithmic spiral (associated with a golden rectangle)

- Start with a golden rectangle ABCD, with length $AB = \Phi$ and length $AD = 1$ (see **Construction 1**).
  We have previously shown in [6] that Φ is a root of $\lambda^2 - \lambda - 1 = 0$ [5]. Therefore, we have $\Phi^2 - \Phi - 1 = 0 \Rightarrow \Phi^2 = \Phi + 1 \Rightarrow \Phi = 1 + \frac{1}{\Phi}$ [9]
- Using [9], we divide the golden rectangle ABCD into a square AFED whose sides have been reduced by a factor of $\frac{1}{\Phi}$. (Since $AB = \Phi, AF = 1$ and $\Phi = 1 + \frac{1}{\Phi} \Rightarrow FB = \frac{1}{\Phi}$).
  This process can be continued indefinitely, thus we obtain a sequence of nested golden rectangles which have a single point O in common (below).

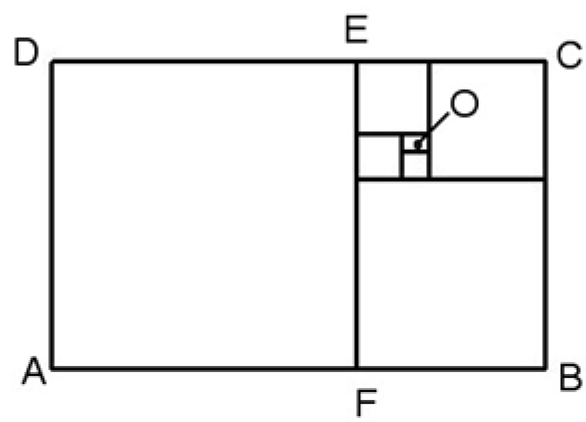

Figure 4.1

The successive points dividing the golden rectangle into squares lie on a logarithmic spiral, see below. [Mat2]

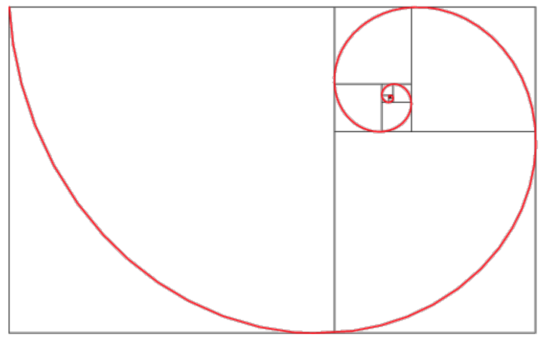

Figure 4.2

Construction 5: Spiral based on a golden isosceles triangle

We can also associate the logarithmic spiral with a golden isosceles triangle, similar to **construction 4**.

- Begin with a golden isosceles triangle.
- Bisecting the angle at vertex C, we will produce a similar isosceles triangle BCD with sides reduced by a factor of $\frac{1}{\Phi}$.
- Bisecting the angle at vertex D produces a similar isosceles triangle DEF.

  Continue this process to obtain a nested sequence of triangles which have the point O in common (below).

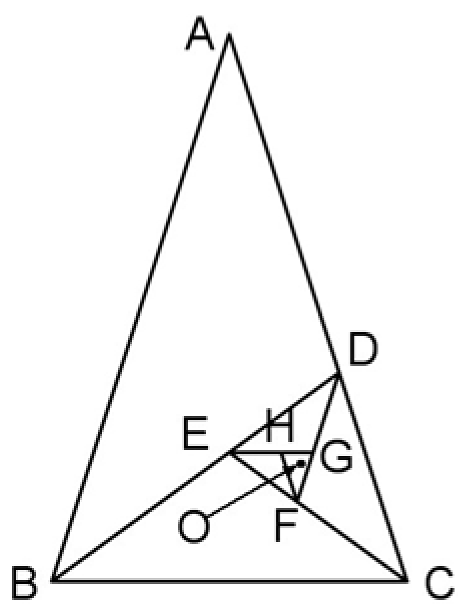

Figure 5.1

- By using polar coordinates with O as a pole and OA as the initial line, a logarithmic spiral can be drawn through the vertices A,B,C,D,E,F,G,H.
- Let the length of OA be equal to c. It follows that these coordinates are $A = (c, 0)$, $B = (c\Phi^{-1}, \frac{3\pi}{5})$, $C = (c\Phi^{-2}, 2 * \frac{3\pi}{5})$, $D = (c\Phi^{-3}, 3 * \frac{3\pi}{5})$, etc.
  In general, the transforms of A are:
  $$r = c\Phi^{-\theta}, \theta = t\frac{3\pi}{5}, t = 0, 1, 2, 3, ...$$
- Letting t be any real number, all the points lie on the logarithmic spiral:
  $$r = \Phi^{\frac{-5\theta}{3\pi}}$$
  This spiral passes through each vertex of the nested triangles (below).

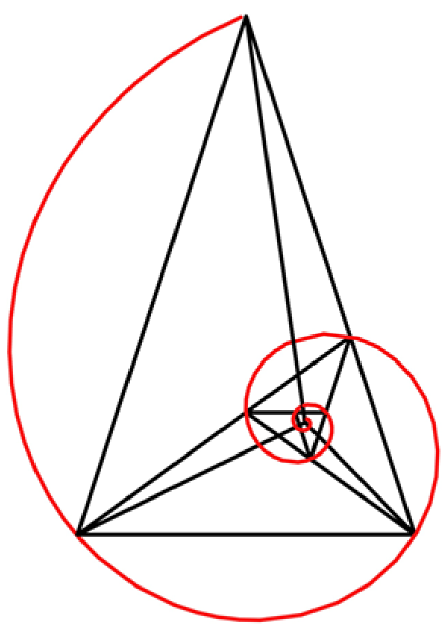

Figure 5.2

4. NATURE

The Fibonacci numbers and the Golden Ratio frequently occur in nature's geometry, playing a fundamental role in the growth of many biological systems [KHH]. In most plants, the spiral arrangements of leaves on a stem (phyllotaxis), number of petals and spirals in flower heads represent successive numbers in the Fibonacci sequence. Phyllotactic patterns have been described for centuries, but the mechanisms that initiate these patterns remain undefined [Kla]. This is related to the mathematical properties of the Golden Ratio in a few different ways: symmetry, optimal spacing and Fibonacci growth spirals.

Symmetry:

Biological systems exhibit a wide variety of symmetry characteristics, having features with dimensions that are related to the golden ratio. The most common types are two-dimensional fivefold symmetry and three-dimensional icosahedral symmetry. The fivefold symmetry can be observed in flowering plants (appendix Fig. 6, Fig. 7) or animals such as sea urchins and starfish (appendix Fig. 8). The icosahedral symmetry can be observed in some types of viruses, whose proliferations are related to the icosahedron (appendix Fig. 9). [Dun]

Optimal spacing:

By cutting a circle into two segments, where the ratio of the arcs is equal to the golden ratio, an angle of 137.5 degrees is produced. In some plants and flowers, this is the angle at which each consecutive leaf along a stem is placed apart from the previous leaf. This allows sunlight to reach the optimum surface area of the plant for it to photosynthesise efficiently (appendix Fig. 10).

Fibonacci growth spirals:

The Fibonacci spirals can be directly recognised among the shells of snails and some other mollusks. In other cases, they describe the growth angles of discrete components which yield a spiral structure: for example, in pinecones and sunflower seeds.

Pine cones (appendix Fig 11) are found both in the "dexter" form, in which most spirals run clockwise, and in the "sinister" form, in which anticlockwise spirals predominate. The growth angles of the scales in pinecones are a rational approximation of the golden ratio.

Fibonacci spirals can also be seen in the arrangement of sunflower seeds, packed very tightly to maximise possibility of pollination. The numbers of clockwise and anticlockwise spirals are successive Fibonacci numbers, arranged in two sets of spiral rows, one curving to the left and the other to the right. Thus, if 34 seed rows curve clockwise, there will be either 21 or 55 anticlockwise spirals on the sunflower head. [Kla]

Arguably, the most striking example of spiral growth is the shell of *Nautilus pompilius*. As the animal grows, it constructs larger and larger chambers in the form of a spiral, sealing off the smaller, unused chambers (appendix Fig 12). The relative volumes of two of these consecutive chambers is related to the golden ratio.

CONCLUSION

The Fibonacci numbers and the Golden Ratio are two fundamental concepts closely associated with the notion of harmonious proportions, which is expressed throughout history in different areas of science and art. Ever since

Kepler began to prove what Pythagoras had conjectured 2160 years before him - that the universe is determined by identical norms of proportionality - scientists have begun discovering that the same holds true for other components of growing nature; the principle of harmonic ratios also defines the structure of atoms, molecules, and, ultimately of all living organisms, and we can use mathematics to understand it.

APPENDIX

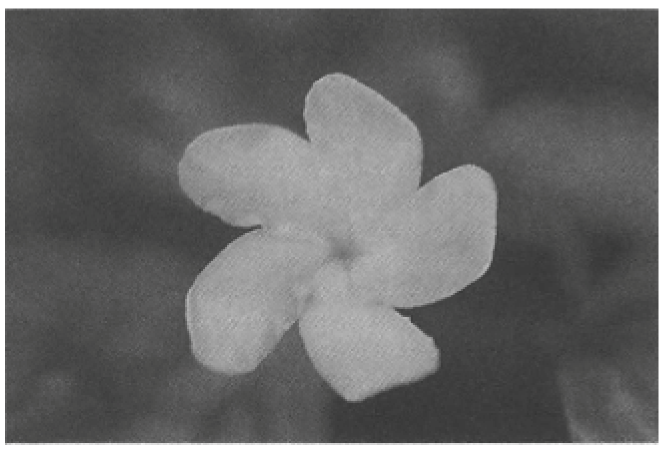

Figure 6: Rotational symmetry only [Dun, Fig 13.1 a]

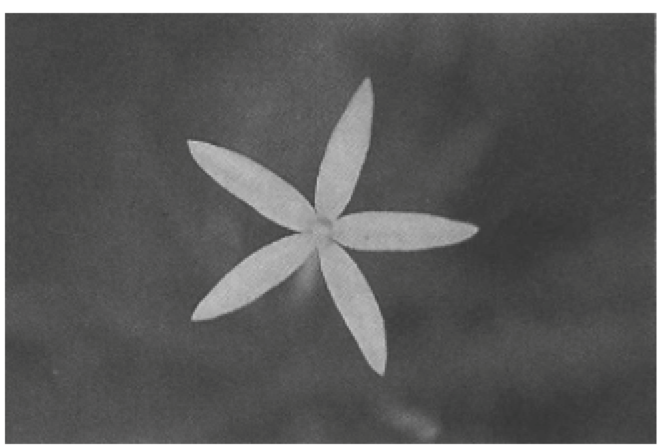

Figure 7: Rotational and inversion symmetry [Dun, Fig 13.1 b]

Figure 8: A starfish exhibiting fivefold symmetry [Dun, Fig 13.3]

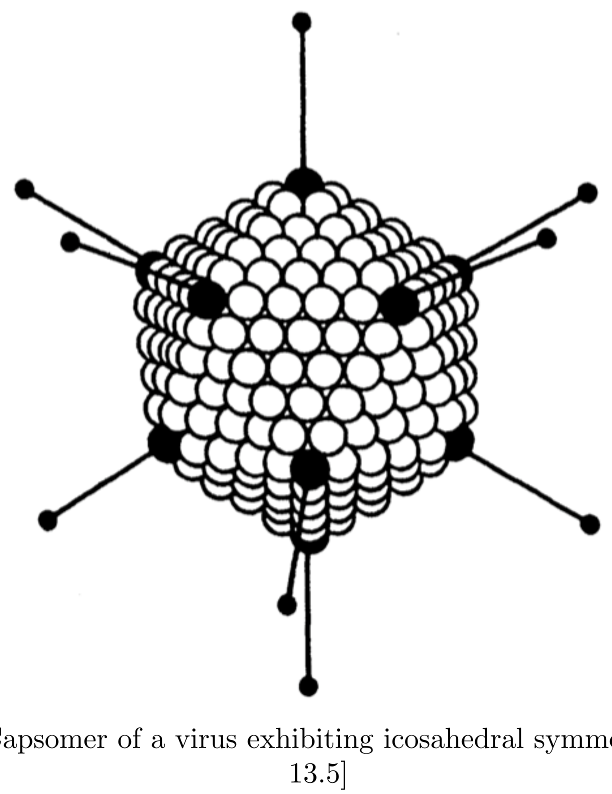

Figure 9: Capsomer of a virus exhibiting icosahedral symmetry [Dun,Fig 13.5]

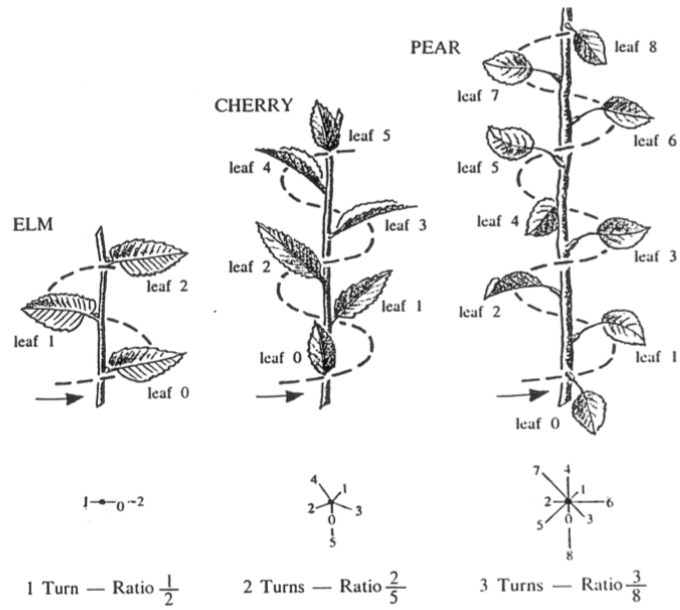

Figure 10: Arrangement of leaves of a stem [VHAM, Fig 21]

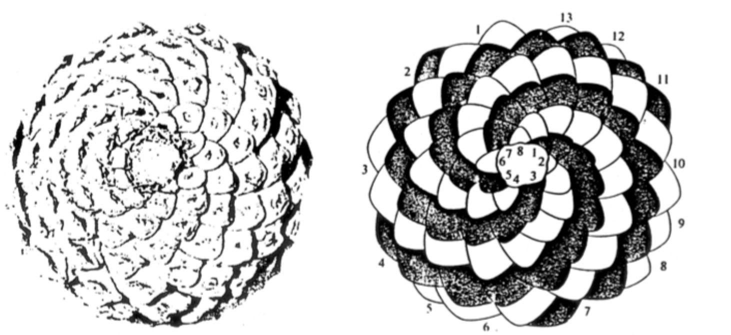

Figure 11: Spirals seen on pine cones [VHAM, Fig 22]

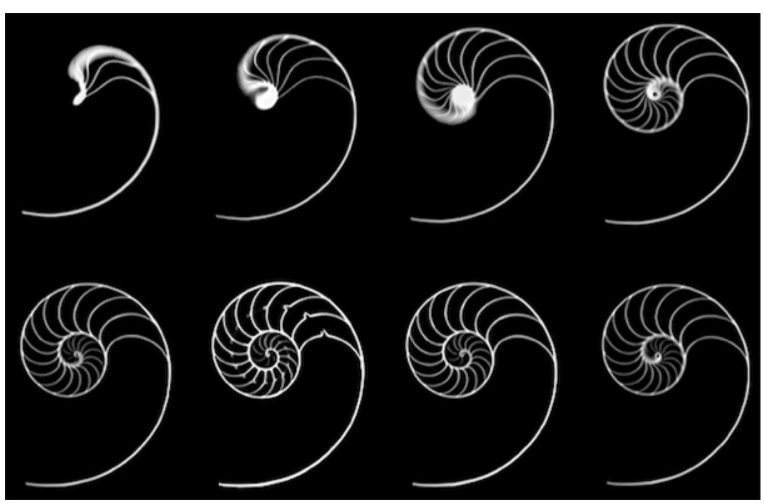

Figure 12: Section cut of a nautilus shell [Wik, Florian Elias Rieser]

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